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Fundamentals of the projective differential geometry of complexes and congruences of lines

Note II by correspondent GUIDO FUBINI

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5. Linear complexes. If we use homogeneous coordinates whose first one x is equal to 1 then we will have $x_{rs} = X = 0$, and from (16), $b_{rs} = \lambda\, c_{rs}$, where $\lambda = -\xi$. Suppose we choose parameters u_1, u_2, u_3 such that the last three projective coordinates are p, q, r. Recalling the definition of the covariant derivative, we will have from (16):

$$\binom{h\,k}{1} = -\,p_{hk} = [a_{hk}\,P + e_{hk}\,(\pi + \lambda\,p)],$$

$$\binom{h\,k}{2} = -\,a_{hk}\,P + e_{hk}\,(\kappa + \lambda\,q),$$

$$\binom{h\,k}{3} = -\,a_{hk}\,R + e_{hk}\,(\rho + \lambda\,r)],$$

$$\frac{\partial^2 y}{\partial u_h\,\partial u_k} = y_{hk} + \sum \binom{h\,k}{3} y_r = a_{hk}\,[Y - p\,y_1 - q\,y_2 - r\,y_3]$$
$$+ c_{hk}\,[\eta + \lambda\,r - (p + l\,p)\,y_1 - (\kappa + \lambda\,q)\,y_2 - (\rho + \lambda\,r)\,y_3],$$

which we write, with a self-explanatory notation, as:

$$\frac{\partial^2 y}{\partial u_h\,\partial u_k} = a_{hk}\,\bar{Y} + c_{hk}\,\bar{\eta}, \text{ and analogously } \quad \frac{\partial^2 z}{\partial u_h\,\partial u_k} = a_{hk}\,\bar{Z} + c_{hk}\,\bar{\zeta}\,. \tag{23}$$

Proof of that: *The necessary and sufficient condition for a complex to be linear is that the form χ must be zero.* (16 cont.) proves that the condition is necessary. Conversely, if $\chi = 0$ – i.e., $c_{hk} = 0$ – then (23) will show that if $\bar{Y} = 0$ then the second derivatives of y will be zero; i.e., y will be a linear function of $p = u_1$, $q = u_2$, $r = u_3$, $x = 1$, and the complex will be linear. However, if $\bar{Y} \neq 0$ then (23) will show that one can find a quantity μ such that $\frac{\partial^2 z}{\partial u_h\,\partial u_k} - \mu\,\frac{\partial^2 y}{\partial u_h\,\partial u_k} = 0$. Differentiating with respect u_l and subtracting the result that is obtained by permuting k with l, one will find that

$\mu_l \frac{\partial^2 y}{\partial u_h \, \partial u_k} - \mu_k \frac{\partial^2 y}{\partial u_h \, \partial u_l} = 0$; i.e., from (23), and from the fact that $\bar{Y} \neq 0$, one will deduce that $\mu_l \, a_{hk} - \mu_l \, a_{hk} = 0$, and since $\Delta \neq 0$, one will have $\mu_l = 0$, i.e., μ = const. Hence, $z - \mu \, y$ is a linear function of the u_i, 1 – i.e., of p, q, r, x – and the complex will again be linear.

A very easy study to carry out in general would be to study all of the congruences that are contained in a given complex using our methods and to find the relationships between them that are projectively invariant. We content ourselves by proving that: *If all of the congruences of a given complex are W then the complex will be linear.* A congruence of the complex will be obtained by taking u_3 to be equal to a function $f(u_1, u_2)$ of u_1, u_2 : It will be *W if it determined from the x,* $\frac{dx}{du_r}$, $\frac{d^2 x}{du_h \, du_k}$ (for $h, k = 1, 2$) *is zero.* We write d, instead of ∂, in order to recall that one must consider u_3 to be a function of u_1, u_2 in the derivatives. One will have:

$$(24) \quad \begin{cases} \dfrac{dx}{du_k} = \dfrac{\partial x}{\partial u_k} + x_3 \dfrac{\partial f}{\partial u_k} \quad \text{(and analogously for } y, z, \text{etc.)} \quad (h, k = 1, 2) \\[2ex] \dfrac{d^2 x}{du_h \, du_k} = \dfrac{\partial^2 x}{\partial u_h \, \partial u_k} + \dfrac{\partial^2 x}{\partial u_3^0} \dfrac{\partial f}{\partial u_h} \dfrac{\partial f}{\partial u_k} + \dfrac{\partial^2 z}{\partial u_h \, \partial u_3} \dfrac{\partial f}{\partial u_k} + \dfrac{\partial^2 z}{\partial u_k \, \partial u_3} \dfrac{\partial f}{\partial u_h} + \dfrac{\partial z}{\partial u_3} \dfrac{\partial^2 f}{\partial u_h \, \partial u_k}. \end{cases}$$

