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Theory of hypercomplex numbers. II

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In my paper "Theorie der hyperkomplexen Grössen" (which will be cited as *H*, in what follows), I mainly treated the properties of DEDEKIND groups in connection with my investigation into the determinants of finite groups. As MOLIEN has shown (*H*, § 11, II), any group (ε) with a principal unit is homomorphic to a DEDEKIND group ($\mathcal{D}$) whose determinant is divisible by each prime factor in the determinant of the entire group. If (η) is the invariant subgroup of (ε) that consists of nothing but *roots of zero* – I call it the *radical* of (ε) – then one will have (ε) = ($\mathcal{D}$), mod (η); i.e., ($\mathcal{D}$) will be the group (ε), when one considers any two quantities in it to be equal when their difference is contained in (η).

If the first m of the n independent basic numbers $\varepsilon_1, \dots, \varepsilon_n$ define a group ($\mathcal{D}$) and the last $n - m$ define a group (η) then I will call (ε) = ($\mathcal{D}$) + (η) the *sum* of these two subgroups. If one of them (η) is an invariant subgroup of (ε) then the other one ($\mathcal{D}$) will be a group that is homomorphic to (ε), since:

$$\varepsilon_1 x_1 + \dots + \varepsilon_m x_m + \varepsilon_{m+1} x_{m+1} + \dots + \varepsilon_n x_n \equiv \varepsilon_1 x_1 + \dots + \varepsilon_n x_n \qquad (\text{mod } \eta).$$

If two subgroups are invariant then the product of each quantity in ($\mathcal{D}$) and each one in (η) will vanish, and (ε) will *decompose* into the two groups ($\mathcal{D}$) and (η). (*H*, § 9).

A group (η) that is contained in (ε) is called an *invariant subgroup* of (ε) if xy and yx are quantities in (η) whenever y is any quantity in (η) and x is any quantity in (ε). If $n - m$ is the order of (η) then (ε) (mod η) will be homomorphic to a subgroup ($\mathcal{D}$) of (ε) of the given kind such that (ε) = ($\mathcal{D}$) + (η).

In his paper "Sur les groupes bilinéaires et les systèmes de nombres complexes," Ann. de Toulouse **13** (1898) (which will be cited by *C* in what follows), CARTAN gave a more precise statement to MOLIEN's theorem, which I would like to derive in order to complete the presentation of the results of my first paper:

Any group with a principle unit is the sum of its radical and a DEDEKIND group whose determinant is divisible by every prime factor of the determinant of the entire group.

However, whereas the invariant subgroup (η) that I have called the *radical* of (ε) is determined completely, the subgroup $(\mathcal{D})$ can be chosen in an infinitude of ways, at best.

If a power of a number vanishes then I will call it a *root of zero* (*pseudo-null*, for CARTAN, no. 21, *nilpotent* for PEIRCE). I will call a group that consists of nothing but roots of zero a *root group*.

If a group (ε) is the sum of a DEDEKIND group $(\mathcal{D})$ of order m and a root group (η) of order $n - m$ that is an invariant subgroup of (ε) then (η) will be the radical of (ε). Hence, since (ε) and $(\mathcal{D})$ are homomorphic, the determinant of the group (ε) will be divisible by the group $(\mathcal{D})$ (*H*, § 9). The linear rank of the determinant of a DEDEKIND group $(\mathcal{D})$ is equal to its order m. As a result, the linear rank of the determinant of (ε) will be $\bar{m} > m$. The radical $(\bar{\eta})$ of (ε) order $n - \bar{m}$. Therefore, $m = \bar{m}$ and $(\eta) = (\bar{\eta})$.

Before I go on to the proof of CARTAN theorem (§ 5), I would like to derive the main properties of root groups.

§ 2.

If neither of the two determinants $| S(x)$ and $| T(x) |$ vanishes identically then the group (ε) will possess a *principal unit* e, and for any quantity x in (ε) one will have:

$$e\,x = x\,e = x. \tag{1}$$

Conversely, as CARTAN showed: If $ex = x$ for any quantity x then one will have $| S(e) | = E$, so $| S(e) | = 1$, and if $xe = e$ then $T(e) = E$, so $| T(e) | = 1$. If one then seeks to determine x in such a way that $ex = 0$ then one will get n homogeneous, linear equations for the coordinates $x_1, \ldots, x_n$ from the matrix $S(e)$. However, one would then have $x = ex = 0$, so these equations would be satisfied by only the system of values $x_1 = \ldots = x_n = 0$, and as a result, its determinant $| S(e) |$ would be non-zero. Now, since $S(e)^2 = S(e^2) = S(e)$, one will have $S(e) = E$.

