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EXTERIOR FORMS IN MECHANICS

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TABLE OF CONTENTS

INTRODUCTION

The mechanics of parametric systems that is traditionally developed from the ideas of Lagrange always encounters some significant difficulties when it desires to enter into the questions of friction between solids (viz., impossibility and indeterminacy) or the general notion of constraint (Béghin's servo), and on the other hand, the Lagrangian form of the equations of motion gives us no indication about the nature of the integration problem.

In his celebrated lectures on integral invariants, Élie Cartan showed that all of the properties of the differential equations of the dynamics of holonomic systems result from the existence of the integral invariant $\int \omega$, $\omega = p_i\, dq^i - H\, dt$. Therefore, any holonomic system whose forces are derived from a force function is associated with a form ω such that the equations of motion are the characteristics of the exterior form $d\omega$. In the course of the last ten years, under the influence of the topologists, the theory of exterior forms on differentiable manifolds has been erected on foundations that seem definitive. It is therefore natural to wonder whether classical mechanics might not benefit greatly from that line of thought if one constructs it by basing it upon an exterior form of degree two, whether the notion of a constraint might not be envisioned from a more intelligible angle thanks to the notion of manifolds, whether the indeterminacies and impossibilities that seem paradoxical in the Lagrangian context might not have a natural explanation, and finally if it is not possible to consider the problem of integrating equations of motion in a new light when the latter are generated by a form Ω of degree two.

In order to achieve those diverse objectives, it seems to me useful to recall in Chapter I the study of the logical bases upon which Galilean mechanics is erected. In § I, I will then show that when one proposes to find some generating forms for the equations of motion of a material point that are invariant under the transformations of the Galilean group, the most interesting form is an exterior form of degree two that is defined on a manifold $V_7 = E_3 \otimes E_3 \otimes T$ (E_3 is Euclidian space, T is the temporal number line) ([1]). In § II, it will be shown that any holonomic parametric system with n degrees of freedom is associated with a form Ω of degree two of rank $2n$ that is defined on a differentiable manifold whose characteristics are the equations of motion ([2]). That form can be expressed, if one so wishes, by means of $2n$ Pfaff forms and dt, and the Hamiltonian form is only a simple particular case of it. In § III, I shall give an overview of how one can be liberated from the servitude to coordinates in the study of dynamical systems and the important role that is played by H. Cartan's ([3]) anti-derivation operator i ( ), since the characteristic field E of the form Ω is defined by the relation i (E) $\Omega = 0$.

Having laid these foundations, in Chapter II, we shall embark upon the general theory of one constraint that is imposed upon a material system. A constraint that is imposed upon a material system is composed of two distinct:

([1]) KRAVTCHENCKO has presented that concept at the VIII^{e} Congrés de Mécanique.

([2]) In 1946, in tome LXX of the Bulletin des Sciences Mathématiques, pp. 90, LICHNEROWICZ introduced exterior forms for the formation of the equations holonomic and linearly non-holonomic systems.

([3]) H. CARTAN, *Colloque de Topologie, Bruxelles*, 1950, Masson, Paris, 1951.

1. One arbitrary relation a (p_i , q^i, t) = 0 links the position parameters q^i and p_i , which define a submanifold of V_{2n+1} .

2. An action of forces that can be applied to the system in order to realize that constraint, which is an action of forces that, in the language of manifolds, translates into a constraint field E_i that is defined in the tangent space to V_{2n+1} .

The form da and the constraint field E_i are not independent, since they are coupled by the condition:

$$\text{(II.I)} \qquad i\,(\mathrm{E} + \mathrm{E}_i)\,da = 0 \qquad \text{or} \qquad i\,(\mathrm{E}_i)\,da + i\,(\mathrm{E})\,da = 0.$$

An important class of constraint is the one in which the field E has the form $\lambda\, e$, where λ is a numerical function on V_{2n+1} , and e is a direction field that is known *a priori* (a convention will always permit one to reduce to this case). Indeed, that category includes contact between solids, with or without friction (cf., § **6**), zero-power constraints, whose general definition we shall give in § **5**, and Béghin's servo constraint. For those constraints, (II. I) will show that the factor λ is the quotient of two scalar invariants i (E) da and i (e) da . One then sees that one of the great advantages of H. Cartan's operator i ( ) is that it permits one to determine the classical reactions independently of the coordinates and to solve the equations of motion. In addition, for i (e) da = 0, i (E) $da \neq 0$, the postulate of solid rigidity that is imposed upon a material system translates into i (e) $da \neq 0$. Finally, in § **7**, we shall study the determination of the equations of motion by means of characteristics of a form Ω_s of degree two of rank 2 (n – 1), to which, one appends a Pfaff form, and the existence of that form is a consequence of the notion of compatibility of constraints.

Chapter III is devoted to the study of sets of p constraints in the previously-given sense of that word and the compatibility of that set. The possibility of determining the factors in the constraints by means of H. Cartan's operator independently of the equations of motion is also discussed, along with determining the equations of motion as characteristics of a form of degree two and rank 2 (n – p) that is joined with p Pfaff forms. Various concrete examples that show the generality of the method will be given.

Chapters IV and V will study the following problem for a special class of constraints that include the classical unilateral constraints: The signs are imposed *a priori* on the factors of the constraints and the forms da, and the initial conditions are given for any possible motions of the system. The operator i ( ) immediately permits one to define p equations whose right-hand sides depend upon only the initial conditions and some forces besides the constraint forces. A geometric interpretation of the system will permit one to transform the problem into a problem in *analysis situs* for a family of 2^p p-hedra that is composed by taking a vector in each column of the matrix:

$$\begin{matrix} \mathbf{a}^1, & \mathbf{a}^2, & \ldots, & \mathbf{a}^p, \\ \mathbf{A}^1, & \mathbf{A}^2, & \ldots, & \mathbf{A}^p. \end{matrix}$$

One establishes that the necessary and sufficient for 2^p p-hedra with the same summit to not have any common interior points is that the $(2^p - 1)$ diagonal minor determinants that are extracted from the matrix $\| r^{hk} \|$ $(\mathbf{A}^k = \| - r^{hk} \| \; \breve{\mathbf{a}}^h)$ must all be positive. In particular, that condition implies that for Appell's constraints, which includes the classical holonomic and linearly non-holonomic constraints as particular cases, the initial conditions, combined with the sign convention that was imposed *a priori* on the constraint factors and the forms da, will be sufficient to determine the final motion. For the other types of constraints, the initial conditions might not be sufficient. Therefore, the case of sliding friction, which has been well-known since the work of Painlevé, exhibits nothing exceptional from that standpoint. It then results from this that it is not the laws of Coulomb friction that must be in question, since any other law would give rise to some impossibilities and indeterminacies that depend upon only the sign of the invariant $i\,(e)\,da$ at the point M_0 of the image manifold $[i\,(e)\,da < \mathrm{o}]$.

Chapter VI is devoted to the study of the differential systems of dynamics when they are considered to be characteristics of a form Ω of degree two that is defined on a differentiable manifold V_{2n+1} . That study is carried out by means of endomorphisms in the exterior algebra, which are endomorphisms that lead to H. Cartan's anti-derivation $i\,(\,)$ and derivation $\theta\,(\,)$ operators. In regard to that, permit me to recall the text of his celebrated 1950 talk at the *Colloque de Topologie* in Brussells in order to facilitate the reader's comprehension of that viewpoint. Instead of writing the differential equations in an arbitrary analytical form, which always has the inconvenience of making a coordinate system that is more or less adapted to the question come into play, one argues solely on the basis of the generating form Ω. The major role that is played by infinitesimal transformations in the integration of a differential system has been known since the work of Sophus Lie. That role is marvelously illuminated by the operator $\theta\,(\mathrm{X})$, since if the field X is the infinitesimal generator for Ω then $\theta\,(\mathrm{X})\,\Omega = 0$. Two big cases appear immediately:

A) $d\,\Omega = 0$. Any infinitesimal transformation corresponds to a first integral, and conversely. One can integrate by quadratures only if one knows a sub-ring of n functions in involution. It will then result that for $\Omega = dp_i \wedge dq^i - dH \wedge dt$, finding the cases of integrability amounts to studying the construction of that sub-ring. In the case of mechanics, H will be quadratic, and one indicates the construction of H relative to the existence of p algebraically-generic elements of that sub-ring that permit one to recover the known cases of integrability by a general method and construct some others.

B) $d\,\Omega \neq 0$. One supposes that r fields are known that generate infinitesimal transformations. The integration will then decompose into two phases:

1. The integration of a completely-integrable Pfaff system of rank $(2n - r)$.

2. The integration of a system of r invariant Pfaff forms, and the results of that are already found to be stated implicitly in Élie Cartan's lectures on integral invariants.

In the mechanical applications, the order of the completely-integrable system is found to reduce by $(p + q)$ units if, on the one hand, one knows p first integrals, and on the other hand, q constraints, in the sense of Chapter II. One can integrate by quadratures if $2n - r = p + q$, and if the r invariant forms are closed modulo the integrals of the completely-integrable system.

Some examples will illustrate that theory.

CHAPTER ONE

DIFFERENTIAL FORMS ASSOCIATED WITH A MATERIAL SYSTEM

Since the primary objective of mechanics is to form equations of motion for a material system, it is interesting to have a method that permits one to obtain them in an arbitrary coordinate system. That is why it is useful to study the possibility of associating a system with one or more differential forms that generate the equations, while those forms are invariant under the transformations that one specifies.

§ I. – Cartan exterior form associated with a material point.

The invariant forms that we propose to seek have their origins in the four postulates of Newtonian mechanics.

1) The mass of a body is an invariable positive number, and the mass of a material ensemble is a completely additive function of the ensemble.

2) Time t is an absolute magnitude that is defined up to an additive constant.

3) A point M of mass m that is animated with a velocity $\mathbf{v}$ and subject to a force $\mathbf{F}$ will take on an acceleration $d\mathbf{v} / dt$ such that $m\,(d\mathbf{v} / dt) = \mathbf{F}$ with respect to any Galilean trihedron.

4) The $\mathbf{F}$ is independent of the Galilean reference trihedron.

If two orthonormal Galilean frames are in uniform rectilinear motion with respect to each other then let:

x^j, t	denote the coordinates of M with respect to the first (j = 1, 2, 3)
v^j	denote the components of the velocity $\mathbf{v}$ of M with respect to the first
X^j	denote the components of the force $\mathbf{F}$ with respect to the first
a^j	denote the components of the velocity of translation $\boldsymbol{a}$ of the second frame with respect to the first one
ξ^σ, τ	denote the coordinates of M with respect to the second (σ = 1, 2, 3)
α^σ	denote the components of the velocity $\boldsymbol{\alpha}$ of M with respect to the second

Ξ^{σ} denote the components of the force **F** with respect to the second.

Consider the seven-dimensional space that is the tensor product of the spaces $E_3 \otimes E_3 \otimes T$ ($x^j \in E_3$ is a Euclidian space, $v^i \in E_3$ is a Euclidian space, $t \in T$ is the number line). The transformations or changes in the Galilean frame form a ten-dimensional Lie group G whose finite equations are:

$$\begin{cases} x^j = a^j_\sigma \xi^\sigma + a^j \tau + b^j, \\ t = \tau + t_0, \\ u^i = a^j_\sigma \alpha^\sigma + a^i \end{cases} \qquad i, j, \sigma \text{ are equal to 1, 2, 3.}$$

$\| a^j_\alpha \| = A$ denotes an orthogonal matrix of rank three that depends upon three rotation parameters that one can specify by using the Cayley representation (4) $A = (E - S) \times (E + S)^{-1}$, in which S denotes a skew-symmetric matrix of rank three and E denotes the identity matrix of the same rank.

There exist ten invariant Pfaff forms (viz., the *Maurer-Cartan* forms) for the group G, when it is "prolonged holohedrally" [in the sense of Élie Cartan (5)] by means of an arbitrary orthogonal matrix of rank three, $L = \| L^j_\alpha \|$.

$$\omega^{\rho} = L^{\rho}_j dv^j = \Lambda^{\rho}_{\sigma} d\alpha^{\sigma}, \qquad \rho \text{ varies from 1 to 3,}$$

$$\omega^{\rho'} = L^{\rho}_j (dx^j - v^j dt) = \Lambda^{\rho}_{\sigma} (d\xi^{\sigma} - \alpha^{\sigma} d\tau), \qquad \rho' \text{ varies from 1 to 3,}$$

$$\omega^{\gamma} = dt = d\tau.$$

The product $\| dL \| \cdot \| L^{-1} \| = \| d\Lambda \| \cdot \| \Lambda \|^{-1}$ will give rise to three Pfaff forms when one sets $\Lambda = \| L \| \cdot \| A \|$.

Among those ten forms, the first six of them are independent of the differentials of the three rotation parameters. It will then result that for a material point that is not subject to any force, the differential forms that are the generators of the differential equations (6) of motion that we seek and are invariant under the transformations of the group G are obtained by eliminating the three rotation parameters from those six forms.

That elimination can be carried out by using either ordinary algebra or exterior algebra (7), and the classical properties of the orthogonal matrices:

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(4) Cf., Hermann WEYL, *The Classical Groups*, Princeton, 1946, pp. 56 and 62.

(5) Cf., Élie CARTAN, *La théorie des groupes continus et finis*, Gauthier-Villars, 1937, pp. 121 and 124.

(6) In anticipation of what follows, it is possible to associate a system of differential equations with forms that belong to graded algebras that generate the system by means of anti-derivations.

(7) Cf., N. BOURBAKI, *Algébre multilinéaire*, Actualité scientifiques, no. 1044, Hermann and Co., Paris, 1948, pp. 53 and 76.

$$\sum_{\rho=1}^{3} l_j^\rho \cdot l_i^\rho = 0 \qquad \text{for } i \neq j, \qquad\qquad \sum_{\rho=1}^{3} l_j^\rho \cdot l_i^\rho = 1 \quad \text{for } i = j\,.$$

a) Upon utilizing ordinary algebra, one will get:

(I)
$$\begin{cases} (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2 = \sum_{j=1}^{3} (dx^j - v^j dt)^2, \\ (\omega^4)^2 + (\omega^5)^2 + (\omega^6)^2 = \sum_{j=1}^{3} (dv^j)^2, \end{cases}$$

(II)
$$\omega^1\omega^4 + \omega^2\omega^5 + \omega^3\omega^6 = \sum_{j=1}^{3} (dx^j - v^j dt) \cdot dv^j\,.$$

b) Upon utilizing exterior algebra:

(III)
$$\omega^1 \wedge \omega^4 + \omega^2 \wedge \omega^5 + \omega^3 \wedge \omega^6 = \sum k_{ij}\, dv^j \wedge (dx^j - v^j dt)\,.$$

$$k_{ij} = \sum_{\rho=1}^{3} \begin{vmatrix} l_i^\rho & 0 \\ 0 & l_j^\rho \end{vmatrix} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

(IV)
$$\omega^1 \wedge \omega^4 \wedge \omega^2 \wedge \omega^5 + \omega^2 \wedge \omega^5 \wedge \omega^3 \wedge \omega^6 + \omega^3 \wedge \omega^6 \wedge \omega^1 \wedge \omega^4$$

(V)
$$\begin{cases} \omega^1 \wedge \omega^2 \wedge \omega^4 \\ \omega^4 \wedge \omega^5 \wedge \omega^6 \end{cases}$$

One notes that in the sense of exterior algebra, the IV$^{\text{th}}$ form is the square of the III$^{\text{rd}}$ one, up to a factor of 1 / 2!, and that the two forms in (V) can be replaced with their product, which is the cube of the III$^{\text{rd}}$ one, up to a factor of 1 / 3! . Whereas in exterior algebra, one is led to only one generating form for the differential equations of motion, in ordinary algebra, one would be led to either two forms of type I or one form of type II.

In the case where the material point M of mass m is subject to a force **F**, the postulate (3), namely, $\mathbf{F} = m\, d\mathbf{v} / dt$, leads one to replace dx^i with ($m\, dv^i - X^i\, dt$) in ω^ρ (ρ = 1, 2, 3), where X^i are the components of the forms with respect to the first frame.

The following theorem results from the preceding study:

Theorem I:

There exist three types of differential forms that generate equations of motion for a material point and are invariant under the transformations of the Galilean group:

$$(A)\qquad \begin{cases} s = \dfrac{1}{2m}\displaystyle\sum_{i=1}^{3}(m\,dv^i - X^i dt)^2, \\ e = \dfrac{m}{2}\displaystyle\sum_{i=1}^{3}(dx^i - v^i dt)^2, \end{cases}$$

$$(B)\qquad f = \sum_{i,j=1}^{3}\delta_{ij}(dx^i - v^i dt)(m\,dv^j - X^j dt)\,,$$ δ_{ij} are the Kronecker symbols,

$$(C)\qquad \omega = \sum_{i,j=1}^{3}k_{ij}(m\,dv^i - X^i dt)\wedge(dx^j - v^j dt)\,,$$ k_{ij} are the Kronecker symbols.

The differential equations of motion are obtained by annulling the first-order partial derivatives of the preceding forms with respect to the differentials dx^i, dv^j of the position and velocity parameters.

As far as exterior algebra is concerned, recall that if:

$$\Omega = A_{i_1\cdots i_r}\,dx^{i_1}\wedge dx^{i_2}\wedge\cdots\wedge dx^{i_r}\,,$$ where $i_1, \ldots, i_r$ are r indices that vary from 1 to n,

then

$$\frac{\partial\,\Omega}{\partial(dx^{i_p})} = (-1)^{p+1}A_{i_1\cdots i_r}\,dx^{i_1}\wedge dx^{i_2}\wedge\cdots dx^{i_{p-1}}\wedge dx^{i_{p+1}}\cdots\wedge dx^{i_r}\,.$$

Choice of three types of differential forms that are invariant under the transformations of the Galilean groups. Cartan's exterior form. – In principle, the preceding three types of forms are generators of the differential equations of motion. When one performs a change of variables T that acts upon the set of all position and velocity parameters [$x^i = x(\rho^\alpha, t)$, $v^j = v^j(\rho^\alpha, t)$, ($\alpha$ varies from 1 to 6)], the three types will have the expressions:

$$(A)\qquad \begin{cases} s = \tfrac{1}{2}s_{\alpha\beta}\,d\rho^\alpha d\rho^\beta - Q_{\alpha 0}d\rho^\alpha dt + \tfrac{1}{2}s_{00}\,dt^2, \\ e = \tfrac{1}{2}g_{\alpha\beta}\,d\rho^\alpha d\rho^\beta - \Gamma_{\alpha 0}d\rho^\alpha dt + \tfrac{1}{2}g_{00}\,dt^2, \end{cases}$$

in which one sets:

$$\begin{cases} s_{\alpha\beta} = m\,\delta_{ij}\dfrac{\partial x^i}{\partial\rho^\alpha}\cdot\dfrac{\partial v^j}{\partial\rho^\beta}, \\ Q_{\alpha 0} = \delta_{ij}\left(X^i\dfrac{\partial v^j}{\partial\rho^\alpha} - m\dfrac{\partial v^i}{\partial\rho^\alpha}\dfrac{\partial v^j}{\partial t}\right), \\ s_{00} = \dfrac{1}{m}\delta_{ij}X^i X^j + m\,\delta_{ij}\dfrac{\partial v^i}{\partial t}\dfrac{\partial v^j}{\partial t} - \delta_{ij}X^i\dfrac{\partial v^j}{\partial t}, \end{cases}$$

$$
\begin{cases}
g_{\alpha\beta} = m\,\delta_{ij}\dfrac{\partial x^i}{\partial\rho^\alpha}\cdot\dfrac{\partial v^j}{\partial\rho^\beta}, \\[2ex]
\Gamma_{\alpha 0} = m\,\delta_{ij}\left(v^i\dfrac{\partial v^j}{\partial\rho^\alpha} - \dfrac{\partial x^i}{\partial\rho^\alpha}\dfrac{\partial x^j}{\partial t}\right), \\[2ex]
g_{00} = m\,\delta_{ij}\left(v^i v^j + \dfrac{\partial x^i}{\partial t}\dfrac{\partial x^j}{\partial t} - \dfrac{\partial x^i}{\partial t}v^j\right),
\end{cases}
$$

which are expressions in which δ_{ij} denotes the Kronecker symbol:

(*B*) $$f = f_{\alpha\beta}\,d\rho^\alpha\,d\rho^\beta - f_{\alpha 0}\;d\rho^\alpha\,dt + f_{00}\,dt^2 .$$

In f, the ($f_{\alpha\beta}$, $f_{\alpha 0}$, f_{00}) define a symmetric tensor that is a function of (ρ^α, t) and has the expression:

$$f_{\alpha\beta} = m\,\delta_{ij}\frac{\partial x^i}{\partial\rho^\alpha}\frac{\partial x^j}{\partial\rho^\beta},$$

$$f_{\alpha 0} = m\,\delta_{ij}v^i\frac{\partial v^j}{\partial\rho^\alpha} + \delta_{ij}X^i\frac{\partial x^j}{\partial\rho^\alpha} - m\,\delta_{ij}\frac{\partial x^i}{\partial t}\frac{\partial v^j}{\partial\rho^\alpha} - m\,\delta_{ij}\frac{\partial x^i}{\partial\rho^\alpha}\frac{\partial v^j}{\partial t},$$

$$f_{00} = \delta_{ij}\,v^i X^j - m\,\delta_{ij}v^i\frac{\partial v^j}{\partial t} + m\,\delta_{ij}\frac{\partial x^i}{\partial t}\frac{\partial v^j}{\partial t} - \delta_{ij}\frac{\partial x^i}{\partial t}X^j,$$

and

(*C*) $$\omega = k_{\alpha\beta}\,(d\rho^\alpha\wedge d\rho^\beta) - k_{\alpha 0}\,d\rho^\alpha\wedge dt ,$$

where ($k_{\alpha\beta}$, $k_{\alpha 0}$) is an antisymmetric function of (ρ^α, t) that has the expression:

(I.1)
$$
\begin{aligned}
k_{\alpha\beta} &= m\,k_{ij}\begin{vmatrix}\dfrac{\partial v^i}{\partial\rho^\alpha} & \dfrac{\partial v^i}{\partial\rho^\beta}\\[2ex] \dfrac{\partial x^j}{\partial\rho^\alpha} & \dfrac{\partial x^j}{\partial\rho^\beta}\end{vmatrix}, \\
k_{\alpha 0} &= -\,m\,k_{ij}\begin{vmatrix}\dfrac{\partial v^i}{\partial\rho^\alpha} & \dfrac{\partial v^i}{\partial t}\\[2ex] \dfrac{\partial x^j}{\partial\rho^\alpha} & \dfrac{\partial x^j}{\partial t}\end{vmatrix} + m\,k_{ij}v^i\begin{vmatrix}\dfrac{\partial v^i}{\partial\rho^\alpha} & \dfrac{\partial v^i}{\partial t}\\ 0 & 1\end{vmatrix} - k_{ij}X^i\begin{vmatrix}\dfrac{\partial x^j}{\partial\rho^\alpha} & \dfrac{\partial x^j}{\partial t}\\ 0 & 1\end{vmatrix}.
\end{aligned}
$$

We remark that the two forms of type (*A*) are not generally expressed directly as functions of the differentials of the first integrals of motion, while that property is immediate for the two forms of type (*B*) and *C*) : (*B*) is a quadratic form with zero discriminant, by its very origins, so it will be expressed in terms of the six differentials, which be the differentials of those integrals for a convenient choice of the first integrals. (C) is an exterior form that is reducible to a sum of three exterior products of two differentials, and as a result, it can be written:

$$\omega = \bar{k}_{\alpha\beta}\ dC^\alpha \wedge dC^\beta,$$

for a convenient choice of first integrals C^α, in which $\bar{k}_{\alpha\beta}$ is an antisymmetric tensor that is a function of the C^α and t.

Traditionally, mechanics is constructed by starting from the quotient by dt^2 of the particular form s of type A that is obtained by imagining the particular transformations:

$$x^i = x^i\,(q^k, t), \qquad v^j = \frac{\partial x^j}{\partial q^k}\dot{q}^k + \frac{\partial x^j}{\partial t}.$$

That leads to the Gauss-Appell principle and to the Lagrange equations: The equations of motion render the inhomogeneous quadratic form in $\ddot{q}$:

$$\frac{s}{dt^2} = \tfrac{1}{2} s_{hk}\,\ddot{q}^h\,\ddot{q}^k + s_{kl}\,\ddot{q}^k\,\dot{q}^l - Q_k\,\ddot{q}^k$$

a minimum.

That form does not give us any immediate information about the nature of the problem of integrating the motion, since it depends upon a point-like pseudogroup x^i (q^k, t) ([8]). On the other hand, that Gauss-Appell form is practical for only holonomic and linearly non-holonomic constraints. In Chapter II, we shall see that the notion of constraint is susceptible to considerable extension, and the calculations will lead us to envision the most general transformations that act upon the set of all position and velocity variables. For those reasons, the forms (B) and (C) prove to be more interesting.

There is an important difference between the forms (B) and (C): (C) is bilinear, while (B) is quadratic, so the calculations are simpler for (C) than they are for (B). Furthermore, the form (C) immediately leads to the kinetic integral invariant of Élie Cartan.

The fact that only the differentials of the first integrals enter into the differentials in the form (C) can be further established as follows: Consider a second differential system δ. One can associate the form ω with the bilinear form $\omega(\delta, d)$:

$$\omega(\delta, d) = \delta\rho^\alpha \wedge \frac{\partial \omega}{\partial(d\rho^\alpha)} + \delta\rho^\beta \wedge \frac{\partial \omega}{\partial(d\rho^\beta)} + \delta t \wedge \frac{\partial \omega}{\partial(dt)}$$

$$= k_{\alpha\beta}\,[\delta\rho^\alpha \wedge d\rho^\beta - \delta\rho^\beta \wedge d\rho^\alpha] - k_{\alpha 0}\,[\delta\rho^\alpha \wedge dt - \delta t \wedge d\rho^\alpha].$$

([8]) The pseudogroup that is involved is that of the manifold V_{2n+1}.

The differential equations of motion E annul $\frac{\partial \omega}{\partial (d\rho^\alpha)}$, $\frac{\partial \omega}{\partial (d\rho^\beta)}$, and $\frac{\partial \omega}{\partial (dt)}$ (as a consequence of the preceding) when one considers an arbitrary one-dimensional manifold γ in seven-dimensional space, along which the differential is δ and which lies on the two-dimensional manifold V_2 that is generated by the integral lines of the differential system of the equations of motion that pertain to γ: viz., $\omega(\delta, d) = 0$. The form $\omega(\delta, d)$ is linear with respect to the differential system, since d is zero only on the integral lines of the equations of motion that belong to the sub-module of differentials of the first integrals; in other words, $\omega(\delta, d)$ is expressed in terms of the first integrals. Now, if one takes $\delta = d$ then since $\omega = \frac{1}{2}\,\omega(d, d)$, the form ω of degree two will be expressed in terms of solely the first integrals C^α of E.

(I.2) $$\omega = \bar{k}_{\alpha\beta}\; dc^\alpha \wedge dc^\beta .$$

$\bar{k}_{\alpha\beta}$ is an antisymmetric tensor that is a function of the C^α and t.

The preceding result can be further expressed as follows: The integral $\int_{V_2} \omega(\delta, d)$ will be zero for any one-dimensional manifold in the two-dimensional manifold V_2 that is generated by the integral lines of the differential system E that pertains to γ.

(I.3) $$\int_{V_2} \omega(\delta, d) = 0 .$$

With Lichnerowicz ([9]), we say that ω generates an *absolute integral invariance relation* for the system of differential equations of the mechanics of material points.