Set:

$$\alpha_{hk} = a_{hk} + a_{33} \frac{\partial f}{\partial u_h} \frac{\partial f}{\partial u_k} + a_{3h} \frac{\partial f}{\partial u_k} + a_{3k} \frac{\partial f}{\partial u_h},$$

$$\gamma_{hk} = c_{hk} + c_{33} \frac{\partial f}{\partial u_h} \frac{\partial f}{\partial u_k} + c_{3h} \frac{\partial f}{\partial u_k} + c_{3k} \frac{\partial f}{\partial u_h},$$

so, from (23), one will have:

$$\frac{d^2 x}{du_h \, du_k} = \alpha_{hk} \, \bar{Y} + \alpha_{hk} \, \bar{\eta} + \frac{\partial f}{\partial u_3} \frac{\partial^2 f}{\partial u_h \, \partial u_k} \qquad \text{(and analogously in } z\text{)}.$$

If $x = 1$, $p = u_2$, $q = u_3$, $r = f$ then the determinant that must be zero in order for the congruence to be W will reduce to:

$$\begin{vmatrix} \dfrac{\partial^2 f}{\partial u_1^2} & \dfrac{\partial^2 f}{\partial u_1 \, \partial u_2} & \dfrac{\partial^2 f}{\partial u_2^2} \\[2ex] \dfrac{\partial^2 f}{\partial u_1^2} & \dfrac{d^2 y}{du_1 \, du_2} & \dfrac{d^2 z}{du_2^2} \\[2ex] \dfrac{\partial^2 z}{\partial u_1^2} & \dfrac{d^2 z}{du_1 \, du_2} & \dfrac{d^2 z}{du_2^2} \end{vmatrix} = \begin{vmatrix} \dfrac{\partial^2 f}{\partial u_1^2} & \alpha_{11} & \gamma_{11} \\[2ex] \dfrac{\partial^2 f}{\partial u_1^2} & \alpha_{12} & \gamma_{12} \\[2ex] \dfrac{\partial^2 z}{\partial u_1^2} & \alpha_{23} & \gamma_{22} \end{vmatrix} \cdot \begin{vmatrix} 1 & 0 & 0 \\ y_3 & \bar{Y} & \bar{\eta} \\ z_3 & \bar{Z} & \bar{z} \end{vmatrix}.$$

If the first factor in the right-hand side is zero *for any f* (i.e., for any congruence) then the α will be proportional to the γ, the a, to the e, and φ, to χ. From the conjugacy relations (18), one will have $\chi = 0$, so the complex *will be linear.* However, if one makes the second factor equal to zero then from (23), it will follow that the $\frac{\partial^2 y}{\partial u_i \, \partial u_j}$ (with the crooked ∂) will be proportional to the $\frac{\partial^2 z}{\partial u_i \, \partial u_j}$, and as above, one will again find that the complex is linear.

The formula that was exhibited here can be utilized for the study of the congruences that are contained in a given complex.

6. Congruences. We have two complexes ξ, ξ' that are defined by (12):

(12′) $\xi = D_2 x + x \,\mathrm{S}\, X\, D_2 x$ (in which one sets $du_s = R_s$) (and analogously for η, etc.)
(12″) $\xi' = D_2 x + x \,\mathrm{S}\, X\, D_2 x$ (in which one sets $du_s = R'_s$).

If the $x, y, \ldots$ are not normalized then one will note that the ξ, ξ' will still be multiplied by ρ when the $x, y, \ldots$ are multiplied by a factor ρ.