If $| S(x) | = \Theta$ and $| T(x) | = \Theta'$ are the two antistrophic determinants of the group ε then one can determine two quantities y and z whose coordinates are *entire* functions of $x_1, \ldots, x_n$ in such a way that $xy = \Theta e$ and $zx = \Theta' e$. One will then have $T(y)\,T(x) = T(xy) = \Theta\,T(e)$, and therefore $| T(y) |\,\Theta' = \Theta''$, and likewise $| S(z) |\,\Theta = \Theta'^{\alpha}$. As a result, every prime factor of one of the two determinants Θ and Θ' is also contained in the other one (*H*, § 3).

Since I have considered groups without principal units many times, I shall make only the following remarks about them: If the relations:

$$\sum_{\kappa} a_{\kappa\alpha\beta}\, a_{\gamma\kappa\delta} = \sum_{\kappa} a_{\kappa\beta\delta}\, a_{\gamma\alpha\kappa} \tag{2}$$

exist between $(n-1)^3$ quantities $a_{\alpha\beta\gamma}$ ($\alpha, \beta, \gamma = 1, 2, \ldots, n-1$) then there is always a group $(\bar{\varepsilon})$ with $n-1$ linear-independent basic numbers $\varepsilon_1, \ldots, \varepsilon_{n-1}$, between which the relations:

$$\varepsilon_\beta\, \varepsilon_\gamma = \sum_\kappa a_{\kappa\beta\gamma}\varepsilon_\kappa \tag{3}$$

exist. If (ε) possesses a principal unit then I showed this in *H*, § 2 (cf., also STUDY in the Enzyklopädie). In the other cases, one sets:

$$a_{\alpha 0 \alpha} = a_{\alpha\alpha 0} = 1 \qquad (\alpha = 0, 1, \ldots, n-1), \tag{4}$$

although otherwise one will have $a_{\alpha\beta\gamma} = 0$ whenever one of the indices is 0. Equations (2) are also true then for the n^3 quantities $a_{\alpha\beta\gamma}$ ($\alpha, \beta, \gamma = 1, 2, \ldots, n$). If one further denotes the matrix $(a_{\alpha\lambda\beta})$ for a certain λ by E_λ then one will have $E_0 = E$ and:

$$E_\beta\, E_\gamma = \sum_n a_{\alpha\beta\gamma}\, E_\alpha\,. \tag{5}$$

Moreover, if one had $\sum c_k\, E_k = 0$ then one would have $\sum c_\lambda\, a_{\alpha\lambda 0} = 0$, so one would have $c_\alpha = 0$. Therefore $E_1, \ldots, E_{n-\lambda}$ would define a representation of the group $(\bar{\varepsilon})$ of order $n-1$ by matrices of degree *n*.

This group $(\bar{\varepsilon})$ is an invariant subgroup of a group (ε) whose basis is $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-1}$, and for which, ε_0 is the principal unit. In its two antistrophic matrices $S(x)$ and $T(x)$, one has $s_{00}(x) = t_{00}(x) = x_0$, but $s_{\alpha\beta}(x) = t_{\alpha\beta}(x) = 0$ (and $s_{\alpha 0}(x) = t_{\alpha 0}(x) = x_\alpha$). If one omits the first row and column and sets $x_0 = 0$ in the $(n-1)^{\text{th}}$-degree matrices thus-obtained then one would obtain the two antistrophic matrices $\bar{S}(x)$ and $\bar{T}(x)$ of $(\bar{\varepsilon})$. Since at least one of its determinants vanishes identically, one of the two determinants $|\, S(x) \,|$ or $|\, T(x) \,|$ will be divisible by x_0^2.

Conversely, let (ε) be a group of order *n* with the principal unit *e* that contains an invariant subgroup $(\bar{\varepsilon})$ of order $n-1$. Let $\varepsilon_1, \ldots, \varepsilon_{n-1}$ be a basis of $(\bar{\varepsilon})$, and let ε_0 be a quantity in (ε) that is not contained in $(\bar{\varepsilon})$. $|\, S(x) \,|$ and $|\, T(x) \,|$ are then divisible by x_0. According to whether neither of these two determinants or at least one of the two is divisible by x_0^2, $(\bar{\varepsilon})$ will or will not possess a principal unit $(\bar{e})$, respectively. If *e* were contained in $(\bar{\varepsilon})$ then, by the definition of an invariant subgroup, one would also have $e\,\varepsilon_0 = \varepsilon_0$. Therefore, one can always choose ε_0 to be the principal unit *e*.

§ 3.