Special case. – If $d\omega = 0$ then ω will be expressed in terms of only the differentials of the first integrals of E, and conversely.

ω is expressed in canonical form by grouping the first integrals together pair-wise:

$$\omega = \sum_{\alpha=1}^{3} k_{\alpha\alpha'}\, dc^\alpha \wedge dc^{\alpha'} , \qquad \text{with} \qquad \alpha^* = \alpha + 3 .$$

$d\omega = 0$ implies the fact that a coefficient $k_{\alpha\alpha'}$ will be a function of only the two integrals C^α, $C^{\alpha'}$. Indeed, if u denotes one of the first integrals or time then $d\omega$ will be a sum of terms of the form:

$$\frac{\partial k_{\alpha\alpha'}}{\partial u}\, du \wedge dc^\alpha \wedge dc^{\alpha'}.$$

Since $d\omega$ is zero, $\partial k_{\alpha\alpha'} / \partial u = 0$, so $k_{\alpha\alpha'}$ will be a function of only C^α and $C^{\alpha'}$. By a change of first integrals of the form:

([9]) Cf., Lichnerowicz, Bull. Sci. Math. (2) **70** (1946).

$$C^{\alpha} = C^{\alpha}\,(\bar{C}^{\alpha'}, \bar{C}^{\alpha}), \qquad\qquad C^{\alpha'} = C^{\alpha'}(\bar{C}^{\alpha'}, \bar{C}^{\alpha}),$$

ω will become:

$$\text{(I.4)} \qquad \omega = \sum_{\alpha=1}^{3} d\bar{c}^{\alpha'} \wedge d\bar{c}^{\alpha},$$

and as a result, it is expressed in terms of solely differentials of the first integrals of E.

Élie Cartan's kinetic integral invariant. – Since $d\omega$ is zero, ω will be a closed form, and there will exist a Pfaff form ϖ^1 in the space that is homeomorphic to $\mathbb{R}^7$ such that $\omega = d\varpi^1$. From (I.4), $\varpi^1 = \sum_{\alpha=1}^{3} \bar{c}^{\alpha'} \cdot d\bar{c}^{\alpha}$ will admit ω for its differential, and it will be expressed in terms of only first integrals of the motion and their differentials. Hence, ϖ^1 will generate an absolute integral invariant for the differential equations E of motion. If one considers the two-dimensional manifold that is generated by the integral curves of E that pertain to an arbitrary one-dimensional manifold γ_0 then the integral (3) will become:

$$0 = \int_{V_2} -\omega(\delta, d) = \int_{V_2} d[\varpi^1(\delta)] = \int_{F(V_2)} \varpi^1(\delta).$$

$F(V_2)$ denotes the frontier of V_2, which is composed of an arc of the curve γ_0 MM', the arcs of the integrals of E, $M_0 M$, $M'_0 M'$, which issue from the points M_0 and M'_0, and the arc MM'. Along the arcs MM_0 and $M'_0 M'$, since $\varpi^1(\delta)$ is a linear form in the differentials of the first integrals of E, it will be zero. It will then result that $\int_{F(V_2)} \varpi^1(\delta)$ reduces to two integrals that are taken along γ_0 and γ, where γ is deduced from γ_0 by means of the trajectories that are solutions to E ; hence:

$$\text{(I.5)} \qquad \int_{M_0 M'_0} \varpi^1(\delta) = \int_{MM'} \varpi^1(\delta).$$

The preceding equality expresses the idea that ϖ^1 generates an absolute integral invariant.

If one returns to the expression for ω, namely:

$$\omega = \sum_{i,j,=1}^{3} k_{ij}(m\,dv^i - x^i dt) \wedge (dx^j - v^j dt) = m\,k_{ij}\,dv^i \wedge dx^j - m\,k_{ij}\,v^i dv^j \wedge dt + k_{ij}\,X^i dx^j \wedge dt.$$

$d\omega = 0$ imposes the condition on the Pfaff form $k_{ij}\,X^i\,dx^j$ that it must be closed, and as a result, it will reduce to the differential of a function U, so:

$$\text{(I.6)} \qquad \omega = m\,k_{ij}\,dv^i \wedge dx^j - dH \wedge dt.$$

With $H = T - U$, where $T = \frac{1}{2}\sum_{i=1}^{3} m(v^i)^2$ is one-half the *vis viva*, and U a force function, (I.6) will show that ω is the exterior derivative of:

$$\omega^1 = \sum_{i=1}^{3} m\, v^i dv^i - H\, dt\,.$$

The form ω^1 generates Élie Cartan's integral invariant (10). It differs from the form $\bar{\omega}^1$ in (I.4) by a closed form.

The fact that the only differentials that enter into the form ω of degree two are differentials of first integrals, and that ω is the exterior derivative of the form that generates Élie Cartan's kinetic integral invariant in the case where $d\omega$ is zero will lead one to choose it as the generating form for the differential equations of motion of a material point and to place it at the foundations of Newtonian mechanics.

We shall summarize the preceding study in the following theorem:

Theorem II:

Any material point of mass m whose coordinates are x^i that is animated with a velocity whose components are v^i and is subject to a force **F** *whose components are X^i with respect to an orthonormal Galilean trihedron can be associated with a second-order Cartan exterior differential form ω that possesses the following properties:*

1. *ω is invariant under the transformations of the Galilean group; in other words, it has the same expression with respect to any orthonormal Galilean frame.*

2. *The differential equations of motion of the point are the associated equations to ω, in the Cartan sense.*

3. *ω is unique.*

4. *ω is expressed as a function of only the differentials of first integrals of the equations of motion whose coefficients are an antisymmetric tensor that is a function of the first integrals and one variable – for example, t. If $d\omega = 0$ then ω is expressed in terms only first integrals and their differentials.*

Recall that when one uses the usual variables (x^j, v^i, t), ω can be written as follows in its developed form:

(I.7) $$\omega = m\, k_{ij}\, dv^i \wedge dx^j - m\, k_{ij}\, v^i\, dv^j \wedge dt + k_{ij}\, X^i\, dx^j \wedge dt,$$

in which k_{ij} is the Kronecker symbol.

The associated equations to ω are the classical Newton equations:

$$\frac{\partial\omega}{\partial(dx^j)} = -\,m\,(dv^i - X^i\,dt) = 0,$$

$$\frac{\partial\omega}{\partial(dv^j)} = m\,(dx^i - v^i\,dt) = 0 \qquad (i, j = 1, 2, 3),$$

which will signify that:

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F}, \qquad \frac{d\overrightarrow{OM}}{dt} = \mathbf{V},$$

when they are interpreted geometrically in Euclidian space E_3. The existence of ω and the associated equations constitute the natural analytical translation of the four postulates of Newtonian mechanics for a material point ([10]).

Remarks:

1. ω is composed of two parts: a kinetic part ω_c:

$$\omega_c = m\,k_{ij}\,dv^i \wedge dx^j - m\,k_{ij}\,v^i\,dv^j \wedge dt\,,$$

which is a closed form, since:

$$\omega_c = d\,(k_{ij}\,m\,v^i\,dv^j - T\,dt)\,,$$

in which T denotes one-half the *vis viva* $\frac{1}{2}\sum_{i=1}^{3} m\,(v^i)^2$, and a dynamic part ω_d:

$$\omega_d = k_{ij}\,X^i\,dx^j \wedge dt,$$

which is the exterior product of the elementary work done by the force that acts upon the point with the differential of time.

2. It is essential to remark that the associated equations to ω link the position parameters to the velocity parameters, as one sees, in particular, from the equations:

$$\frac{\partial\omega}{\partial(dv^i)} = dx^j - v^j\,dt = 0\,.$$

3. We point out that the relativistic mechanics of material points can be constructed like Newtonian mechanics in terms of an exterior form ω_1 that is invariant under the Lorentz group, and that the generating form of the of the equations of Newtonian

([10]) Cf., Élie CARTAN, *Leçons sur les Invariants intégraux*, Paris, 1922, pp. 1 to 6.

mechanics is the limit of the form ω_1 when one makes the ratio $\beta = v / c$ tend to zero, where v denotes the velocity of the material point, and c is the speed of light ([11]).

4. It is likewise important to point out that the form ω can be expressed in terms of the six Pfaff forms that are constructed from the differentials of the position, velocity, and time parameters, which are forms that are introduced logically in the problems of integration and in the study of certain constraints:

$$\omega = K_{\alpha\beta}\, \omega^{\alpha} \wedge \omega^{\beta}, \qquad K_{\alpha\beta} \text{ is an antisymmetric tensor } (\alpha, \beta = 1 \text{ to } 6).$$

Examples:

1. A gravitating point that is launched along the vertical ascendant and subject to a resistance that is a function of the velocity $m\, f(v)$.

If m is the mass of that point, v is its velocity, g is the intensity of gravity, and z is the height then ω can be written:

$$\frac{\omega}{m} = dv \wedge dz - [v\, dv + g\, dz + f(v)\, dz] \wedge dt,$$

or rather:

$$\frac{\omega}{m} = [g + f(v)] \left[\frac{dv}{g + f(v)} + dt\right] \wedge \left[dz + \frac{v\, dv}{g + f(v)}\right].$$

ω / m is then expressed in terms of the two forms $\dfrac{dv}{g + f(v)} + dt$, $dz + \dfrac{v\, dv}{g + f(v)}$, and when they are annulled, from the preceding theorem, that will give the equations of motion, since the exterior derivative of each of them will be zero; in other words, each of them is closed, so the problem is reduced to quadratures.

In this simple example, one knows the procedure that will permit one to study the case in which the equations of motion are integrable.

2. A point moving under the action of given forces X, Y, Z, which are functions of x, y, z, along with a resistance of the medium $m\, f(v)$ that opposes the velocity. Upon taking the velocity parameters to be the spherical coordinates of the velocity vector v, ψ, θ, the position parameters to be the coordinates x, y, z of the point with respect to a fixed trihedron, and m to be the mass of the point, ω / m will be written:

$$\begin{aligned}\frac{\omega}{m} = {} & (dv \operatorname{con} \theta \cos \psi + v \cos \theta \cos \psi\, d\theta - v \sin \theta \sin \psi\, d\psi) \wedge dx \\ & + (dv \sin \theta \sin \psi + v \cos \theta \sin \psi\, d\theta + v \sin \theta \cos \psi\, d\psi) \wedge dy \\ & + (dv \cos \theta - v \sin \theta\, d\theta) \wedge dz - (v\, dv - X\, dx - Y\, dy - Z\, dz) \wedge dt \\ & - f(v)\, (\sin \theta \cos \psi\, dx + \sin \theta \sin \psi\, dy + \cos \theta\, dz) \wedge dt .\end{aligned}$$

([11]) Cf., F. GALLISOT, Ann. de l'Inst. Fourier **3** (1951), pp. 277 to 285.

That form, which appears to be more complicated than the classical equations, will exhibit the first integrals of motion for a certain choice of X, Y, Z. Hence, for a gravitating point $X = 0$, $Y = 0$, $Z = - m\, g$:

$$\begin{aligned}\frac{\omega}{m} &= [dv + f(v)\, dt + g \cos\theta\, dt] \wedge [\sin\theta \cos\psi\, dx + \sin\theta \sin\psi\, dy + \cos\theta\, dz - v\, dt] \\ &\quad + [v\, d\theta - g \sin\theta\, dt] \wedge [\cos\theta \cos\psi\, dx + \cos\theta \sin\psi\, dy - \sin\theta\, dz] \\ &\quad + v\, d\psi \wedge [-\sin\theta \sin\psi\, dx + \sin\theta \cos\psi\, dy].\end{aligned}$$

The associated equations are obtained by annulling the six forms that are placed in brackets; in particular:

$$d\psi = 0, \qquad dv + [f(v) + g\cos\theta]\, dt = 0, \qquad v\, d\theta - g \sin\theta\, dt = 0,$$

whose geometric integration is immediate: It is the projection of the forces onto the tangent and the normal to the trajectory.

3. Gravitating body moving with friction on an inclined plane.

Let i be the angle of inclination of the horizontal plane, let f be the coefficient of friction, let m be the mass of the point, let Ox be the axis that points along the line of greatest slope in the plane, and let Oy be directly perpendicular to it. Take the velocity parameters to be the polar coordinates v, α of the velocity v and the position parameters to be x, y, $\omega = \omega_c + \omega_d$:

$$\omega_c = d\left[mv\cos\alpha\, dx + mv \sin\alpha\, dy - \tfrac{1}{2} mv^2 dt\right],$$

where d is the symbol of the exterior derivative, and:

$$\omega_c = mg \sin i\, dx - f\, mg \cos i \cos\alpha\, dx - f\, mg \cos i \sin\alpha\, dy] \wedge dt,$$

so

$$\begin{aligned}\frac{\omega}{m} &= dv \wedge (\cos\alpha\, dx + \sin\alpha\, dy) + v\, d\alpha \wedge (-\sin\alpha\, dx + \cos\alpha\, dy) \\ &\quad - v\, dv \wedge dt + [g \sin i\, dx - f\, g \cos i\, (\cos\alpha\, dx + \sin\alpha\, dy)] \wedge dt,\end{aligned}$$

or

$$\begin{aligned}\frac{\omega}{m} &= [d(v \sin\alpha) + f\, g \cos i \sin\alpha\, dt] \wedge [dy - v \sin\alpha\, dt] \\ &\quad + [d(v\cos\alpha) + (f\, g \cos i \cos\alpha - g \sin i)\, dt \wedge [dx - v\cos\alpha\, dt].\end{aligned}$$

ω / m is expressed in terms of four independent forms that will give the differential equations of motion when they are equated to zero, and their geometric interpretation is immediate. The integration of those equations will be performed in Chapter VI as an application of the general method.

§ II. – Exterior form associated with a parameterized material system.

With Brelot ([12]), we consider a material system S to be something that carries:

1. A mass distribution (≥ 0) or measure (≥ 0) that is denoted by m (e) and is a finite function (≥ 0) on a bounded Borel subset that is completely additive.

2. A force distribution, which is a vector field of measure $\mathbf{F}$ (e).

Since functions and Borel subsets are supposed to be bounded, we shall show that one can associate a parameterized system with an exterior form $\Omega = \Omega_c + \Omega_d$.

a) Calculating Ω_c . – Consider a bounded Borel set D in a three-dimensional Euclidian space, and the variable bounded Borel set Δ, which is in bijective correspondence with D for every value of the $n + 1$ real parameters q^i (i varies from 0 to n, with $q^0 = t$) by means of the Borel vector function at M :

$$\overrightarrow{O\mu} = \boldsymbol{f}\,(M, q^i) . \tag{II.1}$$

The domain Δ is determined geometrically when one is given the point q whose local coordinates are q^i in the parameter space. Upon supposing the existence of the first derivatives of $\boldsymbol{f}$ with respect to the bounded q^i for M that varies in D and the bounded q^i by differentiating (II.1):

$$d\overrightarrow{O\mu} = \frac{\partial \boldsymbol{f}}{\partial q^i} dq^i .$$

We saw in § **1** that the velocity of a material point can be defined arbitrarily and that the associated equations to the form ω indicate the manner by which the velocity parameters are coupled with the position parameters of the point. It then results that we define the velocity in the following manner. Let:

$$\mathbf{V}_\mu = \mathbf{V}\,(M, q^i, \rho^\alpha) \tag{II.2}$$

be another Borel vector function at M of the (n + 1) parameters q^i and n other bounded parameters ρ^α that admits bounded first derivatives with respect to the q^i and the ρ^α for any M in D. $\mathbf{V}_\mu$ will be called the *velocity* of the point μ . When one differentiates (II.2), one will get:

$$d\,\mathbf{V}_\mu = \frac{\partial \mathbf{v}}{\partial q^i} dq^i + \frac{\partial \mathbf{v}}{\partial \rho^\alpha} d\rho^\alpha .$$

([12]) *Principes mathématiques de la mécanique classique*, Arthaud, Grenoble, pp. 10-19.

The projections of $d\overrightarrow{O\mu}$ and $d\,\mathbf{V}m$ onto the three axes of an orthonormal Galilean frame define dx^j and dv^i. The point M of D that corresponds to the point μ of Δ is associated with the form ω_c :

$$\omega_c = k_{ij}\, dv^i \wedge dx^j - k_{ij}\, v^i\, dv^j \wedge dt$$

$$= k_{ij} \frac{\partial v^i}{\partial \rho^\alpha} \frac{\partial f^j}{\partial q^k} d\rho^\alpha \wedge dq^k + k_{ij} \frac{\partial v^i}{\partial q^i} \frac{\partial f^j}{\partial q^k} dq^i \wedge dq^k - k_{ij} v^i \left(\frac{\partial v^i}{\partial \rho^\alpha} d\rho^\alpha + \frac{\partial f^j}{\partial q^k} dq^k \right) \wedge dt\,.$$

ω_c is then a form of degree 2 that is defined on the fibered manifold V_{2n+1} of $(2n + 1)$ parameters q^i, t, ρ^α, whose base manifold is the space-time configuration manifold V_{n+1}. The material set Δ is associated with the form $\Omega_c = \int_D \omega_c\, \delta m$ that is defined on the manifold V_{2n+1}, where the $\int$ symbol refers to the Radon integral:

$$\Omega_c = k_{\alpha i}\, d\rho^\alpha \wedge dq^i + k_{lk}\, dq^l \wedge dq^k - k_{\alpha 0}\, d\rho^\alpha \wedge dt - k_{k0}\, dq^k \wedge dt\,.$$

with

$$k_{\alpha i} = \int_D k_{jk} \frac{\partial v^j}{\partial \rho^\alpha} \cdot \frac{\partial f^k}{\partial q^i} \delta m, \qquad k_{lk} = \int_D k_{ij} \left(\frac{\partial v^i}{\partial q^l} \frac{\partial f^j}{\partial q^k} - \frac{\partial v^i}{\partial q^k} \frac{\partial f^j}{\partial q^l} \right) \delta m,$$

$$k_{\alpha 0} = \int_D k_{ij} v^i \frac{\partial v^j}{\partial \rho^\alpha} \delta m - \int_D k_{ij} \frac{\partial v^i}{\partial \rho^\alpha} \frac{\partial f^j}{\partial t} \delta m,$$

$$k_{k0} = \int_D k_{ij} v^i \frac{\partial v^j}{\partial \rho^\alpha} \delta m - \int_D k_{ij} \left(\frac{\partial v^i}{\partial t} \frac{\partial f^j}{\partial q^k} - \frac{\partial v^i}{\partial q^k} \frac{\partial f^j}{\partial t} \right) \delta m\,.$$

We remark that we can likewise calculate Ω_c as the exterior derivative of:

$$\int_D \left(\mathbf{V}_\mu \cdot d\overrightarrow{O\mu} \right) \delta m - \tfrac{1}{2} dt \int_D (\mathbf{V}_\mu)^2\, \delta m\,,$$

and that this calculation can be performed by replacing the differentials of the position parameters dq^i with the Pfaff forms that are constructed by means of those differentials, and the coefficients of those forms are Borel functions of the parameters q^i at M. One will then have that the kinetic part Ω_c of Ω is:

$$\Omega_c = k_{\sigma\pi} (\omega^\sigma \wedge \omega^\pi - k_{\sigma 0} (\omega^\sigma \wedge dt),$$

in which ω^σ, ω^π denote $2n$ Pfaff forms that are constructed on the differentials of the position parameters q^i and the velocity parameters ρ^α, while $(k_{\sigma\pi}\,,\ k_{\sigma 0})$ define an antisymmetric tensor that is a function of the q^i and ρ^α.

b) *Calculating* Ω_d . – Calculating the dynamical part of Ω presents a complication that has not been resolved up to now and is based upon the fact that it is not possible to define a measure of the internal forces in a system ([13]), ([14]). We shall then proceed axiomatically.

Let $\mathbf{F}_e$ be a vector measure of the force that is defined on Δ and which we, with Brelot, will call a *dyname*. The presentation that follows conforms to the one that was adopted by the aforementioned author.

Let $\mathbf{w}$ be a vector field that is defined on Δ and will be called the *field of virtual velocities*. Set:

$$P_{\mathbf{w}} = \int_\Delta \mathbf{w} \cdot \delta \mathbf{F}_e \,.$$

By definition, $P_{\mathbf{w}}$ is the power delivered by the *dyname* $\mathbf{F}_e$ relative to the virtual field $\mathbf{w}$. The elementary work $\mathcal{T}_e$ done by $\mathbf{F}_e$ relative to $\mathbf{w}$ during the time dt will be:

$$\mathcal{T}_e = P_{\mathbf{w}} \cdot dt \,.$$

If one takes $\mathbf{w} = \dfrac{d\overrightarrow{O\mu}}{dt} = \dfrac{\partial \boldsymbol{f}}{\partial q^i}\dot{q}^i$ (i vary from 1 to n) then the power delivered by $P_{\mathbf{w}}$ will have the expression:

$$P_{\mathbf{w}} = \left[\int \frac{\partial \boldsymbol{f}}{\partial q^i}\,\delta \mathbf{F}_e\right]\dot{q}^i = Q_i\,\dot{q}^i$$

when one sets:

$$Q_i = \int \frac{\partial \boldsymbol{f}}{\partial q^i}\,\delta \mathbf{F}_e \,.$$

Now consider the Pfaff form $\pi = Q_i\,dq^i$.

From the preceding definition, π will be equal to the elementary work done by the force measure $\mathbf{F}_e$ in the field $\mathbf{w} = \dfrac{\partial \boldsymbol{f}}{\partial q^i}\dot{q}^i$.

Powerless dyname: If the power delivered $P_{\mathbf{w}}$ is zero then one will say that $\mathbf{F}_e$ is *powerless* for $\mathbf{w}$. Hence, if $\mathbf{w}$ reduces to a field of moments then one will have several fields of moments that are defined on Δ, and $\mathbf{F}_e$ will then be powerless whenever the system of forces $\mathbf{F}_e$ is a system of vectors that is equivalent to 0.

Having posed those definitions, for each subset σ of Δ, consider the dyname $\mathbf{F}_e$ to be the sum of an equivalent dyname $\mathbf{F}_{1e}$, which is called the *external dyname*, and a dyname $\mathbf{F}_{2e}$ that is equivalent to 0 that is called the *internal dyname*. In the applications that we propose to develop, the system S will be composed of a set of solid bodies. We assume the following postulate:

([13]) Cf., BRELOT, Annales de l'Université de Grenoble **19** (1943); *ibid.* **20** (1944).

([14]) R. DE POSSEL, "Sur les principes mathématiques de la mécanique classique," Gazeta de Matematica **29** (1946), Lisbon.

For any field of moments, the power delivered by the internal force is zero. – The following consequences result from the postulate:

1. The power delivered by the external forces is calculated for each solid body by taking **w** to be a field of moments that is defined by each of them.

Therefore, the dynamical part Ω_d of Ω will be:

$$\Omega_d = Q_i \, dq^i \wedge dt \, .$$

The Q_i are functions of the (q^i, ρ^h, t), where the q^i are the position parameters and the ρ^h are the velocity parameters.

2. The kinetic part Ω_c of Ω must be calculated with the same field of moments, since $d\,\overrightarrow{O\mu} = \dfrac{\partial \boldsymbol{f}}{\partial q^i} dq^i$ enters into Ω_c since Ω_c is the exterior derivative of:

$$\int_D \left(\mathbf{V}_\mu \, d\,\overrightarrow{O\mu}\right) \delta m - \tfrac{1}{2} dt \int_D (\mathbf{V}_\mu)^2 \, \delta m \, .$$

Upon taking $d\,\overrightarrow{O\mu} = \dfrac{\partial \boldsymbol{f}}{\partial q^i} dq^i$, $\mathbf{V}_\mu = \dfrac{\partial \boldsymbol{f}}{\partial q^k} \rho^k$, one will be led to the two classical expressions for Ω_c :

$$\int_D \left(\mathbf{V}_\mu \, d\,\overrightarrow{O\mu}\right) \delta m = \left[\int_D \frac{\partial \boldsymbol{f}}{\partial q^k} \cdot \frac{\partial \boldsymbol{f}}{\partial q^i} \, \delta m \right] \rho^k dq^i = g_{ki} \, \rho^k \, dq^i,$$

upon setting $g_{ki} = \displaystyle\int \frac{\partial \boldsymbol{f}}{\partial q^k} \cdot \frac{\partial \boldsymbol{f}}{\partial q^i} \, \delta m$, which is a symmetric covariant tensor of order 2:

$$T = \tfrac{1}{2} \int_D (\mathbf{V}_\mu)^2 \, \delta m = \tfrac{1}{2} \left[\int \frac{\partial \boldsymbol{f}}{\partial q^k} \cdot \frac{\partial \boldsymbol{f}}{\partial q^i} \, \delta m \right] \rho^i \rho^k = \tfrac{1}{2} g_{ki} \, \rho^k \, \rho^i,$$

in which d denotes the symbol of exterior derivation:

$$\Omega_c = d\left(g_{ki} \, \rho^k dq^i - \tfrac{1}{2} g_{ki} \, \rho^k \rho^i dt\right).$$

Hamiltonian form of Ω_c . – Since the choice of velocity parameters ρ^h is arbitrary, we can set:

$$p_i = g_{hi} \, \rho^h \, ,$$

which is always possible, since $2T = g_{hi} \, \rho^h \, \rho^i$ is a positive-definite form of rank n (det $g_{hi} \neq 0$). In order to simultaneously imagine the cases in which the holonomic constraints do or do not depend upon time, we shall vary the indices i and h from 0 to n with $dq^0 = dt$, $\rho_0 = 1$:

$$p_0 = g_{h0}\,\rho^h,$$

$$T = T_2 + T_1\,\rho^0 + T_0\,(\rho^0)^2,$$

$$p_0 = T_1 + 2\,T_0\,,$$

so

$$\begin{aligned}\Omega_c &= d\left[\sum_{i=1}^{n} p_i\,dq^i + p_0\,dt - (T_2 + T_1 + T_0)\,dt\right] \\ &= d\left[\sum_{i=1}^{n} p_i\,dq^i - (T_2 - T_0)\,dt\right].\end{aligned}$$

We call:

(II.3) $$\Omega_c = \sum_{i=1}^{n} dp_i \wedge dq^i - d(T_2 - T_0) \wedge dt$$

the *Hamiltonian expression* for Ω_c. In T_2, which is the part of T that is quadratic when expressed in terms of the ρ^k, one must then replace the ρ^k with their values as functions of the p, when calculated by way of the equations:

$$p_i = g_{ki}\,\rho^k + g_{i0}\,.$$

If the constraints do not depend upon time t then:

$$\Omega_c = \sum_{i=1}^{n} dp_i \wedge dq^i - dT \wedge dt\,.$$

Lagrangian form of Ω_c. – Upon remarking that:

$$g_{ki}\,\rho^k = \frac{\partial T}{\partial \rho_i}\,,$$

one will get:

(II.4) $$\Omega_c = d\left(\frac{\partial T}{\partial \rho_i} dq^i - T\,dt\right)$$

$$= \frac{\partial^2 T}{\partial \rho^i\,\partial \rho^k} d\rho^k \wedge dq^i + \left(\frac{\partial^2 T}{\partial \rho^i\,\partial \rho^k} - \frac{\partial^2 T}{\partial \rho^k\,\partial \rho^i}\right) dq^k \wedge dq^i - \frac{\partial T}{\partial \rho^k} d\rho^k \wedge dt - \frac{\partial T}{\partial q^i} dp^i \wedge dt\,.$$

That is the form for Ω_c that leads to the Lagrange equations.