The expression ([1]):

$$W = \frac{1}{\Delta^2}(x, x_1, x_2, x_{11}, x_{12}, x_{22}) \tag{25}$$

is zero for any W-congruence, and only for them, when one does not change the coordinate variables u_r (i.e., it is *intrinsic*) and will be multiplied by $1 / \rho^2$ when one multiplies the $x, y, \ldots$ by ρ, and therefore, φ by ρ^2 and Δ, by ρ^4. *Except for the W-congruences* (which are regarded as *abnormal*), the calculations will present themselves in a simple way if we choose ρ *in such a way that W will be unity.* We will have complete determinacy if the *normal* coordinates of a line of the congruence have *the form of* φ*, which, when it is assumed to be a linear element, will define a metric that is completely specific to the congruence and will be preserved under projective transformations.* Obviously the determinant:

([1]) We call that expression W precisely because it is zero for the W-congruences (viz., congruences of lines whose coordinates all satisfy the same second-order, homogeneous linear equation.).

$$\frac{1}{\sqrt{\Delta}}(x, x_1, x_2, x_1, X, \xi, \xi') = \frac{\Delta^2 W}{\sqrt{\Delta}}\begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{A_{11}}{2} & A_{12} & \frac{A_{22}}{2} \\ 0 & 0 & 0 & R_1^2 & 2R_1R_2 & R_2^2 \\ 0 & 0 & 0 & R_1'^2 & 2R_1'R_2' & R_2'^2 \end{vmatrix}$$

$$= \frac{\Delta^2}{\sqrt{\Delta}} W(R_1R_2' - R_2R_1')(A_{11}R_2R_2' - A_{12}[R_1R_2' + R_2R_1'] + A_{22}R_2R_2') = i\,W,$$

by virtue of (14). That is:

$$\left[\frac{1}{\sqrt{\Delta}}(x, x_1, x_2, X, \xi, \xi')\right]^2 = -W^2 = -1. \tag{26}$$

One immediately recognizes that (from the rule that pertains to forming the square of a determinant), and by virtue of (1), (2), (4), (10), (13), the left-hand side is equal to a determinant that one can easily calculate to be – (S ξ^2 S ξ'^2 – [S $\xi\xi'$]^2). Therefore:

(26, cont.) $$\text{S}\,\xi^2\,\text{S}\,\xi'^2 - [\text{S}\,\xi\xi']^2 = 1.$$

Set:

(27) $\bar{\xi} = \xi\,\text{S}\,\xi\xi' - \xi\,\text{S}\,\xi'^2$, $\quad \bar{\xi}' = \xi'\,\text{S}\,\xi\xi' - \xi\,\text{S}\,\xi'^2$, and analogously in $\bar{\eta}$, etc.

(If the x, y are not normalized then one sets $\bar{\xi} = \dfrac{\xi\,\text{S}\xi\xi' - \xi'\,\text{S}\xi^2}{\sqrt{\text{S}\xi^2\text{S}\xi'^2 - (\text{S}\xi\xi')^2}}$, etc. The $\bar{\xi}$, $\bar{\xi}'$, thus-defined, like the ξ, ξ', will still be multiplied by ρ when one multiplies the x, etc. by ρ.) From (26) and (27), we will have:

(28) $$\text{S}\,\bar{\xi}^2 = \text{S}\,\xi^2, \quad \text{S}\,\bar{\xi}'^2 = \text{S}\,\xi'^2, \quad \text{S}\,\bar{\xi}\,\bar{\xi}' = \text{S}\,\xi\,\xi', \qquad \text{S}\,\xi\,\bar{\xi} = \text{S}\,\xi'\,\bar{\xi}' = 0.$$