If *A* and *B* are two matrices of degree *n* then the characteristic functions of *AB* and *BA* will be equal to each other. If $|\, B \,|$ is non-zero then $B^{-1}(BA)B = AB$ will be similar to the matrix *BA*. Thus, if the elements of *A* and *B* are variable quantities then the equation $|\, aE$

$- AB \mid = \mid aE - BA \mid$ will be true for all values of these variables for which $\mid B \mid$ is non-zero, and as a result, it will be true identically.

If y and z are two quantities in the group (e) then the coordinates of their product $x = yz$ will be:

$$x_\alpha = \sum_{\beta,\gamma} a_{\alpha\beta\gamma}\, y_\beta\, z_\gamma = \sum_\gamma s_{\alpha\gamma}(y)\, z_\gamma\,. \tag{1}$$

Now, if $S(y) = 0$, so $y_1, \ldots, y_n$ satisfy the linear equations $s_{\alpha\gamma}(y) = 0$, then one will have $yz = 0$ when z is an arbitrary quantity of (ε). Thus, if (ε) possesses a principal unit then one will have $y = 0$, although one will have $y^2 = 0$ in any case.

More generally, if x is a quantity in (ε) for which $S(x)^n = S(x^n) = 0$ then one will have $x^n\, z = 0$ and $x^{n+1} = 0$. Therefore, in order for x to be a root of zero, it is necessary and sufficient that the matrix $S(x)$ (or $T(x)$) be a root of zero, so the characteristic roots of $S(x)$ must all vanish. If zy is a root of zero then yz will also be one. The characteristic roots of the matrix $S(zy) = S(z)\, S(y)$ will all be zero then, and consequently, the roots of the matrix $S(y)\, S(z) = S(yz)$ will, as well.

A quantity y in a group (ε) is called a *root quantity* when yz is always a root of zero whenever z is an arbitrary quantity in (ε). If x and z are any two quantities of (ε) then $y\,(zx) = (yz)\, x$ will also be a root of zero, so $x\,(yz)$ will be one, as well. Thus, if y is a root quantity of (ε) then the same thing will be true for xy, yz, and xyz.

The characteristic roots of the matrix $S(yz)$ then vanish, and as a result, their sum $\sigma(yz)$, as well. Conversely, let $\sigma(yz) = 0$ for any quantity z in (ε). If one replaces z with $z(yz)^{\kappa-1}$ then one will get $\sigma((yz)^\kappa) = 0$. Thus, [H, § 4, (5)] the sum of the κ^{th} powers of the characteristic roots of the matrix $S(yz)$ will then vanish, and thus those roots themselves, and as a result yz will be a root of zero. The coordinates $y_1, \ldots, y_n$ of the root quantities y will then be found by solving the homogeneous, linear equations that one obtains when one sets the derivatives of the bilinear form $\sigma(yz)$ with respect to $z_1, \ldots, z_n$ (or the quadratic form $\sigma(y^2)$ with respect to $y_1, \ldots, y_n$) equal to zero. Therefore, the root quantities of (ε) will be reproduced by addition and multiplication by ordinary quantities. Moreover, yz and zy will be root quantities when y is one. As a result, the root quantities of (ε) will define an invariant subgroup that I will call the *radical* of (ε).

I. *The radical of a group* (ε) *is defined by all quantities y whose coordinates satisfy the linear equations* $\sigma(yz) = 0$ (*or* $\tau(yz) = 0$) *for every quantity z.*

One infers the fact that one also has:

$$\sigma(xy) = \sigma(yx) \tag{2}$$

for groups without principal unit [H, § 4, (9)] from § 2, (2) when one sets $\gamma = \delta$ and sums over γ [MOL, § 3, (4)].

A root group can also be defined as a group that is equal to its radical. If one calls a non-zero quantity x that satisfies the equation $x^2 = x$ a *unit* then PEIRCE has shown ("Linear Associative Algebra," no. 51, American Journal of Math., v. 4) that:

II. *In order for a group to be a root group, it is necessary and sufficient that it contain no unit.*

If m is the smallest exponent for which $x^m = 0$ then one cannot have $x = x^2$, since otherwise one would have $x^{m-1} = x^m = 0$. That will be the equation of the lowest degree for which $px^0 + qx + rx^2 + \ldots = 0$ or $px + qx^2 + rx^2 + \ldots = 0$ according to whether (ε) does or does not have a unit $x^0 = e$, respectively. Let $y(u) = (u - a)^n\, \chi(u)$ and $\chi(a)$ be non-zero. One determines ("Über vertauschbare Matrizen," Sitzungsberichte, 1896, § 3) an entire function $f(u)$ in such a way that $f(u) - 1$ is divisible by $(u - a)^n$. If (e) possesses a principal unit, so one lets $f(u)$ be divisible by $\chi(u)$, moreover, then this will not be the case, and if a is non-zero then one will let $f(u)$ be divisible by $u\, \chi(u)$. One will then have that $f(u)^2 - f(u)$ will be divisible by $y(u)$, but not $f(u)$ itself. Thus, if $y = f((x))$ then one will have $y^2 = y$ and y will be non-zero. If (ε) is not a root group then it will contain a unit.