3. Definition of a holonomic parameterized system. – A *holonomic parameterized system* is a set of solid bodies and material points such that the kinetic part of Ω can be given the Hamiltonian or Lagrangian form.

4. Riemannian manifold and natural frame R. – The Riemannian manifold V_{n+1} that is associated with a holonomic system is the configuration space-time that is endowed with the metric:

$$d\sigma^2 = g_{ih}\, dq^i\, dq^h + g_{0i}\, dq^i\, dt + g_{00}\, dt^2 \qquad (i, h \text{ vary from 1 to } n).$$

The frame $\mathcal{R}$ at a point M (q^i, t) of that manifold is defined by (n + 1) vectors $\mathbf{e}_0$, $\mathbf{e}_i$ such that $\mathbf{e}_i \cdot \mathbf{e}_k = g_{ik}$, $\mathbf{e}_0 \cdot \mathbf{e}_i = g_{0i}$, $(g_{00})^2 = g_{00}$.

We call the system of n vectors $\mathbf{e}_i$ the *natural frame* R at M ; R is then deduced from $\mathcal{R}$ by suppressing the vector $\mathbf{e}_0$. The point that is the image of the system on the manifold V_{n+1} has a velocity $\mathbf{v}$ that is a vector whose (n + 1) contravariant components $(\dot{q}^i, 1)$ are $\mathbf{v} = \mathbf{e} + \dot{q}^i \mathbf{e}_i$.

For the purposes of mechanics, when one says that the system depends upon n velocity parameters q, one then considers $\mathbf{v}$ to be in the subspace R, so $\mathbf{v} = \mathbf{e}_i\, \dot{q}^i$, where the $\dot{q}^i$ are the contravariant components of $\mathbf{v}$ in R. One must likewise recall that a vector $\mathbf{X}$ whose contravariant components with respect to R are X^i will have n covariant components $X_i = g_{ik}\, X^k$, and that one can pass from the covariant components to the contravariant components by the formulas:

$$X^k = g^{ki}\, X_i\,, \qquad \text{with} \qquad g^{ki} = \frac{\text{minor of } g_{ik} \text{ in } \det|\, g_{ik}\,|}{\det|\, g_{ik}\,|}.$$

5. Generalized force. – We have characterized a force that is applied to a system by the power that it delivers $P = Q_i\, \dot{q}^i$ (i varies from 1 to n). The Q_i are then the covariant components of the *generalized force* with respect to the frame. We remark that the real power is $\mathcal{P} = Q_i\, \dot{q}^i + Q_0$. That is then the power delivered with respect to the Riemannian frame $\mathcal{R}$ when we characterize the system of external forces by the dynamical part of Ω, namely, $\Omega_d = \pi \wedge dt$, where $\pi = Q_i\, dq^i$ is a Pfaff form that is defined on a manifold V_{n+1} or V_n (for a system that is independent of time).

6. Equations of motion. – The equations of motion of the system are the characteristic equation of Ω.

Hamiltonian form:

$$\Omega = dp_i \wedge dq^i - [d\,(T_2 - T_0) - Q_i\, dq^i] \wedge dt,$$

$$\begin{cases} \dfrac{\partial\Omega}{\partial(dq^i)} = -dp_i - \dfrac{\partial(T_2 - T_0)}{\partial q^i}dt + Q_i dt = 0, \\ \dfrac{\partial\Omega}{\partial(dp_i)} = dq^i - \dfrac{\partial(T_2 - T_0)}{\partial q^i}dt = 0. \end{cases}$$

Lagrangian form:

$$\Omega = \frac{\partial^2 T}{\partial\rho^i\partial\rho^k}d\rho^k\wedge dq^i + \left(\frac{\partial^2 T}{\partial\rho^i\partial q^k} - \frac{\partial^2 T}{\partial\rho^k\partial q^i}\right)dq^k\wedge dq^i$$
$$- \frac{\partial T}{\partial\rho^k}d\rho^k\wedge dt - \frac{\partial T}{\partial q^i}dq^i\wedge dt + Q_i\,dq^i\wedge dt\,,$$

$$\frac{\partial\Omega}{\partial(d\rho^k)} = \frac{\partial^2 T}{\partial\rho^i\partial\rho^k}dq^i + \frac{\partial^2 T}{\partial\rho^k}dt = 0,$$

$$\frac{\partial\Omega}{\partial(dq^i)} = -\frac{\partial^2 T}{\partial\rho^i\partial\rho^k}d\rho^k + \frac{\partial^2 T}{\partial\rho^i\partial q^k}dq^k + \frac{\partial^2 T}{\partial\rho^k\partial q^i}dq^k + \frac{\partial T}{\partial q^i}dt + Q_i\,dt = 0,$$

but $\dfrac{\partial T}{\partial\rho^i} = g_{ki}\,\rho^k$, so $\dfrac{\partial^2 T}{\partial\rho^k\partial\rho^i} = g_{ki}$, and it will result that the first n equations can be further written $\dfrac{\partial\Omega}{\partial(d\rho^h)} = g_{ki}\,(dq^i - \rho^i\,dt) = 0$ and $dq^i = \rho^i\,dt$.

When one takes into account the fact that $dq^i = \rho^i\,dt$, the last n will take the Lagrange form when one points out that:

$$\frac{\partial^2 T}{\partial\rho^k\partial q^i}dq^k = \frac{\partial^2 T}{\partial\rho^k\partial q^i}\rho^k dt = \frac{\partial T}{\partial q^i}dt\,,$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\rho^i}\right) - \frac{\partial T}{\partial\rho^i}dt - Q_i\,dt = 0.$$

Those n equations keep the same form under the transformations of the point-like "pseudo-group" $q^i = q^i\,(r^k,\,t)$ that acts upon only the position variables q^i, which are traditionally placed at the foundations for rational mechanics. They have the inconvenience that they prove to be less manageable in the study of the integrable cases or the topological properties of trajectories. That is why it is preferable to consider the equations of motion to be the characteristics of a form of degree two.

General form. – If one performs an arbitrary change of variables on the position parameters q^i and the velocity parameters p_i :

$$p_i = p_i\,(x^\alpha, t), \qquad\qquad q^i = q^i\,(x^\alpha, t),$$

in which α varies from 1 to $2n$, then upon letting ω^α denote n Pfaff forms in dx^α, Ω will be written:

$$\Omega = k_{\alpha\beta}\,\omega^\alpha \wedge \omega^\beta - k_{\alpha 0}\,\omega^\alpha \wedge dt\,,$$

so $(k_{\alpha\beta}, k_{\alpha 0})$ will be an antisymmetric tensor of order 2 that is a function of the x^α, t. The equations take the general form:

$$\frac{\partial\Omega}{\partial\omega^\alpha} = k_{\alpha\beta}\,\omega^\beta - k_{\alpha 0}\,dt = 0.$$

7. In order for the theorem thus-constructed to make sense from the physical standpoint, it is obviously necessary in the applications for the applied forces to be bounded in magnitude and for the rigidity of the internal constraints to be respected.

Example. – Calculate Ω for a solid body that moves around one of its fixed points O.

Oxyz denotes a trihedron that is invariably coupled with the body (moving trihedron) $\Omega_c + \Omega_d = \Omega$:

$$\Omega_c = \int_c (k_{ij}\,dv^i \wedge dx^j)\,\delta m - \left[\int_c k_{ij}\,v^i dv^j\,\delta m\right] \wedge dt\,, \qquad (i, j = 1, 2, 3),$$

$$\Omega_d = \left[\int_c \delta\mathbf{F} \cdot d\overrightarrow{OM}\right] \wedge dt$$

with respect to any Galilean trihedron. In order to use the moving axes, one must then calculate the absolute differentials of the position and velocity parameters with respect to those axes.

If x, y, z denote the coordinates of the point M, which is fixed with respect to the moving trihedron, and **i**, **j**, **k** are the unit vectors of the moving axes then:

$$\overrightarrow{OM} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}\,,$$
$$d\,\overrightarrow{OM} = x\,d\,\mathbf{i} + y\,d\,\mathbf{j} + z\,d\,\mathbf{k}\,.$$

If ω^1, ω^2, ω^3 denote three differentiable Pfaff forms that are constructed from the differentials of the parameters that characterize the displacement of the trihedron *Oxyz* (Euler angles or any other system) then:

$$\begin{cases} d\,\mathbf{i} = \omega^3\mathbf{j} - \omega^2\mathbf{k}, \\ d\,\mathbf{j} = \omega^1\mathbf{k} - \omega^3\mathbf{i}, \\ d\,\mathbf{k} = \omega^2\mathbf{i} - \omega^1\mathbf{j}, \end{cases}$$

$$d\,\overrightarrow{OM} = \mathbf{i}\,(z\,\omega^2 - y\,\omega^3) + \mathbf{j}\,(x\,\omega^3 - z\,\omega^2) + \mathbf{k}\,(y\,\omega^1 - x\,\omega^2)\ .$$

Take the velocity parameters to be the components p, q, r with respect to the moving axes of the instantaneous rotation vector:

$$\mathbf{V} = (qz - ry)\,\mathbf{i} + (rx - pz)\,\mathbf{j} + (py - qx)\,\mathbf{k}\ .$$

The kinetic part Ω_c of Ω is calculated as the exterior derivative of the scalar form:

$$\int_c (\mathbf{V}_M \cdot d\,\overrightarrow{OM})\,\delta m - \tfrac{1}{2}dt \int_c (\mathbf{V}_M)^2\,\delta m\,,$$

with

$$\int_c (\mathbf{V}_M \cdot d\,\overrightarrow{OM})\,\delta m$$

$$= \int_c [(qz - ry)(\omega^2 z - \omega^3 y) + (rx - pz)(\omega^3 x - \omega^1 z) + (py - qx)(\omega^1 y - \omega^2 x)]\,\delta m$$

$$= p\,\omega^1 \int (y^2 + z^2)\,\delta m + q\,\omega^2 \int (z^2 + x^2)\,\delta m + r\,\omega^3 \int (x^2 + y^2)\,\delta m$$
$$-(r\,\omega^1 + q\,\omega^3)\int yz\,\delta m - (p\,\omega^3 + r\,\omega^1)\int zx\,\delta m - (q\,\omega^1 + p\,\omega^2)\int xy\,\delta m$$

Upon choosing the trihedron $Oxyz$ to be the principal trihedron of inertia at O and using the classical rotations:

$$A = \int (y^2 + z^2)\,\delta m\,, \quad B = \int (z^2 + x^2)\,\delta m\,, \quad C = \int (x^2 + y^2)\,\delta m\,,$$

$$\int_c (\mathbf{V}_M \cdot d\,\overrightarrow{OM})\,\delta m = A\,p\,\omega^1 + B\,q\,\omega^2 + C\,r\,\omega^3,$$

$$\tfrac{1}{2}\int_c (\mathbf{V}_M)^2\,\delta m = \tfrac{1}{2}(A\,p^2 + B\,q^2 + C\,r^2)\ .$$

If d denotes the symbol of the exterior derivative then:

$$\Omega_c = A\,dp \wedge \omega^1 + B\,dq \wedge \omega^2 + C\,dr \wedge \omega^3$$
$$+ A\,p\,d\omega^1 + B\,q\,d\omega^2 + C\,r\,d\omega^3 - (A\,p\,dp + B\,q\,dq + C\,r\,dr) \wedge dt\ .$$

The calculation of $d\omega^1$, $d\omega^2$, $d\omega^3$ results from exterior differentiating the vector relations:

$$d\,\mathbf{i} = \omega^3\,\mathbf{j} - \omega^2\,\mathbf{k}, \qquad d\,\mathbf{j} = \ldots, \qquad d\mathbf{k} = \ldots$$

$$0 = d\omega^3\,\mathbf{j} - d\omega^2\,\mathbf{k} - \omega^3 \wedge (\omega^1\,\mathbf{k} - \omega^3\,\mathbf{i}) + \omega^2 \wedge (\omega^2\,\mathbf{i} - \omega^2\,\mathbf{j}),$$

or

$$(d\omega^3 + \omega^1 \wedge \omega^2)\,\mathbf{j} - (d\omega^2 + \omega^3 \wedge \omega^1)\,\mathbf{k} = 0, \quad \text{and analogous ones;}$$

i.e.:

$$d\omega^3 = -(\omega^1 \wedge \omega^2)\,, \quad d\omega^1 = -(\omega^2 \wedge \omega^3)\,, \quad d\omega^2 = -(\omega^3 \wedge \omega^1)\,,$$

which are the structure equations of the group of displacements around O.

It results that the expression for Ω_c is:

$$\Omega_c = A\,dp \wedge \omega^1 + B\,dq \wedge \omega^2 + C\,dr \wedge \omega^3 - A\,p\omega^2 \wedge \omega^3 - B\,q\omega^3 \wedge \omega^1 - C\,r\omega^1 \wedge \omega^2$$
$$- (A\,p\,dp + B\,q\,dq + C\,r\,dr) \wedge dt\,.$$

Let us calculate the dynamical part Ω_d of Ω:

The external forces are defined by the power that they deliver:

$$P = L\,p + M\,q + N\,r,$$

so one has the Pfaff form:

$$\pi = L\omega^1 + M\,\omega^2 + N\omega^3,$$

and one will have

$$\Omega_d = (L\omega^1 + M\,\omega^2 + N\omega^3) \wedge dt$$

when one lets L, M, N denote the components of the resultant moment with respect to the Ox, Oy, Oz of the forces that are applied to the body; hence:

$$\Omega = A\,(dp \wedge \omega^1) + B\,(dq \wedge \omega^2) + C\,(dr \wedge \omega^3) - A\,p\omega^2 \wedge \omega^3 - B\,q\omega^3 \wedge \omega^1 - C\,r\omega^1 \wedge \omega^2$$
$$- [A\,p\,dp + B\,q\,dq + C\,r\,dr - (L\omega^1 + M\,\omega^2 + N\omega^3)] \wedge dt\,.$$

Differential equations of motion:

$$\frac{\partial\Omega}{\partial\omega^1} = -A\,dp + B\,q\,\omega^3 - C\,r\,\omega^2 + L\,dt = 0,$$

$$\frac{\partial\Omega}{\partial\omega^2} = -B\,dq + C\,r\,\omega^1 - A\,p\,\omega^3 + M\,dt = 0,$$

$$\frac{\partial\Omega}{\partial\omega^3} = -C\,dr + A\,q\,\omega^2 - B\,q\,\omega^1 + N\,dt = 0,$$

$$\frac{\partial\Omega}{\partial(dp)} = A\,(\omega^1 - p\,dt) = 0,$$

$$\frac{\partial\Omega}{\partial(dq)} = B\,(\omega^2 - q\,dt) = 0,$$

$$\frac{\partial\Omega}{\partial(dr)} = C\,(\omega^3 - r\,dt) = 0\,.$$

When one takes the last three equations into account, the first three equations will be the equations of motion that Euler gave.

Calculating Ω_c in the reference trihedron that moves with the body and in space. – For the sake of applications, it is interesting to know the expression for Ω_c for a solid body that moves around one of its arbitrary points O when one uses a trihedron that moves in both the body and in space for the reference trihedron. Let **i**, **j**, **k** be the unit vectors in the reference trihedron, let $\Omega^1, \Omega^2, \Omega^3$ be three Pfaff forms that are constructed from the parameters that characterize the position of the trihedron that moves around the point fixed point O, let x, y, z be the coordinates of a point M of the solid with respect to the moving trihedron, let $\omega^1, \omega^2, \omega^3$ be three Pfaff forms that are constructed from the differentials of the parameters that characterize the absolute displacement of the solid, and let p, q, r be the components of the absolute rotation of the body with respect to the moving trihedron. Hence:

$$\begin{aligned}
\overrightarrow{OM} &= x\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k},\\
d\,\overrightarrow{OM} &= (\omega^2 z - \omega^3 y)\,\mathbf{i} + (\omega^3 x - \omega^2 z)\,\mathbf{j} + (\omega^1 y - \omega^2 x)\,\mathbf{k},\\
\mathbf{V} &= (q\,z - r\,y)\,\mathbf{i} + (r\,x - p\,z)\,\mathbf{j} + (p\,y - q\,x)\,\mathbf{k}\,.
\end{aligned}$$

Calculate the differential $d\mathbf{V}$ while taking into account the expressions for the differentials $d\,\mathbf{i}$, $d\,\mathbf{j}$, $d\,\mathbf{k}$:

$$d\,\mathbf{i} = \Omega^3\,\mathbf{j} - \Omega^2\,\mathbf{k}\,, \qquad d\,\mathbf{j} = \Omega^1\,\mathbf{k} - \Omega^3\,\mathbf{i}\,, \qquad d\,\mathbf{k} = \Omega^2\,\mathbf{i} - \Omega^1\,\mathbf{j}\,,$$

so:

$$\begin{aligned}
d\,\mathbf{V} = {} & [z\,dq - y\,dr + q\,dz - r\,dy + (rx - pz)\,(-\Omega^3) + (py - qx)\,\Omega^2]\,\mathbf{i}\\
& + [x\,dr - z\,dp + r\,dx - p\,dz + (py - qx)\,(-\Omega^1) + (qy - ry)\,\Omega^3]\,\mathbf{j}\\
& + [y\,dp - x\,dq + p\,dy - q\,dx + (qz - ry)\,(-\Omega^2) + (rx - pz)\,\Omega^3]\,\mathbf{k}\,.
\end{aligned}$$

dx, dy, dz are the differentials of the coordinates of M with respect to the moving axes and have the values:

$$\begin{aligned}
dx &= (\omega^2 - \Omega^2)\,z - (\omega^3 - \Omega^3)\,y\,,\\
dy &= (\omega^3 - \Omega^3)\,x - (\omega^1 - \Omega^1)\,z\,,\\
dz &= (\omega^1 - \Omega^1)\,y - (\omega^2 - \Omega^2)\,x\,,
\end{aligned}$$

and one will then have the following expression for $d\,\mathbf{V}$:

$$\begin{aligned}
d\,\mathbf{V} = {} & \{-x\,(q\,\omega^2 + r\,\omega^3)\\
& \qquad + y\,[p\,\Omega^2 + q\,(\omega^1 - \Omega^1) - dr] + z\,[p\,\Omega^3 + r\,(\omega^1 - \Omega^1) - dq]\}\,\mathbf{i}\\
& + \{x\,[q\,\Omega^1 + p\,(\omega^2 - \Omega^2) + dr]\\
& \qquad - y\,(r\,\omega^3 + p\,\omega^1) + z\,[q\,\Omega^3 + r\,(\omega^2 - \Omega^2) - dp]\}\,\mathbf{j}\\
& + \{x\,[r\,\Omega^1 + p\,(\omega^3 - \Omega^3) - dq]\\
& \qquad - y\,[r\,\Omega^2 + q\,(\omega^3 - \Omega^3) + dp] - z\,(p\,\omega^1 + q\,\omega^2)\}\,\mathbf{k}\,.
\end{aligned}$$

Form the expression $k_{ij}\ dv^i \wedge dx^j$ in seven-dimensional space, where k_{ij} is the Kronecker symbol:

$$k_{ij}\,dv^i \wedge dx^j$$

$$= x^2\,[dr \wedge \omega^3 + dq \wedge \omega^2 + 2p\ \omega^2 \wedge \omega^3 - p\,\Omega^2 \wedge \omega^3 + q\,\Omega^1 \wedge \omega^3 - r\,\Omega^1 \wedge \omega^2 + p\,\Omega^3 \wedge \omega^2]$$

$$+ y^2\,[dp \wedge \omega^1 + dr \wedge \omega^3 + 2q\ \omega^3 \wedge \omega^1 - q\,\Omega^3 \wedge \omega^1 + r\,\Omega^2 \wedge \omega^1 - p\,\Omega^2 \wedge \omega^1 + q\,\Omega^1 \wedge \omega^3]$$

$$+ z^2\,[dq \wedge \omega^2 + dp \wedge \omega^1 + 2r\ \omega^1 \wedge \omega^2 - r\,\Omega^1 \wedge \omega^2 + p\,\Omega^3 \wedge \omega^2 - q\,\Omega^3 \wedge \omega^1 + r\,\Omega^2 \wedge \omega^1]$$

$$+ yz\,[- dr \wedge \omega^2 - dq \wedge \omega^3 + 2q\ \omega^1 \wedge \omega^2 + 2r\ \omega^3 \wedge \omega^1 + p\,(\Omega^2 \wedge \omega^2 - \Omega^3 \wedge \omega^3) - q\,\Omega^1 \wedge \omega^2 + r\,\Omega^1 \wedge \omega^3]$$

$$+ zx\,[- dp \wedge \omega^3 - dr \wedge \omega^1 + 2r\ \omega^2 \wedge \omega^3 + 2p\ \omega^1 \wedge \omega^2 + q\,(\Omega^3 \wedge \omega^3 - \Omega^1 \wedge \omega^1) - r\,\Omega^2 \wedge \omega^2 + p\,\Omega^2 \wedge \omega^1]$$

$$+ xy\,[- dq \wedge \omega^1 - dp \wedge \omega^2 + 2p\ \omega^3 \wedge \omega^1 + 2q\ \omega^2 \wedge \omega^3 + r\,(\Omega^1 \wedge \omega^1 - \Omega^2 \wedge \omega^2) - p\,\Omega^3 \wedge \omega^1 + q\,\Omega^3 \wedge \omega^2]\ .$$

Likewise calculate $k_{ij}\,v^i\,dv^j$:

$$\begin{aligned} k_{ij}\,v^i\,dv^j = {} & x^2\,[r\,dr + q\,dq + pr\,(\omega^2 - \Omega^2) - pq\,(\omega^3 - \Omega^3)] \\ & + y^2\,[p\,dp + r\,dr + qp\,(\omega^3 - \Omega^3) - qr\,(\omega^1 - \Omega^1)] \\ & + z^2\,[q\,dq + p\,dp + rq\,(\omega^1 - \Omega^1) - rp\,(\omega^2 - \Omega^2)] \\ & - yz\,[r\,dq + q\,dr + (q^2 - r^2)(\omega^1 - \Omega^1) - pq\,(\omega^2 - \Omega^2) + rp\,(\omega^3 - \Omega^3)] \\ & - zx\,[p\,dr + r\,dp + (r^2 - p^2)(\omega^2 - \Omega^2) - qr\,(\omega^3 - \Omega^3) + pq\,(\omega^1 - \Omega^1)] \\ & - xy\,[q\,dp + p\,dq + (p^2 - q^2)(\omega^3 - \Omega^3) - rp\,(\omega^1 - \Omega^1) + qr\,(\omega^2 - \Omega^2)]\ . \end{aligned}$$

Let:

$$a = \int x^2 \delta m\,, \quad b = \int y^2 \delta m\,, \quad c = \int z^2 \delta m\,,$$

$$D = \int y\,z\,\delta m\,, \quad E = \int z\,x\,\delta m\,, \quad F = \int x\,y\,\delta m\,,$$

and note that these quantities are generally variables that vary in time, since the axes move in the body and in space. One will then deduce that:

$$\Omega_c = \int \omega_c\,\delta m = \int (k_{ij}\,dv^i \wedge dx^j - k_{ij}\,v^i dv^j \wedge dt)\,\delta m\,,$$

so

$$\Omega_c =$$

$$\begin{aligned} & a\,[dq \wedge \omega^2 + dr \wedge \omega^3 + 2p\ \omega^2 \wedge \omega^3 - p\,\Omega^2 \wedge \omega^3 + q\,\Omega^1 \wedge \omega^3 - r\,\Omega^1 \wedge \omega^2 + p\,\Omega^3 \wedge \omega^2] \\ & + b\,[dr \wedge \omega^3 + dp \wedge \omega^1 + 2q\ \omega^3 \wedge \omega^1 - q\,\Omega^3 \wedge \omega^1 + r\,\Omega^2 \wedge \omega^1 - p\,\Omega^2 \wedge \omega^3 + q\,\Omega^1 \wedge \omega^3] \\ & + c\,[dp \wedge \omega^1 + dq \wedge \omega^2 + 2r\ \omega^1 \wedge \omega^2 - r\,\Omega^1 \wedge \omega^2 + p\,\Omega^3 \wedge \omega^2 - q\,\Omega^3 \wedge \omega^1 + r\,\Omega^2 \wedge \omega^1] \end{aligned}$$

$$- D\,[dq \wedge \omega^3 + dr \wedge \omega^2 - 2q\,\omega^1 \wedge \omega^2 - 2r\,\omega^1 \wedge \omega^1 - p\,(\Omega^2 \wedge \omega^2 - \Omega^3 \wedge \omega^3) + q\,\Omega^1 \wedge \omega^2 - r\,\Omega^1 \wedge \omega^3]$$

$$- E\,[dr \wedge \omega^1 + dp \wedge \omega^3 - 2r\,\omega^2 \wedge \omega^3 - 2p\,\omega^1 \wedge \omega^2 - q\,(\Omega^3 \wedge \omega^3 - \Omega^1 \wedge \omega^1) + r\,\Omega^2 \wedge \omega^3 - p\,\Omega^2 \wedge \omega^1]$$

$$- F\,[dp \wedge \omega^2 + dq \wedge \omega^1 - 2p\,\omega^3 \wedge \omega^1 - 2q\,\omega^2 \wedge \omega^3 - r\,(\Omega^3 \wedge \omega^2 - \Omega^2 \wedge \omega^3) + p\,\Omega^3 \wedge \omega^1 - q\,\Omega^3 \wedge \omega^2]$$

$$- a\,[q\,dq + r\,dr + pr\,(\omega^2 - \Omega^2) - pq\,(\omega^3 - \Omega^3)] \wedge dt$$
$$- b\,[r\,dr + p\,dp + qp\,(\omega^3 - \Omega^3) - qr\,(\omega^1 - \Omega^1)] \wedge dt$$
$$- c\,[p\,dq + q\,dq + rq\,(\omega^1 - \Omega^1) - rp\,(\omega^2 - \Omega^2)] \wedge dt$$

$$+ D\,[r\,dq + q\,dr + (q^2 - r^2)(\omega^1 - \Omega^1) - pq\,(\omega^2 - \Omega^2) + rp\,(\omega^3 - \Omega^3)] \wedge dt$$
$$+ E\,[p\,dr + r\,dp + (r^2 - p^2)(\omega^2 - \Omega^2) - qr\,(\omega^3 - \Omega^3) + pq\,(\omega^1 - \Omega^1)] \wedge dt$$
$$+ F\,[q\,dp + p\,dq + (p^2 - q^2)(\omega^3 - \Omega^3) - rp\,(\omega^1 - \Omega^1) + qr\,(\omega^2 - \Omega^2)] \wedge dt\,.$$

Special cases:

1. Axes fixed in the body:

$$a + b = C, \qquad b + c = A, \qquad c + a = B, \qquad \Omega^1 = \omega^1, \qquad \Omega^2 = \omega^2, \qquad \Omega^3 = \omega^3,$$

$$\Omega_c = A\,dp \wedge \omega^1 + B\,dq \wedge \omega^2 + C\,dr \wedge \omega^3 - Ap\,\omega^2 \wedge \omega^3 - Bq\,\omega^3 \wedge \omega^1 - Cr\,\omega^1 \wedge \omega^2$$

$$- D\,[dq \wedge \omega^3 + dr \wedge \omega^2 - q\,\omega^1 \wedge \omega^2 - r\,\omega^3 \wedge \omega^1]$$
$$- E\,[dr \wedge \omega^1 + dp \wedge \omega^3 - r\,\omega^2 \wedge \omega^3 - p\,\omega^1 \wedge \omega^2]$$
$$- F\,[dp \wedge \omega^2 + dq \wedge \omega^1 - p\,\omega^3 \wedge \omega^1 - q\,\omega^2 \wedge \omega^3]$$

$$- [Ap\,dp + Bq\,dq + Cr\,dr - D\,(q\,dr + r\,dq) - E\,(r\,dp + p\,dr) - F\,(p\,dq + q\,dp)] \wedge dt.$$

A, B, C, D, E, F are constants.

