"Über eine mögliche geometrische Deutung der relativistischen Quantentheorie," Zeit. f. Phys. **54** (1929), 798-802.

On a possible geometric interpretation of relativistic quantum theory

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With the help of Dirac's four-rowed matrices γ_ν, the linear expression:

$$ds = \sum_{\nu=1}^{4} \gamma_\nu dx_\nu$$

is defined, which is regarded as a geometric operator for the line element *ds*. This conception makes possible a geometric interpretation of the Dirac equations. The reasoning can be carried over to the general theory of relativity, where the connection between the variable matrices γ_ν that Tetrode introduced with Einstein's "vierbeins" $h_{\nu\alpha}$ comes to light.

In the last two decades, repeated attempts have been made to express the association of physical laws with geometric concepts. In the realm of gravitation and classical mechanics, these attempts have found their culmination in Einstein's general theory of relativity. Up to now, however, quantum theory has found no place in geometric structures; research in this direction (e.g., Klein, Fock) has produced no result. When Dirac presented his equations for the electron, the path to further research in this direction first seemed to be created.

1. The Dirac theory is based upon a linear Hamiltonian operator that is a component of the impulse, which enters into these quantities in place of a quadratic Hamiltonian function. This transition from a quadratic expression to a linear one must have a counterpart in the realm of purely geometric concepts.

A further essential path to relativistic quantum theory is the knowledge that the four continuously varying world coordinates *x*, *y*, *z*, *t* do not allow one to arrive at the formulation of the simplest mechanical problem, and that one must introduce a fifth discontinuous quantity ζ (the index of the wave function). This fact must likewise be capable of a geometric interpretation.

The linearity of relativistic Hamiltonian operators is closely related to the idea that one must base *ds* upon a linear expression:

$$ds = \sum_{\nu} \gamma_\nu dx_\nu \; , \tag{1}$$

instead of the basic quadratic metric form, where γ_ν are Dirac's four-rowed matrices. How is this expression for ds – which is not a number, but a four-rowed matrix – to be interpreted geometrically? A simple (if also entirely naïve) interpretation is the following one: We suggest that we are dealing with the representation of the concept of distance in a space with four continuous dimensions and one discontinuous one. The position of a point in such a space is established by the given of four coordinates x, y, z, t and the whole number ζ (ζ = 1, 2, 3, 4). Here, we are concerned with two points x_1, y_1, z_1, t_1 and x_2, y_2, z_2, t_2. There are 16 combinations of the whole numbers ζ_1, ζ_2, and correspondingly, 16 possible values for the distance between two points, whose continuously-varying coordinates differ by infinitely small quantities. The composition of the 16 values into one matrix equation then yields formula (1). Note that one can choose the γ_ν in such a way that all 16 values happen to be real.

The form of the Dirac equations, or also formula (1), seems to show that the matrices γ_ν have a purely geometric meaning. We consider an expression:

$$u = \sum_\nu \gamma_\nu u_\nu \tag{2}$$

that is constructed from the components of an arbitrary vector.

If we regard the components u_ν as numbers then we have a geometric operator before us here whose eigenvalue, up to sign, coincides with the absolute value of the vector. This conception is a purely geometric one – i.e., it is not quantum-theoretic. If, by contrast, we regard the u_ν as quantum-theoretic operators then (2) is the quantum-theoretic operator for the physical quantity whose classical image is the vector with the components u_ν.

As an example of the presence of such "quantum-geometric" structures in the Dirac theory, one can mention the impulse-momentum, in addition to the Hamiltonian operator.

We now again consider the expression (1) for ds. If we divide it by $d\tau$ – viz., the differential of proper time – then we obtain a "linear-geometric" expression for the vector of four-velocity:

$$\frac{ds}{d\tau} = \sum_\nu \gamma_\nu v_\nu \,. \tag{3}$$

If we regard the v_ν as numbers here then the eigenvalues of this "geometric" operator are equal to $\pm c$. If we now go over from "linear geometry" to "quantum geometry" then we would like to regard the components v_ν as quantum-theoretic operators:

$$v_\kappa = \frac{1}{m}\left(p_\kappa + \frac{e}{c}A_\kappa\right) = \frac{1}{m}\left(\frac{h}{2\pi i}\frac{\partial}{\partial x_\kappa} + \frac{e}{c}A_\kappa\right).$$

The "quantum-theoretic" operator (3) then becomes:

$$\frac{ds}{d\tau} = \frac{1}{m}\sum_{\kappa=1}^{4} \gamma_\kappa \left(p_\kappa + \frac{e}{c}A_\kappa\right). \tag{4}$$

The corresponding classical quantity – viz., the magnitude of the four-velocity – is equal to c. If we correspondingly set the operator (4) equal to c:

$$\frac{1}{m}\sum_{\kappa=1}^{4}\gamma_\kappa\left(p_\kappa+\frac{e}{c}A_\kappa\right)\psi = c\ \psi \tag{5}$$

then we obtain the Dirac wave equation.