If (ε) has a principal unit, and if:

$$\psi(u) = (u - a)^{\alpha}\, (u - b)^{\beta}\, (u - c)^{\gamma} \ldots$$

then the various roots $a, b, c, \ldots$ of $\psi(u)$ in the given manner might correspond to entire functions $f(u)$, $g(u)$, $h(u)$, … Thus, $f(u)\, g(u)$ and $f(u) + g(u) + h(u) + \ldots = 1$ will be divisible by $\psi(u)$, and one will then have:

$$f((x))^2 = f((x)), \qquad f((x))\, g((x)) = 0 \tag{3}$$

and

$$f((x)) + g((x)) + h((x)) + \ldots = e. \tag{4}$$

In this way, the principal unit e can be decomposed into just as many independent units as the equation $y(u) = 0$ has distinct roots (C, 17).

III. *If the prime factors of the determinants of a group with a principal unit are all linear then its radical will consist of all of the roots of zero that are contained in the group.*

If x and y are two quantities in the group (e) then:

$$|\, S(ux + vy + we)\, | = \Pi\, (uu_{\alpha} + vv_{\alpha} + w)$$

will be a product of prime factors. If one sets $v = 0$ or $u = 0$ then one will recognize that $u_1, \ldots, u_n$ are characteristic roots of $S(x)$ and $v_1, \ldots, v_n$ are ones of $S(y)$. If y is a root of zero then one will have $v_1 = \ldots = v_n = 0$. If $z = g((y))$ then the characteristic roots of $S(z)$ will all be equal to $g(0)$. Thus, if v is non-zero and $z = u(y + ve)^{-1}$ then all of them will be equal to u / v. As a result:

$$|\, S(x + u(y + ve)^{-1})\, | = \Pi \left(u_{\alpha} + \frac{u}{r} \right), \qquad |\, S(y + ve)\, | = v^{\alpha},$$

so since:

$$S(x + u(y + ve)^{-1})\, S(y + ve) = S(xy + vx + ue),$$

one will have:

$$|\, S(xy + vx + ue)\,| = \Pi\,(u_\alpha v + u).$$

Since both sides of this equation are entire functions of v, it will also be true for the value $v = 0$, which has been excluded up to now. Therefore, one will have $|\, S(xy + ue)\,| = u^n$, so (C, 28) xy will be a root of zero and y will be a root quantity in (ε).

The converse of the theorem can be deduced from the results that are contained in § 5, so if $r > 1$ there then one will have $\varepsilon_{12}^2 = 0$, but ε_{12} will not belong to the radical of (ε).

§ 4.

I. *In any root group there is a non-zero quantity* x *that satisfies the equation* $xy = yx = 0$ *whenever* y *is an arbitrary quantity in the group.*

CARTAN (no. 31-24) gave an incisive, but somewhat roundabout, proof of this theorem, which I will replace here with one that is essentially simpler. One can also express the theorem as:

II. *Every root group contains an invariant subgroup of order* 1.

The fact that both theorems are identical follows from a Lemma that was employed by CARTAN (C, 29), in which x and y meant two quantities of a group that does or does not have a principal unit:

III. *If* $xy = ax$ (*or* $yx = ax$), *where* a *is an ordinary quantity, and if* y *is a root of zero then one will have either* $x = 0$ *or* $a = 0$.

By assumption, there is a number k such that $y^k = 0$, so there is also a number l such that $xy^l = 0$. Let $x \neq 0$, and let m be the smallest number of that sort. Thus, $m > 0$ and $xy^m = 0$, but xy^{m-1} (in which, one understands that for $m = 1$, one will have x) is non-zero. It will then follow from $xy^m = axy^{m-1}$ that $a = 0$. One will arrive at the same result when one multiplies the equation $x\,(y - a) = 0$ on the right with $y^{m-1} + y^{m-2}a + \ldots + a^{m-1}$. The theorem can be generalized as:

IV. *If* $x_1, x_2, \ldots, x_m$ *are quantities of a root group, and if the product* $x_1\, x_2 \ldots x_m$ *is non-zero then the* m *quantities:*

$$x_1, \qquad x_1\, x_2\,, \qquad x_1\, x_2\, x_3\,, \qquad \ldots, \qquad x_1\, x_2 \ldots x_m$$

will be linearly-independent.