One verifies that:

$$\Omega_c = d\,[Ap\,\omega^1 + Bq\,\omega^2 + Cr\,\omega^3 - D\,(q\omega^3 + r\omega^2) - E\,(r\omega^1 + p\omega^3) - F\,(p\omega^2 + q\omega^1) - \tfrac{1}{2}(A\,p^2 + Bq^2 + Cr^2 - 2Dqr - 2Erp - 2Fpq)\,dt]\,.$$

2. Axes fixed in space:

$$a + b = C, \qquad b + c = A, \qquad c + a = B, \qquad \Omega^1 = \Omega^2 = \Omega^3 = 0,$$

$$\Omega_c = A\,dp \wedge \omega^1 + B\,dq \wedge \omega^2 + C\,dr \wedge \omega^3 + (B + C - A)\,\omega^2 \wedge \omega^3 + (C + B - A)\,\omega^3 \wedge \omega^1 + (A + B - C)\,\omega^1 \wedge \omega^2$$

$$- D\,[dq \wedge \omega^3 + dr \wedge \omega^2 - 2q\,\omega^1 \wedge \omega^2 - 2r\,\omega^3 \wedge \omega^1]$$

$$- E\,[dr \wedge \omega^1 + dp \wedge \omega^3 - 2r\,\omega^2 \wedge \omega^3 - 2p\,\omega^1 \wedge \omega^2]$$
$$- F\,[dp \wedge \omega^2 + dq \wedge \omega^1 - 2p\,\omega^3 \wedge \omega^1 - 2q\,\omega^2 \wedge \omega^3]$$

$$- \{Ap\,dp + Bq\,dq + Cr\,dr - D\,(q\,dr + r\,dq) - E\,(r\,dp + p\,dr) - F\,(p\,dq + q\,dp)$$
$$+ (C - A)\,pr\,\omega^2 + (A - B)\,qp\omega^3 + (B - C)\,rp\omega^1 - D\,[(q^2 - r^2)\omega^1 - pq\omega^2 + rp\,\omega^3]$$
$$- E\,[(r^2 - p^2)\omega^2 - qr\omega^3 + pq\omega^1] - F\,[(p^2 - q^2)\omega^3 - rp\omega^1 + qr\omega^2]\} \wedge dt.$$

3. Body of revolution: If the Oz axis of the trihedron is the axis of revolution, while Ox and Oy move in the body and in space, then:

$$a = \tfrac{1}{2}C, \qquad c = A - \tfrac{1}{2}C,$$

$\Omega_c =$

$$A\,dp \wedge \omega^1 + A\,dq \wedge \omega^2 + C\,dr \wedge \omega^3 + Cp\,\omega^2 \wedge \omega^3 + Cq\,\omega^3 \wedge \omega^1 + (2A - C)\,r\,\omega^1 \wedge \omega^2$$
$$- Ar\,\Omega^1 \wedge \omega^2 + Cp\,\Omega^2 \wedge \omega^3 - Aq\,\Omega^3 \wedge \omega^1 + Cq\,\Omega^1 \wedge \omega^3 + Ar\,\Omega^2 \wedge \omega^1 + Ap\,\Omega^3 \wedge \omega^2$$
$$- [Apdp + Aq\,dq + Cr\,dr + (C - A)\,pr\,(\omega^2 - \Omega^2) - (C - A)\,qr\,(\omega^1 - \Omega^1)] \wedge dt.$$

§ III. – The basic principle in the study of mechanical systems without coordinates.

The preceding presentation, by the way that it was developed, can give the impression that it is necessary to use coordinates in order to study the properties of dynamical systems. One will easily liberate oneself from the servitude of coordinates by envisioning the question from the following angle:

A holonomic system is characterized by a form Ω of degree 2 and rank $2n$ that is defined on a manifold V_{2n+1} ([15]): Since the space-time manifold V_{n+1} is fibered with V_n for its fiber and the number line t for its base, V_{2n+1} will be the space of vectors tangent to the fibers of V_{n+1} ([16]). Let T be the tangent space to the manifold V_{2n+1}, let T' be the dual space to T, and let $A\,(T')$ be the exterior algebra that is constructed over T and is the direct sum of the vector subspaces A^r of varying degrees $r \geq 0$ that are generated by the forms of degree r.

Henri Cartan's operator $i\,(x)$ ([17]). – If x is a field that is an element of T then, with Henri Cartan, one calls an endomorphism of $A\,(T')$ of degree – 1 that, on the one hand, maps an element of A^r to A^{r-1}, and on the other hand, for:

$$a \in A^p, \qquad b \in A^q, \qquad ab \in A^r \qquad (r = p + q)$$

satisfies:

([15]) Cf., Ch. EHRESMANN, "Espaces fibrés associés à une varieté différentiable," C. R. Acad. Sci., t. 216, pp. 626.

([16]) That conception of V_{2n+1} is justified because at a point M of V_{n+1}, the n directions in the fibers define a frame R that will coincide with the frame R that was defined in § II.4 when one endows V_{n+1} with a metric.

([17]) Cf., H. CARTAN, *Colloque de Topologie*¸ Brussels 1950, Masson and Co., Paris, 1950, pp. 15-27.

$$i\,(x)\,a \cdot b = (i\,(x)\,a)\,b + (-1)^p\,a \cdot (i\,(x)\,b)$$

the *anti-derivation that is defined by the operator* $i\,(x)$. Since the algebra $A\,(T')$ is generated in the multiplicative sense by its elements of degree 0 and 1 (A^0 is identified with the ring of numerical functions), the anti-derivation $i\,(x)$ is determined when it is known on A^0 and $A^1 = T'$; it is zero on A^0, and on A^1 it reduces to the scalar product that defines the duality between T and T'.

For:

$$x \in T, \quad x' \in T', \qquad i\,(x) \cdot x' = \langle x, x' \rangle .$$

It will then result that when the operator $i\,(x)$ is applied to an element of degree r in $A\,(T')$ that is denoted by $(x'_1 \wedge x'_2 \wedge \cdots \wedge x'_r)$, it can be written as:

$$i\,(x)\;(x'_1 \wedge x'_2 \wedge \cdots \wedge x'_r) = \sum_{1 \le k \le r} (-1)^{k+1} < x, x'_k > x'_1 \wedge \cdots \wedge \widehat{x'_k} \wedge \cdots \wedge x'_r .$$

The symbol $\widehat{\quad}$ signifies that the term beneath it must be suppressed.

One should further note that the operator $i\,(x)$ has square zero, since its square is zero on A^0 and A^1.

Having recalled those essential generalities that are extracted from the conference talk by H. Cartan ([17,cont.]), the operator $i\,(x)$ will map, in particular, the form Ω of degree 2 into the vector subspace T' of Pfaff forms.

$$i\,(x)\,\Omega = \pi,$$

which is a symbol that will give:

$$i\,(x)\,\Omega = k_{\alpha\beta}\,(x^\alpha\,d\rho^\beta - x^\beta\,d\rho^\alpha) ,$$

when V_{2n+1} is referred to a coordinate system ρ^α ($\alpha = 0$ to $2n$), since $\Omega = k_{\alpha\beta}\,d\rho^\alpha \wedge d\rho^\beta$, where $k_{\alpha\beta}$ is an anti-symmetric tensor that is a function of the ring A^0, x has x^α for its coordinates, and we suppose that the scalar product is defined by the Kronecker symbol.

In particular, with a Hamiltonian coordinate system:

$$\Omega = dp_i \wedge dq^i - dH \wedge dt + Q_i\,dq^i \wedge dt,$$

$$i\,(x)\,\Omega = x_i\left(dq^i - \frac{\partial H}{\partial p_i}\,dt\right) - x^i\left(dp_i + \frac{\partial H}{\partial q_i}\,dt - Q_i\,dt\right) + x^0\left(dH - \frac{\partial H}{\partial t}\,dt - Q_i\,dq^i\right).$$

The differential system Σ of the characteristics of Ω can be written in an infinitude of ways in terms of $2n$ independent fields $(x^1, x^2, \ldots, x^{2n})$; i.e., such that $x^1 \wedge x^2 \wedge \ldots \wedge x^{2n} \neq$

([17,cont.]) The viewpoint that is adopted here is more restrictive than that of H. Cartan, to whom the reader is requested to refer; all of this application is developed while preserving the ideas and notations of H. Cartan.

0 . When one writes the characteristic equations of Ω in the form $\partial\,\Omega\,/\,\partial\rho^{\alpha} = 0$ ($\alpha = 1$ to $2n$), that will amount to taking the x to be the $2n$ special fields (0, 0, …, 1, …, 0).

We remark that for an arbitrary $x \in T$, the Pfaff form $i\,(x)\,\Omega$ will not be an arbitrary form in T', but it will belong to the sub-module of characteristic forms of Ω.

Characteristic field E. – Since the form Ω of degree 2 that is defined on V_{2n+1} has rank $2n$, there will exist an element $\mathrm{E} \in T$ that is defined up to a factor (viz., a numerical function) and maps Ω to the zero in the space of forms. E is called the *characteristic field*, and one will have:

$$i\,(\mathrm{E})\,\Omega = 0.$$

Since the goal of our study is to construct the theory of constraints of an arbitrary nature for parameterized systems, the following theorems will play an important role:

Theorem I:

The necessary and sufficient condition for a Pfaff form π to belong to the sub-module of characteristic forms of Ω is that $i\,(\mathrm{E})\,\pi = 0$.

1. The condition is necessary because if π belongs to the sub-module of characteristic forms then there will exist $x \in T$, modulo E, such that $i\,(x)\,\Omega = \pi$.

$$i\,(\mathrm{E})\,\pi = i\,(\mathrm{E}) \cdot i\,(x)\,\Omega = -\,i\,(x) \cdot i\,(\mathrm{E})\,\Omega,$$
$$i\,(\mathrm{E})\,\Omega = 0 \qquad \text{implies that} \qquad i\,(\mathrm{E})\,\pi = 0\,.$$

2. The condition is sufficient. Indeed, for $a \in T$, $\pi \in T'$, it will suffice to remark that the condition $i\,(a)\,\pi = 0$ signifies that π belongs to a sub-module of T'. Apply the indicated proposition that π belongs to a sub-module of characteristic forms of Ω to a = E.

Theorem II:

Let f be a numerical function on V_{2n+1}. The numerical value of f on a characteristic line of Ω is a function of the parameter t whose first derivative with respect to t is $i(\mathrm{E}) \cdot df$, and its n^{th} derivative is $(i\,(\mathrm{E}) \cdot df)(n)$.

Before proving that theorem, we point out that if f is a numerical function on V_{2n+1} then when the operator i (E) is applied to the form df, that will make it correspond to a new numerical function $f_E^1 = i\,(\mathrm{E}) \cdot df$. In the particular case where E has $2n$ zero components, while the first one equals 1, relative to a basis x^i, one will have $f_{x^i}^{(1)} = \partial f\,/\,\partial x^i$. That will justify the following definition:

Definition. – *First derived function of a numerical function with respect to a field.* – If f is a numerical function on V_{2n+1} then we shall call the function $f_{\mathrm{E}}^{(1)} = i\,(\mathrm{E})\,df$ the *first* *derivative* of the function f *relative to the field* E. The function $f_{\mathrm{E}}^{(1)}$ is itself a numerical function on V_{2n+1}, so one can apply the operator i (E) to the form $df_{\mathrm{E}}^{(1)}$ and call $f_{\mathrm{E}}^{(2)} = i(\mathrm{E})\cdot df_{\mathrm{E}}^{(1)}$ the *second derivative* of f *with respect to* E, which we shall denote symbolically by $(i\ (\mathrm{E})\cdot df)^{(2)}$. More generally, the n^{th} derivative of the function f with respect to E is the function $f_{\mathrm{E}}^{(n)} = i\ (\mathrm{E})\cdot\ df_{\mathrm{E}}^{(n-1)} = (i\ (\mathrm{E})\cdot df)^{(n)}$.

Remarks:

1. If f is a product of two functions $u\cdot v$ then one can prove in a classical manner that:

$$(u\cdot v)_{\mathrm{E}}^{(n)} = u_{\mathrm{E}}^{(n)}\cdot v + \ldots + C_n^p\, u_{\mathrm{E}}^{(n-p)}\cdot v_{\mathrm{E}}^{(p)} + \ldots + u\cdot v_{\mathrm{E}}^{(n)}.$$

2. One can define a derivative $i\ (\mathrm{E}_2)\ d\ (i\ (\mathrm{E}_1)\cdot df)$ for numerical functions relative to two fields that are taken in the order E_1, E_2. Having posed those generalities, one can prove Theorem II. Let f and g be two numerical functions on V_{2n+1}. Consider the two functions $i\ (\mathrm{E})\cdot df$ and $i\ (\mathrm{E})\cdot dg$. If f is a function of g then df will be proportional to dg. Since the operator i (E) is linear and homogeneous:

$$\frac{df}{dg} = \frac{i(\mathrm{E})\cdot df}{i(\mathrm{E})\cdot dg}.$$

Upon taking $g = t$ and choosing the numerical function that the field E, which is defined to be a solution of $i\ (\mathrm{E})\ \Omega = 0$ such that $i\ (\mathrm{E})\ dt = 1$, depends upon to be arbitrary:

$$\frac{df}{dt} = i\ (\mathrm{E})\ df = f_{\mathrm{E}}^{(1)}.$$

In the right-hand side, one must consider the function $f_{\mathrm{E}}^{(1)}$ to be a function of t along the characteristic line.

Upon repeating the argument that was made for f with the function $f_{\mathrm{E}}^{(1)}$, we shall write:

$$f_{\mathrm{E}}^{(2)} = (i(\mathrm{E})\cdot df)_{\mathrm{E}}^{(2)}$$

along a characteristic line of the field E, and by recurrence:

$$f_{\mathrm{E}}^{(n)} = (i(E)\cdot df)_{\mathrm{E}}^{(n)}.$$

Application. – If one chooses $2n$ independent functions f_i that have no singularities in the neighborhood of a point M then one can represent the characteristic lines of the field E by the system of $2n$ differential equations that one writes:

$$\frac{df_i}{dt} = i\,(\mathrm{E}) \cdot df_i\,.$$

That system will permit one to study the local aspect of the motion in the neighborhood of a point M_0 of the manifold, since one can calculate all of the successive derivatives at M_0 by means of the operator i (E) .

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CHAPTER TWO

THEORY OF A SINGLE CONSTRAINT IMPOSED UPON A MATERIAL SYSTEM

§ I. – Generalities.

Let a material system S be composed of points and solid bodies that depend upon n parameters q^i. In that system, there exist holonomic constraints that one accounts for by saying that the system depends upon n parameters. It can be visualized by a manifold V_{2n+1} of the kind that was specified in Chapter I, § III.

Definition. – We say that one *imposes a new constraint* on the holonomic system S when:

1. The image of S is a submanifold of V_{2n+1}, with a (M) = 0, M $\in V_{2n+1}$.

2. One extends the characteristic field E by way of a constraint field E_l, which is a field that is due to the forces that are necessary in order to realize that constraint.

When one employs a particular system of variables for position q^i, velocity $\dot{q}^i$, and time t, that definition will translate into:

1. A relation $a(q^i, \dot{q}^i, t) = 0$ (i varies from 1 to n) that admits partial derivatives $\partial a / \partial \dot{q}^i \neq 0$ for at least one i (if the relation is holonomic then we agree to replace it with its derivative with respect to t).

2. A generalized force **L** to be appended to the forces applied to S, and whose covariant components L_i with respect to natural frame R (Chapter I, § II.4) are defined by the relative power $P = \sum_{i=1}^{n} L_i \, \dot{q}^i$.

When one characterizes the system S by a form Ω on V_{2n+1}, restricting S by means of a new constraint will amount to replacing Ω with $\Omega + \Omega_l$, $\Omega_l = L_i \, dq^i \wedge dt$, with the condition that $da = 0$.

In the expression for P, the L_i are functions of the q^i, $\dot{q}^i$, t, where the index i takes all possible values; one does not take the relation $a = 0$ into account in the expression for P.

Remarks:

1. There exists a relation that couples the form da to the field E_l that will translate into a relation that couples the partial derivatives of the function a to the L_i when one uses coordinates. The study of that relation will be the subject of § II in this chapter.

2. From the standpoint that one assumes when one considers a solid body S that is restricted to rolling without slipping on a given surface, that condition will translate into three relations, and the solid body S will be considered to be restricted by three constraints of the aforementioned type. In order to make that more precise, take a homogeneous sphere S of radius a that is restricted to rolling without slipping on a fixed plane. With respect to a fixed frame that is composed of a tri-rectangular trihedron, let:

ξ, η, ζ	denote the coordinates of the center of S,
p, q, r	denote the components of the instantaneous rotation vector,
u, v, w	denote the components of the velocity of the point of contact.

The constraints translate into:

$$\begin{cases} u = \dot{\xi} - aq = 0, \\ P_1 = Xu, \end{cases} \qquad \begin{cases} v = \dot{\eta} + ap = 0, \\ P_2 = Yv, \end{cases} \qquad \begin{cases} w = \dot{\zeta} = 0, \\ P_3 = Zw. \end{cases}$$

On the contrary, a sphere that slides on the plane is restricted by only one constraint $\zeta - a = 0$, $P = N\,(\dot{\zeta} - f\sqrt{u^2 + v^2})$ (see the theory in § **6** in of this chapter).

§ II. – Relation between the form *da* and the field E_l .

Since the constraint relation is defined by a numerical function $a = 0$ on V_{2n+1} , the form da must be zero on the integral curves of the equations of motion, so it will belong to the sub-module of characteristic forms of the form $\Omega + \Omega_l$. From the Theorem I in § III, Chapter I, the necessary and sufficient condition for that to be true is that:

(II.1) $$i\,(\mathrm{E} + \mathrm{E}_l) \cdot da = 0,$$

which can also be written:

(II.2) $$i\,(\mathrm{E}_l) \cdot da + i\,(\mathrm{E}) \cdot da = 0.$$

Let us make the condition (II.1) more explicit by taking the variables to be the Hamiltonian variables. The form Ω that characterizes the system S mechanically is:

$$\Omega = dp_i \wedge dq^i - [d\,(T_2 - T_0) - Q_i\, dq^i] \wedge dt\,.$$

The constraint is defined by $a\,(p_i\,, q^i, t) = 0$, $\Omega_l = L_i\, dq^i \wedge dt$.

The characteristic equations of the form $\Omega + \Omega_l$ are:

$$\frac{\partial(\Omega + \Omega_l)}{\partial(dp_i)} = dq^i - \frac{\partial(T_2 - T_0)}{\partial p_l}\, dt\,,$$

$$\frac{\partial(\Omega+\Omega_l)}{\partial(dq^i)} = -\,dp_i + \left[Q_i + L_i - \frac{\partial(T_2 - T_0)}{\partial q^i}\right] dt\,.$$

The form $da = \dfrac{\partial a}{\partial p_i} dp_i + \dfrac{\partial a}{\partial q^i} dq^i + \dfrac{\partial a}{\partial t} dt$ must belong to the sub-module of characteristic forms of $\Omega + \Omega_l$, which has:

$$\frac{\partial(\Omega+\Omega_l)}{\partial(dp_i)}, \qquad \frac{\partial(\Omega+\Omega_l)}{\partial(dq^i)}, \qquad da = -\frac{\partial a}{\partial p_i}\cdot\frac{\partial(\Omega+\Omega_l)}{\partial(dq^i)} + \frac{\partial a}{\partial q^i}\cdot\frac{\partial(\Omega+\Omega_l)}{\partial(dp_i)}$$

for a basis, so one has the condition:

$$\frac{\partial a}{\partial p_i}\left[Q_i + L_i - \frac{\partial(T_2 - T_0)}{\partial q^i}\right] + \frac{\partial a}{\partial q^i}\cdot\frac{\partial(T_2 - T_0)}{\partial p_i} + \frac{\partial a}{\partial t} = 0\,.$$

Interpretation of the condition (II.1). – The condition (II.1) can be interpreted in two different ways.

First viewpoint. – Since the fields E and E_l are known, the condition (II.1) will be a first-order partial differential equation that defines the function *a*. It expresses the necessary and sufficient for a = const. to be a first integral of the system of differential equations of the characteristics of the form $\Omega + \Omega_l$. As the general integral of a first-order linear partial differential equations that depends upon an arbitrary function, $\mathcal{F}(a)$ = const. is also a first integral, but it does not constitute a distinct integral.

Consequence: The constraint relation $a = 0$ cannot be chosen arbitrarily, so it will be a particular integral of the partial differential equation. That constraint relation is also represented analytically by $\mathcal{F}(a) = 0$, with $\mathcal{F}(0) = 0$. That is why we say that a constraint relation is defined analytically only up to an arbitrary function for a holonomic system *S*.

Second viewpoint. – As (II.2) shows, if the submanifold $a = 0$ of V_{2n+1} is given then the condition (II.1) will be linear in the field E_l . There will then exist an infinitude of fields E_l that satisfy (II.2). One can give a geometric interpretation to (II.2) by using the natural frame *R*(Chapter I, § II.4). At the point *M* of the manifold V_{n+1}, consider the vector **L** (whose covariant components are L_i) and the vector **a** (whose contravariant components are $\partial a / \partial p_i$). $i\,(\mathrm{E}) \cdot da = \sum_{i=1}^{n} \dfrac{\partial a}{\partial p_i} L_i$ is the scalar product of those two vectors. The numerical function $i\,(\mathrm{E}) \cdot da$, which we denote by α, does not depend upon the constraint force. Equation (II.2) will then take the geometric form $\mathbf{a} \cdot \mathbf{L} + \alpha = 0$, and will signify that the extremity of the vector **L** whose origin is *M* is situated in a hyperplane that is defined by the equation:

$$\sum_{i=1}^{n} \frac{\partial a}{\partial p_i} L_i + \alpha = 0.$$

That equation determines how to get one of the components of L_i when one supposes that the other (n – 1) are known.

§ III. – Distinction between a first integral of motion of *S* and a constraint imposed upon *S*.

If i (E) · da = 0 at any point of V_{2n+1} then, from Theorem I (Chapter I, § III), that will signify that da belongs to the sub-module of characteristic forms of Ω ; in other words, that a = 0 is a particular first integral of the system of characteristics of Ω. That constraint can be realized, in particular, without appending any new forces to the mechanical system S, since one can satisfy (II.1) by taking the field E_l to be zero. That will permit us to make the notion of a constraint that is imposed upon a mechanical system S more precise.

Definition. – A constraint is imposed upon a system S when a = 0 is not a particular first integral of the equations of motion of S . That definition translates into the condition i (E) · $da \neq 0$, and when one employs the coordinates p_i , q^i, t, it can be written:

$$\sum_{i=1}^{n} \frac{\partial a}{\partial p_i}\left(Q_i - \frac{\partial (T_2 - T_0)}{\partial q^i}\right) + \sum_{i=1}^{n} \frac{\partial a}{\partial q^i} \cdot \frac{\partial (T_2 - T_0)}{\partial p_i} + \frac{\partial a}{\partial t} \neq 0 .$$

§ IV. – Constraint of the type a = 0, $\lambda\, e$. Compatibility.

For a great number of the usual constraints, the mechanism of constraint is such that one knows, *a priori*, the direction of the generalized constraint force **L** ; i.e., the constraint field $\mathbf{E}_l$ has the form $\lambda\, e$ of a field with a known direction, where λ is a numerical function to be determined.

Examples:

1. Two solid bodies in contact without friction, so the constraint force is directed along the common normal.

2. Two solid bodies that slide over each other with friction, and the constraint force obeys either Coulomb's law or a generalized Coulomb law. One then knows the direction of the constraint force, since it is situated, on the one hand, in the half-plane that is defined by the common normal and the vector that is opposite to the sliding velocity $\mathbf{V}_g$, and on the other hand, makes an angle φ with the common normal in that half-plane such that tan $\varphi = f$. (f = const. is the ordinary Coulomb law. When f is a function of the sliding velocity and the normal component N of the pressure between the two solid

bodies, that is the generalized Coulomb law.) The same thing will result when one takes into account the couple of the resistance to rolling and pivoting.

3. Zero-power constraint. One will see how to deal with this in § V.

4. Béghin's servo constraint (†), whose direction of action is known.

Upon exhibiting the direction e of the constraint force $E_l = \lambda\, e$, equation (II.2) can be written:

$$\lambda\, i\,(e) \cdot da + i\,(\mathrm{E}) \cdot da = 0\,. \tag{II.3}$$

By hypothesis, in the case of one constraint imposed upon S, $i\,(\mathrm{E}) \cdot da \neq 0$, the preceding equation will determine a finite value for the factor λ when $i\,(e) \cdot da \neq 0$. The hypotheses $i\,(\mathrm{E}) \cdot da \neq 0$, $i\,(e) \cdot da = 0$ lead one to attribute an infinite value to λ. Under the action of a force whose magnitude becomes infinite, the invariable constraints that one must necessarily take into account when one exhibits the equations of the parameterized system must cease to be. One must then imagine the deformation of those constraints. In the sense of the theory of invariable solid bodies, one cannot begin to study the case $i\,(e) \cdot da = 0$, for which the postulate in Chapter I, § II (viz., a zero-power system of internal forces) is not valid. We shall encounter an example of that situation in the case of friction due to sliding, rolling, and pivoting at the end of this chapter. The foregoing leads us to the notion of a compatible constraint.

Definition. – *Compatibility of a constraint of type $a = 0$, $\lambda\, e$.* – A constraint imposed on a mechanical system S of type $a = 0$, $\lambda\, e$ is called *compatible* when $i\,(e) \cdot da \neq 0$.

Remark:

If one has simultaneously $(i\,(e) \cdot da)_0 = 0$, $(i\,(\mathrm{E}) \cdot da)_0 = 0$ at a point M_0 of V_{2n+1} that belongs to the submanifold $a = 0$ then λ will not be determined by equation (II.3). One can then determine it by means of the value of $a^{(2)}$ that is calculated by applying Theorem II to Chapter I, § III:

$$a^{(2)} = (i\,(\mathrm{E} + le) \cdot da)^{(2)} = 0,$$

$$\lambda^2\,(i\,(e)\,da)^{(2)} + \lambda\,[(i\,(e)\,d\lambda) \cdot (i\,(e)\,da) + (i\,(\mathrm{E})\,d\lambda) \cdot (i\,(e)\,da) + i\,(e)\,d\,(i\,(\mathrm{E})\,da) \\ + i\,(\mathrm{E})\,d\,(i\,(e)\,da)] + (i\,(\mathrm{E})\,da)^{(2)} = 0,$$

which is a relation that reduces to:

$$\lambda_0^2\,(i(e)\,da)_0^{(2)} + \lambda_0\,[i\,(e)\,d\,(i\,(\mathrm{E})\,da) + i\,(\mathrm{E})\,d\,(i\,(e)\,da)]_0 + (i(e)\,da)_0^{(2)}$$

(†) Translator: The French phrase *liaison d'asservissement* means literally "constraint of servitude (or slavery," but nowadays such constraints are referred to as "servo-constraints," so I have chosen the modern terminology.

at M_0 . That equation has degree two in λ_0 , so it will generally determine that number, and otherwise one will use the first expression for $(i(\mathrm{E}+\lambda e)\cdot da)_0^{(n)}$, which is not identically zero.

