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Fundamentals of the projective differential geometry of complexes and congruences of lines

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1. The scope of the present study is to give a system of first-order differential forms that are invariant under collineations and to give the "*intrinsic* significance" (i.e., independent of the choice of coordinate variables) that would distinguish a congruence or complex of lines, as well as to geometrically interpret the results thus-obtained and write down the differential equations whose aforementioned forms would allow one to get back to the congruence or complex. The integrability conditions for them are *the analogue of the Gauss-Codazzi equations* in the metric geometry of surfaces that pertains to the projective geometry of line systems. There are three such forms of degree two that are coupled by certain conjugacy conditions (viz., apolarity), or, if one prefers, in the case of congruences, there will be just two forms, one of which has degree two and the other of which has degree four. *The search for all the projective invariants of a congruence or complex is thus reduced to the search* (as would one perform using classical methods) *for the invariants of such a system of forms.* The first two forms that relate to a complex are preserved, not only by collineations, but also *by projective deformations* and can be defined in metric geometry to be *completely determined by the complex and to be preserved under all such transformations.* Those two quadratic forms determine three systems of ∞^1 lines of the complex that are the analogues of the lines of curvature of a surface. It is not difficult to extend the notions of geodetic, curvature, etc., to both complexes and congruences, and to thus develop a theory for them that is analogous to the theory that was developed for surfaces. The method that I have proposed in the papers that I published in the Annali di Matematica can serve, e.g., to determine all complexes with a continuous Lie group of projective deformations into themselves, etc.

We use the algorithm of Ricci's absolute calculus and contravariant differentials ([1]), about which, we recall only that if x is a function of u_r then the elementary formula $d^2x = \sum x_i\, d^2u_i + \sum x_{rs}\, du_r\, du_s$ will continue to be true if one writes the covariant derivatives and contravariant differentials in place of x_{rs} and d^2u, resp.

The method that is proposed can be applied to *all problems that relate to geometric entities whose coordinates are coupled by a quadratic relation:* for example, to hypersurfaces and systems of spheres or hyperspheres with respect to the conformal group of a Euclidian space ([2]).

([1]) Cf., a paper that I published in the Atti della R. Accad. d. Scienza di Torino.

([2]) One will find nothing new in regard to surfaces in ordinary space. Cf., the final paragraph of my paper: "Applicabilità proiettiva di due superficie," Rend. Circ. Matem. di Palermo **41** (1916).

2. Line systems. We denote the projective coordinates of a line by x, y, z, p, q, r, which are assumed to be coupled by ([1]):

$$\text{S}\, x^2 = x^2 + y^2 + z^2 + p^2 + q^2 + r^2 = 0. \tag{1}$$

However, it is not pointless to introduce complex quantities, because we will prefer to compare only the squares of p, q, r, and their derivatives.

A *line system* is defined by giving the x, etc. (i.e., the x, y, …, r) as functions of n parameters a_i ($i = 1, \dots, n$). $n = 1$ for ruled surfaces, which we shall not study, and $n = 2$ for congruences; $n = 3$ for complexes. Obviously, it follows from (1) that:

$$\text{S}\, x\, x_j = 0 \qquad (j \leq n). \tag{2}$$

Set:

$$\varphi = \sum a_{js}\, du_j\, du_s = \text{S}\, dx^2. \tag{3}$$

Suppose that *the discriminant* $\Delta \neq 0$ (which will exclude the congruences with coincident focal surfaces and the complexes of tangents to a surface). Let A_{js} denote the algebraic complement of a_{js} in Δ, divided by Δ, and let $\left\{ \begin{matrix} i\,k \\ l \end{matrix} \right\}$ denote the Christoffel symbols of the second kind. If one uses the second covariant derivatives of φ then (2), (3) will imply that:

$$a_{js} = \text{S}\, x_j\, x_s = -\,\text{S}\, x\, x_{js}, \qquad \text{S}\, x_j\, x_{st} + \text{S}\, x_s\, x_{jt} = -\,\text{S}\, x_t\, x_{js} - \text{S}\, x\, x_{jst} = a_{jst} = 0 \tag{4}$$

(since the covariant derivatives of a_{js} are zero). One deduces immediately that:

$$\text{S}\, x_j\, x_{st} = \text{S}\, x\, x_{jst} = 0. \tag{5}$$

Set:

$$\Delta_2 x = \sum A_{js}\, x_{js}, \qquad \Delta_1 x = \sum A_{js}\, du_j\, du_s,$$
$$D_2\, x = \sum A_{js}\, x_{js}, \qquad D_3\, x = \sum x_{jst}\, du_j\, du_s\, du_t. \tag{6}$$