The complexes $\bar{\xi}$, $\bar{\xi}'$ are complexes of the pencil that is determined by the complexes ξ, ξ' and are in involution with ξ and ξ'. Therefore, the complexes ξ, ξ', $\bar{\xi}$, $\bar{\xi}'$ belong to the same pencil in involution with the complexes x, x_1, x_2, X. There exist complexes $\xi + i\bar{\xi}$ in that pencil that are obviously special. That is, from (28), one will have S $(\xi \pm i\bar{\xi})^2$ = 0, *so the two lines whose coordinates are $\xi \pm i\bar{\xi}$ will be the directices of the congruence that is common to the pencil of complexes ξ, ξ', $\bar{\xi}$, $\bar{\xi}'$ and will then be the* *two lines that are common to the complexes* x, x_1, x_2, X. *They are the lines* [1] *that are conjugate tangents at* x *to the two focal sheets* (viz., principal focal lines), *and the lines* $x + \beta\,(\xi \pm i\bar{\xi})$ *will describe the two pencils of focal lines* (viz., tangents to the focal surfaces) *when one varies* β. *The two conjugate pencils – i.e., the pencils of lines that have their centers at the center of one focal pencil and the plane of the other one for their planes – are so-called "central pencils." The complexes* $\bar{\xi}$ *and* $\bar{\xi}'$ *are the complexes that one calls "satellites" or "accompanying," which contain the central pencils that correspond to a point of one focal sheet and the points that are infinitely close to it. The involution that is determined in the focal sheets by the intersections of the two complexes* $\sum x_{ij}\, R_i\, du_j$ *and* $\sum x_{ij}\; R_i'\, du_j$ *is the involution of the conjugate tangents to the focal sheets. (The focal pencils constitute the intersections of the complexes* x, x_1, x_2). This fact can help one write down to the differential equations of the asymptotes to the focal sheets. (The problem of writing all of the other formulas that relate to the focal sheets is more complicated, but not difficult.)

Along with W, we have found some additional invariants for a congruence in the form of $\sum \xi^2$, $\sum \xi'^2$, $\sum \xi\,\xi'$ [which are coupled by (26, cont.)]. However, in order to find the complete system of such invariants and their mutual relationships, we must now pursue a more analytic path.

We can search for the formula $x_{rs} - a_{rs}\, X = b_{rs}\, \xi + c_{rs}\, \xi' + g_{rs}\, x$, as in the case of complexes, and then seek to deduce the derivatives of the X, ξ, ξ'. Along this path, we would find the forms $\sum b_{rs}\, du_r\, du_s$, $\sum c_{rs}\, du_r\, du_s$, $\sum g_{rs}\, du_r\, du_s$, in addition to φ, for the definition of a congruence. We prefer to follow a different route and to obtain *just one* form Φ of degree four, in addition to φ, and if one so desires then it would not be difficult to substitute a form of degree *two* for Φ. In fact, any form of degree four, with the process of *covariant division* that I already studied elsewhere, can be written in one and only one way in the form $\Phi = I\,\varphi^2 + \psi\,\varphi + \chi\bar{\chi}$, where ψ, χ, $\bar{\chi}$ are forms of degree *two* that are conjugate to φ, and the χ, $\bar{\chi}$ are conjugate to each other. (For example, if $\varphi = 2a_{12}\, du_1\, du_2$ – i.e., if the u_1, u_2 are the developables of the congruence – then ψ will have the type $b_{11}\, du_1^2 + b_{22}\; du_2^2$ and the χ, $\bar{\chi}$ will have the type $c_{11}\, du_1^2 \pm c_{22}\; du_2^2$.) Hence, instead of giving Φ, one can give the invariant I and the two forms ψ, χ. (The invariant I will be determined from $W^2 = 1$ when one is given the forms φ, ψ, χ.)

7. The form Φ of degree four. Recalling (8), we will set:

$$\Phi = \sum k_{rspq}\, du_r\, du_p\, du_s\, du_q = \mathrm{S}\; dx\, D_2 x = -\,\mathrm{S}\,(D_2 x)^2. \tag{29}$$

From the rules of differentiation with contravariant differentials, one has:

$$\mathrm{S}\; dx\, D_2 x = \sum a_{rs}\, du_r\, d^2u_s + \Phi, \qquad \mathrm{S}\,(D_2 x)^2 = \sum a_{rs}\, d^2u_r\, d^2u_s - \Phi. \tag{30}$$

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[1] For these theorems, cf., **Wälsch**, Wiener Sitzungsberichte IIA **100** (1891). The complexes $\bar{\xi}$, $\bar{\xi}'$ here are what **Wälsch** calls the *Begleitcomplexe* (auxiliary complexes).