§ V. – Zero-power constraint.

That category of constraints is easily characterized when one takes the variables to be the Lagrangian variables $(q^i, \dot{q}^i, t)$. The power delivered by a force that is defined by $L_i \dot{q}^i$, like the velocity itself, depends upon the chosen frame. It will thus exhibit two distinct aspects according to the frame (Chap. I, § II.4) that one envisions:

1. With respect to the natural frame R, $P = \sum_{i=1}^{n} L_i \dot{q}^i$.

2. With respect to the frame $\mathcal{R}$ in the $(n + 1)$-dimensional Riemann space:

$$\mathcal{P} = \sum_{i=0}^{n} L_i \dot{q}^i, \quad \text{with} \quad \dot{q}_0 = 1,$$

which is a frame that one is obliged to introduce when the implicit constraints that one takes into account in order to conclude with the notion of a parameterized system depend upon time t. Note that the first expression will be deduced from the second one when one introduces the usual convention of saying "the power at const t " and that the two expressions will coincide with the implicit constraints do not depend upon time.

For the sake of generality in our arguments in this paragraph, we shall use the power $\mathcal{P} = \sum_{i=0}^{n} L_i \dot{q}^i$ with respect to the Riemannian frame $\mathcal{R}$.

One is liberated from the use of coordinates in the following way: The power $\mathcal{P}$ corresponds to the form $\pi = \mathcal{P}\, dt = \sum_{i=0}^{n} L_i \dot{q}^i$ that is defined on V_{n+1} . Along an integral curve Σ of the equations of motion that relate to the field $\mathrm{E} + \mathrm{E}_l$, the value of the form π will be $i\,(\mathrm{E} + \mathrm{E}_l)\ \pi$, while the value of the form dt is set to $i\,(\mathrm{E} + \mathrm{E}_l) \cdot dt = 1$, by convention (cf., Theorem II, Chap. I, § III), so:

$$\mathcal{P} = \frac{\pi}{dt} = i\,(\mathrm{E} + \mathrm{E}_l)\ \pi .$$

If one now considers the set of integral curves Σ then $i\,(\mathrm{E} + \mathrm{E}_l)\ \pi$ will be a numerical function on V_{2n+1} .

Definition. – We say that a constraint has *zero power* when $i\,(\mathrm{E} + \mathrm{E}_l)\,\pi = \mathcal{P}$ is zero on the set of integral curves of the motion of S when it is restricted by that constraint.

Theorem:

A zero-power constraint has $\sum_{i=0}^{n} l_i\,\dot{q}^i$ *for its analytical expression, where the quantities* l_i *are functions of* q^i, $\dot{q}^i$, t.

Since the two equations $a = 0$, $\mathcal{P} = 0$ do not have to constitute two distinct constraint relations, they are two equivalent analytical forms for the constraint that is imposed on S in the sense that was given to that expression in § II. One can then take the analytical expression for the constraint to be $\mathcal{P} = i\,(\mathrm{E} + \mathrm{E}_l)\,\pi = 0$. The last equation shows that π is defined only up to a factor that is a numerical function on V_{2n+1} . When one takes the variables to be Lagrangian, it will result that a zero-power constraint is defined by:

$$\sum_{i=0}^{n} l_i\,\dot{q}^i = 0, \qquad \mathcal{P} = \lambda \sum_{i=0}^{n} l_i\,\dot{q}^i \;.$$

Particular cases:

1. The l_i are functions of only the q^i . These are the classical linearly non-holonomic constraints:

$$\sum_{i=0}^{n} l_i\,\dot{q}^i + l_0 = 0.$$

2. The l_i are the partial derivatives of a function $a\,(q^i, t)$. These are holonomic constraints.

3. The l_i are partial derivatives with respect to the of the same homogeneous function of degree m with respect to the $\dot{q}^i$, namely, $a\,(q^i, \dot{q}^i, t)$:

$$l_i = \frac{\partial a}{\partial q^i}\,, \qquad \sum_{i=1}^{n} \frac{\partial a}{\partial \dot{q}^i}\,\dot{q}^i = ma \;.$$

These are the constraints that Appell ([18]) discovered.

Remark:

The Appell constraints contain the previous two as special cases: $m = 1$ gives the linearly non-holonomic constraints, while $m = 1$, $l_i = \partial a / \partial q^i$.

([18]) Cf., P. APPELL, C. R. Acad. Sci. **152** (1911), 1197-1199.

Value of the factor λ for a zero-power constraint. – Equation (II.3), with $a = l_i \dot{q}^i$, becomes:

$$\lambda \sum_{i=1}^{n} \left(\frac{\partial a}{\partial p_i} l_i \right) + \alpha = 0, \qquad \alpha = i \text{ (E) } da,$$

but:

$$\frac{\partial a}{\partial p_i} = \frac{\partial a}{\partial \dot{q}^i} \cdot \frac{\partial \dot{q}^i}{\partial p_i} = \frac{\partial a}{\partial \dot{q}^i} g^{ij} .$$

(g^{ij} is the metric tensor on V_n or V_{n+1} .) $l_i\, g^{ij} = l^j$ are the contravariant components of the direction of action of the constraint force with respect to the frame R. λ will then be determined from:

(II.4) $$\lambda \sum_{j=1}^{n} \left(\frac{\partial a}{\partial \dot{q}^j} l^j \right) + \alpha = 0 .$$

Particular case. – Appell constraint. $\partial a / \partial \dot{q}^j = l^j$ (II.4) becomes:

(II.5) $$\lambda \sum_{j=1}^{n} \left(l_j \cdot l^j \right) + \alpha = 0 .$$

Since $\sum l_j \cdot l^j$ is the square of the magnitude of the vector $\boldsymbol{l}$ with respect to the frame R, the coefficient of λ will always be a positive quantity for Appell constraints. That fact is very important for the uniqueness of the motions of a system S that is subject to constraints of class U (unilateral constraints), which will be studied in Chapters IV and V.

§ VI. – Study of the constraint that consists of contact between two solid bodies that slide, roll, and pivot with respect to each other.

Consider a moving frame that is composed of a tri-rectangular trihedron $MXYZ$ that has its origin at the point of contact between the two solid bodies S_1 and S_2 , and its axes are the common normal that points towards S_1 , while MX and MY are located in the common tangent plane and are coupled with a system of two orthogonal curves that are traced on S_2 .

Some notations that relate to that trihedron are:

$\boldsymbol{\Omega}$ (P^1, P^2, P^3)	the rotation of the coordinate trihedron
$\boldsymbol{\omega}_1$ (p^1, p^2, p^3)	the absolute rotation of the body S_1
$\boldsymbol{\gamma}_{AG_1}$ $(\alpha^1, \alpha^2, \alpha^3)$	the absolute velocity of G_1, which is the center of gravity of S_1
$\overrightarrow{MG_1}$ (g^1, g^2, g^3)	the coordinates of G_1 ,

$\mathbf{V}_{AG_2}(\alpha^4, \alpha^5, \alpha^6)$	the absolute velocity of G_2 ,
$\overrightarrow{MG_2}(g^4, g^5, g^6)$	the coordinates of G_2 ,
$\mathbf{R}$	the resultant of the action of S_2 on S_1
$\mathbf{K}$	the resultant moment with respect to M of the action of S_2 on S_1
$\mathbf{V}_{MS_1}$	the velocity of the point of S_1 that contacts S_2 at M
$\mathbf{V}_{MS_2}$	the velocity of the point of S_2 that contacts S_1 at M

The vector $\mathbf{V} = \mathbf{V}_{MS_1} - \mathbf{V}_{MS_2}$ represents the kinematical state of S_1 with respect to S_2 . That vector will have the components:

$$
\begin{aligned}
u^1 &= \alpha^1 - \alpha^4 + (g^2 p^3 - g^3 p^2) - (g^5 p^6 - g^6 p^3) , \\
u^2 &= \alpha^2 - \alpha^5 + (g^3 p^1 - g^1 p^3) - (g^6 p^4 - g^4 p^6) , \\
u^3 &= \alpha^3 - \alpha^6 + (g^1 p^2 - g^2 p^1) - (g^4 p^5 - g^5 p^4) .
\end{aligned}
$$

The condition of contact between S_1 and S_2 translates into $u^3 = 0$. It constitutes the constraint relation in the sense of paragraph I.

$$a = \alpha^3 - \alpha^6 + (g^1 p^2 - g^2 p^1) - (g^4 p^5 - g^5 p^4) = 0 . \tag{II.11}$$

Let us evaluate the power delivered by the system of constraint forces.

The vector that characterizes the relative rotation of S_1 with respect to S_2 is $\boldsymbol{\omega}_2 - \boldsymbol{\omega}_1$, whose components are:

$$\left.\begin{aligned} &p^1 - p^4, \\ &p^2 - p^5, \end{aligned}\right\} \quad \text{on the common tangent plane } \boldsymbol{\omega}_t ,$$

$$p^3 - p^6 \qquad \text{on common normal } \boldsymbol{\omega}_n .$$

The power delivered by the constraint forces is:

$$P = \mathbf{R}(\mathbf{V}_{MS_1} - \mathbf{V}_{MS_2}) + \mathbf{K}(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2) .$$

Under the hypothesis that $\mathbf{V}_{MS_1} \neq \mathbf{V}_{MS_2}$, $\boldsymbol{\omega}_1 \neq \boldsymbol{\omega}_2$, a non-zero P will be determined only when one specifies the directions and magnitudes of $\mathbf{R}$ and $\mathbf{K}$. That is the subject of the Coulomb laws that we shall recall.

1) The laws concerning $\mathbf{R}$: $\mathbf{V}_{MS_1} - \mathbf{V}_{MS_2} = \mathbf{V}$, whose normal components is zero if S_1 and S_2 are in contact, will have a tangential component $\mathbf{V}_g$ that is the sliding velocity of S_1 with respect to S_2 . $\mathbf{R}$ has a tangent component $\mathbf{T}$ and a normal component $\mathbf{N}$.

a) The tangential component $\mathbf{T}$ is collinear with $\mathbf{V}_g$, but points in the opposite direction.

b) $|\,\mathbf{T}\,|$ is coupled $|\,\mathbf{N}\,|$ by way of the relation $|\,\mathbf{T}\,| = f\,|\,\mathbf{N}\,|$, where f is the coefficient of friction, so:

$$\mathbf{R}\,(\mathbf{V}_{MS_1} - \mathbf{V}_{MS_2}) = \mathbf{N}\,u^3 - f\,|\,\mathbf{N}\,\mathbf{V}_g\,|\,.$$

2) Laws concerning $\mathbf{K}$: $\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2$ has a tangential component $\boldsymbol{\omega}_r$, which characterizes the rolling of S_1 with respect to S_2 , and a normal component that characterizes the pivoting of S_1 with respect to S_2 . $\mathbf{K}$ has a tangential component $\mathbf{K}_t$ and a normal component $\mathbf{K}_n$.

a) The tangential component $\mathbf{K}_t$ is collinear with $\boldsymbol{\omega}_2$, but points in the opposite direction.

b) $|\,\mathbf{K}_t\,|$ is coupled to $|\,\mathbf{N}\,|$ by the relation $|\,\mathbf{K}_t\,| = \delta\,|\,\mathbf{N}\,|$, where δ is a coefficient that is called the *resistance to rolling parameter*.

c) $\mathbf{K}_n$ is collinear with $\boldsymbol{\omega}_n$, but points in the opposite direction, and is coupled to $\mathbf{N}$ by the relation $|\,\mathbf{K}_n\,| = \varpi\,|\,\mathbf{N}\,|$, where ϖ is a coefficient that is called the *resistance to pivoting parameter*.

It results from these laws that:

$$\mathbf{K}\,(\boldsymbol{\omega}_1 - \boldsymbol{\omega}_2) = -\,|\,\mathbf{N}\,|\,[\,\delta\,|\,\boldsymbol{\omega}_r\,| + \varpi\,|\,\boldsymbol{\omega}_n\,|]\,.$$

The power delivered by the system of constraint forces in the sense of the definition in Paragraph I is:

$$P = N\,[u^3 - f\,|\,\mathbf{V}_g\,| - \delta\,|\,\boldsymbol{\omega}_r\,| - \varpi\,|\,\boldsymbol{\omega}_n\,|\,], \qquad N > 0\,, \tag{II.6}$$

or:

$$P = N\left[u^3 - f\sqrt{(u^1)^2 + (u^2)^2} - \delta\sqrt{(p^1 - p^4)^2 + (p^2 - p^5)^2} - \varpi\,\varepsilon(p^3 - p^6)\right], \tag{II.7}$$

$$\varepsilon = \pm\,1, \qquad \varepsilon\,(p^3 - p^6) > 0.$$

The expression in brackets is a function of the 12 parameters q^i that characterize the position of the two solid bodies in space and their first derivatives $\dot{q}^i$. We remark that the quantities $u^1, u^2, u^3, p^1, \ldots, p^6$ are homogeneous linear forms in the $\dot{q}^i$. In order to put

P into the classical form $\sum_{i=1}^{n} L_i \, \dot{q}^i$, it will suffice to apply Euler's theorem to each of the homogeneous functions:

$$\sqrt{(u^1)^2 + (u^2)^2} = \frac{u^1}{\sqrt{(u^1)^2 + (u^2)^2}} \sum_{i=1}^{12} \frac{\partial u^1}{\partial \dot{q}^i} \dot{q}^i + \frac{u^2}{\sqrt{(u^1)^2 + (u^2)^2}} \sum_{i=1}^{12} \frac{\partial u^2}{\partial \dot{q}^i} \dot{q}^i ,$$

$$\begin{aligned}\sqrt{(p^1 - p^4)^2 + (p^2 - p^5)^2} &= \frac{p^1 - p^4}{\sqrt{(p^1 - p^4)^2 + (p^2 - p^5)^2}} \sum_{i=1}^{12} \frac{\partial (p^1 - p^4)}{\partial \dot{q}^i} \dot{q}^i \\ &+ \frac{p^2 - p^5}{\sqrt{(p^1 - p^4)^2 + (p^2 - p^5)^2}} \sum_{i=1}^{12} \frac{\partial (p^2 - p^5)}{\partial \dot{q}^i} \dot{q}^i .\end{aligned}$$

The expression for the L_i will result from that, and likewise that of l_i, since N enters as a factor in the L_i. It is important to point out that this calculation is valid under the hypothesis that the coefficients f, δ, ϖ are functions of N, which is the pressure between the two solid bodies, the sliding velocity, and the rotation ω_r or ω_n.

Special case. – f, δ, ϖ are either constants or functions of N, which is the normal pressure, when considered to be a parameter. If one sets:

(II.8) $$\Psi = u^3 - f \sqrt{(u^1)^2 + (u^2)^2} - \delta \sqrt{(p^1 - p^4)^2 + (p^2 - p^5)^2} - \varpi \varepsilon (p^3 - p^6),$$

where the expression (II.7) shows that $P = N\,\Psi$. Since Ψ is homogeneous and has degree 1 with respect to $\dot{q}^i$, the classical form of P will then be (II.9), which will permit one to state the theorem:

(II.9) $$P = N \sum \frac{\partial \Psi}{\partial \dot{q}^i} \dot{q}^i .$$

Theorem I:

When solid bodies contact each other by rubbing, sliding, or pivoting over each other, if the coefficients f, δ, ϖ are constants or functions of the parameter N (normal pressure) then the covariant components of the constraint force with respect to the natural frame R will be proportional to the partial derivatives of a function Ψ (q^i, $\dot{q}^i$) with respect to the $\dot{q}^i$.

Calculating N. – Since the constraint that is defined by contact between two solid bodies has type $a = 0$, λ e, the calculation of N will be achieved by means of equation (II.3), which is (II.10):

(II.10) $$N\,(i\,(e) \cdot da) + i\,(\mathrm{E}) \cdot da = 0.$$

If f, δ, ϖ are independent then that equation will be linear in N. In the opposite case, it will represent an equation that defines N implicitly.

The effective determination of N demands that one must calculate i (E) · da and i (e) · da. i (E) · da can be calculated only when one is given the system of external forces that act upon the two solid bodies. Since the latter is arbitrary, i (E) · da will be an arbitrary numerical function on V_{2n+1}. As for i (e) · da, it will be determined when one knows the constraint relation and the power P.

Theorem II:

When two solid bodies in contact slide, roll, or pivot over each other, the invariant $i(e)\cdot da$ *is given by the formula:*

(II.12) $$i\,(e)\cdot da = A + f\,(B \sin s + C \cos s) + \delta(D \cos \sigma + F \sin \sigma) + \varepsilon\,\varpi\,G\,,$$

in which f, δ, ϖ denote the coefficient of friction, and the resistance to rolling and the resistance to pivoting parameters, resp., s and σ are the angles that the sliding vector $\mathbf{V}_g$ *and the rotation of rolling vector* $\boldsymbol{\omega}_r$*, resp., make with an arbitrarily-chosen direction in the tangent plane that is common to the two solid bodies, and A, B, C are coefficients that depend upon only the distribution of the masses in the two solid bodies at the instant t.*

We remark that the constraint relation and power are expressed simply by means of the velocity parameters $\alpha^1, \ldots, \alpha^6, p^1, \ldots, p^6$:

(II.11) $$a = \alpha^3 - \alpha^6 + g^1p^2 - g^2p^1 - g^4p^5 + g^5p^4 = 0,$$

(II.13)[*sic*] $P =$

$$= N\left[\frac{\partial u^3}{\partial \beta^\rho}\beta^\rho - f\,\frac{\partial\sqrt{(u^1)^2+(u^2)^2}}{\partial(\beta^\rho)}\beta^\rho - \delta\frac{\partial\sqrt{(p^1-p^2)^2+(p^2-p^5)^2}}{\partial(\beta^\rho)}\beta^\rho - \varpi\,\varepsilon(p^3-p^6)\right],$$

in which β^ρ denotes either of the parameters β^ρ or p^ρ. The expression (II.13) for P involves the partial derivatives with respect to u^1, u^2, and $p^1 - p^4$, $p^2 - p^5$, which have a geometric significance. If s denotes the angle that the sliding velocity $\mathbf{V}_g$ makes with the MX axis then:

$$\cos s = \frac{u^1}{\sqrt{(u^1)^2+(u^2)^2}}\,, \qquad \sin s = \frac{u^2}{\sqrt{(u^1)^2+(u^2)^2}}\,.$$

If σ denotes the angle that the rolling rotation vector $\boldsymbol{\omega}_r$ makes with MX:

$$\cos \sigma = \frac{p^1-p^4}{\sqrt{(p^1-p^4)^2+(p^2-p^5)^2}}\,, \quad \sin \sigma = \frac{p^2-p^5}{\sqrt{(p^1-p^4)^2+(p^2-p^5)^2}}\,,$$

which will then give the new expression for the power:

(II.14) $$P = N\, l_\rho\, \beta^\rho,$$

with

$$\begin{cases} l_{\alpha_1} = -f\cos s, \\ l_{\alpha_2} = -f\sin s, \\ l_{\alpha_3} = 1, \\ l_{p_1} = -g^2 - f\sin s\, g^3 - \delta\cos\sigma, \\ l_{p_2} = \quad g^2 + f\sin s\, g^1 - \varepsilon\varpi, \\ l_{p_3} = -f\cos s\, g^2 - f\sin s\, g^3 - \delta\cos\sigma, \end{cases} \qquad \begin{cases} l_{\alpha_4} = f\cos s, \\ l_{\alpha_5} = f\sin s, \\ l_{\alpha_6} = -1, \\ l_{p_4} = \quad g^5 - f\sin s\, g^6 + \delta\cos\sigma, \\ l_{p_5} = -g^4 + f\sin s\, g^6 + \delta\cos\sigma, \\ l_{p_6} = \quad f\cos s\, g^5 - f\sin s\, g^4 + \varepsilon\varpi. \end{cases}$$

Everything comes down to establishing the formula that permits one to calculate the invariant $i(e)\ da$ in terms of the parameters β^ρ. Now, when one uses Hamiltonian variables $i\ (e)\cdot da = \sum l_i \dfrac{\partial a}{\partial p_i}$; when one uses Lagrangian variables, if g^{ij} is the fundamental metric tensor then:

$$i\ (e)\cdot da = \sum_{i,j} g^{ij} l_i \frac{\partial a}{\partial \dot q^j}.$$

If one uses arbitrary velocity parameters that are coupled to the parameters by the linear relations $\beta^\rho = \mu_i^\rho \dot q^i$ then:

$$l_i = \mu_i^\rho l_\beta, \qquad \frac{\partial a}{\partial \dot q^j} = \frac{\partial a}{\partial \beta^\rho}\frac{\partial \beta^\rho}{\partial \dot q^j} = \frac{\partial a}{\partial \beta^\rho}\cdot \mu_j^\rho,$$

$$i\ (e)\cdot da = g^{ij}\mu_i^\rho \mu_j^\sigma l_\rho \frac{\partial a}{\partial \beta_\sigma} = \gamma^{\rho\sigma} l_\rho \frac{\partial a}{\partial \beta^\sigma},$$

in which $\gamma^{\rho\sigma} = g^{ij}\mu_i^\rho \mu_j^\sigma$ is the contravariant expression for the fundamental tensor in the new system of parameters. The contravariant tensor $\gamma^{\rho\sigma}$ is deduced from the covariant tensor $\gamma_{\rho\sigma}$, which is known since it is given by the coefficient of the absolute *vis viva* in the system of axes *MXYZ*. Let:

$$\begin{aligned} 2T = m_1\,[(\alpha^1)^2 + (\alpha^2)^2 + (\alpha^4)^2] + A_1\,(p^1)^2 + B_1\,(p^2)^2 + C_1\,(p^3)^2 \\ - 2D_1\, p^2\, p^3 - 2E_1\, p^3\, p^1 - 2F_1\, p^1\, p^2 \\ + m_2\,[(\alpha^4)^2 + (\alpha^5)^2 + (\alpha^6)^2] + A_2\,(p^4)^2 + B_2\,(p^5)^2 + C_2\,(p^6)^2 \\ - 2D_2\, p^5\, p^6 - 2E_2\, p^6\, p^4 - 2F_2\, p^4\, p^5, \end{aligned}$$

in which m_1, m_2 denote the masses of the two solid bodies, $A_1, \ldots, F_1$ are the coefficients if the ellipsoid of inertia of S_1 with respect to axes that are parallel to $MXYZ$ and issue from G, while $A_1, \ldots, F_1$ are the analogues for S_2 . It result from this that:

$$\gamma^{\alpha_1\alpha_1} = \gamma^{\alpha_2\alpha_2} = \gamma^{\alpha_3\alpha_3} = \frac{1}{m_1}; \qquad \gamma^{\alpha_4\alpha_4} = \gamma^{\alpha_5\alpha_5} = \gamma^{\alpha_6\alpha_6} = \frac{1}{m_2};$$

$$\gamma^{\alpha_i\alpha_j} = 0, \qquad \text{for} \qquad i \neq j,$$

$$\gamma^{p_1p_1} = a^1, \quad \gamma^{p_2p_2} = b^1, \quad \gamma^{p_3p_3} = c^1, \quad \gamma^{p_2p_3} = d^1, \quad \gamma^{p_3p_1} = e^1, \quad \gamma^{p_1p_2} = f^1,$$

$$\gamma^{p_4p_4} = a^2, \quad \gamma^{p_5p_5} = b^2, \quad \gamma^{p_6p_6} = c^2, \quad \gamma^{p_5p_6} = d^2, \quad \gamma^{p_6p_4} = e^2, \quad \gamma^{p_4p_5} = f^2.$$

$i\,(e) \cdot da = \gamma^{\rho\sigma} l_\rho \dfrac{\partial a}{\partial \beta^\sigma}$ will give formula (II.12), with values for A, B, C, D, F, G that depend upon only the distribution of the masses inside of the two solid bodies at the instant t.

$$A = \frac{1}{m_1} + \frac{1}{m_2} + a^1(g^2)^2 + b^1(g^1)^2 - 2f^1g^1g^2 + a^2(g^4)^2 + b^2(g^5)^2 - 2f^2g^4g^5,$$

$$B = a^1g^2g^4 + d^1(g^1)^2 - e^1g^1g^2 - f^1g^1g^3 + a^2g^5g^6 + d^2(g^4)^2 - e^1g^4g^5 - f^2g^4g^6,$$

$$C = b^1g^1g^3 - d^1g^1g^2 + e^1(g^2)^2 - f^1g^2g^3 + b^2g^4g^5 - d^2g^4g^5 + e^2(g^5)^2 - f^2g^5g^6,$$

$$D = \quad a^1g^2 - f^1g^1 + a^2g^5 - f^2g^4,$$

$$F = -b^1g^1 + f^1g^2 - b^2g^4 + f^2g^5,$$

$$G = -d^1g^1 + e^1g^2 - d^2g^4 + e^2g^5.$$

In the case where one supposes that f, δ, ϖ are independent of N, N will be finite for $i\,(\mathrm{E}) \cdot da \neq 0$ only when $i\,(e) \cdot da \neq 0$.

That question presents a certain practical interest, so we shall show briefly that the distribution of matter in the two solid bodies must be very special in order for $i\,(e) \cdot da = 0$ to be realized for values of $f < 1$.

In order to simplify this, take the case of a solid of revolution that slides with friction on a plane, while the axis of the solid remains in a plane with vertical symmetry. In the preceding calculations, one must take:

$$g^2 = 0, \qquad p^1 = 0, \qquad p^3 = 0, \qquad p^4 = p^5 = p^6 = 0,$$

$$g^2 = g^5 = g^6 = 0, \qquad d^1 = 0, \qquad f^1 = 0, \qquad b^1 = \frac{1}{m_1 k^2},$$

and thus, the condition:

$$k^2 + (g^1)^2 + f\, g^1\, g^3 = 0, \qquad \text{if} \qquad \cos s = 1.$$

The sliding takes place in the positive sense along the X-axis, so since g^3 is positive, one must have that g^1 is negative. Take the reference system to be parallel axes that issue from G. The coordinates of M with respect to O are $\xi = -g^1$, $\zeta = -g^3$. If k and f are given then one can deduce that the contact point M of the solid body is along the branch of the hyperbola:

$$(\xi)^2 + f\,\xi\,\zeta = -k^2$$

with respect to G, so the minimum distance from G to M will be given by the length of the transverse semi-axis $\dfrac{k\sqrt{2}}{\sqrt{\sqrt{1+f^2}-1}}$. Hence, one has the inequality: $\dfrac{k}{GM} \leq \dfrac{\sqrt{\sqrt{1+f^2}-1}}{\sqrt{2}}$. If the body reduces to a homogeneous bar then $\dfrac{k}{GM} = \dfrac{1}{\sqrt{3}}$, so for a homogeneous body $\dfrac{k}{GM} \geq \dfrac{1}{\sqrt{3}}$. The double inequality is verified only when $f > 4 / 3$.

§ VII. – General method for forming equations of motion for a mechanical system that is subject to one constraint.

Everything comes down to the exterior form of degree two that is associated with the mechanical system S, because that form can be expressed by means of Pfaff forms.