From (5), one will have:

$$-\,\text{S}\, x_{jh}\, x_{st} = \text{S}\, x_j\, x_{sth}, \tag{7}$$

so

$$-\,\text{S}\, (D_2\, x)^2 = \text{S}\, dx\, D_3\, x. \tag{8}$$

The coordinates x, y, … are *determined up to a factor*. What will happen if one substitutes $\bar{x} = \rho\, x$, $\bar{y} = \rho\, y$, etc.? The new value $\bar{\varphi}$ from (3) will obviously be $\rho^2 \varphi$; the $\bar{x}_r$ will be the covariant derivatives of $\bar{x} = \rho\, x$ with respect to not only the φ, but also $\bar{\varphi} = \rho^2 \varphi$. If one sets $\varepsilon_{hh} = 1$, $\varepsilon_{hk} = 0$ for $h \neq k$ and denotes the new values of our expressions by an overbar then one will easily find that:

([1]) A sum will be denoted by S or $\sum$ according to whether its addends are obtained from each other by substituting the y, z, etc., for x or by varying the indices.

(9)
$$\begin{cases} \bar{x} = \rho x, \ \bar{\varphi} = \rho^2 \varphi, \ A_{hk} = \dfrac{1}{\rho^2} a_{hk}, \ \bar{\Delta} = \rho^3 \Delta, \ \bar{a}_{rs} = \rho^2 a_{rs}, \\ \overline{\begin{pmatrix} h\,k \\ l \end{pmatrix}} = \begin{pmatrix} h\,k \\ l \end{pmatrix} + \varepsilon_{hl} (\log \rho)_k + \varepsilon_{hl} (\log \rho)_k - a_{hk} \, \underset{m}{\mathrm{S}} \, A_{lm} (\log \rho)_m, \\ \bar{x}_{rs} = \rho x_{rs} + x\left(\rho_{rs} - \dfrac{2}{\rho} \rho_r \rho_s + \dfrac{1}{\rho} a_{rs} \Delta_1 \rho \right) + a_{rs} \, \underset{l,m}{\mathrm{S}} \, A_{lm} \rho_l x_m \\ \overline{\Delta_2 x} = \dfrac{1}{\rho} \Delta_2 x + x\left(\dfrac{1}{\rho^2} \Delta_2 \rho + \dfrac{a(?)2}{\rho} \Delta_1 \log \rho \right) + \dfrac{a}{\rho^2} \sum A_{lm} \rho_l x_m \\ \overline{D_2 x} = \rho D_2 x + x\left(D_2 \rho - \dfrac{2}{\rho} d\rho^2 \right) + \dfrac{\varphi}{\rho} x \Delta_1 \rho + \varphi \sum A_{lm} \rho_l x_m, \end{cases}$$

(and analogously for y, etc. Note that ρ_{rs} is calculated from φ.)

We deduce some immediate consequences. If we take:

(10)
$$X = \frac{1}{n} \Delta_2 x \qquad \text{then} \qquad \mathrm{S}\, X\, x = -1, \qquad \mathrm{S}\, X\, x_j = 0.$$

For any value of r, s, the linear complex whose coordinates are $\bar{x}_{rs} - \bar{a}_{rs} \bar{X}$ *, etc. will vary in the pencil of complexes that is determined by the complexes whose coordinates are* $x_{rs} - X\, a_{rs}$ *, etc., and the* (special) *complex whose coordinates are x, etc.* (with the noted exception of the quantities that are deduced by substituting $y, z, p, \ldots$ for x).

(11)
$$\bar{x}_{rs} - \bar{a}_{rs} \bar{X} = \rho\,(x_{rs} - X\, a_{rs}) + x \left[\rho_{rs} - \frac{2}{\rho} \rho_r \rho_s + \frac{2}{n} a_{rs} \Delta_1 \rho - \frac{a_{rs}}{n} \Delta_2 \rho \right].$$

The complex whose coordinates are $\bar{X}$ *, etc., varies in the linear system that is defined by the complexes* x*,* x_s*,* X*. That line system is really* ∞^{n+1} *systems, since the* x*,* x_s*,* X *define* $n + 2$ *linearly-independent systems.* If one had:

$$ax + \sum_s b_s x_s + cX = 0 \qquad \text{(and analogously for } y, \ldots),$$

multiplied that by x and summed over the analogous expressions then one would deduce from (2) and (10) that $c = 0$. If one multiplies by x_t and sums over the analogous expressions then one will have $\sum_s b_s a_{st} = 0$ for any value of t, and since $\Delta \neq 0$, one will also have $b_s = 0$; it will then follow that $a = 0$.