Summing (30) gives $\frac{1}{2} d^2\varphi = \sum a_{rs}\, du_r\, d^2u_s + \sum a_{rs}\, d^2u_r\, d^2u_s$, which is an identity that I already found in *loc. cit.* Any of equations (30) will give a new definition for Φ. All of those definitions will show that Φ, like φ, is *an intrinsic form;* hence, *it is determined completely* when the x, y, … are normal coordinates. If the coordinates are not normalized then upon multiplying $x, y, \ldots$ by ρ, Φ will change into:

(9, cont.) $$\bar{\Phi} = \rho^2\,\Phi + \varphi^2\,\Delta_1\rho + 2\varphi\,(\rho\, D_2\rho - 2\Delta\rho^2).$$

Let (rs, hk) denote the four-index Riemann symbols for φ, so a known formula of Ricci in the absolute calculus will give the identity:

(31) $$x_{rst} - x_{rts} = -\sum_{p,q}(st, rp) A_{pq} x_q\,.$$

Setting [cf., (21)]:

(32) $$h_{rspq} = -\,\mathrm{S}\; x_{rs}\, x_{pq} = \mathrm{S}\; x_r\, x_{pqs}\,,$$

one will deduce: *The h_{rspq} are symmetric in the four indices, except for $h_{1122} = h_{2311}$ and $h_{1212} = h_{2112} = h_{2121} = h_{2121} = h_{2112}$, for which, one will have:*

(33) $$h_{1212} - h_{1122} = (21, 21).$$

Comparing this with (29), and recalling that the k are *symmetric* in the four indices, one deduces that:

(34) $$k_{iiij} = h_{iiij} \qquad (i, j = 1, 2), \qquad h_{1212} = k_{1122} + \tfrac{1}{3}(21, 21), \qquad h_{1122} = k_{1122} - \tfrac{2}{3}(21, 21).$$

Given the forms Φ, φ, all of the h will be determinate.

One deduces the value of W^2 (which we have set equal to 1) immediately from (25):

(35) $$W^2 = \frac{1}{\Delta^2}\begin{vmatrix} 0 & a_{11} & a_{12} & a_{22} \\ a_{11} & h_{1111} & h_{1112} & h_{1122} \\ a_{12} & h_{1211} & h_{1212} & h_{1222} \\ a_{22} & h_{2211} & h_{2212} & h_{2222} \end{vmatrix}.$$

I also say that:

(35, cont.) $$W^2 = \begin{vmatrix} 0 & A_{11} & A_{12} & A_{22} \\ a_{11} & H_{1111} & H_{1112} & H_{1122} \\ a_{12} & H_{1211} & H_{1212} & H_{1222} \\ a_{22} & H_{2211} & H_{2212} & H_{2222} \end{vmatrix},$$

in which:

(36) $$H_{rspq} = \sum_{i,j} A_{ri} A_{sj} h_{ijpq}\,.$$

In fact, if one multiplies the right-hand side of (35, cont.) by:

$$\Delta^2 = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & a_{11}^2 & 2a_{11}a_{12} & a_{12}^2 \\ 0 & a_{11}a_{12} & a_{11}a_{22}+a_{12}^2 & a_{12}a_{22} \\ 0 & a_{12}^2 & 2a_{22}a_{12} & a_{22}^2 \end{vmatrix}$$

then one will find the determinant of the right-hand side of (35).

One will then also find the preceding invariants [coupled by (26, cont.) when $W^2 = 1$]:

(37) $$\mathrm{S}\xi^2 = -\sum h_{rspq}\, R_r\, R_s\, R_p\, R_q\,, \quad \mathrm{S}\xi\xi' = -\sum h_{rspq}\, R_r\, R_s\, R'_p R'_q\,,$$

and analogously for $\mathrm{S}\xi'^2$.

Since $W \neq 0$, for a congruence that is not W, the x, x_s, x_{rs}, etc. will define six linearly-independent systems. We can write six analogous quantities – e.g., the x_{rst}, x_{rst}, etc. – as linear combinations of those systems, and we write:

(38) $$x_{rst} = b_{rst}\, x + \sum_{i,j} h_{rstj} A_{ji} x_i + \sum_{p,q,i,j} l_{rstpq} A_{pi} A_{qj} x_{ij}\,,$$

(and analogously for y, …).

Multiplying (38) by x_p and summing over the analogous expressions, one will find that $h_{rstp} = \mathrm{S}\, x_p\, x_{rst}$. Thus, the h that appears in (38) will coincide with the h that we defined previously and calculated with (34). Note that it follows from (38) that b_{rst} will be symmetric in its three indices, while l_{rstpq} will be symmetric in r, s, t, as well as in p, q.