Then, let:

$$ax_1 \ldots x_l + b\, x_1 \ldots x_l\, x_{l+1} + c\, x_1 \ldots x_l\, x_{l+1}\, x_{l+2} + \ldots = 0,$$

where a is the first non-zero coefficient. One will then have that:

$$b\, x_{l+1} + c\, x_{l+1}\, x_{l+2} + \dots = -\, y$$

is a quantity in the root group (η), and thus a root of zero, and that:

$$x_1 \dots x_l\, (a - y) = 0,$$

so $x_1 \dots x_l = 0$, and therefore one will also have $x_1 \dots x_m = 0$.

V. *The product of any n quantities in a root group of order n* – 1 *will vanish.*

Thus, if $x_1 \dots x_n$ were non-zero then the n quantities:

$$x_1, \qquad x_1\, x_2\,, \qquad x_1\, x_2\, x_3\,, \qquad \dots, \qquad x_1\, x_2 \dots x_n$$

would be linearly-independent, so the order of the group would be $\geq n$.

From this theorem, for any root group (η) there is an invariant number m such that the product of any m quantities of (η) vanishes, but not the product of $m - 1$. If x is a non-vanishing product of $m - 1$ quantities in (η) then xy and yx will be products of m quantities, and therefore zero. The repeated application of that theorem yields (C, 31):

VI. *The basic numbers* $\eta_1, \eta_2, \dots, \eta_{\alpha-1}$ *in a root group of order* $n - 1$ *can be chosen in such a way that the product* $\eta_\alpha\, \eta_\beta$ *becomes a linear combination of the basic numbers whose index is greater than both* α *and* β.

From Theorem V, there is, moreover, an invariant number l ($< m$) for a root group (η) that is the smallest one for which the l^{th} power of any quantity in the group vanishes. For a commutative group, one will have $l = m$. Furthermore, one has the theorem:

VII. *If there are* $n - 1$ *quantities in a root group of order* $n - 1$ *whose product* $x_1\, x_2 \dots x_{n-1}$ *is non-zero then* x_1^{n-1} *will also be non-zero, and the group will be the commutative group that is defined by all entire functions of* x_1 .

From Theorem IV, the $n - 1$ quantities:

$$x_1, \qquad x_1\, x_2\,, \qquad x_1\, x_2\, x_3\,, \qquad \dots, \qquad x_1\, x_2 \dots x_n$$

are the linearly-independent then, so they define a basis for the group (η). Therefore:

$$x_\kappa = a_{\kappa 1}\, x_1 + a_{\kappa 2}\, x_1\, x_2 + \dots + a_{\kappa, n-1}\, x_1\, x_2 \dots x_{n-1} \qquad (\kappa = 1, 2, \dots, n - 1).$$

Since the product of any n quantities in the group (η) vanishes, one will get:

$$x_1\, x_2 \dots x_{n-1} = a_{11}\, a_{21} \dots a_{n-1,\,1}\, x_1^{n-1}$$

upon multiplying these $n - 1$ equations, so x_1^{n-1} will be non-zero. Therefore, when one sets $x_1 = x$:

$$x, x^2, \ldots, x^{n-1}$$

will also define a basis for (η). If one extends it by way of $x^0 = e$ to a group (ε) of order n that consists of the quantities $z = z\, x^0 + z_1\, x + \ldots + z_{n-1}\, x^{n-1}$ then its determinant $|\, S(z)\, | = z^n$ will take on only an elementary divisor.

§ 5.

I shall now turn to the proof of CARTAN's theorem. Let (ε) be a group of order n with a principal unit, let (η) be its radical, and let $n - m$ be its order. m will then be the linear rank of the determinant of (ε):

$$\Theta(x) = |\, S(x)\, | = \Pi\, \Phi^s. \tag{1}$$

Any of its k prime factors can be brought into the form of a determinant:

$$\Phi(x) = |\, x_{\alpha\beta}\, | \qquad (\alpha, \beta = 1, 2, \ldots, r) \tag{2}$$

by a suitable choice of basic numbers. The:

$$m = r^2 + r'^2 + r''^2 + \ldots \tag{3}$$

elements:

$$x_{\alpha\beta}, \quad x'_{\alpha\beta}, \quad x''_{\alpha\beta}, \ldots \tag{4}$$

of the k different determinants Φ, Φ', Φ'', … of degrees r, r', r'', …, resp., are all independent variables. The determinant of the DEDEKIND group ($\mathcal{D}$) that is homomorphic to (ε) is $\Pi\, \Phi^r$, where the exponent r of Φ is equal to the degree of the prime factor.