The complex whose coordinates are:

(12) $$\xi = D_2 x + x \, \mathrm{S} \, X \, D_2 x \qquad \text{(and analogously for } \eta, z, \pi, x, \rho\text{)},$$

in which the du_i *are consider to be parameters coupled by* $\varphi = 0$, *will remain unchanged, and for their coordinates one will have simply* $\bar{\xi} = \rho\,\xi$. From (2), (5), (10), one will have:

(13) $$\mathrm{S}\,\xi x = \mathrm{S}\,\xi x_r = \mathrm{S}\,\xi X = 0.$$

Any complex ξ *is in involution with the* ∞^{n+1} *linear systems that were considered previously. If* $n = 2$ *then because there are two such complexes* ξ *they will be denoted by* ξ *and* ξ'. *If* $n = 3$ *then there will be only one such complex* ξ, which will be confirmed in the following calculations (although it would seem that there should be ∞^1 of them that depend upon three parameters du_s that are coupled by $\varphi = 0$). The fact that there are two complexes ξ for $n = 2$ is obvious from the fact that $\varphi = 0$ is a second-degree equation. Indeed, let $R_1 : R_2$ and $R_1' : R_2'$ be the two values of $du : dv$ that annul φ. We can assume that the R_i, as well as the R_i', transform like the du_i – i.e., they define a contravariant system. The R, R' are determined up to a contact (*tattore*), and we will find a further indeterminacy upon observing that $\sqrt{\Delta}(R_1 R_2' - R_2 R_1')$ will remain invariant under changes of the coordinate variables u_j, if we demand that the expression should be equal to $i = \sqrt{-1}$ ([1]). With that convention, it will follow immediately that:

(14) $$\begin{cases} \sqrt{\Delta}(R_1 R_2' - R_2 R_1') = i : R_? R_?' = \frac{1}{2} A_? : R_1 R_2' + R_2 R_1' = A_{12}, \\ \sum a_{hk} R_h R_k' = 1 : A_{11} R_2 R_2' - 2 A_{12} (R_1 R_2' + R_2 R_1') + A_{22} R_1 R_1' = \dfrac{1}{\Delta}. \end{cases}$$

3. Line complexes. (13) determines the ξ, etc., up to a common factor. In order to determine those coordinates *intrinsically*, we can set them equal to the complements of the ξ in the determinant $(x, x_1, x_2, x_3, X, \xi)$ (the quantities in parentheses are written in the first row, while the other ones are deduced by substituting the y, z, … for the x), divided by $\sqrt{\Delta}$. (If $\Delta < 0$ and the complex is real then one can get a real entity by dividing by $\sqrt{-\Delta}$.) Note that: *The* ξ, η, … , *thus-defined, remain invariant under not only changes of the variables* u_s, *but also under multiplication of the* x, y, … *by an arbitrary factor*.

From the rule for squaring a matrix, (2), (4), (10) will give:

([1]) That expression is imaginary [*illegible*], since $R_1 : R_2$ and $R_1' : R_2'$ are complex conjugates in this case. I shall not give the real entities here, and especially since, as one will see, the essential part of this study will be concerned with expressions that are always *real* for real complexes.

$$\text{(15)} \qquad \mathrm{S}\,\xi^2 = \frac{1}{\Delta}\begin{vmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & a_{11} & a_{12} & a_{13} & 0 \\ 0 & a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{31} & a_{32} & a_{33} & 0 \\ -1 & 0 & 0 & 0 & \mathrm{S}\,X^2 \end{vmatrix} = -1,$$

which makes it obvious that $\bar{\xi} = \xi$.