Multiplying (38) by – x and summing over the analogous expressions, one will find that:

(39) $$0 = \sum_{p,q} l_{rstpq} A_{pq}\,.$$

Multiplying (38) by – x_{rs} and summing over the analogous expressions, one will find that:

(39, cont.) $$-\,\mathrm{S}\, x_{pq}\, x_{rst} = b_{rst}\, a_{pq} + \sum_{\pi,\sigma,i,j} l_{rst\pi\sigma} A_{\pi i} A_{\sigma j} h_{ijpq}\,.$$

The left-hand sides of (39, cont.) *are determined completely by* φ, Φ. In fact, taking the covariant derivative of (33), one will deduce all of the $\mathrm{S}\, x_{rs}\, x_{pqt} + \mathrm{S}\, x_{pq}\, x_{rst}$, and [since, from (31), $\mathrm{S}\, x_{pq}\, x_{rst}$ is symmetric in its indices p, q, as well as in the indices r, s, t] one will also deduce that the $\mathrm{S}\, x_{pq}\, x_{rst}$ are *all* given as functions of the covariant derivatives of the h, and thus, from (34), they will be known when one is given φ, Φ. If one is given φ, Φ then (39) and (39, cont.) will constitute a system of four linear equations in the four unknowns b_{rst}, l_{rst11}, l_{rst12}, l_{rst22}, such that the determinant of the coefficients of the unknowns will be, from (35, cont.), equal to $W^2 = 1$. The b, l are also determined in a simpler way when one is given the forms φ, Φ. *If one is given those* *forms then one will know the system of equations* (38), *which permits one to solve for the congruence. The integrability conditions for those equations are the analogues of the Gauss-Codazzi equations in our case.*

Observation. – The preceding study applies to congruences W in the places where one can suppose that $W \neq 0$. Let the forms φ, Φ, which are *not* normalized by the preceding methods, be constructed for such a congruence. If:

(40) $$A\,x_{11} + 2B\,x_{12} + C\,x_{22} + \alpha\,x_1 + \beta\,x_2 + \gamma x = 0$$
(and analogous expressions in y, …)

is the equation (written with covariant derivatives) that is satisfied by the line coordinates then if one multiplies (40) by x or by x_r and sums over the analogous expressions then one will find that:

(41) $$A\,a_{11} + 2\,B\,a_{12} + C\,a_{22} = 0,$$

and that $a_{r1}\,\alpha + a_{r2}\,\beta = 0$. Hence, if $\Delta \neq 0$ then $\alpha = \beta = 0$, and (40) will have the type of:

(40, cont.) $$A\,x_{11} + 2\,B\,x_{12} + C\,x_{22} + \gamma x = 0.$$

Other relations are $A\,h_{11ij} + 2\,B\,h_{12ij} + C\,h_{22ij} = 0$, which one finds upon multiplying by x_{ij} and summing over analogous expressions. One can take a ruled surface u_1, u_2 or the developable of the congruence, or the characteristic of (40, cont.). In the last case, $A = C = 0$. Thus, $a_{12} = 0$, $h_{12ij} = 0$, $\varphi = a_{11}\ du_1^2 + a_{22}\ du_2^2$: $\Phi = k_{1111}\ du_1^4 + k_{2222}\ du_2^4 - 2\,(21, 21)\ du_1^2\,du_2^2$. Thus, Φ has the type $\chi\bar{\chi} + \frac{1}{2}\frac{(21,21)}{-a_{12}^2}\varphi^2$ (cf., § 6. One has $I = \frac{1}{2}\frac{(21,21)}{-a_{12}^2} = \frac{1}{2}$ curvature of φ). Giving Φ is equivalent to giving just the quadratic form χ that is conjugate to φ; the form ψ in § 6 is zero. In ordinary derivatives, (40, cont.) is then:

$$\frac{\partial^2 x}{\partial u_1\,\partial u_2} - \frac{\partial \log\sqrt{a_{11}}}{\partial u_2}\frac{\partial x}{\partial u_1} - \frac{\partial \log\sqrt{a_{22}}}{\partial u_1}\frac{\partial x}{\partial u_2} + \gamma x = 0.$$

It seems that for a complete study of the congruence W (*abnormal*), one must also examine the form $\mathrm{S}\ d\xi^2$, in which the ξ are defined by $\mathrm{S}\ \xi x = \mathrm{S}\ \xi x_r = \mathrm{S}\ \xi x_{rs} = 0$; i.e., if they are the coordinates of the osculating complex.

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