This transition to the quantum-theoretic equation (5) is a purely formal one. One gets a physically intuitive interpretation for the wave equation when one introduces the statistical concept of expectation value. In fact, the integral:

$$\int_{t_1}^{t_2} dt\iiint\overline{\psi}\left\{\frac{1}{m}\sum_{\kappa=1}^{4}\gamma_\kappa\left(p_\kappa+\frac{e}{c}A_\kappa\right)-c\right\}\psi\,dx\,dy\,dz\,dt\,, \tag{6}$$

whose variation provides the Dirac equations, up to a factor, according to Darwin (*), is equal to the time mean of the expectation value of the quantity $ds\ /\ d\tau - c$, whose classical value is equal to zero.

The argument that has led us to the interpretation of the Dirac wave equation came about in three steps: The first step was the introduction of the purely geometric operators γ_ν and the definition of the linear line element ds. The second step consisted in the fact that we replaced the components of the velocity with the associated quantum-mechanical operators. The last step consisted in the use of the mathematical concept of expectation value. Perhaps, similar reasoning is also justified in the formulation of complicated problems – e.g., the two-body problem – when starting from a heuristic picture.

Our considerations can also be carried over to the general theory of relativity. Here, we would like to restrict ourselves to the "first step." The "linear-geometric" ds of formula (1) also remains valid here, if we understand the γ_ν to mean the covariant matrices that Tetrode (**) introduced. It is noteworthy that there is a connection between the γ_ν with the "vierbeins" that were introduced by Ricci (***), and which were denoted by $h_{\nu a}$ by Einstein (****).

In fact, we can set:

$$\gamma_\nu = \sum_a h_{\nu a}\gamma_a^0\,, \tag{7}$$

where γ_a^0 denote the constant Dirac matrices. If we then remain in the realm of geometry and regard the dx as commutable numbers then the substitution of (7) in (1) and squaring yields:

(*) C. Darwin, "The wave equations of the electron," Proc. Roy. Soc. (A) **118** (1928), 654.

(**) H. Tetrode, "Allgemein-relativistische Quantentheorie des Elektrons," Zeit. f. Phys. **50** (1928), 336.

(***) Ricci and Levi-Civita, "Méthodes de calcul différentiel absolu et leurs applications," Math. Ann. **54** (1900).

(****) A. Einstein, "Riemann-Geometrie mit Aufrechterhaltung des Begriffes des Fernparallelismus," Berl. Ber. **17** (1928), 217.

$$ds^2 = \sum_a \left(\sum_\nu h_{\nu a} dx_\nu \right)^2 , \tag{8}$$

in agreement with Einstein's formula. Analogously, one expresses γ^ν in terms of h_a^ν by:

$$\gamma^\nu = \sum_a h_a^\nu \gamma_a^0 . \tag{9}$$

Conversely, $h_{\nu a}$ or h_a^ν can be expressed in terms of γ_ν (γ^ν, resp.) using (7) or (9):

$$h_{\nu a} = \tfrac{1}{2}(\gamma_\nu \gamma_a^0 + \gamma_a^0 \gamma_\nu) , \tag{10}$$

$$h_a^\nu = \tfrac{1}{2}(\gamma^\nu \gamma_a^0 + \gamma_a^0 \gamma^\nu) . \tag{11}$$

The rotational invariance that was demanded by Einstein corresponds to the fact that the γ_a^0 are established only up to a canonical transformation.

The close relationship between the γ_ν and the $h_{\nu a}$ seems to prove that a bridge exists between two domains that were distantly separated up to now – viz., gravitation and quantum theory – in the form of "linear geometry."

Added in proof. Einstein's new theory of gravitation is closely related to the known theory of orthogonal congruences of curves that goes back to Ricci. In order to make the comparison between the two theories easier, the notation of Ricci and Levi-Civita, as it was given in the well-known textbook "Der absolute Differentialkalkul" of Levi-Civita, might be combined with that of Einstein.

Levi-Civita: $p_{i/\nu}$, p_i^ν , γ_{ikl} ,

Einstein: $h_{\nu i}$, h_i^ν , $\Delta_{\mu\nu}^\rho - \Gamma_{\mu\nu}^\rho$.

The following relations exist between the Levi-Civita symbols and those of Einstein:

$$p_{i/\nu} = h_{\nu i} , \qquad p_i^\nu = h_i^\nu ;$$

$$\Delta_{\mu\nu}^\rho - \Gamma_{\mu\nu}^\rho = \sum_{ikl} \gamma_{ikl} p_i^\rho p_{k/\mu} p_{l/\nu} = \sum_i p_i^\rho \nabla_\nu p_{i/\mu} ,$$

$$\Lambda_{\mu\nu}^\rho = \Delta_{\mu\nu}^\rho - \Delta_{\nu\mu}^\rho = \sum_{ikl} (\gamma_{ikl} - \gamma_{ilk}) p_i^\rho p_{k/\mu} p_{l/\nu} ,$$

$$\varphi_\mu = \sum_\nu \Lambda_{\mu\nu}^\nu = \sum_{ik} \gamma_{iki} p_{k/\mu} .$$

The tensor $\Delta^{\rho}_{\mu\nu} - \Gamma^{\rho}_{\mu\nu}$ then possesses the invariants γ_{ikl}, the tensor $\Lambda^{\rho}_{\mu\nu}$ has the invariants $\gamma_{ikl} - \gamma_{ilk}$, and the vector φ_{μ} has the invariants $\sum_i \gamma_{iki}$.