With our hypothesis that $\Delta \neq 0$, *our complex* ξ *will never be special.* We see the geometric significance of this quite quickly. The complexes whose coordinates are x, x_r, X, ξ are linearly independent. That is because if one had $ax + bX + c\xi + \sum h_r\, x_r = 0$, and analogously for $y, \ldots$, and one multiplied by ξ and summed over the analogous expressions then one would find from (13), (15) that $c = 0$. One would therefore also have that $a = b = h_r = 0$, which would then show us that the x, X, x_r define a linearly-independent system. Thus, any six quantities – in particular, the x_{rs}, y_{rs}, etc. – can be written in the form:

$$x_{rs} = \alpha_{rs}\, X + b_{rs}\, x + c_{rs}\, \xi + \sum_t l_{rs}^t x_t \qquad \text{(and analogously in } y, \ldots),$$

in which, α, b, l are quantities to be determined. If one multiplies this by x and sums over the analogous expressions then one will find from (1), (4), (10), (13) that $\alpha_{rs} = a_{rs}$. If one multiplies by x_h and sums then one will find from (5) that $\sum_t l_{rs}^t a_{ht} = 0$ for any r, s, h. Since $\Delta \neq 0$, one will have $l_{rs}^t = 0$. Therefore:

$$\text{(16)} \qquad x_{rs} = a_{rs}\, X + b_{rs}\, x + c_{rs}\, \xi, \qquad \text{namely,} \qquad D_2 x = \varphi\, X + x\,\psi + \xi\,\chi,$$

where

$$\text{(16, cont.)} \qquad \chi = \sum c_{rs}\, du_r\, du_s = -\frac{1}{\sqrt{\Delta}}(x, x_1, x_2, x_3, X, D_2\, x), \qquad \psi = \sum b_{rs}\, du_r\, du_s\,.$$

These are *the fundamental formulas* that allow one to solve for the complex – e.g., for the forms φ, ψ, χ. If one multiplies the $x, y, \ldots$ by the same factor ρ then (9), (11), (15), (16) will give:

$$\text{(17)} \qquad \bar{\varphi} = \rho^2\,\varphi, \qquad \bar{\chi} = \rho\chi, \qquad \bar{\psi} = \rho\,\psi + D_t\,\rho - \frac{2}{\rho}\,d\rho^2 + \varphi\left(\frac{2}{3\rho}\Delta_1\rho - \frac{1}{3}\Delta_2\rho\right).$$

We see how one can *remove the indeterminacy in* φ, ψ, χ. The function ψ, which has the most complicated behavior, is *the least important*, as one sees. *As for the rest of them, one can make them proportional to* χ, if one desires, by choosing one of the line coordinates to be equal to 1. (For example, let $x = 1$ and $x_{rs} = D_2 x = X = 0$. From (16), ψ will be equal to $\lambda\chi$, where $\lambda = -\xi$.) Multiplying (16) by A_{rs} and summing the results that are obtained varying the indices r, s, one will find that *the forms* ψ, χ *are conjugate to the reciprocal of* φ *; i.e.:*

(18) $$\sum A_{rs}\, b_{rs} = \sum A_{rs}\, c_{rs} = 0.$$

However, it does not seem appropriate to make, e.g., $x = 1$, since that equality would not be preserved under projective transformations. Rather, consider the third-degree equation in ω that is obtained when one makes the determinant $|\, \omega\, a_{rs} - c_{rs} \,|$ (viz., the discriminant of $\omega\varphi - \chi$) equal to zero. If the three roots of that equation are zero then if one thinks of the du_r as representing homogeneous coordinates of the points in a plane σ then $\varphi = 0$, $\psi = 0$ will represent two conics C_φ , C_ψ , the second of which is *conjugate* to the first one, which is thought of as its envelope. Therefore, *if the three roots* ω_s *are zero for every s then either the form c is identically zero* (in which case, the complex must be *linear*) *or* C_χ *will degenerate into a line that is tangent to* C_φ *and another line that passes through the point of contact.* That case, which we shall call the *abnor.mal* case, must be studied separately. In the general case (viz., the *normal* case), the roots ω_i will change to $\bar{\omega}_i = \omega_i / \rho$ when one multiplies the x, y, … by ρ. We can determine ρ in a *rational and intrinsic* manner by demanding that a symmetric function of the ω_i [e.g., the one that presents itself as the denominator in the formula that results from solving equations (19)] *should be equal to unity*. *The other two symmetric functions*, independently of the preceding, *will be two projective invariants of the complex* (which I believe have not been noticed up to now), and which can be called *the projective curvature* of the complex. Fixing ρ, it will remain for us to determine, in an intrinsic way, the *coordinates* that one calls *normal* of a line of the complex, *which are subjected to only orthogonal transformations with constant coefficients and unity determinant under collineations.* The forms φ, ψ, χ also remain determinate, each of which define a *metric geometry that is completely special to the complex and invariant under collineations.* (So far, we had generalized only the notion of *angle:* It was defined by the metric that had φ for its linear element.)

