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On the electrodynamics of point-like electrons

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With one illustration

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A conclusive argument against the usual representation of extended electrons will be presented, and the foundations of a theory of electrodynamics that is free of that representation will be suggested that considers electrons to be point-like force centers. It will imply that the **Maxwell-Lorentz** differential equations admit special solutions that are different from the "retarded" and "advanced" potentials and which can perhaps be meaningful in quantum theory. Furthermore, the equations of motion of an electron and the related problem of *mass* will be discussed. In Part Three, a reshaping of the concept of energy is indicated that follows from the non-existence of "self-forces." In that way, the usual *total* electromagnetic energy (and energy current) will be replaced with the corresponding quantities that determine the *mutual* action of various electrons on each other.

Introduction. – In a tentative notice, I referred to a fundamental complication in the electromagnetic theory of mass and the inadmissibility of the usual representation of spatially-extended electrons that is connected with it ([1]). The aforementioned difficulty lies in the high mass defect of helium (or also heavier atoms) relative to hydrogen. It is known that when protons and electrons are packed together into an atomic nucleus, each proton will lose 0.008 of its mass. When the mass of a proton or electron is a consequence of the action of its infinitely-small elements on each other, the mass defect that is a consequence of the interaction of protons and electrons can never exceed twice the mass of the lighter particles for each pair. However, in reality, the mass of an electron is about sixteen times smaller than the aforementioned mass defect.

It seems to me that there is only one way around that difficulty, namely, to consider the electrons and protons to be geometrically, but not physically, indivisible things. However, only a *point* is geometrically indivisible. For that reason, we would like to treat the electrons and protons as *point charges* and consider their masses to be a primary property that is independent of charge.

The goal of this paper consists of suggesting the changes to the main principles of electrodynamics that are required by our picture.

([1]) Naturwissenschaften **12** (1924), 882. At this point, allow me to express my heartfelt thanks to Herrn Prof. **P. Ehrenfest** for his support.

I. – The electromagnetic field of a point-like electron ([1]).

§ 1. – The **Maxwell-Lorentz** equations for the electromagnetic field can remain unchanged when the electric charge density is set to zero everywhere except for "singular" points where one finds electrons. In so doing, the integral $\int E_n \, dS$, which is extended over a closed surface S that encloses an isolated point, must remain equal to $4\pi e$ ($\pm c$ = the elementary charge). Obviously, one can consider that to be a limiting case of the usual distribution for which the charge density ρ keeps a finite value in all of a finite space that is defined to be the volume V of the electron. If one passes to the limit $V \to 0$ with the condition that $\int \rho \, dV = e$ = const. then one will get the well-known **Liénard-Wiechert** expressions for the electromagnetic potential of a moving point-charge:

$$\varphi = \frac{e}{\left[R\left(1-\frac{v_r}{c}\right)\right]_0} = \frac{e}{\left[R\left(1+\frac{1}{c}\frac{dR}{dt'}\right)\right]_{t'=t'_0}}, \qquad A = e\frac{v'_0}{c}, \tag{1}$$

in which R_0 means the distance from the electron and v'_0 is its velocity vector at the moment:

$$t'_0 = t - \frac{R(t'_0)}{c} \, . \tag{2}$$

In the derivation of (1), one usually starts from the formulas:

$$\varphi = \int \frac{\rho'}{R} dV', \qquad A = \int \frac{\rho' v'}{c R} dV', \tag{3}$$

which correspond to a continuously-distributed "electrical fluid" ([2]). We would like to choose another path along which the same result can be obtained directly from the differential equations:

$$\Gamma^2\varphi - \frac{1}{r^2}\frac{\partial^2\varphi}{\partial t^2} = -4\pi\rho, \qquad \Gamma^2 A - \frac{1}{r^2}\frac{\partial^2 A}{\partial t^2} = -4\pi\rho\frac{v}{c}, \qquad \operatorname{div} A + \frac{1}{r}\frac{\partial\varphi}{\partial t} = 0, \tag{4}$$

without needing to appeal to the fiction of an electrical fluid. One can consider that path ([3]) to be a generalization of the usual process for ascertaining the electrostatic potential of a point-charge at rest.

