“Über eine kovariante Gestalt der Differentialgleichungen der Bahnkurven allgemeiner mechanischer Systeme,” Math. Zeit. **21** (1924), 154-159.

On a covariant form for the differential equations of the trajectories of general mechanical systems.

By

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The **Lagrange** equations of motion:

(1) $$\frac{d}{dt}\frac{\partial K}{\partial \dot{q}_i} - \frac{\partial K}{\partial q_i} = Q_i \qquad (i = 1, 2, \ldots, n)$$

define the n coordinates $q_1, q_2, \ldots, q_n$ of a mechanical system with n degrees of freedom as functions of time t. The Q_i in them are the covariant force components, which are given functions of the q_i. K, namely, the kinetic energy, is a homogeneous quadratic function of the contravariant velocity components $\dot{q}^i$:

(2) $$\dot{q}^i = \frac{dq_i}{dt}$$

with coefficients that depend upon only the q_i .

For many investigations, it is desirable to eliminate time t from eq. (1) and derive differential equations by which the q_i are coupled to each other directly. If we symbolize the motion of the system by a representative point in n-dimensional space with the coordinates $q_1, \ldots, q_n$ then the solutions of the differential equations thus-obtained will be the equations of the trajectories of the representative point. We would also like to call them the trajectories of our system, in a general sense. If we would like to give the representation a covariant form then it would be best to regard the q_i as functions of an arbitrary parameter u. We could then treat time on an equal footing with the coordinates and look for differential equations that would allow us to determine the $q_1, \ldots, q_n$, and t as functions of u. The solutions of those equations would then determine, on the one hand, the $q_1, \ldots, q_n$ as functions of u – viz., the *trajectories* of the system – and on the other hand, t as a function of u, namely, the temporal course of motion along the trajectories. The differential equations would take an especially simple and intuitive form if we were to introduce the kinetic energy K, and then seek to derive a system of simultaneous differential equations for the $n + 1$ quantities $q_1, \ldots, q_n, K$ as functions of the parameter u from eq. (1).

We denote the derivatives with respect to the parameter u with a prime and those with respect to t by a dot, so:

$$q'^i = \frac{dq_i}{du}, \qquad t' = \frac{dt}{du}, \qquad \dot{u} = \frac{du}{dt}, \qquad \text{etc.} \tag{3}$$

Moreover, let k be what K will become when one replaces the $\dot{q}^i$ with the q'^i, so:

$$\dot{q}^i = q'^i \dot{u}, \qquad q'^i = \dot{q}^i t', \qquad k = K t'^2. \tag{4}$$

Furthermore, let K_i be the covariant impulse components, and analogously:

$$k_i = \frac{\partial k}{\partial q'^i}, \qquad \text{so} \qquad k_i = t' K_i. \tag{5}$$

The **Lagrange** expressions that relative to the independent variable t shall be denoted by Λ_i, and the ones that relative to u shall be denoted by λ_i, so:

$$\Lambda_i(K) \equiv \frac{d}{dt}\frac{\partial K}{\partial \dot{q}^i} - \frac{\partial K}{\partial q_i}, \qquad \lambda_i(k) \equiv \frac{d}{dt}\frac{\partial k}{\partial q'^i} - \frac{\partial k}{\partial q_i}. \tag{6}$$

In order to eliminate t from eq. (1), we start from the relation $\sqrt{k} = t'\sqrt{K}$ that follows from (4), which reads:

$$\int \sqrt{K}\, dt = \int_{u_0}^{u_1} \sqrt{K}\, t'\, dt = \int_{u_0}^{u_1} \sqrt{k}\, du \tag{7}$$

in integrated form. The integral in that is extended between two fixed values. If we form the first variation of that integral, but while varying only the q_i, but not the t (so we will get only the geometric form of the trajectory, but not the temporal course of motion), and we introduce variations δq^i that vanish at the integration limits, then it will follow from eq. (7) in the known way that:

$$\int_{u_0}^{u_1} \sum_{i=1}^{n} \Lambda_i\left(\sqrt{K}\right)\delta q^i\, t'\, du = \int_{u_0}^{u_1} \sum_{i=1}^{n} \lambda_i\left(\sqrt{k}\right)\delta q^i\, du, \tag{8}$$

and due to the arbitrariness in the δq^i :

$$t'\,\Lambda_i\left(\sqrt{K}\right) = \lambda_i\left(\sqrt{k}\right) \qquad (i = 1, 2, \ldots, n). \tag{9}$$