In addition to the *abnormal* case, we also exclude the one in which the conics C_ψ, C_χ *are bitangents;* those cases are quite simple, but they must be studied separately. In the other cases, one can show that (16) is equivalent to *system of total differential equations*. Indeed, if one lets (st, rp) denote the four-index Riemann symbols for φ then from a formula of Ricci's in the absolute calculus, the integrability conditions for (16) will be:

$$x_{rst} - x_{rts} = -\sum_{p,q} (st, rp)\, A_{pq}\, x_q \,,$$

which will become:

(19) $$a_{rs}\, X_t - a_{rt}\, X_s + c_{rs}\, \xi_t - c_{rt}\, \xi_s = \\ = (c_{rts} - c_{rst})\, \xi + (c_{rts} - c_{rst})\, \xi = b_{rs}\, x_t + b_{rt}\, x_s - \sum_{p,q} (st, rp)\, A_{pq}\, x_q$$

in the present case, and when that is solved for x_s, x_t, X_s, X_t, it will give them *as linear combinations of the* x, x_p, ξ (1). (16), (19) *then constitute a system of total differential equations that permit one to determine a complex of given form* φ, ψ, χ. *The integrability conditions* (which I shall not write down, for the sake of space limitations) *are the analogues of the equations of the Gauss-Codazzi equations for the metric geometry of surfaces in the projective geometry of complexes.* The expression:

$$\text{(16, cont.)} \qquad \chi = -\frac{1}{\sqrt{\Delta}}\,(x, x_1, x_2, x_3, X, D_2\,x),$$

when squared, will give a simple expression for χ^2 that one can also deduce from (16), if one recalls (15):

$$\text{(20)} \qquad \begin{aligned}\chi^2 &= -\,\mathrm{S}\,(D_2\,x - \varphi\,X - x\psi)^2 = -\,\mathrm{S}\,(D_2\,x - \varphi\,X)^2 \\ &= -\,\mathrm{S}\,(D_2\,x)^2 + 2\varphi\,\mathrm{S}\,X\,D_2\,x - \varphi^2\,\mathrm{S}\,X^2\end{aligned}$$

since $0 = \mathrm{S}\,x^2 = \mathrm{S}\,x(D_2\,x - \varphi\,X)$.

In order to compare this with the theory of congruences, one notes that if one sets:

$$\text{(21)}\begin{cases} h_{rspq} = -\mathrm{S}\,x_{rs}x_{pq} & \text{then one will have:}\ -\mathrm{S}(D_2x)^2 = \sum h_{rspq}du_r du_s du_p du_q, \\ -\sum X^2 = \tfrac{1}{9}\sum h_{rspq}A_{rs}A_{pq} & \qquad -\mathrm{S}\,X\,(D_2x)^2 = -\tfrac{1}{3}\sum A_{rs}h_{rspq}du_p du_q. \end{cases}$$

4. Geometric interpretation. Projective deformation of a complex. Let $ax + by + cz + lp + mq + sr = 0$ (a, b, c, l, m, s = const.) be a linear complex Γ that is tangent to the given complex C along a certain line r. One will then have not only $\mathrm{S}\,ax = 0$ (which is just a concise way of writing down the complex Γ), but also $\mathrm{S}\,ax_r = 0$. The complex Γ will cut the given complex C at another line that is infinitely close to r and is determined by $\mathrm{S}\sum a\,x_{rs}\,du_r\,du_s = 0$; i.e., $\varphi\,\mathrm{S}\,aX + \chi\,\mathrm{S}\,a\xi = 0$. If one thinks of the a as the coordinates of a point in a three-dimensional space σ then that equation will determine a pencil of quadric cones whose vertex is the imaginary point of r. Each of those cones