If we introduce a rectangular coordinate system X_1, X_2, X_3 and set $i\,c\,t = x_4$ ($i = \sqrt{-1}$) then the differential equations for φ (the components A_1, A_2, A_3 of A, resp.) will assume the following form:

([1]) In what follows, we refer to protons as "positive electrons."
([2]) **Abraham**, *Theorie der Elektrizität*, II.
([3]) That was first attempted by **Herglotz** (Götting. Nachr., 1904).

$$\frac{\partial^2\psi}{\partial x_1^2}+\frac{\partial^2\psi}{\partial x_2^2}+\frac{\partial^2\psi}{\partial x_3^2}+\frac{\partial^2\psi}{\partial x_4^2}=0\,. \tag{5}$$

In that way, the point $P'(x_1', x_2', x_3')$ where the electron in question is found at the moment $t' = x_4'/ic$ is treated as a *singular point* of the function ψ at the time $t = t'$. If we regard x_1, x_2, x_3, x_4 as the rectangular coordinates of a point Q of the four-dimensional "world" then we can say that the world-line of the electron:

$$x_1' = f_1(t')\,,\quad x_2' = f_2(t')\,,\quad x_3' = f_3(t')\,,\quad x_4' = ict'\,, \tag{6}$$

defines a singular line of the function ψ. Now we can easily show that this function can be represented as a superposition of the solutions $\psi_{Q'}$ of (5) that correspond to a continuous sequence of isolated points Q' along the aforementioned line. However, in so doing, we must include not only real points but *imaginary points* that are given by (6) for complex values of time t', as will be shown directly ([1]).

Obviously, the function $\psi_{Q'}$ must depend upon only the four-dimensional distance from the pole Q' to the world-point considered Q (x_1, x_2, x_3, x_4) . If we denote that distance by S, where:

$$S^2 = (x_1-x_1')^2+(x_2-x_2')^2+(x_3-x_3')^2+(x_4-x_4')^2 = R^2-c^2(t-t')^2\,, \tag{7}$$

then we will have:

$$\frac{\partial\psi_{Q'}}{\partial x_k} = \frac{d\psi_{Q'}}{dS}\frac{\partial S}{\partial x_k} = \frac{d\psi_{Q'}}{dS}\frac{x_k-x_k'}{S}\,, \qquad \frac{\partial^2\psi_{Q'}}{\partial S^2} = \frac{d^2\psi_{Q'}}{dS^2}\frac{(x_k-x_k')^2}{S^2}+\frac{d\psi_{Q'}}{dS}\frac{S^2-(x_k-x_k')^2}{S^3}\,,$$

and as a result:

$$\sum_{k=1}^{4}\frac{\partial^2\psi_{Q'}}{\partial x_k^2} = \frac{d^2\psi_{Q'}}{dS^2}+\frac{3}{S}\frac{d\psi_{Q'}}{dS} = 0\,,$$

from which, we easily get:

$$\psi_{Q'} = \frac{\alpha}{S^2}\,. \tag{8}$$

(The additive integration constant can be set equal to zero.)

Now in order to use (8) to arrive at the desired "line solution" of (5) for A_1, A_2, A_3, and $A_4 = ic$, from which should satisfy the condition that:

([1]) In that way, it is assumed that the functions f_1, f_2, f_3, which are indeed given for only the real values of t', admits a direct analytic continuation to the entire t'-plane. We would not like to go into the corresponding restrictions on the type of motion here.

$$\operatorname{div} A + \frac{1}{c}\frac{\partial \varphi}{\partial t} = 0 ,$$

i.e.:

$$\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} + \frac{\partial A_4}{\partial x_4} = 0 , \tag{9}$$

one must set α proportional to:

$$dx'_k = \frac{dx'_k}{dt'} dt' \qquad (k = 1, 2, 3, 4)$$

and *integrate the corresponding expression over any closed curve in the complex* t'*-plane while considering the relations* (6).

In fact, if one generally sets:

$$A_k = \int \frac{\alpha'_k \, dx'_k}{S^2} = \int \frac{\alpha'_k \dfrac{dx'_k}{dt'}}{S^2} dt' , \tag{10}$$

in which α'_k ($k = 1, \ldots, 4$) mean still-undetermined functions of t', then one will have:

$$\frac{\partial A_k}{\partial x_k} = \int \alpha'_k \frac{\partial}{\partial x_k}\left(\frac{1}{S^2}\right) dx'_k = - \int \alpha'_k \frac{\partial}{\partial x'_k}\left(\frac{1}{S^2}\right) dx'_k ,$$

and as a result:

$$\sum_{k=1}^{4} \frac{\partial A_k}{\partial x_k} = - \int \sum_{k=1}^{4} \alpha'_k \frac{\partial}{\partial x'_k}\left(\frac{1}{S^2}\right) \frac{dx'_k}{dt'} dt' .$$

If that expression is to vanish identically, i.e., independently of the relations (6), then the condition:

$$\alpha'_1 = \alpha'_2 = \alpha'_3 = \alpha'_4 = \text{const.} \tag{10.a}$$

must be fulfilled and the path of integration must be closed, since it is *only in that case* will one have:

$$\sum_{k=1}^{4} \frac{\partial A_k}{\partial x_k} = - \text{const.} \int \frac{d}{dt'}\left(\frac{1}{S^2}\right) dt' \equiv 0 .$$

When the constant in (10.a) is denoted by $\dfrac{\beta e}{2\pi i}$ (e means the charge of the electron considered and b is a still-undetermined constant):

$$A_k = \beta e \frac{1}{2\pi i} \oint \left(\frac{dx'_k}{dt'} \Big/ S^2 \right) dt' , \tag{11}$$

i.e., A_k is equal to the product of βe with sum of the residues of the function $\frac{dx'_k}{dt'}\Big/S^2$ relative to those roots of the equation:

$$S^2 \equiv R^2 - c^2(t-t')^2 = 0 \tag{11.a}$$

that are enclosed by the integration curve.