We suppose that the constraint l is defined by $a(p_i, q^i, t) = 0$, $\Omega_l = \Lambda\, l_i\, dq^i \wedge dt$. The case where $\Omega_l = L_i\, dq^i \wedge dt$, in which all of the L_i are known except for one of them, can be considered to be a special case of this, since it will suffice to group all of the known L_i with the Q_i that correspond to external forces in order to reduce it to the preceding case, and the unknown function L_i will play the role of Λ. The form Ω that is associated to the system that is restricted by the constraint l can be written:

$$\Omega = dp_i \wedge dq^i - d(T_2 - T_0) \wedge dt + Q_i\, dq^i \wedge dt + \Lambda\, l_i\, dq^i \wedge dt .$$

By hypothesis, the constraint imposed on S is compatible (cf., Chapter II, § IV), so $i(e)\,da = \sum_{i=1}^{n} \dfrac{\partial a}{\partial p_i} l^i \neq 0$. It will then result that one can always replace the two associated differentials dp_i, dq^i with the two Pfaff forms π and σ, which are defined by:

$$p = l_i\, dq^i,$$

$$\sigma = \frac{\partial a}{\partial p_i} dp_i + \frac{\partial a}{\partial q^i} dq^i + \frac{\partial a}{\partial t} dt ,$$

which can be solved for the two associated differentials. To fix ideas, let dp_1, dq^1 be:

$$dp_1 = \frac{\sigma - \sum_{i=2}^{n} \frac{\partial a}{\partial p_i} dp_i - \sum_{i=1}^{n} \frac{\partial a}{\partial q^i} dq^i - \frac{\partial a}{\partial t} dt}{\frac{\partial a}{\partial p_1}} ,$$

$$dq^1 = \frac{\pi - \sum_{i=2}^{n} l_i \, dq^i}{l_1} .$$

The exterior form that is associated with the system can be written:

$$\Omega = k_{12} (\sigma \wedge \pi) + k_{1\alpha} (\sigma \wedge \omega^\alpha) + k_{2\alpha} (\sigma \wedge \omega^\alpha) + k_{\alpha\beta} \, \omega^\alpha \wedge \omega^\beta$$
$$- (k_{\alpha 0} \, \omega^\alpha + k_{10} \, \pi + k_{20} \, \pi - \Lambda \, \pi) \wedge dt ,$$

when it is expressed in terms of a system of the $2n$ Pfaff forms σ, π, ω^α, where α varies from 3 to $2n$.

The differential equations of motion are the associated equations to Ω.

(1) $$\frac{\partial \Omega}{\partial \omega^\alpha} = k_{\alpha\beta} \, \omega^\beta - k_{1\alpha} \, \sigma - k_{2\alpha} \, \pi - k_{1\alpha} \, dt = 0$$ [which are 2 (n – 1) in number],

(2) $$\frac{\partial \Omega}{\partial \pi} = - k_{12} \, \sigma - k_{2\alpha} \, \omega^\alpha - (k_{20} - \Lambda) \, dt = 0,$$

(3) $$\frac{\partial \Omega}{\partial \sigma} = \quad k_{12} \, \pi + k_{1\alpha} \, \omega^\alpha - k_{10} \, dt = 0,$$

to which one adds the constraint $a = 0$ and the defining relations for the forms.

Reduced form Ω_r . – We point out that if one takes the constraint into account when annulling s then one will get a reduced form:

$$\Omega_r = k_{2\alpha} (\sigma \wedge \omega^\alpha) + k_{\alpha\beta} (\omega^\alpha \wedge \omega^\beta) - (k_{\alpha 0} \, \omega^\alpha + k_{20} \, \pi - \Lambda \, \pi) \wedge dt .$$

Compare the first ($2n$ – 1) equations that are associated with Ω_r and with Ω, when one sets $\sigma = 0$:

(1′) $$\frac{\partial \Omega_r}{\partial \omega^\alpha} = k_{\alpha\beta} \, \omega^\beta - k_{2\alpha} \, \pi - k_{\alpha 0} \, dt = 0$$ [which are 2 (n – 1) in number], (2′) $$\frac{\partial \Omega_r}{\partial \pi} = k_{2\alpha}\, \varpi^\alpha - (k_{20} - \Lambda)\, dt = 0\,.$$

One confirms that these first $(2n - 1)$ equations are the same. The last equation in the former system, which will be missing when one uses Ω_r , can be replaced with an equation that is independent of the previous ones. Upon observing that:

$$\frac{\partial \Omega}{\partial \sigma} = \frac{\partial \Omega}{\partial (dp_1)} \cdot \frac{\partial (dp_1)}{\partial \sigma} = 0,$$

$\frac{\partial \Omega}{\partial \sigma} = 0$ is equivalent to $\frac{\partial \Omega}{\partial (dp_1)} = 0$, and conversely. One then deduces that it will suffice to append to equations (1′) and (2′) the equation that is written:

$$dq^1 - \frac{\partial T_2}{\partial p_1} dt = 0.$$

If one replaces p_1 in the set of equations that was just written with its value that is calculated by means of the equation $a = 0$ then one will have the theorem:

Theorem I:

If one is given an n-parameter mechanical system that is restricted by a compatible constraint $a\,(p^i, q_i\,, t) = 0$, $\Omega_l = \Lambda\; l_i\; dq^i \wedge dt$ *then one can obtain the equations of motion and the constraint factor* Λ *by appending to the associated equations to* Ω_r *, which are from* Ω *by taking the constraint into account, on the one hand* $\pi = l_i\, dq^i$*, and on the other hand, one of the equations* $dq^i - \frac{\partial T_2}{\partial p_i} dt = 0$*, where i is such that* $\frac{\partial a}{\partial p_i} l_i \neq 0$*, and* p_i *is replaced with its value that is calculated by means of the relation* $a = 0$ *in the equations.*

Remark:

In practice, if Ω is expressed by means of the Lagrangian variables q^i, $\dot{q}^i$, t, and their differentials then that theorem will translate into the following rule:

Rule. – In order to obtain the differential equations of motion, one replaces the two associated differentials dq^1, $d\dot{q}^1$ in Ω with, for example, the two forms π and σ. Ω will reduce to Ω_r by setting $\sigma = 0$ and replacing $\dot{q}^1$ with its value that is calculated by means of the equation $a\,(q^i, \dot{q}^i, t) = 0$. The associated equations to Ω_r , to which one adds, on the one hand $dq^1 - \dot{q}^1 dt = 0$, and on the other hand, $\pi = l_i\, dq^i$, determine the equations of motion and the constraint factor.

Reduced form Ω_s that gives the differential equations of motion uniquely. – As was just pointed out, the form Ω_r depends upon the 2 (n – 1) forms ω^α, dt, and π:

$$\Omega_r = k_{2\alpha}\,\pi \wedge \omega^\alpha + k_{\alpha\beta}\,\omega^\alpha \wedge \omega^\beta - (k_{\alpha 0}\,\omega^\alpha + k_{20}\,\pi - \lambda\,\pi) \wedge dt\,.$$

Since $\pi = P_u\,dt$ (where P is the power delivered by the forces that are necessary in order to realize the constraint, so $P = \lambda\,P_u$, $P_u = l_i\,\dot{q}^i$), replace π with $P_u\,dt$ in Ω_r . One will then get a form:

$$\Omega_s = k_{2\alpha}\,P_u\,dt \wedge \omega^\alpha + k_{\alpha\beta}\,\omega^\alpha \wedge \omega^\beta - k_{\alpha 0}\,\omega^\alpha \wedge dt$$

$$= k_{\alpha\beta}\,\omega^\alpha \wedge \omega^\beta - (P_u\;k_{2\alpha} + k_{\alpha 0})\,\omega^\alpha \wedge dt\,,$$

into which λ does not enter. Upon comparing the associated equations to Ω_s and Ω_r , when one replaces π with $P_u\,dt$, namely:

$$\frac{\partial\Omega_s}{\partial\omega^\alpha} = k_{\alpha\beta}\,\omega^\beta - (P_u\;k_{2\alpha} + k_{\alpha 0})\,dt = 0 \qquad \text{[which are 2 (}n - 1\text{) in number],}$$

$$\frac{\partial\Omega_r}{\partial\omega^\alpha} = k_{\alpha\beta}\,\omega^\beta - k_{2\alpha}P_u\,dt - k_{\alpha 0}\,dt = 0 \qquad \text{[which are 2 (}n - 1\text{) in number],}$$

one will confirm that they are the same. One can then get the equations of motion independently of λ by means of Ω_s , in which all that appear are 2 (n – 1) differentials dq^i, dp_i and (2n – 1) variables p_i , q^i, where p_1, for example has been replaced with its value that one infers from the constraint a = 0. The equation that must be appended to the characteristic equations of Ω_s is $dq^1 - \dfrac{\partial T_2}{\partial p_1}dt = 0$.

Constructing Ω_s . – Ω_s is obtained by replacing two associated differentials dp_1 , dq^1 in Ω with the values:

$$dq^1 = P_u\,dt - \sum_{i=2}^{n} l_i\,dq^i\,,$$

$$dp_1 = -\,\frac{\displaystyle\sum_{i=2}^{n}\frac{\partial a}{\partial p_i}dp_i + \sum_{i=1}^{n}\frac{\partial a}{\partial q^i}dq^i + \frac{\partial a}{\partial t}dt}{\dfrac{\partial a}{\partial p_1}}\,.$$

Theorem II:

If one is given an n-parameter mechanical system that is restricted with one compatible constraint a (p_i, q^i, t) = 0, $\Omega_l = \lambda\,l_i\,dq^i \wedge dt$ *then one can obtain the* *differential equations of motion as the characteristics of a form* Ω_s *of rank* 2 (n – 1) *that is deduced from* Ω *by replacing the two associated differentials* dp_i , dq^i *with the values that are calculated, in the first case, by means of the differential of the equation of constraint, and in the second case, by means of the equation* $\sum_{i=1}^{n} l_i\, dq^i - P_u\, dt = 0$. *If the variable* p_i *is replaced with its value that one extracts from* $a = 0$ *then one will append the form* $dq^i - \frac{\partial T_2}{\partial p_i} dt = 0$ *to the system of characteristic equations of* Ω_s .

Remarks:

1. Zero-power constraint: $\mathcal{P} = 0$. Since $\mathcal{P} = P_u + l_0$, one replaces P_u with $- l_0$ in the preceding formulas.

2. Zero-power constraint independent of time: $P = 0$. dq^i is a linear function of the other (n – 1) differentials.

3. Holonomic constraint that is independent of time: One pair of the variables p_i , q^i and their differentials will no longer enter into Ω_s . The differential equations of motion are the characteristics of an exterior form of degree two and rank 2 (n – 1).

4. If one variable q^i does not enter into either Ω, the constraint relation, or P_u then there will be an obvious advantage to eliminating dq^i, because Ω_s will depend upon only 2 (n – 1) variables and their differentials. The differential equations of motion are the characteristics of a form of degree two and rank 2 (n – 1), so the variable q^i will be determined by a quadrature.

CHAPTER THREE

MATERIAL SYSTEM SUBJECT TO *p* CONSTRAINTS

§ I. – Introduction.

Definition. – A holonomic material system S that is characterized by an exterior form Ω of degree two and rank $2n$ that is defined on a differentiable manifold is *subject to p constraints if:*

1. The image of S is a submanifold of V_{2n+1} that is defined by:

$$a^1 (M) = 0, \qquad \ldots, \qquad a^p (M) = 0, \qquad M \in V_{2n+1} .$$

2. One appends p constraint fields $E^1, \ldots, E^p$ to the characteristic field E that is defined by i (E) $\Omega = 0$, which are fields that are determined by the forces that are necessary to realize that constraint.

When one employs a particular coordinate system, the p constraints will be defined:

a) From the theoretical standpoint by:

1. p relations $a^h (p_i , q^i, t) = 0$ (h varies from 1 to p).

2. p forms $\Omega^h = L_i^h \, dq^i \wedge dt$ that one must add to Ω (i varies from 1 to p).

b) From the practical standpoint by:

1. p relations $a^h (q^i, \dot{q}^i , t) = 0$ (h varies from 1 to p).

2. p powers $P^h = L_i^h \, \dot{q}^i$ (i varies from 1 to n).

Remarks:

1. The set of p constraint relations $a^1 = 0, \ldots, a^p = 0$ can also be defined by an arbitrary set of p functions of class C^∞, viz., $f_k (a^1, \ldots, a^p) = 0$ (k varies from 1 to p), that are zero at the point O, where the Jacobian is non-zero. A set of p relations then constitutes a sub-ring of numerical functions in the ring of numerical functions, and p generic elements are defined by any choice of $a^1, \ldots, a^p$ that are independent of each other. There is obviously much interest in choosing those p elements to be as simple as possible in practice, but that choice is not necessary in theory.

2. One can always suppose that the field E^h that related to the h^{th} constraint has the form $E^h = \lambda_h e^h$, where e^h is a known direction field and λ_h is a numerical function on V_{2n+1}, because if $(n - 1)$ of the n components of E^h are known and the n^{th} one is unknown then one can write $E^h = E_c^h + \lambda_h e^h$ and consider the new field $\overline{E} = E + E_c^h$, which amounts to including the known part of the constraint forces in the external forces that are applied to S from the mechanical standpoint. It will then result that, with no loss of generality, one can always consider, on the one hand, the power of the set of constraints in the form $P = \lambda_h l_i^h \dot{q}^i$, and on the other hand, the form Ω_l, which is the sum of the Ω_h that are added to Ω and is written $\Omega_l = \lambda_h l_i^h dq^i \wedge dt$, where the λ_h are p numerical functions on V_{2n+1} to be determined, and the l_i^h are np known numerical functions.

§ II. Compatibility of p constraints.

The characteristic field for the material system that is subject to p constraints is E + $\sum_{h=1}^{p} \lambda_h e^h$. The forms $da^1, \ldots, da^p$, which will be zero on the integral curves of the equations of motion, must belong to the sub-module of characteristic forms of the form $\Omega + \Omega_l$. From Theorem I of Chapter I, § III, the necessary and sufficient conditions for that to be true are:

$$\text{(III.1)} \qquad i\left(E + \sum_{h=1}^{p} \lambda_h e^h\right) da^k = 0 \qquad (h, k = 1, \ldots, p),$$

which are conditions that can also be written as:

$$\text{(III.2)} \qquad \lambda_h \, i(e^h) \, da^k + i(E) \, da^k = 0.$$

We point out that if we change the analytical representation of the constraints then the conditions that one will obtain will be only linear combinations of the p equations (III.2), because $df_j = \frac{\partial f_j}{\partial a^k} da^k$, and:

$$i(E + \lambda_h e^h) \, df_j = \lambda_h \, i(e^h) \, df_j + i(E) \, df_j = \sum_{k=1}^{p} \frac{\partial f_j}{\partial a^k} [\lambda_h \, i(e^h) \, da^k + i(E) \, da^k] .$$

The p conditions (III.2) constitute a linear system in p unknowns $\lambda_1, \ldots, \lambda_p$. The λ are determined only if $\det | i(e^h) \, da^k | \neq 0$. That condition is obviously independent of the variables that are used to characterize the system of p constraints. We shall call it the *compatibility condition*, because if it is realized then the constraint factors λ_h will be neither indeterminate nor infinite. As we pointed out before in the preceding chapter, the postulate of "a system of zero-power internal forces" is valid. For that reason, we pose the following definition:

p constraints that are imposed upon a material system are *compatible* if $\det\left| i(e^h)\, da^k \right| \neq 0$.

Remark. – That notion of compatibility of the constraints leaves out an important class of motions in which the constraints are indeterminate in the sense of the mechanics of rigid solid bodies, but that notion is indispensible if one would like to avoid some paradoxical results.

For example, take a heavy bar whose extremities *A* and *B* slide without friction on a vertical circle of radius *R*. For $AB = 2R$, apart from the constraints, a uniform rotational motion of the bar around the center *O* of the circle might be possible. If we consider the reactions at *A* and *B* then a similar motion will be impossible, because those reactions cannot equilibrate the action of gravity. We can show that two such constraints are incompatible, from our theory.

Let *M* be the mass of the bar, and let *k* be its radius of gyration around its center of gravity. When one takes the polar axis to be the descending vertical that issues from the center of the circle, the polar coordinates of the center of gravity of the bar (which is assumed to be homogeneous) to be r, θ, and the angle of rotation of the bar to be φ, the form Ω will be:

$$\frac{\Omega}{M} = d\dot{r} \wedge dr + r^2\, d\dot{\theta} \wedge d\theta + 2r\,\dot{\theta}\, dr \wedge d\theta + k^2 d\dot{\varphi} \wedge d\varphi$$
$$- (\dot{r}\, d\dot{r} + r^2\,\dot{\theta}\, d\dot{\theta} + r\,\dot{\theta}^2\, dr + k^2\dot{\varphi}\, d\dot{\varphi}) \wedge dt + g\,(\cos\theta\, dr - r\sin\theta\, d\theta) \wedge dt.$$

The characteristic field E has the components:

$$r^2\,\dot{\theta} + g\cos\theta, \qquad \frac{-1}{r}(2\dot{r}\,\dot{\theta} + g\sin\theta), \quad 0, \qquad \dot{r}\,; \qquad \dot{\theta}, \qquad \dot{\varphi}, \qquad 1.$$

Suppose that the bar has a length of $2R \sin u$ with $u \leq \pi / 2$, so classically the two constraints translate into the two relations $r = R \cos u$, $\varphi - \theta = \pi / 2$, and from our point of view, they are $\dot{r} = 0$, $\dot{\varphi} - \dot{\theta} = 0$, and if the bar slides without friction then they will contribute zero power $P^1 = \lambda_1 \dot{r}$, $P^2 = \lambda_2 (\dot{\varphi} - \dot{\theta})$. One then deduces:

1. The two forms Ω^1 and Ω^2:

$$\Omega^1 = \lambda_1\, dr \wedge dt, \qquad \Omega^2 = \lambda_1\, (d\varphi - d\theta) \wedge dt\,.$$

2. The components of the two direction fields for the constraints e^1 and e^2:

$$e^1 = (1, 0, 0, 0, 0, 0, 0), \qquad e^2 = \left(0, \frac{-1}{r^2}, \frac{1}{k^2}, 0, 0, 0, 0\right).$$

Since we would like to examine what happens when $r = R \cos u$ for $u = \pi / 2$, the two fields E and e^2 will become infinite for that value, and we take the components of the fields to be:

$$\mathrm{E} = (r^2(r\dot{\theta}^2 + g\cos\theta), \quad -2r(\dot{r}\,\dot{\theta} + g\sin\theta), \quad 0, \quad r^2\dot{r}, \quad r^2\dot{\varphi}, \quad r^2),$$

$$e^1 = (r^2, 0, 0, 0, 0, 0, 0), \qquad e^2 = \left(0, -1, \frac{r^2}{k^2}, 0, 0, 0, 0\right).$$

One immediately gets:

$$i(e^1)\,d\dot{r} = r^2, \quad i(e^2)\,d\dot{r} = 0, \quad i(e^1)\,d(\dot{\varphi} - \dot{\theta}) = 0, \qquad i(e^2)\,d(\dot{\varphi} - \dot{\theta}) = 1 + \frac{r^2}{k^2}.$$

The compatibility condition can be written:

$$i(e^1)\,d\dot{r} \cdot i(e^2)\,d(\dot{\varphi} - \dot{\theta}) = r^2\left(1 + \frac{r^2}{k^2}\right) = R^2\cos^2 u\left(1 + \frac{R^2\cos^2 u}{k^2}\right) \neq 0.$$

For $u = \pi / 2$, the two constraints are incompatible.

Geometric interpretation of the scalar function $\det \mid i\,(e^h)\,da^k \mid$. – The numerical function $\det \mid i\,(e^h)\,da^k \mid$ can be interpreted geometrically on V_{2n+1} by considering, on the one hand, the field $e^1 \wedge e^2 \wedge \ldots \wedge e^p$, which has order p and is constructed from the p fields $e^1, e^2, \ldots, e^p$ in the space T that is tangent to V_{2n+1}, and on the other hand, from the form $da^1 \wedge da^2 \wedge \ldots \wedge da^p$, which has order p and is constructed from the p forms da^1, $da^2, \ldots, da^p$ in the space T' that is dual to T. At any point M on the manifold, the field of order p and the form of order p will give rise to a p-vector and a p-form, respectively, whose interior product will be $\det \mid i\,(e^h)\,da^k \mid$ ([19]). The compatibility condition $\det|\,i(e^h)\,da^k\,| \neq 0$ then imply the non-vanishing of that p-vector and that p-form, which means the following things:

1. The non-vanishing of the p-vector means that the directions of the forces of constraint at a point M of the manifold V_{2n+1} form a p-hedron, and are consequently independent.

2. The non-vanishing of the p-form implies that the p functions that define the constraint relations are independent.

From the analytical standpoint, the compatibility condition translates into:

If, on the one hand, one lets A denote the matrix with p rows whose elements in a row are the components of one of the p forms $da^1, da^2, \ldots, da^p$ with respect to the basis (dp^i,

(19) Cf., N. BOURBAKI, *Algèbre multilinéaire*, Actualités scientifques, no. 1044, Hermann, Paris, 1948, pp. 106.

dq_i , dt), and on the other hand, one lets L denote the matrix with p columns whose elements in one column are the components of one of the p fields e^1, …, e^p then the product of A with L will define a square matrix of order p whose determinant is the interior product of the p-vector and the p-form.

$$A = \left\| \begin{matrix} \dfrac{\partial a^1}{\partial p^1} & \cdots & \dfrac{\partial a^1}{\partial p^n} & \dfrac{\partial a^1}{\partial q^1} & \cdots & \dfrac{\partial a^1}{\partial q^n} & \dfrac{\partial a^1}{\partial t} \\ \dfrac{\partial a^p}{\partial p^1} & \cdots & \dfrac{\partial a^p}{\partial p^n} & \dfrac{\partial a^p}{\partial q^1} & \cdots & \dfrac{\partial a^p}{\partial q^n} & \dfrac{\partial a^p}{\partial t} \end{matrix} \right\|, \qquad L = \left\| \begin{matrix} l_1^2 & \cdots & l_1^p \\ l_n^1 & \cdots & l_n^p \\ 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{matrix} \right\|.$$

The components $L_{i_1 \cdots i_p}$ of the p-vector are formed by means of determinants of order p that are extracted from the matrix L. The components $A^{i_1 \cdots i_p}$ of the p-form are formed by means of the determinants of order p that are extracted from the matrix A:

$$A^{i_1 \cdots i_p} L_{i_1 \cdots i_p} = \det | A \cdot L |.$$

§ III. – Reduced form Ω_r that is associated with S subject to p constraints that gives the equations of motion and the constraint factors.

We shall show the existence of the p-vector $L_{i_1\cdots i_p}$ and the p-form $A^{i_1\cdots i_p}$ whose interior product is not zero permits us to construct a reduced form Ω_r whose $(2n - p)$ associated equations, combined with p conveniently-chosen equations, will give the equations of motion, on the one hand, and the p constraint factors, on the other.

Set:

$$l_i^h \, dq^i = \pi^h \qquad (h = 1, \ldots, p) .$$

The existence of the p-vector permits us to calculate p differentials dq^i as functions of the π^h.

Set:

$$\frac{\partial a^h}{\partial p_i} dp_i + \frac{\partial a^h}{\partial q^i} dq^i + \frac{\partial a^h}{\partial t} dt = \sigma^h \qquad (h = 1, \ldots, p).$$

The existence of the p-form $A^{i_1\cdots i_p}$ permits us to calculate p differentials dp^i as functions of the σ^h.

If the interior product of the p-vector and the p-form is not zero then one can choose the two systems of differentials to be dq^i, dp_i, which one can express as functions of the forms π^h, σ^h, and with the same partition of the indices (from 1 to p, to fix ideas), which is always possible with an appropriate choice of notations. It will then result that the form Ω that is associated with the system can be expressed by means of the $2n$ Pfaff forms π^h, σ^h (h varies from 1 to p), ω^α (α varies from $2p + 1$ to $2n$) can be written:

$$\Omega = k_{ih} (\sigma^i \wedge \pi^h) + k_{\alpha i} (\omega^\alpha \wedge \sigma^i) + k_{\alpha h} (\omega^\alpha \wedge \pi^h) + k_{\alpha\beta} (\omega^\alpha \wedge \omega^\beta)$$

$$+ (k_{\alpha 0} \, \omega^\alpha + k_{i\,0} \, \sigma^i + k_{h0} \, \pi^h - \lambda_h \, \pi^h) \wedge dt .$$

The equations of motion and the constraint factors are given by the characteristics of Ω :

(1) $$\frac{\partial \Omega}{\partial \omega^\alpha} = k_{\alpha i} \, \sigma^i + k_{\alpha h} \, \pi^h + k_{\alpha\beta} \, \omega^\beta - k_{\alpha 0} \, dt = 0,$$ which are $2(n - p)$ in number,

(2) $$\frac{\partial \Omega}{\partial \pi^h} = - k_{ih} \, \sigma^i - k_{\alpha h} \, \omega^\alpha - k_{h\,0} \, dt + \lambda_h \, dt = 0,$$ which are p in number,

(3) $$\frac{\partial \Omega}{\partial \sigma^i} = k_{ih} \, \pi^h - k_{\alpha i} \, \omega^\alpha - k_{i\,0} \, dt = 0,$$ which are p in number,

to which one adds the constraint relations $a^h = 0$ and the defining equations of the forms with the p conditions $\sigma^h = 0$.

Reduction of Ω. – Let Ω_r (which will become Ω when one annuls the σ^h) be:

$$\Omega_r = k_{\alpha h}\,(\omega^\alpha \wedge \pi^h) + k_{\alpha\beta}\,(\omega^\alpha \wedge \omega^\beta) - (k_{\alpha 0}\,\omega^\alpha + k_{h0}\,\pi^h - \lambda_h\,\pi^h) \wedge dt.$$

The first $(2n - p)$ equations that are associated with Ω_r are:

(1′) $\dfrac{\partial \Omega_r}{\partial \omega^\alpha} = k_{\alpha h}\,\pi^h + k_{\alpha\beta}\,\omega^\beta - k_{\alpha 0}\,dt = 0,$ which are $2\,(n - p)$ in number,

(2′) $\dfrac{\partial \Omega_r}{\partial \pi^h} = -\,k_{\alpha h}\,\omega^\alpha - k_{h0}\,dt + \lambda_h\,dt = 0,$ which are p in number.

One notes that the set of $(2n - p)$ equations (1′) and (2′) are nothing but equations (1) and (2) when one annuls the σ^h. In order to complete that system, one must theoretically append the p equations $\dfrac{\partial \Omega}{\partial \sigma^i} = 0$ or p equivalent equations to (1′) and (2′). Now, if Ω is expressed in terms of the variables p_i, q^i then one will have the following n equations for the associated equations:

$$\frac{\partial \Omega}{\partial (dp_i)} = dq^i - \frac{\partial T_2}{\partial p_i}\,dt = 0.$$

When one performs the change of forms:

$$dp_i = \alpha_{ik}\,\sigma^k, \qquad \text{with} \quad \det|\,\alpha_{ik}\,| \neq 0, \qquad \frac{\partial \Omega}{\partial (\sigma^h)} = \frac{\partial \Omega}{\partial (dp_i)} \cdot \alpha_{ik}\,,$$

in such a way that the p equations $\dfrac{\partial \Omega}{\partial (dp_i)} = 0$ will imply $\dfrac{\partial \Omega}{\partial \sigma^h} = 0$, and conversely. It will then suffice to append the p equations $dq^i - \dfrac{\partial T_2}{\partial p_i}\,dt = 0$ to (1′) and (2′), in which one replaces p of the quantities p_i with their values that are calculated by means of the p constraint equations $a^h(p_i, q^i, t) = 0$. One can then state the theorem:

Theorem I:

If one is given an n-parameter mechanical system that is subject to p compatible constraints of the type $a^h(p_i, q^i, t) = 0$, $P^h = \lambda_h\, l_i^h\, \dot{q}^i dt$ *then one can obtain the differential equations of motion and the p constraint factors* λ_h *by appending to the first* $(2n - p)$ *equations that are associated with the form* Ω_r *, on the one hand, the p equations* $dq^i - \dfrac{\partial T_2}{\partial p_i}\,dt = 0$, *and on the other hand, the p equations* $l_i^h\, dq^i = \pi^h$, *where p of the* p_i *are calculated by means of the p constraints* $a^h = 0$.