(1) In order to see that (19) can be solved for *the generic line* $u_i = u_i^0$, one can, e.g., reduce the φ, χ to some canonical form for $u_i = u_i^0$. If the conics C_χ, C_φ have just one point in common (e.g., the point $du_1 = du_2 = 0$) then φ will reduce to the form $du_2^2 + 2\,du_1\,du_3$, while χ will be of the type $\beta\,(du_2^2 + 2\,du_1\,du_3) + \alpha\,du_{(?)}^2$, where, from (18), (?) = 0; this is then the *abnormal* case. If the two conics have the point $du_2 = du_3 = 0$ in common then χ will be of the type $\alpha\,du_2^2 + 2\beta\,du_{(?)}\,du_2$ (± ?) $2\gamma\,du_{(?)}\,du_{(?)} + 2\lambda\,du_2\,du_3$. If the point $du_1 = du_2 = 0$ is the point of contact then the line $\frac{1}{du_2}(\chi - \gamma\varphi) = 0$ [i.e., $(\alpha - \gamma)\,du_2 + 2\beta\,du_1 + 2\lambda\,du_3 = 0$] must pass through it; thus, $\lambda = 0$. Hence, from (18), $\alpha + 2\gamma = 0$ (while they are in the abnormal case, for which the conics are bitangents) and $\alpha \neq 0$, $\beta \neq 0$. One sees easily that (19) are soluble in that case – i.e., that C_φ, C_χ also have four distinct intersections. As one sees, in that case, one can suppose that (for $u_i = u_i^0$) $\varphi = du_1^2 + du_2^2 + du_3^2$, $\psi = \alpha\,du_2^2 + \beta\,du_3^2 + \gamma\,du_3^2$ [*illegible*] $\alpha \neq \beta \neq \gamma \neq \delta$.

corresponds to a linear complex Γ that is tangent to C at r, and vice versa. *The complex ξ is geometrically the complex that corresponds to a quadric cone $\chi = 0$ that is apolar or conjugate to the cone $\varphi = 0$, which is thought of as its envelope. The linear system of complexes that is defined by the complexes x, x_r, X is the system of complexes that is in involution with the complex ξ.*

There is another complex C^0 along the line x^0, y^0, …, which are functions of the same parameters a_r – i.e., there is a one-to-one correspondence with C. The two complexes can be projectively mapped to a pair of homologous lines r, r^0, so one can transform one of them with a suitable collineation in such a manner that along r, r^0, one will have:

$$x = \rho\, x^0, \qquad x_s = \rho\,(x_s^0 + m_s\, x^0), \tag{22}$$
$$\frac{\partial^2 x}{\partial u_i\, \partial u_j} = \rho\left(\frac{\partial^2 x^0}{\partial u_i\, \partial u_j} + \mu_i \frac{\partial x^0}{\partial u_j} + \mu_j \frac{\partial x^0}{\partial u_i} + h_{ij}\, x^0\right),$$

with suitable values of ρ, m, μ, h. One deduces directly that along the lines r, r^0, *the forms φ, φ^0 of the two complexes will be proportional.* Supposing that *this condition is satisfied for all values of u* (*so, if one multiplies x^0 by a convenient factor then one can assume that $\varphi = \varphi^0$ identically) is a necessary and sufficient condition for C, C^0 to be mapped projectively to two homologous lines r, r^0 and for the two forms χ, χ^0 to be equal on them.*

In fact, if one sets $\varphi = \varphi^0$ identically then (22) will become, in covariant coordinates:

(22, cont.) $\quad x = x^0,\ \ x_s = x_s^0 + m_s\, x^0, \qquad x_{ij} = x_{ij}^0 + \mu_i\, x_j^0 + \mu_j\, x_i^0 + h_{ij}\, x^0.$

It is enough to recall the value (16, cont.) of χ in order to see that $\chi = \chi^0$.

Conversely: Let $\varphi = \varphi^0$ identically. If one has $\chi = \chi^0$ for the lines r, r^0 then we can transform C^0 with a collineation such that one will have $x = x^0$, $x_r = x_r^0$, $\xi = \xi^0$ for the line considered, and then the expressions S x^2, S $x_r\, x_s$, S ξ^2, S $\xi\, x$, S $\xi\, x_r$, S $x\,\xi$ will have the same values for both complexes; the complex X^0 will belong to the pencil of the two complexes X, x. If one writes down (16) for the two complexes then one will see that for the lines r, r^0, (22, cont.) will give $m_r = \mu_r = 0$ for $\rho = 1$. Q. E. D.

The forms φ, χ collectively constitute the *projective linear element* of the complex. The problem of the projective deformation of a complex – i.e., of determining the forms ψ that are compatible with φ, χ – then reduces to the study of the integrability conditions for (16), (19).

One first studies the case in which χ is identically zero.