If one denotes one of those roots by $t' = t'_n$ then the corresponding residue of $\frac{dx'_k}{dt'}\Big/S^2$ will be equal to:

$$\left(\frac{dx'_k}{dt'}\Big/\frac{dS^2}{dt'}\right)_{t'=t'_n},$$

or, since:

$$\frac{dS^2}{dt'} = 2\left[R\frac{dR}{dt'} + c^2(t-t')\right] \qquad \text{and} \qquad R_n = \pm c(t-t'_n),$$

one will have:

$$\operatorname*{\mathfrak{Res}}_{t'=t'_n}\left(\frac{dx'_k}{dt'}\Big/S^2\right) = \frac{\frac{1}{c}\frac{dx'_k}{dt'}}{2R\left(\frac{1}{c}\frac{dR}{dt'} \pm 1\right)} \qquad (t' = t'_n), \tag{12}$$

and for a positive sense of traversal of the circuit, the upper sign corresponds to the equations:

$$R - c(t-t') = 0 \tag{12.a}$$

and the lower sign corresponds to the equation:

$$R + c(t-t') = 0 \tag{12.b}$$

If the electron moves with subluminal velocity then those equations will have *only one real root*, which we will denote by t'_0 (t''_0, resp.). Now in order to obtain the **Liénard-Wiechert** formulas from (11), one must obviously choose the integration curve in such a way that it includes only real roots of (12.a), so t'_0, and therefore set:

$$\beta = 2\,. \tag{13}$$

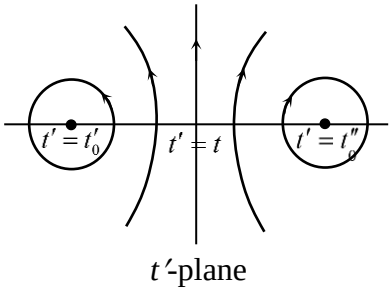

Figure 1.

In the case of an electron at rest or in a state of uniform rectilinear motion, the equation $S^2 = 0$ has no imaginary roots. One can then deform the path of integration into an arbitrary closed or infinite curve that runs upward between the points $t' = t'_0$ and $t' = t''_0$ (Fig. 1). The simplest of those curves is the

line that goes through the point $t' = t$ in the direction of the imaginary axis. If it were chosen to be the path of integration then one would get:

$$\varphi = \frac{A_1}{i} = -\frac{e}{\pi}\int_{t'=-i\infty}^{t'=+i\infty}\frac{i\,c\,dt'}{R^2+(x_4'-x_4)^2} = +\frac{e}{\pi}\int_{-\infty}^{+\infty}\frac{d(x_4'-x_4)}{R^2+(x_4'-x_4)^2} = \frac{e}{R}$$

in the case of an electron at rest (R = const.), so the **Coulomb** potential. If the electron moves in the positive X_1-direction with a velocity $v < c$ then when one sets:

$$x_1' = vt' = -i\beta x_1' \qquad \left(\beta = \frac{v}{c}\right), \qquad x_2' = x_3' = 0 \quad \text{and} \quad t = 0\,,$$

one will get:

$$S^2 = x_1^2 + x_2^2 + x_3^2 + 2i\beta x_1 x_4'(1-\beta^2)$$

$$= (1-\beta^2)\left(x_4' + \frac{i\beta x_1}{1-\beta^2}\right)^2 + \frac{x_1^2 + (1-\beta^2)(x_2^2+x_3^2)}{1-\beta^2}\,.$$

If one denotes $\sqrt{x_1^2 + (1-\beta^2)(x_2^2+x_3^2)}$ by R' (viz., the effective distance) and introduces the variable:

$$u = (1-\beta^2)\left(x_4' + \frac{i\beta x_1}{1-\beta^2}\right)^2$$

in place of t' then one will get:

$$\varphi = -\frac{e}{\pi}\int_{+\infty}^{-\infty}\frac{du}{R'^2+u^2} = \frac{e}{R'} \qquad \text{and} \qquad A_1 = \varphi\frac{v}{c}\,,$$

which is the known **Heaviside** result.