It will follow from eq. (6) that:

(10) $$\Lambda_i\left(\sqrt{K}\right) = \frac{1}{2\sqrt{K}}\Lambda_i\left(\sqrt{K}\right) - \frac{1}{4\sqrt{K^2}}\frac{dK}{dt}K_i\ .$$

Eq. (1) implies the energy equation:

(11) $$\frac{dK}{dt} = \sum_{h=1}^{n} Q_h\,\dot{q}^h$$

in a known way. With the help of eq. (1), $\Lambda_i(K_i) = Q_i$, eqs. (9), (4), (5), (11), it will follow from (10) that:

(12) $$\frac{\lambda_i(\sqrt{k})}{\sqrt{k}} = \frac{1}{2K}\left(Q_i - \frac{k_i}{2k}\sum_{h=1}^{n} Q_h\,q'^h\right) \qquad (i = 1, 2, \ldots, n).$$

If we set:

(13) $$\frac{\lambda_i(\sqrt{k})}{\sqrt{k}} = G_i\,, \qquad Q_i - \frac{k_i}{2k}\sum_{h=1}^{n} Q_h\,q'^h = P_i$$

then the P_i will be functions of the q_i and their first derivatives with respect to u, while the G_i are functions of the same quantities and the second derivatives. Therefore, one can also replace the equations of motion (1) with eq. (12) and the equation that follows immediately from (11):

(14) $$\frac{dK}{du} = \sum_{h=1}^{n} Q_h\,q'^h\ .$$

Eq. (14) and

(15) $$G_i = \frac{1}{2K}P_i \qquad (i = 1, 2, \ldots, n)$$

then define a system of $n + 1$ simultaneous differential equations for the $n + 1$ quantities $q_1, \ldots, q_n$, K as functions of u. The identities:

(16) $$\sum_{i=1}^{n} P_i\,q'^i = 0\,, \qquad \sum_{h=1}^{n} G_h\,q'^h = 0$$

then exist between the left-hand side of (15) and the right-hand side, due to eq. (13), so only $n - 1$ of the equations in (15) will be mutually independent, in general. However, yet another differential equation enters into them as the definition of the parameter u, whose addition will first make the coordinates of the trajectory into single-valued functions of u. The simplest covariant assumption for that equation would be $k = 1$.

The G_i can also be given a form that will illuminate the second series of identities (16) immediately. We make the Ansatz for k that:

$$k = \tfrac{1}{2}\sum_{l,m} a_{lm}\, q'^{l} q'^{m} , \tag{17}$$

in which the a_{lm} are functions of q_i. It is then known that:

$$\lambda_i\,(k) \equiv \sum_{l=1}^{n} a_{il}\, q''^{l} + \sum_{l,m=1}^{n} \begin{bmatrix} l\ m \\ i \end{bmatrix} q'^{l}\, q'^{m} , \tag{18}$$

in which the $\begin{bmatrix} l\ m \\ i \end{bmatrix}$ are the **Christoffel** symbols of the first kind that are constructed from the a_{lm} .

If we form the equation for $\lambda_i\left(\sqrt{k}\right)$ that is analogous to (10) then that will yield:

$$G_i \equiv \frac{\lambda_i\left(\sqrt{k}\right)}{\sqrt{k}} \equiv \frac{1}{2k}\left(\lambda_i - \frac{k_i}{2k}\sum_{h=1}^{n} \lambda_h\, q'^{h}\right) , \tag{19}$$

in which the λ_i mean the expressions (18). The identities (16) for the G_i follow immediately from eq. (19).

The geometric meaning of eqs. (14), (15) will become immediately clear when one forms them for the simplest case of the motion of a material point in the plane and rectangular coordinates. Here, one has:

$$q_1 = x , \qquad q_2 = y , \qquad Q_1 = X , \qquad Q_2 = Y , \qquad K = \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2) = \tfrac{1}{2} m v^2 .$$