One can choose n basic numbers of (ε) such that $\varepsilon_{m+1}, \ldots, \varepsilon_n$ define the basis for the radical (η), and the quantities (4) define the coordinates of $\varepsilon_1, \ldots, \varepsilon_m$. I therefore denote these m basic numbers by:

$$\varepsilon_{\alpha\beta}, \quad \varepsilon'_{\alpha\beta}, \quad \varepsilon''_{\alpha\beta}, \ldots, \tag{5}$$

and the $n - m$ ones by $\eta_1, \ldots, \eta_{n-m}$, such that:

$$x = \sum x_{\alpha\beta}\, \varepsilon_{\alpha\beta} + \sum x'_{\alpha\beta}\, \varepsilon'_{\alpha\beta} + \sum x''_{\alpha\beta}\, \varepsilon''_{\alpha\beta} + \ldots + \sum y_r\, h_r \tag{6}$$

is an arbitrary quantity of (ε). The characteristic determinant of $S(x)$ is then:

(7) $$| S(u\,e - x) | = \Pi\, | u\, e_{\alpha\beta} - x_{\alpha\beta} |^s.$$

The relations:

(8) $$\varepsilon_{\alpha\beta}\, \varepsilon_{\beta\gamma} = \varepsilon_{\alpha\gamma}, \quad \varepsilon_{\alpha\delta}\, \varepsilon_{\beta\gamma} = 0, \qquad \varepsilon_{\alpha\delta}\, \varepsilon'_{\beta\gamma} = 0 \qquad (\text{mod } \eta),$$

where β and δ are assumed to be different in the second one, exist between the basic numbers (5). One must show that the basic numbers (5) mod (η) can be changed in such a way that these equations are true absolutely. Let:

$$z = a_1\, \varepsilon_{11} + \ldots + a_r\, \varepsilon_{rr} + a_{r+1}\, \varepsilon'_{11} + \ldots + a_{r+r'}\, \varepsilon'_{r'r'} + \ldots,$$

or when one sets:

(9) $$\varepsilon_{11} = \varepsilon_1, \ldots, \qquad \varepsilon_{rr} = \varepsilon_r, \qquad \varepsilon'_{11} = \varepsilon_{r+1}, \ldots, \varepsilon'_{r'r'} = \varepsilon_{r+r'}, \ldots$$

$z = \sum \varepsilon_\lambda\, a_\lambda$. The:

(10) $$p = r + r' + r'' + \ldots$$

coordinates a_λ can be chosen in any well-defined way, except that they should all be different from each other. Thus, one will have:

$$\varphi(u) = | S(u\, e - z) | = (u - a_1)^s \ldots (u - a_r)^s\, (u - a_{r+1})^{s'} \ldots (u - a_{r+r'})^{s'} \ldots,$$

and when $f(u)$ is a complete function of u, from (8), one will have:

$$f((z)) = \sum \varepsilon_\lambda\, f(a_\lambda) \qquad (\text{mod } \eta)\,.$$

As in § 3, (3), I now determine p entire functions $f_\lambda(u)$ in such a way that $(u - a_1)^s$ $f_1(u)$ is divisible by $f(u)$ and $f_1(u) - 1$ is divisible by $(u - a_1)^s$. One will then have:

(11*) $$f_\lambda((z))^2 = f_\lambda((z)), \qquad f_n((z))\, f_\lambda((z)) = 0$$

and

(12*) $$\sum f_\lambda((z)) = e.$$

Moreover, $f_\lambda((z)) = \varepsilon_\lambda$ (mod η), and therefore one can change ε_λ (mod η) in such a way that one will have $f_\lambda(z) = \varepsilon_\lambda$. One will then have:

(11) $$\varepsilon_{\alpha\alpha}^2 = \varepsilon_{\alpha\alpha}\,, \qquad \varepsilon_{\alpha\alpha}\, \varepsilon_{\beta\beta} = 0, \qquad \varepsilon_{\alpha\alpha}\, \varepsilon'_{\beta\beta} = 0,$$

and

(12) $$\sum \varepsilon_\lambda = e.$$

Now, from (8), one will have:

$$\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\alpha}\, \varepsilon_{\alpha\beta}\, \varepsilon_{\beta\beta}\,, \qquad \varepsilon'_{\alpha\beta} = \varepsilon'_{\alpha\alpha}\, \varepsilon'_{\alpha\beta}\, \varepsilon'_{\beta\beta}\,, \qquad \ldots \qquad (\text{mod } \eta).$$

If $\alpha = \beta$ then the equations will be valid absolutely. If α is different from β then one will replace $\varepsilon_{\alpha\beta}$ with $\varepsilon_{\alpha\alpha}\varepsilon_{\alpha\beta}\varepsilon_{\beta\beta}$. In that way, these basic numbers (mod η) will be changed in such a way that from (11) one will now have:

(13) $$\varepsilon_{\alpha\alpha}\varepsilon_{\alpha\beta} = \varepsilon_{\alpha\beta}, \qquad \varepsilon_{\alpha\beta}\varepsilon_{\beta\beta} = \varepsilon_{\alpha\beta},$$

and

(14) $$\varepsilon_{\alpha\beta}\varepsilon_{\delta\gamma} = 0, \qquad \varepsilon_{\alpha\beta}\,\varepsilon'_{\delta\gamma} = 0,$$

when δ is different from β in the first equation.