§ 2. – In the special case considered, the ordinary "retarded" potential, which corresponds to the moment $t' = t_0'$, coincides with the "advanced" one, which is the negative residue relative to the point $t' = t_0''$. In the general case of an arbitrarily-moving electron, the advanced potential is thought of as "physically meaningless," and for that reason, it is left unobserved. Let it be remarked that as long as we are at a distance from any sort of "ether," the retarded action-at-a-distance through empty space is just as "incomprehensible" as the advanced kind. There are also no logical grounds for preferring the former over the latter. The usual conception of the causality principle, according to which the cause must always precede the effect is entirely illusory, as we easily see.

In fact, from the standpoint of classical mechanics, which is connected with the picture of "momentary" action-at-a-distance (i.e., a momentary communication of force), the *acceleration* of any material point must depend upon the simultaneous position of all other points that act upon it, so it must be considered to be *simultaneous* with its causes. The concept of the motion remaining somewhat delayed relative to the force is based upon the fact that we perceive the motion, not by its acceleration, but by its velocity, or rather, by the corresponding displacement. However, in order to notice a change in velocity or a change in place, a certain time interval must be expected.

We then see that *causa* and *effectum* are considered to be simultaneous in classical mechanics. If that simultaneity were replaced by a delayed force-action then the temporal unity of cause and effect would be perturbed, so it does not seem possible to also assume an advanced force-action.

Nevertheless, if there seem to be conclusive reasons against giving an advanced force-action then they are not of a logical nature, but a purely empirical one. Namely, such an action will be denied by the well-known phenomena of the propagation of light. However, from quantum theory, light is not always emitted: The stationary motions of electrons remain radiation-less and correspond to a constant value of the mechanical energy. It is easy to show that such conservative motions can be explained electrodynamically by saying that the electrical action-at-a-distance is half retarded and half advanced ([1]). In that way, one will get an electromagnetic field that coincides with ordinary standing waves in the case of periodic motion ([2]), so it will not be experimentally observable.

However, along with the retarded and advanced actions, which correspond to the real roots of $S^2 = 0$, there are, in general, a host of other solutions to the electromagnetic field equations that correspond to *complex roots of* (12.a) *or* (12.b) and depend upon the type of motion in an entirely-special "singular" way. It seems that those singular or complex solutions have not been considered up to now. However, physically they can be just as admissible as the general "real" solutions. They correspond to an electrical action-at-a-distance that is neither retarded nor advanced, and which perhaps must take place for stationary motions of electrons. It should be observed that it is precisely for periodic or forced periodic motions that the number of (complex) roots of (12.a) and (12.b) is infinitely large, which corresponds to a still-unobserved arbitrariness in the determination of the electromagnetic field and perhaps might offer the possibility of bringing that determination into agreement with quantum theory.

From (11) and (12), the most general form of the electromagnetic potential of a moving point-charge will be given by:

$$A_k = \left[\sum_{t'_n} \beta'_n \frac{\frac{1}{c}\frac{dx'_k}{dt'}}{R\left(\frac{1}{c}\frac{dx'_k}{dt'} + 1\right)} + \sum_{t''_n} \beta''_n \frac{\frac{1}{c}\frac{dx'_k}{dt'}}{R\left(\frac{1}{c}\frac{dx'_k}{dt'} - 1\right)} \right], \tag{14}$$

([1]) **Hargreaves**, Trans. Camb. Phil. Soc. **22** (1917), 191. Namely, that author has shown that in the development of the **Liénard-Wiechert** potential in negative powers of *c*, the terms of even order correspond to conservative forces.

([2]) **L. Page**, Phys. Rev. **24** (1924).

in which the sums $\sum_{t'_n}$ and $\sum_{t''_n}$ are extended over all roots of (12.a) [(12.b), resp.], while β'_n and β''_n mean numerical coefficients that remain completely undetermined, except for the condition that $\int E_n \, dS = \pm \pi e$ ([1]).

Finding the complex roots of the equation $S^2 = 0$ presents a very difficult problem, even in the case of the simplest periodic motions (e.g., a rectilinear harmonic oscillation). I cannot see any general principle of a rational type for the determination of the coefficients β'_n and β''_n, either. I hope to return to those problems in a later publication.

The tentative arguments above were presented here in order to show the possibilities that exist in classical electrodynamics for explaining observed facts as long as one treats the electromagnetic field of the atom have still not been exhausted, as is often assumed.

II. – The equations of motion of a point-like electron.

§ 3. – In **H. A. Lorentz**'s classical theory of the electron, the equations of a motion were derived from the principle that the resultant of the forces that act externally upon the electron must always be in equilibrium with the forces that the electron exerts upon itself.

That "self-force" will then be regarded as the resultant of the elementary forces that the infinitely-small spatially-separated elements of the electron charge exert upon each other.