When one sets $x'^2 + y'^2 = 1$ in eq. (14), so one chooses the arc-length to be the parameter $u = s$, one will then have:

$$m v \frac{dv}{ds} = m \frac{dv}{dt} = X \frac{dx}{ds} + Y \frac{dy}{ds} ,$$

which are then the equations of motion for the tangential components of the force. Due to the facts that $\lambda_1 = m x''$, $\lambda_2 = m y''$, $k_1 = m x'$, $k_2 = m y'$, as well as eqs. (19) and (18), one will have:

$$G_1 = x'' , \qquad G_2 = y'' , \qquad P_1 = X - x'(X x' + Y y') , \qquad P_2 = Y - y'(X x' + Y y')$$

in eq. (15). Due to the identities (16), the two equations (150 are reducible to one. After a suitable combination of them, it will read:

$$x'' y' - y'' x' = \frac{1}{2K}(X y' - Y x') ,$$

or after introducing the radius of curvature r :

$$-\frac{m v^2}{r} = X \frac{dy}{ds} - Y \frac{dx}{ds}.$$

However, that is the equation of motion for the normal component of the force.

Accordingly, we can also generally regard eqs. (14), (15) in such a way that they correspond to the decomposition of force into a tangential and a normal component ([1]). The first of identities (16) indeed says that the vector P_i is perpendicular to the tangent to the trajectory in the space of the representative point. The vector G_i is then the curvature vector of that curve.

Differential equations that determine only the geometric form of the trajectory can be derived from eqs. (14), (15) by eliminating K. If one would like to perform that elimination symmetrically then one will initially conclude from (15) that:

$$K = \frac{1}{2} \frac{\sum_{i=1}^{n} P_i G^i}{\sum_{i=1}^{n} G_i G^i}. \tag{20}$$

The G^i in that are the contravariant components of the vector G_i ([2]). It will then follow from eq. (14) that:

$$\frac{d}{du}\left(\frac{\sum_i P_i G^i}{\sum_i G_i G^i} \right) = 2 \sum_h Q_h q'^h. \tag{21}$$

That is then combined with the relations that follow from (15):

$$\frac{P_1}{G_1} = \frac{P_2}{G_2} = \ldots = \frac{P_n}{G_n}. \tag{22}$$

The eqs. (21), (22) define a system of n differential equations for the direct determination of the q^i as functions of u. In that way, eq. (22) will include the q_i, q'^i, q''^i, and eq. (21) will also include the third derivatives q'''^i.

The derived form of the differential equations for the trajectories (14), (15) [(21), (22), resp.] allows one to read off many properties of those curves, and due to the geometric analogy with the simple case of the motion of a point in the plane, one can also make them intuitively enlightening.

([1]) A similar equation can also be found in **J. E. Wright**, *Invariants of differential forms*, Cambridge, 1908.

([2]) Moreover, the contravariant components of an arbitrary vector can enter in place of the G in eqs. (20), (21).

That is how one can understand the properties of real trajectories that **Painlevé** ([3]) proved with almost no calculation.

From a mathematical standpoint, one must prove that the possibility of the covariant elimination of time is based, above all, on the homogeneity of K in the $\dot{q}^i$, which especially emerges from eq. (7). All of the considerations can also be implemented when one starts from eq. (1) but assumes that the function K in it is homogeneous of degree m in the $\dot{q}^i$, which would mean that it does not need to be entire and rational.

What will then enter in place of (4) and (5) will be:

$$k = K\,t'^m, \qquad k_i = K_i\,t'^{m-1}. \tag{23}$$

In place of (7), one will then have:

$$\int \sqrt[m]{K}\,dt = \int \sqrt[m]{k}\,du. \tag{24}$$

From (9), one will have $t'\,\Lambda_i\left(\sqrt[m]{K}\right) = \lambda_i\left(\sqrt[m]{k}\right)$. Eq. (10) will become:

$$\Lambda_i\left(\sqrt[m]{K}\right) = \frac{1}{m\,K^{(m-1)/m}}\,\Lambda_i(K) - \frac{m-1}{m^2}\,\frac{1}{K^{(2m-1)/m}}\,\frac{dK}{dt}\,K_i\;. \tag{25}$$

Since the relation:

$$(m-1)\frac{dK}{dt} = \sum_h Q_h\,\dot{q}^h \tag{26}$$

enters in place of (11), it will follow from (25) that:

$$\frac{\lambda_i\left(\sqrt[m]{k}\right)}{\sqrt[m]{k}} = \frac{1}{m\,K}\left(Q_i - \frac{k_i}{m\,k}\sum_h Q_h\,q'^h\right), \tag{27}$$

which is precisely analogous to eq. (12).

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([3]) **P. Painlevé**, "Sur les mouvements et les trajectoires réels des systèmes," Bull. Soc. Math. France **22** (1894), 136-184.