I shall alter the basic numbers $\varepsilon_{\alpha\alpha}$, , $\varepsilon'_{\alpha\alpha}$ … no further, but only the basic numbers $\varepsilon_{\alpha\beta}$, $\varepsilon'_{\alpha\beta}$, …, and only in such a way that the relations that were obtained already will remain valid. If η is an arbitrary quantity in (e) then I will call:

(15) $$\varepsilon_{\alpha}\,\eta\,\varepsilon_{\lambda} = \eta_{\alpha\lambda}$$

a quantity of type (κ; λ). It satisfies the equations:

(16) $$\varepsilon_{\kappa}\,\eta_{\kappa\lambda} = \eta_{\kappa\lambda}, \qquad \eta_{\kappa\lambda}\,\varepsilon_{\lambda} = \eta_{\kappa\lambda},$$

and when μ is different from κ; and ν is different from λ:

(17) $$\varepsilon_{\mu}\,\eta_{\nu\lambda} = 0, \qquad \eta_{\kappa\lambda}\,\varepsilon_{\nu} = 0 .$$

If η is a quantity in the invariant subgroup (η) then $\eta_{\kappa\lambda}$ will also belong to the radical. One can then alter $\varepsilon_{\alpha\beta}$ by an arbitrary root quantity $\eta_{\alpha\beta}$ of type (α, β) without changing equations (13) and (14).

Now, if $\varepsilon_{\alpha\beta}\,\varepsilon_{\beta\alpha} = \varepsilon_{\alpha\alpha}$, so:

(18) $$\varepsilon_{\alpha\beta}\,\varepsilon_{\beta\alpha} = \varepsilon_{\alpha\alpha} = \eta_{\alpha\alpha},$$

in which $\eta_{\alpha\alpha}$ is a quantity in (η), and in fact, from (13), one of type (α, α). Therefore, the right-hand side will equal be to $\varepsilon_{\alpha\alpha}(e - \eta_{\alpha\alpha})$. It follows from the equation:

$$e - x^l = (e - x)\,(x^0 + x + \ldots + x^{l-1})$$

that:

$$e = (e - x)\,(x^0 + x + \ldots + x^{l-1})$$

when $x^l = 0$. Now, if $\eta_{\alpha\alpha}^l = 0$ then one multiplies equation (18) on the right by:

$$\eta_{\alpha\alpha}^0 + \eta_{\alpha\alpha} + \ldots + \eta_{\alpha\alpha}^{l-1} .$$

One then obtains:

$$\varepsilon_{\alpha\beta}\,(\varepsilon_{\beta\alpha} + \eta_{\beta\alpha}) = \varepsilon_{\alpha\alpha},$$

where:

$$\eta_{\beta\alpha} = \varepsilon_{\beta\alpha}\,(\eta_{\alpha\alpha} + \eta_{\alpha\alpha}^2 + \ldots + \eta_{\alpha\alpha}^{l-1})$$

is a root quantity of type (β, α). If one then changes $\varepsilon_{\beta\alpha}$ by $\eta_{\beta\alpha}$ then one will get:

(19) $$\varepsilon_{\alpha\beta}\,\varepsilon_{\beta\alpha} = \varepsilon_{\alpha\alpha} .$$

However, one will also always have (C, 52):

(20) $$\varepsilon_{\beta\alpha}\,\varepsilon_{\alpha\beta} = \varepsilon_{\beta\beta} .$$

One will then have $\varepsilon_{\beta\alpha}\,\varepsilon_{\alpha\beta} = \varepsilon_{\beta\beta} - \eta_{\beta\beta}$, and that will yield $\varepsilon_{\beta\alpha}\,(\varepsilon_{\alpha\beta} + \eta_{\alpha\beta}) = \varepsilon_{\beta\beta}$, as it did just now, so when one multiplies in the left by $\varepsilon_{\alpha\beta}$, one will get $\varepsilon_{\alpha\beta} + \eta_{\alpha\beta} = \varepsilon_{\alpha\beta}$ and therefore $\eta_{\alpha\beta} = 0$.