In the first approximation, the self-force is proportional to the acceleration w and in the opposite direction to it, so it is equal to $- m_0 w$, so for a spherical electron of radius a, $m_0 = \frac{e^2}{c^2 a}$ (k mean a numerical factor that is known to be equal to 2/3 for a surface charge and 3/5 for a volume charge). When one identifies the aforementioned self-force with the mechanical force of inertia, the coefficient m_0 will take on the meaning of the *mass*, which is "explained electromagnetically" in that way.

When one observes the retardation of the electrical action-at-a-distance between the different elements of an electron in the calculation of the self-force (with the usual formulas), along with the inertial force $- m_0 w$, one will get (in the second approximation) a type of "frictional force" $\frac{2}{3}\frac{e^2}{c^3}\frac{dw}{dt}$ that is completely independent of the charge distribution. Therefore, in order to get more exact expressions, one must then consider other force terms that are proportional to the higher derivatives of the acceleration (relative to t) and positive powers of the electron radius. However, that is still not everything. If the velocity of the electron is large enough then one must replace m_0 with a function of the velocity whose form cannot be determined without special assumptions on the dependency of the form of the electron on velocity (and which is, e.g., for the **Lorentz** "deformable" electron, equal to $\frac{m_0}{\sqrt{1 - v^2/c^2}}$ in the case of a transverse acceleration). Similarly,

([1]) If the expression (14) is a complex quantity then one can define A_k to be its real (or imaginary) part.

for higher derivatives of *w*, the coefficients will no longer be constants, but functions of the lower-order derivatives whose exact determination will assume that one knows the change in form that will come about in the moving electron as a result of those derivatives.

In our attempt to ascertain the precise form of the self-force, we will then get a most complicated series that cannot be determined uniquely from its motion then without infinitely-many assumptions on the dependency of the form of an electron.

One must then add to that the fact that the self-force indeed vanishes for an electron at rest, but the mutual repulsion of the various elements remains and can be cancelled by only some sort of non-electric force. The internal equilibrium of an extended electron will then become an insoluble riddle from the standpoint of electrodynamics.

I consider that riddle (and the questions that are connected with it) to be a *purely academic problem*. It arises from an uncritical adaptation to the elementary parts of matter (electrons) of a principle of subdivision that had led to precisely those latter "smallest" parts when it was applied to compound systems (atoms, etc.).

Electrons are not only physically, but also geometrically, indivisible. They have no extension in space at all. There are no internal forces between the elements of an electron because such elements do not exist. The electromagnetic explanation for mass then goes away, but in that way, all complications that are connected with establishing more precise equations of motion for an electron on the basis of the aforementioned (**Lorentzian**) principle will also disappear.

It should be remarked that this principle is entirely arbitrary and occurs in electrodynamics only by means of the concept of mass. In order to achieve the desired equations of motion, one can follow **Einstein** and start from the *principle of relativity*. However, the special theory of relativity gives no unique solution to the problem in question, but only fixes the general form of such solutions. If one replaces the time *t* by the invariant proper time of the electron:

$$\tau = \int_0^t \sqrt{1 - v^2/c^2}\, dt\,,$$

and the spatial vectors of acceleration and force with the corresponding world-vectors with the components $\frac{d^2 S_k}{d\tau^2}$ and F_k (impulse-work for τ-units) then one will get the well-known **Einstein** formula as the simplest form for the equations of motion:

$$m_0 \frac{d^2 S_k}{d\tau^2} = F_k \qquad (k = 1, 2, 3, 4), \tag{15}$$

whose spatial projection reads:

$$\frac{d}{dt} \frac{m_0 v}{\sqrt{1 - v^2/c^2}} = f\,, \tag{16}$$

in which *v* means the usual three-dimensional velocity and *f* is the usual force (= impulse for the *t*-units).

However, instead of (15), one can exhibit a large number of other equations of motion of a general type that satisfy the principle of relativity, e.g.:

$$m_0 \frac{d^2 S_k}{d\tau^2} + \kappa_1 \frac{d^3 S_k}{d\tau^3} + \kappa_2 \frac{d^4 S_k}{d\tau^4} + \cdots = F_k \,, \tag{17}$$

in which κ_1, κ_2, … are undetermined coefficients [the second term on the left-hand side of (17) obviously corresponds to the "force of friction" $\frac{2}{3}\frac{e^2}{c^3}\frac{dw}{dt}$ for $\kappa_1 < 0$].

Experiments show that free electrons (in the form of cathode rays), as well as bound ones (for stationary states of atoms), move according to the simple equation (15). In contrast, that equation seems to lose its validity for transitional motions between two different stationary states. A mechanical explanation for the spontaneous transitional motions that result from a loss of energy (i.e., radiation) would seem to demand the introduction of terms of odd order into (17) that would be similar to the force of friction.

However, it is also not excluded that the aforementioned transition constraints are produced by some strange reshaping of the electromagnetic field ([1]).