Remark. – In practice, if Ω is expressed by means of Lagrangian variables q^i, $\dot{q}^i$, t and their differentials then that theorem will translate into the following rule:

Rule. – In order to obtain the differential equations of motion and the constraint factors, one replaces the p pairs of associated differentials dq^i, $d\dot{q}^i$ in Ω with the $2p$ forms π^h, σ^h, Ω_r that one deduces from Ω by annulling the σ^h and replacing p of the variables $\dot{q}^i$ with their values that one calculates by means of the p constraint equations. The first $(2n - p)$ associated equations to Ω_r, to which one appends, on the one hand, the p equations $dq^i - \dot{q}^i\,dt = 0$, and on the other hand, the p defining equations of the π^h, $\pi^h = l_i^h dq^i$, will determine the equations of motion and the p constraint factors.

§ IV. – Reduced form Ω_s that gives the differential equations of motion uniquely.

When Ω_r is formed in the way that we indicated in the preceding paragraph, it will depend upon the $2(n-p)$ forms ω^α, the p forms π^h, and dt:

$$\Omega_r = k_{\alpha h}\,(\omega^\alpha \wedge \pi^h) + k_{\alpha\beta}\,(\omega^\alpha \wedge \omega^\beta) - (k_{\alpha 0}\,\omega^\alpha + k_{h0}\,\pi^h - \lambda_h\,\pi^h) \wedge dt\,.$$

Since $\pi^h = P_u^h dt$ (where P^h is the power generated by the forces that are necessary to realize the h^{th} constraint $P^h = \lambda_h P_u^h$), one can replace each π^h with $P_u^h dt$ and thus obtain a form:

$$\Omega_s = k_{\alpha h}\,(\omega^\alpha \wedge P_u^h dt) + k_{\alpha\beta}\,(\omega^\alpha \wedge \omega^\beta) - k_{\alpha 0}\,\omega^\alpha \wedge dt\,,$$

in which none of the constraint factors λ_h appear any more. The form Ω_s has rank $2(n - p)$, because its $(n-p)^{\text{th}}$ power is:

$$\Omega_s^{(n-p)} = K\,\omega^1 \wedge \omega^2 \wedge \ldots\,\omega^{2n-2p} \wedge dt \quad (K \text{ is a numerical function}).$$

Upon comparing the $2(n-p)$ characteristic equations of Ω_s and the first $2(n-p)$ associated equations to Ω_r, in which one replaces each π^h with $P_u^h dt$, namely:

$$\frac{\partial \Omega_s}{\partial \omega^\alpha} = k_{\alpha h}\,P_u^h dt + k_{\alpha\beta}\,\omega^\beta - k_{\alpha 0}\,dt = 0,$$

$$\frac{\partial \Omega_r}{\partial \omega^\alpha} = k_{\alpha h}\,P_u^h dt + k_{\alpha\beta}\,\omega^\beta - k_{\alpha 0}\,dt = 0,$$

one will confirm that they are the same. One can then obtain the equations of motion independently of the constraint factors by means of Ω_s, in which all that appears will be $2(n-p)$ differentials dq^i, dp_i, and $(2n-p)$ variables p_i, q^i, where p of the variables p_i have been replaced with their values that one calculates by means of the p constraint equations.

In addition, one must append to the characteristic equations of Ω_s , the p equations $dq^i - \frac{\partial T_2}{\partial p_i} dt$ that correspond to the differentials dq that do not appear in Ω_s .

Forming Ω_s . – Ω_s is deduced from Ω by replacing p pairs of associated differentials dp_i , dq^i with their values that one calculates by means of the two systems:

$$\frac{\partial a^h}{\partial p_i} dp_i + \frac{\partial a^h}{\partial q^i} dq^i + \frac{\partial a^h}{\partial t} dt = 0, \qquad l_i^h dq^i - \mathrm{P}_u^h dt = 0 \qquad (h = 1, \ldots, p).$$

Theorem II:

When one is given an n-parameter mechanical system with p compatible constraints a^h $(p_i , q^i, t) = 0$, $\Omega_l = \lambda_h l_i^h dq^i \wedge dt$ *, one can obtain the differential equations of motion as the characteristics of a form of rank* 2 $(n - p)$ *that is deduced from* Ω *by replacing p pairs of associated differentials* dp_i *,* dq^i *with the values that one calculates for the* dp_i *by means of the p differential equations of constraint, and that one calculates for the* dq^i *by means of the p relations* $l_i^h dq^i - P_u^h dt = 0$*. If the p variables* p_i *are calculated by means of the p constraint equations* $a^h = 0$ *then one can append the p forms* $dq^i - \frac{\partial T_2}{\partial p_i} dt = 0$ *to the characteristic system of* Ω_s *.*

Remarks:

1. For a zero-power constraint: $\mathcal{P}^h = 0$. Since $\mathcal{P}_u^h = P_u^h + l_0^h$, one replaces P_u^h with $-l_0^h$.

2. For zero-power constraints that are independent of time, $P_u^h = 0$, and p of the differentials dq^i are expressed as functions of the other $(n - p)$.

3. Holonomic or pseudo-holonomic constraints that are independent of time: p pairs of the variables p_i , q^i, along with their differentials, will no longer figure in Ω_s . The equations of motion will be characteristics of an exterior form of degree two and rank 2 $(n - p)$.

4. If p variables q^i do not enter into either Ω or the p constraint relations or the powers of the constraints then there will obviously be an advantage to eliminating the p differentials dq^i, because Ω_s will no longer depend upon the 2 $(n - p)$ variables and their differentials. The differential equations of motion will be the characteristics of a form of degree two and rank 2 $(n - p)$, so the p variables q^i will be determined by quadratures, since the p forms $dq^i - \frac{\partial T_2}{\partial p_i} dt$ are closed modulo the characteristics of Ω_s .

§ V. – General method for the study of a mechanical system with *n* parameters. Applications.

One can naturally deduce a general method for the study of mechanical systems that depend upon n parameters from the preceding four paragraphs collectively. Those systems are always composed of sets of solid bodies and material points that are restricted by a certain set of constraints of the type $a(p_i, q^i, t) = 0$, $P = \lambda l_i \dot{q}^i$, which are supposed to be compatible.

1. The constraint factors λ are always determined independently of the motions by the combined use of the form Ω and H. Cartan's operator i ( ). In order to do that, it will suffice to determine the following things for the system that is freed of those constraints:

a) The characteristic field E of the corresponding form Ω, which will be immediate when Ω is written in canonical form.

b) The fields $e^1, \ldots, e^p$ of the directions of the constraint forces.

The p equations (III.2) determine the system of constraint forces completely.

On that subject, we point out that if $e^1, \ldots, e^p$ do not depend upon the λ then those equations will be linear in the λ, so the system (III.2) will constitute a system of implicit equations for the λ. In particular, that is what happens in the case of friction due to sliding, rolling, or pivoting when one considers the coefficients of friction, which are the parameters of resistance to rolling and pivoting, to be functions of the normal pressures.

2. The equations of motion can always be obtained independently of the constraint factors by means of the characteristic equations of a form Ω_s, to which one appends p suitable differential forms (cf., Th. III, § IV). The properties of that differential system will be studied in Chapter VI of this work. First, we shall give some simple examples of the application of the preceding methods.

Example I. – A heavy homogeneous disc of mass M, radius R, and moment of inertia $M k^2$ with respect to its center rolls and slides along the line of greatest inclination an inclined plane that makes an angle of i with the horizontal. One supposes that the coefficient of friction is a function of the sliding velocity v and the normal pressure N, namely, $f(v, N)$.

Upon taking the x-axis to be the line of greatest slope in the plane, pointing downward, the Oy axis to be the perpendicular to it, ξ, η to be the coordinates of the center of the circle, and θ to be its angle of rotation, the form Ω for the free disc will be:

$$\Omega = M\,d\dot{\xi}\wedge d\xi + M\,d\dot{\eta}\wedge d\eta + Mk^2 d\dot{\theta}\wedge d\theta - M(\dot{\xi}\,d\xi + \dot{\eta}\,d\dot{\eta} + k^2\dot{\theta}\,d\dot{\theta})\wedge dt$$
$$+ Mg\,(\sin i\;dx - \cos i\;dh)\wedge dt\,.$$

The characteristic field E has the components:

$$\mathrm{E} = (g \sin i, -g \cos i, 0, \dot{\xi}, \dot{\eta}, \dot{\theta}, 1)\,.$$

The constraint translates into $\dot{\eta} = 0$, so the power is $P = N[\varepsilon f(\dot{\xi} + R\dot{\theta}) + \dot{\eta}]$, with $\varepsilon(\dot{\xi} + R\dot{\theta}) < 0$, hence, $\Omega_l = N\,[\varepsilon f\,(d\xi + R\,d\theta) + d\eta]$. Note that $V = \dot{\xi} + R\dot{\theta}$.

The normal pressure N is given by $N\, i\,(e)\; d\dot{\eta} + i\,(\mathrm{E})\; d\dot{\eta} = 0$:

$$i\,(e)\, d\dot{\eta} = \frac{1}{M}\,, \qquad i\,(\mathrm{E})\; d\dot{\eta} = -g \cos i, \quad \text{so} \qquad N = Mg \cos i\,.$$

In order to obtain the differential equations, form Ω_s , which is deduced from $\Omega + \Omega_l$ by replacing $\dot{\eta}$ with 0 and $d\eta$ with:

$$-\varepsilon f\,(d\xi + R\,d\theta) + [\dot{\eta} + \varepsilon\, f\,(\dot{\xi} + R\dot{\theta})]\,dt\,,$$

so

$$\frac{\Omega_s}{M} = d\dot{\xi} \wedge d\xi + k^2 d\dot{\theta} \wedge d\theta - (\dot{\xi}\, d\dot{\xi} + k^2 \dot{\theta}\, d\dot{\theta}) \wedge dt$$
$$+\; g \sin i\, d\xi \wedge dt \;+ g \cos i\;\varepsilon f\,(dx + R\,dq) \wedge dt\,.$$

The equations of motion are the characteristics of Ω_s . An intuitive choice of four of H. Cartan's operators will permit one to put those equations into the form:

$$i\,(x^1)\,\Omega_s = \frac{dv}{\sin i + \varepsilon\, f\,(v) \cos i \left(1 + \dfrac{R^2}{k^2}\right)} - g\,dt = 0,$$

$$i\,(x^2)\,\Omega_s = \frac{\dot{\theta}\, dv}{\sin i + \varepsilon\, f\,(v) \cos i \left(1 + \dfrac{R^2}{k^2}\right)} \cdot \frac{1}{g} = 0,$$

$$i\,(x^3)\,\Omega_s = d\dot{\theta} \frac{-\varepsilon R \cos i\;\, f\,(v)\, dv}{k^2 \sin i + \varepsilon\, f\,(v) \cos i \left(k^2 + R^2\right)} = 0,$$

$$i\,(x^4)\,\Omega_s = d\xi + R d\theta - \frac{1}{g} \frac{v\, dv}{\sin i + \varepsilon\, f\,(v) \cos i \left(1 + \dfrac{R^2}{k^2}\right)} = 0\,.$$

The existence of the preceding fields $x^1, \ldots, x^4$ amounts to infinitesimal transformations of the form Ω_s, as we showed in Chapter VI, § II. The system is integrable by quadratures.

Rolling without slipping. – There are two zero-power constraints in our theory:

$$\dot{\eta} = 0, \qquad P^1 = N\,\dot{\eta}\,, \qquad \dot{\xi} + R\,\dot{\theta} = 0, \quad P^2 = T(\dot{\xi} + R\,\dot{\theta})\,,$$

$$P_u^1 = \dot{\eta}\,, \qquad P_u^2 = \dot{\xi} + R\,\dot{\theta}\,.$$

The components of the direction fields of the constraints are:

$$e^1 = \left(0, \frac{1}{M}, 0, 0, 0, 0, 0\right), \qquad e^2 = \left(\frac{1}{M}, 0, \frac{R}{Mk^2}, 0, 0, 0, 0\right).$$

First determine the components of the reaction by means of the operators:

$$i(e^1)\,d\dot{\eta} = \frac{1}{M}\,, \qquad i(e^2)\,d\dot{\eta} = 0, \qquad i(\mathrm{E})\,d\dot{\eta} = -\,g\cos i,$$

$$i(e^1)\,d(\dot{\xi} + R\,\dot{\theta}) = 0, \quad i(e^2)\,d(\dot{\xi} + R\,\dot{\theta}) = \frac{1}{M}\left(1 + \frac{R^2}{k^2}\right),$$

$$i(\mathrm{E})\,d(\dot{\xi} + R\,\dot{\theta}) = g\sin i,$$

so

$$N = -\,\frac{i(\mathrm{E})\,d\dot{\eta}}{i(e^1)\,d\dot{\eta}} = Mg\cos i, \qquad T = \frac{-\,i(\mathrm{E})\,d(\dot{\xi} + R\dot{\theta})}{i(e^1)\,d(\dot{\xi} + R\dot{\theta})} = -\,\frac{Mg\sin i}{1 + \dfrac{R^2}{k^2}}.$$

The differential equations of motion that are deduced from Ω_s are obtained by starting from:

$$\frac{\Omega + \Omega_l}{M} = d\dot{\xi} \wedge d\xi + d\dot{\eta} \wedge d\eta + k^2 d\dot{\theta} \wedge d\theta - (\dot{\xi}\,d\dot{\xi} + \dot{\eta}\,d\dot{\eta} + k^2\dot{\theta}\,d\dot{\theta}) \wedge dt$$
$$+\, g\,(\sin i\;d\xi - \cos i\;d\eta) \wedge dt + N\,d\eta \wedge dt + T\,(d\xi + R\,d\theta) \wedge dt$$

and replacing $d\dot{\eta}$ with 0, $d\eta$ with $+\,P_u^1 dt = 0$ (since $P_1 = 0$), $a\,\dot{\theta}$ by $-\dfrac{1}{R}d\dot{\xi}$, and dq with $\dfrac{1}{R}(-d\xi + P_u^2 dt) = -\,\dfrac{d\xi}{R}$ (since $P_u^2 = 0$), namely:

$$\frac{\Omega_s}{M} = \left(1 + \frac{k^2}{R^2}\right) d\dot{\xi} \wedge d\xi - \left(1 + \frac{k^2}{R^2}\right) \dot{\xi}\,d\dot{\xi} \wedge dt - g\sin i\,d\xi \wedge dt\,.$$

One immediately deduces that $d\dot{\xi} = \dfrac{g\sin i}{1 + \dfrac{k^2}{R^2}}\,dt$, $d\xi = \dot{\xi}\,dt$, which are classical results.

Example II. – Homogeneous disc rolling and slipping on a fixed plane curve, while the disc remains in the plane of the curve.

Let *Ox* and *Oy* be two fixed axes, let *m* be the mass of the disc, let *a* be its radius, let mk^2 be its moment of inertia around its center of gravity *G*, let ξ, η be the coordinates of *G*, let θ be the angle of rotation of the disc around *G*, let *X*, *Y* be the components of the resultant of the external forces other than those of the action of contact of the curve on the disc, and let Γ be the resultant moment of the external forces under the same conditions.

The entirely-free disc is associated with the exterior form:

$$\Omega = d\,(m\dot{\xi}\,d\xi + m\dot{\eta}\,d\eta + mk^2\dot{\theta}\,d\dot{\theta} - T\,dt) + (X\,d\xi + Y\,d\eta + \Gamma\,d\theta) \wedge dt\,,$$

in which *T* denotes one-half the *vis viva* of the disc.

The curve *C* on which the disc rolls and slips is supposed to be defined by *x*, *y*, which are functions of the arc-length *s* of the curve. The oriented half-tangent *Pt* to *C* makes an angle of $\alpha = (Ox, Pt)$ at an arbitrary point *P*, so the radius of curvature at *P* will be $\rho = ds\,/\,d\alpha$. It is easy to take moving axes *Pt*, *Pn* with $(Pt, Pn) = \pi / 2$. The coordinates of *G* in this system are, on the one hand, α, and on the other hand, $PG = R$, which are obtained in the following way: One draws a normal *GP* to *C* through a point *G* on the plane, and then the points $P\,(\alpha)$ and $PG = R$ on that oriented normal.

The differentials $d\xi$, $d\eta$ are expressed in terms of $d\alpha$, dR using the formulas:

$$d\xi = (\rho - R)\cos\alpha\,d\alpha - dR\sin\alpha, \qquad d\eta = (\rho - R)\sin\alpha\,d\alpha + dR\cos\alpha\,.$$

Let *u*, *v* be the components of the velocity of *G* with respect to the axes *Pt*, *Pn* :

$$\dot{\xi} = u\cos\alpha - v\sin\alpha, \qquad \dot{\eta} = u\sin\alpha + v\cos\alpha\,,$$

so one has the expressions for *T*, $\dot{\xi}\,d\xi + \dot{\eta}\,d\eta$, and Ω as functions of the chosen variables:

$$T = \tfrac{1}{2}m(u^2 + v^2 + k^2\dot{\theta}^2)\,, \qquad \dot{\xi}\,d\xi + \dot{\eta}\,d\eta = (\rho - R)\,u\,d\alpha + v\,dR,$$

$$\begin{aligned}\Omega = m\,[(\rho - R)\,du \wedge d\alpha - u\,dR \wedge d\alpha + dv \wedge dR + k^2\,d\dot{\theta} \wedge d\theta\,]\\ - m\,(u\,du + v\,dv + k^2\,\dot{\theta}\,d\dot{\theta}) \wedge dt\\ + [(X\cos\alpha + Y\sin\alpha)(\rho - R)\,d\alpha + (Y\cos\alpha - X\sin\alpha)\,dR + \Gamma\,d\theta] \wedge dt\,.\end{aligned}$$

The characteristic field E has the following components under the hypothesis that $r - R \neq 0$:

$$\mathrm{E} = \left(\frac{X\cos\alpha + Y\sin\alpha}{m} + \frac{uv}{\rho - R}, \frac{Y\cos\alpha - X\sin\alpha}{m} - \frac{u^2}{\rho - R}, \frac{\Gamma}{mk^2}, \frac{u}{\rho - R}, v, \dot{\theta}, 1\right).$$

The constraint translates into $v = 0$, so the power is:

$$P = N\,\{v + \varepsilon f\,[(\rho - R)\,\dot{\alpha} + a\,\dot{\theta}\,]\},$$

with $\varepsilon = -1$ if $u + a\dot{\theta} > 0$, $\varepsilon = +1$ if $u + a\dot{\theta} < 0$. One then deduces that:

$$\Omega_l = N\,\{dR + \varepsilon f\,[(\rho - R)\,d\alpha + a\,d\theta]\}\ .$$

Hence, the components of the direction of the constraint field will be:

$$e = \left(\frac{\varepsilon f}{m}, \frac{1}{m}, \frac{\varepsilon f\,a}{mk^2}, 0, 0, 0, 0\right).$$

The value of the normal reaction N is deduced from $N\,i\,(e)\,dv + i\,(\mathrm{E})\,dv = 0$:

$$N = -\frac{i\,(\mathrm{E})\,dv}{i\,(e)\,dv} = X \sin a - Y \cos a + m\frac{u^2}{\rho - a}\,.$$

In order to obtain the differential equations, form Ω_s , which is obtained by starting from Ω and replacing dv with 0 and dR with:

$$-\,\varepsilon f\,[(\rho - R)\,d\alpha + a\,d\theta] + \varepsilon f\,[(\rho - r)\,\dot{\alpha} + a\,\dot{\theta}]\,dt\,,$$

since $v = 0$:

$$\begin{aligned}\Omega_s = {} & m\,(\rho - R)\,du \wedge d\alpha + m\,u\,a\,e\,f\,d\theta \wedge d\alpha + m\,u\,\varepsilon f\,[(\rho - r)\,\dot{\alpha} + a\,\dot{\theta}]\,d\alpha \wedge dt \\ & + mk^2\,d\dot{\theta} \wedge d\theta - m\,(u\,du + k^2\,\dot{\theta}\,d\dot{\theta}) \wedge dt + (X \cos\alpha + Y \sin\alpha)\,(\rho - R)\,d\alpha \wedge dt \\ & - (Y \cos\alpha - X \sin\alpha)\,\varepsilon f\,[(\rho - R)\,d\alpha + a\,d\theta] \wedge dt + \Gamma\,d\theta \wedge dt\,.\end{aligned}$$

The associated equations to Ω_s give:

$$\begin{aligned}\frac{\partial \Omega_s}{\partial(d\alpha)} = {} & -\,m\,(\rho - R)\,du - m\,u\,\varepsilon f\,d\theta + m\,u\,\varepsilon f\,[(\rho - R)\,\dot{\alpha} + a\,\dot{\theta}]\,dt \\ & + (X \cos\alpha + Y \sin\alpha)(\rho - R)\,dt - (Y \cos\alpha - X \sin\alpha)\,\varepsilon f\,(\rho - R)\,dt = 0,\end{aligned}$$

$$\frac{\partial \Omega_s}{\partial(d\theta)} = -\,mk^2 + m\,\varepsilon f\,a\,u\,d\alpha + \Phi\,dt - \varepsilon f\,a\,(Y \cos\alpha - X \sin\alpha)\,dt = 0,$$

$$\frac{\partial \Omega_s}{\partial(du)} = m\,(\rho - R)\,d\alpha - mu\,dt = 0,$$

$$\frac{\partial \Omega_s}{\partial(a\,\dot{\theta})} = mk^2\,(d\theta - \dot{\theta}\,dt) = 0,$$

to which one appends $\dfrac{\partial \Omega_s}{\partial(dv)} = m\,(dR - v\,dt) = 0$. The latter takes into account the given constraint $R = \alpha$. The system can be further written in the form:

$$\frac{1}{2}\frac{du^2}{d\alpha} - \varepsilon f u^2 = \frac{\rho - a}{m}[X\cos\alpha + Y\sin\alpha - \varepsilon f (Y\cos\alpha - X\sin\alpha)],$$

with:

$$\varepsilon = -1, \qquad \text{of } u + a\dot{\theta} > 0,$$
$$\varepsilon = +1, \qquad \text{of } u + a\dot{\theta} < 0,$$

$$mk^2\frac{d\dot{\theta}}{d\alpha} = \Gamma\frac{\rho - a}{u} + m\,\varepsilon f a\, u - \varepsilon f a\,(Y\cos\alpha - X\sin\alpha)\,\frac{\rho - a}{u},$$

$$\frac{d\theta}{d\alpha} = \dot{\theta}\frac{\rho - a}{u}, \qquad dt = \frac{\rho - a}{u}\,d\alpha.$$

Remarks:

1. The points for which $\rho - a = 0$ are singular points whose geometric significance is as follows: The center of gravity of the disc describes a curve Γ that is parallel to *C*, whose radius of curvature is $\rho - a$, so the curvature of Γ will be infinite at the points for which $\rho - a = 0$, and those points will be the points of regression of Γ, in general (so $\rho - a$ will change sign while going through zero). At such a point, the circle must cross the curve *C*, which is materially impossible. Our theory of the compatibility of constraints shows that at similar points, the constraint must be considered to be incompatible. In order to do that, it will suffice to proceed as we did in the example of the bar, for which, certain components of the field E will become infinite, so one begins by multiplying all of the components of E and *e* by $(\rho - a)$; one will then have the compatibility condition: $i(e)\,dv = \dfrac{\rho - a}{m} \neq 0.$

2. If *X*, *Y* are functions of only the position of *G* then they will depend upon only α. When that constraint is realized, the motion of the center *G* of the circle will be the same as that of a material point that slides with friction on a curve that is parallel to *C* and at a distance of *a* from it.

3. If Γ depends upon only α in the preceding conditions, moreover, then the equations of motion can be integrated by quadrature.

Disc rolling without slipping on the curve *C*. – The disc is restricted by two zero-power constraints:

1. $v = 0, \qquad P^1 = N\cdot v, \qquad \Omega^1 = N\,dR \wedge dt,$

2. $u + a\dot{\theta} = 0, \qquad P^2 = T\,(u + a\dot{\theta}), \qquad \Omega^2 = T\,[(\rho - R)\,d\alpha + a\,d\theta] \wedge dt\,.$

The components of the directions of the constraint fields are:

$$e^1 = \left(0, \frac{1}{m}, 0, 0, 0, 0, 0\right), \qquad e^2 = \left(\frac{1}{m}, 0, \frac{a}{mk^2}, 0, 0, 0, 0\right),$$

so

$$i\,(e^1)\,dv = \frac{1}{m}, \qquad i\,(e^2)\,dv = 0, \qquad i\,(\mathrm{E})\,dv = \frac{Y\cos\alpha - X\sin\alpha}{m} - \frac{u^2}{\rho - a},$$

$$i\,(e^1)\,d\,(u + a\dot{\theta}) = 0, \qquad i\,(e^2)\,d\,(u + a\dot{\theta}) = \frac{1}{m} + \frac{a^2}{mk^2},$$

$$i\,(\mathrm{E})\,d\,(u + a\dot{\theta}) = \frac{X\cos\alpha + Y\sin\alpha}{m} + \frac{uv}{\rho - a} + \frac{a\Gamma}{mk^2},$$

and

$$N = X\sin\alpha - Y\cos\alpha + m\frac{u^2}{\rho - a}, \qquad T = -\frac{(X\cos\alpha + Y\sin\alpha)k^2 + a\Gamma}{a^2 + k^2}.$$

The differential equations of motion are the characteristics of Ω_s, which are deduced from $\Omega + \Omega_s$ by replacing dv with 0 and dR with 0 (because $v\,dt$ is zero), R with a, $d\dot{\theta}$ with $-\,du / a$, $\dot{\theta}$ with $-\,du / a$, and $d\theta$ with $-\,(\rho - a) / a\; d\alpha$:

$$\Omega_s = m\,(\rho - a)\left(1 + \frac{k^2}{a^2}\right) du \wedge da - m\left(1 + \frac{k^2}{a^2}\right) u\,du \wedge dt$$
$$+ \left[X\cos\alpha + Y\sin\alpha - \frac{\Gamma}{a}\right](\rho - a)\,da \wedge dt.$$

If X, Y, Γ depend upon only the position of the circle in the plane then when the constraint is realized those functions will depend upon only α. Under those conditions, the form Ω_s will be the same as the one for a material point that moves without friction on a curve that is parallel to C at a distance of a.

Example III. – Rolling with slipping of a sphere on a fixed plane. The sphere is supposed to be homogeneous of mass M, radius a, moment of inertia $M\,k^2$ with respect to one of its diameters. With respect to three fixed rectangular axes, where O is taken in the plane and Oz is directed along the normal to the plane, let:

ξ, η, ζ	be the coordinates of G, the center of the sphere,
p, q, r	be the components of the absolute rotation of the sphere,
ω^1, ω^2, ω^3	be three Pfaff forms that are constructed from the absolute differentials of the parameters that fix the position of a trihedron that is invariably coupled to the sphere
X, Y, Z	be the components of the general resultant of the external forces that are applied to the sphere other than the reaction of the plane.