One now chooses $\varepsilon_{12}, \ldots, \varepsilon_{1\nu}$ in any way (C, 67) according to the relations that were posed up to now, and then chooses $\varepsilon_{21}, \ldots, \varepsilon_{\nu 1}$ in such a way that $\varepsilon_{1\alpha}\,\varepsilon_{\alpha 1} = \varepsilon_{11}$, and as a result one will also have $\varepsilon_{\alpha 1}\,\varepsilon_{1\alpha} = \varepsilon_{\alpha\alpha}$. One will then have the equation:

(21) $$\varepsilon_{\alpha\beta} = \varepsilon_{\alpha 1}\,\varepsilon_{1\beta},$$

when $\alpha = 1$ or $\beta = 1$ or $\alpha = \beta$, so for the cases that we have already imposed upon $\varepsilon_{\alpha\beta}$. However, if α and β are different and both of them are greater than 1 then one will have $\varepsilon_{\alpha\beta} = \varepsilon_{\alpha 1}\,\varepsilon_{1\beta} + \eta_{\alpha\beta}$, where $\eta_{\alpha\beta}$ is a root quantity of type (α, β). One can therefore change $\varepsilon_{\alpha\beta}$ in such a way that the equation (21) is fulfilled. One will then have:

$$\varepsilon_{\alpha\beta}\,\varepsilon_{\beta\gamma} = \varepsilon_{\alpha 1}(\varepsilon_{1\beta}\,\varepsilon_{\beta 1})\,\varepsilon_{1\gamma} = \varepsilon_{\alpha 1}\,\varepsilon_{11}\,\varepsilon_{1\gamma} = \varepsilon_{\alpha 1}\,\varepsilon_{1\gamma} = \varepsilon_{\alpha\gamma} .$$

Equations (8) are now all true absolutely, and therefore, the m quantities (5) will be the basic numbers of a group ($\mathcal{D}$).

From (12), one now has:

$$\eta = (\varepsilon_1 + \ldots + \varepsilon_p)\;\eta\,(\varepsilon_1 + \ldots + \varepsilon_p) = \sum (\varepsilon_\kappa\,\eta\,\varepsilon_\lambda).$$

One can therefore compose all of the root quantities from the $(n - m)\,p^2$ quantities $\varepsilon_\kappa\,\eta\,\varepsilon_\lambda$. If one chooses a system from them independently then each of the basic numbers of the radical will belong to a certain type (κ; λ). In the subgroup ($\mathcal{D}$), $\varepsilon_{\alpha\beta}$ will have type (α, β) when α and β are two of the numbers from 1 to r. By contrast, it will contain, e.g., no quantity of type (1, $r + 1$).

If the n basic numbers of (ε) are chosen in the given way then from (17) all quantities $\xi_{\kappa\lambda}$ of type (κ; λ) would be expressed in terms of only basic numbers that have the same type. The basic numbers of type (1, 1) define the basis for a group, for which $\varepsilon_1 = \varepsilon_{11}$ is the principal unit, and at the same time, the only unit. The remaining basic numbers of type (1, 1) define the basis for its radical. If a and b are two numbers 1, 2, …, r then each of the two equations:

$$\varepsilon_{\alpha\beta}\,\xi_{\beta\lambda} = \xi_{\alpha\lambda} , \qquad \varepsilon_{\beta\alpha}\,\xi_{\alpha\lambda} = \xi_{\beta\lambda} ,$$

would be a consequence of the other one. One would therefore obtain (C, 57) all quantities of type (β, λ) when one multiplied $\varepsilon_{\beta\alpha}$ by all quantities of type (α, λ).

Moreover, all basic numbers of types:

$$(1, 1), (1, r + 1), (1, r + r' + 1), \ldots, (r + 1, 1)\ (r + 1, r + r), (r + 1, r + r' + 1), \ldots,$$
$$(r + r' + 1, 1), (r + r' + 1, r + 1), (r + r' + 1, r + r' + 1), \ldots$$

collectively define the basis for a group (ε'), where the k quantities of ($\mathcal{D}$) that are contained in it, namely, ε_1, ε_{r+1}, $\varepsilon_{r+r'+1}$, …, define the basis for a commutative DEDEKIND group ($\mathcal{D}'$), while the remaining ones define the basis for a root group (η') that is an invariant subgroup of (ε'). From § 1, (η') is then the radical of (ε'). Since ($\mathcal{D}'$) is a commutative group, the determinant of (ε') will decompose into nothing but linear factors. CARTAN (C, 57-60) has explained how one can derive the relations between the basic numbers of the group (ε) from the ones between the basic numbers of (ε') so thoroughly that I would not like to go into that here.