That question must remain open here. However, it is clear that the problem of the motion of a point-like electron does not include any difficulties, in principle. It should be pointed out that from the standpoint of the special theory of relativity, an extended electron is an entirely inconceivable thing, since due to the intrinsic connection between space and time, an invariant definition of a geometrically invariable (i.e., rigid) body is impossible for arbitrary motions (at least, ones that are based upon Euclidian geometry) ([2]).

III. – Force and energy.

§ 5. [*sic*] – The force that acts upon an electron that moves with velocity *v* in a given *external* field E^*, H^* is determined from the well-known **Lorentz** force:

$$f = e\left(E^* + \left[\frac{v}{c}, H^* \right] \right) , \tag{18}$$

or from the corresponding four-dimensional expression:

$$F_i = e\sum_k F_{ik}^* \frac{dx_k}{d\tau} \qquad (i, k = 1, 2, 3, 4), \tag{19}$$

([1]) The generality of **Einstein**'s equations of motion seems to be connected with his theory of gravitation.

([2]) Here, we must cite an attempt by **Born** [Ann. Phys. (Leipzig) **30** (1909), 1] to create such a definition in the form of differential equations. See also **Ehrenfest**, Phys. Zeit. **10** (1909), 918.

in which $F_{ik}^{*} = \frac{\partial A_k^{*}}{\partial x_i} - \frac{\partial A_i^{*}}{\partial x_k}$ mean the rectangular components of the field tensor and $dx_k / d\tau$ means the four-dimensional velocity.

Rather than (19), the theory of extended electrons uses the formula:

$$P_i = \sum_k F_{ik}\, \rho \frac{dx_k}{dt} = \sum_k F_{ik}\, \rho_0 \frac{dx_k}{d\tau}, \tag{20}$$

or its equivalent formulas:

$$p = e\left(E + \left[\frac{v}{c}, H \right] \right), \tag{20.a}$$

$$l = \rho\,(E, v), \tag{20.b}$$

in which the quantities F_{ik} (E and H, resp.) determine the *total* field strengths at the point in question, i.e., the sum of the field strengths that originate in the other electrons, on the one hand, and the neighboring elements to the electron that the point contains, on the other. P_i are the components of the corresponding impulse-work (p, l) per unit volume and time. For that reason, the integral $\int P_i\, dV$, which is taken over the volume of an electron, must be equal to the sum of the external force and proper force (i.e., impulse-work), so from the **Lorentz** principle, it must vanish.

As is known, by means of the basic equations:

$$F_{ik} = \frac{\partial A_k}{\partial x_i} - \frac{\partial A_i}{\partial x_k} \qquad \text{and} \qquad \sum_i \frac{\partial^2 A_k}{\partial x_i^2} = -4\pi\rho \frac{dx_i}{dt}$$

(or the **Maxwell-Lorentz** equations for F_{ik} that are equivalent to them), one can put (20) into the form:

$$P_i = \sum \frac{\partial T_{ik}}{\partial x_k}, \tag{21}$$

where

$$T_{ik} = \frac{\delta_{ik}}{8\pi} \sum\sum_{p<q} F_{pq}^2 - \frac{1}{4\pi} \sum_h F_{ih}\, F_{kh}, \tag{21.a}$$

mean the components of the impulse-energy-tensor ($\delta_{ik} = 1$ when $i = k$ and 0 when $i \neq k$), so in three-dimensional space, they determine the spatial density of the electromagnetic energy and quantity of motion (its current, resp.). Upon integrating (21) over any volume V, one will get the **Poynting** formula:

$$\int (E, \rho v)\, dV = -\frac{d}{dt} \int \frac{E^2 + H^2}{8\pi}\, dV - \int \frac{c}{4\pi} [E, H]_n\, dS, \tag{21.b}$$

and the corresponding relation for the electromagnetic quantities of motion.

From the standpoint of point-like electrons, those formulas and concepts are *physically absurd*, insofar as one can subdivide an electron into infinitely-small elements only by means of *external* forces or the mutual energy of different electrons.

However, one obviously cannot determine the *mutual* action from the *total* field strengths (their derivatives with respect to the coordinates, resp.).

Let us consider, e.g., the simplest case of a system of electrons at rest. One can convert its mutual energy, which is given by the known expression:

$$U^* = \sum\sum_{i<k} \frac{e_i\, e_k}{r_{ik}} = \tfrac{1}{2}\sum_k \varphi_k^*\, e_k\,,$$

as follows: Let φ_k and E_k be the potential (electric field strengths, resp.) of the k^{th} electron at any point (P), and let φ_k' and E_k' be the contributions that originate in all other electrons ($\varphi_k' = \varphi - \varphi_k$). At the point P_k, where one finds the k^{th} electron, one has $\varphi_k' = \varphi_k^*$. Furthermore, let S be a surface that encloses all of those electrons. By means of the formula:

$$e_k = \frac{1}{4\pi}\int (E_k)_n\, dS\,,$$

one will get:

$$\varphi_k^*\, e_k = \frac{1}{4\pi}\int (\varphi_k^*\, E_k)_n\, dS = \frac{1}{4\pi}\int (\varphi_k'\, E_k)_n\, dS - \frac{1}{4\pi}\int [(\varphi_k' - \varphi_k^*) E_k]_n\, dS\ .$$

We convert the last integral using **Gauss**'s theorem:

$$\int [(\varphi_k' - \varphi_k^*) E_k]_n\, dS = \int \operatorname{div}(\varphi_k' - \varphi_k^*) E_k\, dV = \int (\varphi_k' - \varphi_k^*) \operatorname{div} E_k\, dV - \int (E_k, \operatorname{grad}(\varphi_k' - \varphi_k^*))\, dV\,.$$

Since div $E_k = 0$ outside of the point P_k, while:

$$\int \operatorname{div} E_k\, dV = \int (E_k)_n\, dS = 4\pi\, e_k$$

has a finite magnitude, one will have:

$$\int (\varphi_k' - \varphi_k^*) \operatorname{div} E_k\, dV = 0\,.$$

Furthermore, one has:

$$\operatorname{grad}\ (\varphi_k' - \varphi_k^*) = \operatorname{grad}\ \varphi' = -E_k'\ ,$$

and as a result:

$$\int (E_k, \operatorname{grad}(\varphi_k' - \varphi_k^*))\, dV = -\int (E_k, E_k')\, dV\,.$$

We then get the following formula:

$$U^* = \tfrac{1}{2}\sum_k \varphi_k^* e_k = \frac{1}{4\pi}\int\Big(\sum \varphi_k' E_k\Big)_n dS + \frac{1}{4\pi}\int (E_k', E_k)\, dV ,$$

or when the surface *S* is displaced to infinity (since $\sum \varphi_k' E_k$ drops off in inverse proportion to the third or higher power of distance):

$$U^* = \frac{1}{4\pi}\int \sum_k (E_k', E_k)\, dV = \frac{1}{4\pi}\int \sum\sum_{i<k} (E_i, E_k) dV . \tag{22}$$

One can also write that expression in the following form:

$$U^* = \frac{1}{8\pi}\int (E^2 - \sum_k E_k^2)\, dV = U - \sum_k U_k , \tag{22.a}$$

in which *U* means the "total energy" and U_k means the "proper energy" of the k^{th} electron, in the event that those electrons are extended. (However, in reality, those proper energies are infinitely large and physically meaningless.) If one considers the point-like electrons to be a limiting case of the extended ones then one can start immediately from (22.a) in order to calculate the mutual energy, which will perhaps lead to that goal somewhat more simply.

What the above says about electromagnetic energy must also be true for the electro-kinetic or magnetic energy. An isolated electron which might also be in motion, *possesses no magnetic energy at all*, but only a kinetic energy of:

$$T_k = c^2 m_0 \left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1 \right) . \tag{23}$$

The mutual magnetic energy of a system of electrons, which corresponds to their electro-kinetic interactions, is expressed by the integral:

$$T^* = \frac{1}{4\pi}\int \sum_k (H_k, H_k')\, dV = \frac{1}{8\pi}\int \sum\sum_{i<k} (H_k, H_k')\, dV . \tag{23.a}$$

If we would then like to consider the actual mutual actions of different electrons on each other then we must replace the **Poynting** energy equation (21.b) with the following one:

$$\sum_k (f_k^*, e_k v_k) = -\frac{d}{dt}(U^* + T^*) - \int \mathfrak{S}_n^*\, dS \tag{24}$$

in the general case of arbitrarily-moving electrons, in which:

$$\mathfrak{S}^* = \frac{e}{4\pi}\sum_k([E_k', H_k]+[E_k, H_k']) = \frac{e}{4\pi}([E_k', H_k - \sum[E_k, H_k])$$

means the "mutual energy current." One gets an analogous formula for the mutual quantity of motion.

If one sets:

$$(f_k^*, e_k v_k) = \frac{dT_k}{dt}$$

in (24), according to **Einstein**'s equation of motion, then one will have:

$$-\frac{dW^*}{dt} = \int \mathfrak{S}_n^* \, dS\,, \tag{24.a}$$

in which:

$$W^* = \sum_k T_k + T^* + U^* \tag{24.b}$$

means the total energy of the system in question.

In the case of two electrons (1) and (2), one has:

$$\mathfrak{S}^* = \frac{e}{4\pi}([E_1', H_2 - \sum[E_2, H_1])\,.$$

If the first (positive) electron remains at rest, while the second (negative) one orbits it (which is what happens approximately in, e.g., the hydrogen atom) then one will have:

$$H_1 = 0 \qquad \text{and} \qquad \int [E_1, H_2]_n \, dS = 0\,,$$

and as a result:

$$\frac{dW^*}{dt} = 0, \qquad \text{i.e.,} \qquad W^* = \text{const.}$$

The motion of the second electron then happens with no radiation of energy in this case: The total energy of the system remains constant. However, if one would like to consider the first electron (i.e., a proton) to also move around a common center of mass then one will get a radiation of the same types as in the usual theory, but with a magnitude that is, say, 2000 times smaller. In order to turn off the radiation of energy completely, one must replace the retarded potential with another "complex" solution to the corresponding differential equations (see § **3**).

§ 6. – In the treatment of macroscopic phenomena in material bodies, one cares to replace the point-charge, which is at rest or streaming, with an equivalent continuous distribution of the

electric charge or current with finite spatial or surface density. Now, it is easy to show that the interaction of infinitely-small elements of that "substitute distribution" will imply "proper forces" and "proper energy" *that are identical in practice* to the resultant of the actual interactions between the electrons that occur in an excessive number [so they define the volume (surface, resp.) charge] or are required by the associated motion of the current.

Let us consider, e.g., the electrostatic energy of a charged metal body. Let the very large number of excess elementary charges (positive or negative sign) that comprise the aforementioned macroscopic charges be equal to *N*. If we restrict ourselves to those charges then when we calculate their mutual energy, from formula (22), we can extend the double sum $\sum\sum_{i\neq k} (E_i, E_k)$ to the simple one $\sum_k E^2$, at least for external points. In fact, the latter sum is proportional to the number *N*, while the former is proportional to the square of that number, or at least to the square of a number with the same order of magnitude as *N*. The error that is imposed by such an extension is obviously vanishingly small. However, in that way, the part of the electrostatic energy that corresponds to external space will be expressed by the usual integral $\frac{1}{8\pi}\int E^2\, dV$. Now, as far as the interior of the body considered is concerned, in the case of equilibrium, $(\bar{E})^2$, i.e., the square of the *mean* field strength, will vanish along with $\sum_k (E_k', E_k)$. In the general case, the mean value of $\sum_k (E_k', E_k)$ must coincide with $(\bar{E})^2$ in the absence of spatial charge, whereby $\bar{E}$ will correspond to the continuous substitute distribution.

Such a supplementary term (T_0^*) can also appear in the calculation of the magnetic energy T^* of a body with current streaming through it; however, its magnitude will be relatively minor. It should be noted that in that way, the kinetic energy of the associated motion of the streaming electrons will remain vanishingly small in comparison to their mutual magnetic energy. For that reason, one can leave the sum $\sum_k T_k$ out of consideration in (24.b) in the treatment of macroscopic processes, so the total energy will be set equal to ([1]):

$$\frac{1}{8\pi}\int [(\bar{E})^2 + (\bar{H})^2]\, dV$$

as usual.

In conclusion, I would like to mention one point. In the general theory of relativity, the impulse-energy tensor plays a fundamental role since it replaces the ordinary mass and determines the adjoint curvature tensor. In our way of looking at things, that tensor seems to lose its physical

([1]) We imagine, e.g., a ball of radius *R* in a metal body that includes *N* streaming electrons. The mutual mass of those electrons is equal to roughly $\frac{4}{5}\frac{(Ne)^2}{c^2 R}$, while the sum of their masses is equal to $N\, m_0 = \frac{4}{5}\frac{Ne^2}{c^2 a}$, where $a = 2 \cdot 10^{-13}$ means the effective radius of an electron. The ratio of those quantities is equal to $N\, a / R$, so for $R = 1$ mm, it is about 10^8.

meaning since mass can no longer be interpreted electromagnetically. However, it is in precisely that domain that the aforementioned picture gains a new support. As **Nordström** ([1]) first showed, integrating **Einstein**'s differential equations in the vicinity of a spherically-symmetric electron at rest will yield the following expression for ds^2 (in polar coordinates):

$$ds^2 = \frac{1}{h} dr^2 + r^2 (d\Theta^2 + \sin^2 \Theta \, d\varphi^2) - h c^2 dt^2 , \tag{25}$$

in which:

$$h = 1 - \frac{2k m_0}{c^2 r} + \frac{k e^2}{c^4 r^2} \ . \tag{25.a}$$

In that way, e and m_0 are two independent coefficients that have the meaning of *charge* and *mass*.

That shows that an electron can probably be considered to be a *singular point* of the electric and gravitational field. However, that point corresponds to a certain distance $r = \frac{e^2}{2c^2 m_0}$, for which one will have $h = 1$ and which might be defined to the "electron radius," as in the older theory.

Leningrad, Röntgen Physical-Engineering Institute, March 1925.

([1]) Proc. Amsterdam **20** (1918), 1236.