The associated form Ω to the free sphere is:

$$\begin{aligned}\Omega = {} & M\,(d\dot\xi \wedge d\xi + d\dot\eta \wedge d\eta + d\dot\zeta \wedge d\zeta) \\ & + M\,k^2\,(dp \wedge d\omega^1 + dq \wedge d\omega^2 + dr \wedge d\omega^3 + p\ \omega^1 \wedge \omega^3 + q\ \omega^2 \wedge \omega^1 + r\ \omega^1 \wedge \omega^2) \\ & - M\,[(\dot\xi\, d\dot\xi + \dot\eta\, d\dot\eta + \dot\zeta\, d\dot\zeta + k^2\,(p\,dp + q\,dq + r\,dr)] \wedge dt \\ & \qquad\qquad\qquad\qquad + (X\,d\xi + Y\,d\eta + Z\,d\zeta) \wedge dt\,.\end{aligned}$$

(For the kinetic expression for Ω, refer to Chapter I, II, namely, a solid body moving around a fixed point.)

The constraint under the hypothesis of sliding on the sphere translates into:

$$\dot\zeta = 0, \qquad\qquad R = N[\dot\zeta - f\sqrt{(\dot\xi - aq)^2 + (\dot\eta + ap)^2}\,],$$

in which f denotes the coefficient of friction of the sphere on the plane, which is assumed to be constant.

Replace the velocity parameters $\dot\xi$, $\dot\eta$ with the polar coordinates ρ, α of the velocity of sliding of the contact point P of the sphere on the plane:

$$\begin{aligned}&\dot\xi - aq = \rho\cos\alpha, \qquad \dot\xi = \rho\cos\alpha + aq, \qquad d\dot\xi = d\rho\cos\alpha - \rho\sin\alpha\, d\alpha + a\,dq, \\ &\dot\eta + aq = \rho\sin\alpha, \qquad \dot\eta = \rho\sin\alpha - aq, \qquad d\dot\eta = d\rho\sin\alpha + \rho\cos\alpha\, d\alpha - a\,dq\,.\end{aligned}$$

The constraint then translates into $\dot\zeta = 0$, and:

$$P = N\,[\dot\zeta - f\cos\alpha\,(\dot\xi - aq) - f\sin\alpha\,(\dot\eta + ap)],$$

so one has the form:

$$\Omega_l = N\,[d\zeta - f\cos\alpha\,(d\xi - a\ \omega^2) - f\sin\alpha\,(d\eta + a\ \omega^2)] \wedge dt\,.$$

Form Ω_r, which will give the normal component of the reaction and the differential equations of motion when it is combined with $d\zeta - \dot\zeta\,dt = 0$. In order to do that, replace $d\dot\zeta$ with 0 in Ω and $d\zeta$ with:

$$d\zeta = \pi + f\cos\alpha\,(d\xi - a\ \omega^2) + f\sin\alpha\,(d\eta + a\ \omega^1),$$

so

$$\begin{aligned}\Omega_r = {} & M\,[(d\rho\cos\alpha - \rho\sin\alpha\, d\alpha + a\,dq) \wedge d\xi + (d\rho\sin\alpha + \rho\cos\alpha\, d\alpha - a\,dq) \wedge d\eta] \\ & + M\,k^2\,(dp \wedge \omega^1 + dq \wedge \omega^2 + dr \wedge \omega^2 + Mk^2\,[p\,(\omega^2 \wedge \omega^3) + q\,(\omega^2 \wedge \omega^3) + r\,(\omega^2 \wedge \omega^3)] \\ & - M\,(\rho + a\cos\alpha\, q - a\sin\alpha\, p)\, d\rho - a\rho\,(p\cos\alpha + q\sin\alpha)\, d\alpha \\ & \qquad + a\rho\,(\cos\alpha\, dq - \sin\alpha\, dp) + k^2\, r\, dr] \wedge dt - M\,(k^2 + a^2)\,(p\,dp + q\,dq) \wedge dt\,.\end{aligned}$$

The associated equations to Ω_r are then:

$$\frac{\partial \Omega_r}{\partial (d\xi)} = -M\,(d\rho \cos\alpha - \rho \sin\alpha\, d\alpha + a\, dq) + X\, dt + Z f \cos\alpha\, dt = 0,$$

$$\frac{\partial \Omega_r}{\partial (d\eta)} = -M\,(d\rho \sin\alpha + \rho \cos\alpha\, d\alpha - a\, dp) + Y\, dt + Z f \sin\alpha\, dt = 0,$$

$$\frac{\partial \Omega_r}{\partial (d\rho)} = M\,(d\xi \cos\alpha + d\eta \sin\alpha) - M\,(\rho + a\cos\alpha\, q - a\sin\alpha\, p)\, dt = 0,$$

$$\frac{\partial \Omega_r}{\partial (d\alpha)} = M\rho\,(-\sin\alpha\, d\xi + \cos\alpha\, d\eta) + M\,(p\cos\alpha + q\sin\alpha)\, a\, \rho\, dt = 0,$$

$$\frac{\partial \Omega_r}{\partial (dp)} = -M\, a\, d\eta + M\, k^2\, \omega^1 + M\, a\, \rho \sin\alpha\, dt - M\,(k^2 + a^2)\, p\, dt = 0,$$

$$\frac{\partial \Omega_r}{\partial (dq)} = M\, a\, d\xi + M\, k^2\, \omega^2 - M\, a\, \rho \cos\alpha\, dt - M\,(k^2 + a^2)\, q\, dt = 0,$$

$$\frac{\partial \Omega_r}{\partial \omega^1} = M\, k^2\,(dp + q\, \omega^3 - r\, \omega^2) + Z\, a\, f \sin\alpha\, dt = 0,$$

$$\frac{\partial \Omega_r}{\partial \omega^2} = -M\, k^2\,(dq - p\, \omega^3 + r\, \omega^1) - Z\, a\, f \cos\alpha\, dt = 0,$$

$$\frac{\partial \Omega_r}{\partial \pi} = (Z + N)\, dt = 0.$$

One then deduces:

$$\frac{d\xi - a\, q\, dt}{\cos\alpha} = \frac{d\eta + a\, p\, dt}{\sin\alpha} = \rho\, dt,$$

$$\omega^1 = p\, dt, \qquad \omega^2 = q\, dt, \qquad \omega^3 = r\, dt, \qquad \frac{dr}{dt} = 0,$$

$$M\,\frac{d\rho}{dt} = Z\, f\,\frac{a^2 + k^2}{k^2} + X\cos\alpha + Y\sin\alpha, \qquad M\rho\,\frac{d\alpha}{dt} = -X\sin\alpha + Y\cos\alpha,$$

$$M\,\frac{d\rho}{dt} = \frac{Z\, a\, f}{k^2}\sin\alpha, \qquad M\,\frac{dq}{dt} = -\frac{Z\, a\, f}{k^2}\cos\alpha.$$

The equations that give $d\rho / dt$, $d\alpha / dt$ define the hodograph of the sliding velocity.

Special case. – Heavy, homogeneous sphere sliding on an inclined plane. If i denotes the angle of inclination, so $X = Mg \sin i$, $Y = 0$, $Z = - Mg \cos i$, then one will recover Painlevé's result: The equations that give the motion of the center can be integrated by quadratures.

Case of rolling without slipping. – The characteristic field E of the form Ω that corresponds to the completely free sphere has components:

$$\mathrm{E} = \left(\frac{X}{M}, \frac{Y}{M}, \frac{Z}{M}, 0, 0, 0, \dot{\xi}, \dot{\eta}, \dot{\zeta}, p, q, r, 1\right).$$

Rolling without slipping translates into three zero-power constraints in our theory:

$$\dot{\xi} - a\,q = 0, \quad \dot{\eta} + a\,p = 0, \quad \dot{\zeta} = 0,$$

$$P^1 = \lambda\,(\dot{\xi} - a\,q), \qquad P^2 = \mu\,(\dot{\eta} + a\,q), \qquad P^3 = \nu\,\dot{\zeta},$$

$$\Omega^1 = \lambda\,(d\xi - a\,\omega^2) \wedge dt, \qquad \Omega^2 = \mu\,(d\eta + a\,\omega^1) \wedge dt, \qquad \Omega^3 = \nu\,d\zeta \wedge dt\,.$$

The direction fields of the constraints have the components:

$$e^1 = \left(\frac{1}{M}, 0, 0, 0, \frac{-a}{M\,k^2}, 0, 0, 0, 0, 0, 0, 0, 0\right),$$

$$e^2 = \left(0, \frac{1}{M}, 0, \frac{a}{M\,k^2}, 0, 0, 0, 0, 0, 0, 0, 0, 0\right),$$

$$e^3 = \left(0, 0, \frac{1}{M}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\right).$$

The calculation of λ, μ, ν, which are components of the reaction of the plane on the sphere can be carried out by means of H. Cartan's operators:

$$i\,(e^1)\,d\,(\dot{\xi} - a\,q) = \frac{1}{M}\left(1 + \frac{a^2}{k^2}\right), \qquad i\,(e^2)\,d\,(\dot{\xi} - a\,q) = 0,$$

$$i\,(e^3)\,d\,(\dot{\xi} - a\,q) = 0, \qquad i\,(\mathrm{E})\,d\,(\dot{\xi} - a\,q) = \frac{X}{M},$$

$$i\,(e^1)\,d\,(\dot{\eta} + a\,q) = 0, \qquad i\,(e^2)\,d\,(\dot{\eta} + a\,q) = \frac{1}{M}\left(1 + \frac{a^2}{k^2}\right),$$

$$i\,(e^3)\; d\,(\dot\eta + a\,q) = 0, \qquad\qquad i\,(\mathrm{E})\; d\,(\dot\eta + a\,q) = \frac{Y}{M},$$

$$i\,(e^3)\; d\dot\zeta = 0, \qquad i\,(e^2)\; d\dot\zeta = 0, \qquad i\,(e^3)\; d\dot\zeta = \frac{1}{M}, \qquad i\,(\mathrm{E})\; d\dot\zeta = \frac{Z}{M},$$

hence:

$$\lambda = -\frac{X}{1+\frac{a^2}{k^2}}, \quad \mu = -\frac{Y}{1+\frac{a^2}{k^2}}, \quad \nu = -z\,.$$

One wishes to know the differential equations that define the motion of the center of the sphere. One then forms Ω_s, which is deduced from $\Omega + \Omega_l$ by replacing $d\dot\zeta$ with 0, $d\zeta$ with 0, dp with $-d\dot\eta / a$, dq with , ω^1 with $-d\eta / a$, and ω^2 with $d\xi / a$:

$$\Omega_s = M\left(1+\frac{a^2}{k^2}\right)(d\dot\xi \wedge d\xi + d\dot\eta \wedge d\eta) + Mk^2 dr \wedge \omega^3$$

$$+\; M\,\frac{k^2}{a^2}(-\dot\eta\, d\xi \wedge \omega^3 + \dot\xi\, d\eta \wedge \omega^3 - r\, d\eta \wedge d\xi)$$

$$-M\left(1+\frac{a^2}{k^2}\right)(\dot\xi\, d\dot\xi + \dot\eta\, d\dot\eta) \wedge dt - Mk^2\, r\, dr \wedge dt + (X\; d\xi + Y\, d\eta) \wedge dt\,.$$

The associated equations to Ω_s are:

$$\frac{\partial\Omega_s}{\partial(d\xi)} = -M\,\frac{k^2+a^2}{a^2}\,d\dot\xi + M\,\frac{k^2+a^2}{a^2}(-\dot\eta\,\omega^3 + r\,d\eta) + X\;dt \;= 0,$$

$$\frac{\partial\Omega_s}{\partial(d\dot\xi)} = \quad M\,\frac{k^2+a^2}{a^2}(d\xi - \dot\xi\,dt) \;= 0,$$

$$\frac{\partial\Omega_s}{\partial(d\eta)} = -M\,\frac{k^2+a^2}{a^2}\,d\dot\eta + M\,\frac{k^2}{a^2}(\dot\xi\,\omega^3 - r\,d\xi) + Y\,dt \;= 0,$$

$$\frac{\partial\Omega_s}{\partial(d\dot\eta)} = \quad M\,\frac{k^2+a^2}{a^2}(d\eta - \dot\eta\,dt) \;= 0,$$

$$\frac{\partial\Omega_s}{\partial\omega^3} \;= -M\,k^2 dr + M\,\frac{k^2}{a^2}(\dot\eta\,d\xi - \dot\xi\,d\eta) = 0,$$

$$\frac{\partial\Omega_s}{\partial(dr)} = \;M\,k^2\,(\omega^3 - r\,dt) = 0\,.$$

The second two reduce to:

$$M\frac{d\dot{\xi}}{dt} = \frac{a^2}{a^2+k^2}X\,,\quad M\frac{d\dot{\eta}}{dt} = \frac{a^2}{a^2+k^2}Y\,.$$

They show that the center of the sphere moves in the plane $x = a$, like a material point that is subject to the force $\frac{a^2}{a^2+k^2}X$, $\frac{a^2}{a^2+k^2}Y$, which is a special case of a theorem by Routh.

Example IV. – Sliding of a homogeneous sphere of mass M, radius a on two fixed rectangular planes xOy, zOx . One supposes that the external forces reduce to the general resultant whose components with respect to the axes $Oxyz$ are X, Y, Z .

With respect to those axes, let:

ξ, η, ζ	be the coordinates of the center G of the sphere,
p, q, r	be the components of instantaneous rotation of the sphere,
ω^1, ω^2, ω^3	be three Pfaff forms that are constructed from the differentials of the parameters that fix the position of a trihedron invariably on the sphere.

The exterior form that is associated with the free sphere is:

$$\begin{aligned}&M(d\dot{\xi}\wedge d\xi + d\dot{\eta}\wedge d\eta + d\dot{\zeta}\wedge d\zeta)\\
&\quad + M\,k^2\,(dp\wedge\omega^1 + dq\wedge\omega^2 + dr\wedge\omega^3 + p\,\omega^2\wedge\omega^3 + r\,\omega^1\wedge\omega^2) + M\,k^2\,q\,\omega^2\wedge\omega^1\\
&\quad - M\,[\dot{\xi}\,d\dot{\xi} + \dot{\eta}\,d\dot{\eta} + \dot{\zeta}\,d\dot{\zeta} + (p\,dp + q\,dq + r\,dr)]\wedge dt + (X\,d\xi + Y\,d\eta + Z\,d\zeta)\wedge dt.\end{aligned}$$

The corresponding characteristic field has the components:

$$\mathrm{E} = \left(\frac{X}{M},\frac{Y}{M},\frac{Z}{M},0,0,0,\dot{\xi},\dot{\eta},\dot{\zeta},p,q,r,1\right).$$

The first contact constraint between the sphere and the plane xOy translates into:

$$\dot{\zeta} = 0,\qquad P^1 = N_A[\dot{\zeta} - f_A\sqrt{(\dot{\xi}-aq)^2+(\dot{\eta}+ap)^2}\,],$$

in which f_A is the coefficient of friction on xOy, or upon introducing the polar coordinates ρ_A , a of the sliding velocity of the contact point A of the sphere with the plane xOy :

$$\begin{aligned}\dot{\xi}-aq &= \rho_A\cos\alpha,\\ \dot{\eta}+aq &= \rho_A\sin\alpha,\end{aligned}\qquad P^1 = N_A[\dot{\zeta} - f_A\cos\alpha\,(\dot{\xi}-aq) - f_A\sin\alpha\,(\dot{\eta}+ap)],$$

since:

$$\Omega^1 = N_A[d\zeta - f_A \cos\alpha\,(d\xi - a\,\omega^2) - f_A \sin\alpha\,(d\eta + a\omega^1)] \wedge dt\,.$$

The components of the direction field of that constraint are:

$$e^1 = \left(\frac{-f_A\cos\alpha}{M}, \frac{-f_A\sin\alpha}{M}, \frac{1}{M}, \frac{-a\,f_A\sin\alpha}{Mk^2}, \frac{a\,f_A\cos\alpha}{Mk^2}, 0,0,0,0,0,0,0,0\right).$$

The second contact constraint between the sphere and the plane *xOy* translates into:

$$\dot\eta = 0, \qquad P^2 = N_B[\dot\eta - f_B\sqrt{(\dot\zeta - ap)^2 + (\dot\xi + ar)^2}\,],$$

in which f_β is the coefficient of friction on *zOx*, or upon introducing the polar coordinates ρ_β, β of the sliding velocity of the point of contact *B* between the sphere and the plane *zOx* :

$$\begin{aligned}\dot\zeta - ap &= \rho_\beta\cos\beta,\\ \dot\xi + ar &= \rho_\beta\sin\beta,\end{aligned} \qquad P^2 = N_B[\dot\eta - f_B\cos\beta\,(\dot\zeta - ap) - f_B\sin\beta\,(\dot\xi + ar)],$$

so

$$\Omega^2 = N_B\,[d\eta - f_B\cos\beta\,(d\zeta - a\,\omega^1) - f_B\sin\beta\,(d\xi + a\,\omega^3)] \wedge dt\,.$$

The components of the direction of the field of that constraint are:

$$e^2 = \left(\frac{-f_B\sin\beta}{M}, \frac{1}{M}, \frac{-f_B\sin\beta}{M}, \frac{a\,f_B\cos\alpha}{Mk^2}, \frac{-a\,f_B\sin\alpha}{Mk^2}, 0,0,0,0,0,0,0,0\right).$$

Let us determine the normal components of the reactions by means of operators:

$$i(e^1)\,d\dot\zeta = \frac{1}{M}, \qquad i(e^2)\,d\dot\zeta = \frac{-f_B\cos\beta}{M}, \qquad i(\mathrm{E})\,d\dot\zeta = \frac{Z}{M},$$

$$i(e^1)\,d\dot\eta = \frac{-f_A\sin\alpha}{M}, \qquad i(e^2)\,d\dot\eta = \frac{1}{M}, \qquad i(\mathrm{E})\,d\dot\eta = \frac{Y}{M}.$$

The compatibility condition for those two constraints is:

$$i(e^1)\,d\dot\zeta\; i(e^2)\,d\dot\eta - i(e^1)\,d\dot\eta\; i(e^2)\,d\dot\zeta = \frac{1}{M^2}(1 - f_A f_B\sin\alpha\cos\beta) \neq 0\,.$$

Upon supposing that this is satisfied, one then deduces that:

$$N_A = -\frac{Z + f_B\cos\beta\,Y}{1 - f_A\,f_B\sin\alpha\cos\beta}, \qquad N_B = -\frac{Y + f_A\sin\alpha\,Z}{1 - f_A\,f_B\sin\alpha\cos\beta}.$$

In order to obtain the differential equations of motion, one constructs Ω_s by replacing $d\dot{\zeta}$ with 0 and $d\dot{\eta}$ with 0 in Ω, and associating the differentials $d\zeta$ and $d\eta$ with the system:

$$d\zeta - f_A \sin\alpha\, d\eta = f_A \cos\alpha\,(d\xi - a\,\omega^2) + a\,f_A \sin\alpha\,\omega^1 + P_u^1\,dt\,,$$
$$-f_B \cos\beta\, d\zeta + d\eta = f_B \sin\beta\,(d\xi + a\,\omega^3) - a\,f_B \cos\beta\,\omega^1 + P_u^2\,dt\,.$$

Thus:

$$\begin{aligned}\Omega_s = {} & M\,d\dot{\xi}\wedge d\xi \\ & + M\,k^2\,(dp\wedge\omega^1 + dq\wedge\omega^2 + dr\wedge\omega^3 + p\,\omega^2\wedge\omega^3 + q\,\omega^3\wedge\omega^1 + r\,\omega^1\wedge\omega^2) \\ & + k^2\,(p\,dp + q\,dq + r\,dr)]\wedge dt \\ & + \frac{Y}{1 - f_A\, f_B \sin\alpha\cos\beta}[f_B \sin\beta\,(d\xi + a\,\omega^3) + f_A f_B \cos\alpha\cos\beta\,(d\xi - a\,\omega^2) \\ & + a\,f_B\cos\beta\,(1 - f_A\sin\alpha)\,\omega^1]\wedge dt \\ & + \frac{Z}{1 - f_A\, f_B \sin\alpha\cos\beta}[f_A \cos\alpha\,(d\xi - a\,\omega^2) + f_A f_B \sin\alpha\sin\beta\,(d\xi + a\,\omega^3) \\ & + a\,f_A\sin\beta\,(1 - f_B\cos\alpha)\,\omega^1]\wedge dt + X\,d\xi\wedge dt\,.\end{aligned}$$

Hence, one has the associated equations to Ω_s :

$$\frac{\partial\Omega_s}{\partial(d\xi)} = -M\,d\dot{\xi} + X\,dt + \frac{dt}{1 - f_A\, f_B \sin\alpha\cos\beta}$$
$$\times [Y\,(f_B\sin\beta + f_A f_B\cos\alpha\cos\beta) + Z\,(f_A\cos\alpha + f_A f_B \sin\alpha\sin\beta)] = 0,$$

$$\frac{\partial\Omega_s}{\partial\omega^1} = M\,k^2\,(-dp - q\,\omega^3 + r\,\omega^2)$$
$$+ \frac{a\,dt}{1 - f_A\, f_B \sin\alpha\cos\beta}[Y\,f_B\cos\beta\,(1 - f_A\sin\alpha) + Z\,f_A\sin\alpha\,(1 - f_B\cos\beta)] = 0,$$

$$\frac{\partial\Omega_s}{\partial\omega^2} = M\,k^2\,(-dq - r\,\omega^1 + p\,\omega^3)$$
$$+ \frac{-a\,dt}{1 - f_A\, f_B \sin\alpha\cos\beta}[Y\,f_A f_B\cos\alpha\cos\beta + Z\,f_A\cos\alpha] = 0,$$

$$\frac{\partial\Omega_s}{\partial\omega^3} = M\,k^2\,(-dr - p\,\omega^2 + q\,\omega^1)$$
$$+ \frac{a\,dt}{1 - f_A\, f_B \sin\alpha\cos\beta}[Y\,f_B\sin\beta + Z\,f_A\, f_B\sin\alpha\sin\beta] = 0,$$

$$\frac{\partial \Omega_s}{\partial (d\dot{\xi})} = M\,(d\xi - \dot{\xi}\,dt) = 0, \qquad \frac{\partial \Omega_s}{\partial (dp)} = M\,k^2\,(\omega^1 - p\,dt) = 0,$$

$$\frac{\partial \Omega_s}{\partial (dq)} = M\,k^2\,(\omega^2 - q\,dt) = 0, \qquad \frac{\partial \Omega_s}{\partial (dr)} = M\,k^2\,(\omega^3 - r\,dt) = 0,$$

to which one must append $d\zeta - \dot{\zeta}\,dt = 0$, $d\eta - \dot{\eta}\,dt = 0$, and the defining relations of α and β, which take into account that $\zeta = a$, $\eta = a$, and which will permit one to define q and r as functions of $\dot{\xi}$, p, α, β:

$$a\,q = \dot{\xi} - a\,p\cot\alpha, \qquad a\,r = \dot{\xi} - a\,p\tan\beta\,.$$

That system can be further put into the following form:

$$M\,\frac{d\dot{\xi}}{dt} = X + \frac{a\,dt}{1 - f_A\,f_B\sin\alpha\cos\beta}\,[Y\,(f_B\sin\beta + f_A\,f_B\cos\alpha\cos\beta)$$
$$+ Z\,(f_A\cos\alpha + f_A\,f_B\sin\alpha\sin\beta),$$

$$M\,k^2\,\frac{dp}{dt} = \frac{a\,dt}{1 - f_A\,f_B\sin\alpha\cos\beta}\,[Y\,(1 - f_A\sin\alpha)\cos\beta + Z\,(1 - f_B\cos\beta)\,f_A\sin\alpha],$$

$$M\,\frac{k^2}{a^2}\left(\frac{d\dot{\xi}}{dt} - a\,\frac{d(p\cot\alpha)}{dt}\right) = \frac{-f_A\cos\alpha}{1 - f_A\,f_B\sin\alpha\cos\beta}\,(Z + Y\,f_B\cos\beta),$$

$$M\,\frac{k^2}{a^2}\left(\frac{d\dot{\xi}}{dt} + a\,\frac{d(p\tan\beta)}{dt}\right) = \frac{-f_A\sin\alpha}{1 - f_A\,f_B\sin\alpha\cos\beta}\,(Y + Z\,f_A\sin\beta),$$

$$\frac{d\xi}{dt} = \dot{\xi}, \qquad \omega^1 = p\,dt, \qquad \omega^2 = q\,dt, \qquad \omega^3 = r\,dt\,.$$

Example V. – H. Beghin's servo-constraint. – A material plane P can slide without friction by translation on a fixed plane xOy. A heavy, homogeneous sphere Σ of radius a can roll without slipping on that plane. The motion of the plane is governed automatically in such a manner that the center of the sphere turns uniformly around the fixed vertical axis Oz with a given angular velocity ω. Let us calculate the reactions and form the equations of motion of the system.

Let:

u, v	be the coordinates of a point in the plane,
ξ, η, ζ	be the absolute coordinates of the center of the sphere,
p, q, r	be the components of the absolute rotation of the sphere,

$\omega^1, \omega^2, \omega^3$ be three Pfaff forms that are constructed from the absolute differentials of the three parameters that characterize the displacement of a trihedron that is coupled to the sphere,
μ be the mass of the plane,
M be the mass of the sphere,
$M k^2$ be the moment of inertia of the sphere with respect to one of its diameters,
g be the intensity of gravity along the descending vertical.

The associated exterior form to the system without constraints is:

$$\begin{aligned}\Omega = M\ (d\dot{\xi} \wedge d\xi + d\dot{\eta} \wedge d\eta + d\dot{\zeta} \wedge d\zeta)\\ + M k^2 (dp \wedge \omega^1 + dq \wedge \omega^2 + dr \wedge \omega^3 + p\ \omega^2 \wedge \omega^3 + q\ \omega^3 \wedge \omega^1 + r\ \omega^1 \wedge \omega^2)\\ + \mu\ (d\dot{u} \wedge du + d\dot{v} \wedge dv)\\ \\ - M\ [\dot{\xi}\, d\dot{\xi} + \dot{\eta}\, d\dot{\eta} + \dot{\zeta}\, d\dot{\zeta} + k^2 (p\ dp + q\ dq + r\ dr) + g\ d\zeta] \wedge dt\\ - \mu\ (\dot{u}\, du + \dot{v}\, dv) \wedge dt\,.\end{aligned}$$

Constraints. – From our standpoint, the constraints translate into:

1. Contact between the sphere and the plane: $\dot{\zeta} = 0$:

   power: $P^1 = N\dot{\zeta}$, form: $\Omega^1 = N\, d\zeta \wedge dt$.

Rolling without slipping of the sphere on the plane:

2. $\dot{\xi} - aq - \dot{u} = 0$,

   power: $P^2 = X(\dot{\xi} - aq - \dot{u})$, form: $\Omega^2 = X\,(d\xi - a\ \omega^2 - du) \wedge dt$.

3. $\dot{\eta} + ap - \dot{v} = 0$,

   power: $P^2 = Y(\dot{\eta} + ap - \dot{v})$, form: $\Omega^2 = Y\,(d\eta + a\ \omega^1 - dv) \wedge dt$.

Servo-constraint:

4. $\dot{\xi} - \omega\,\eta = 0$:

   power: $P^4 = P\,\dot{u}$, form: $\Omega^4 = P\ du \wedge dt$.

5. $\dot{\eta} + \omega\,\xi = 0$:

power: $P^5 = Q\,\dot{v}$, form: $\Omega^5 = Q\,dv \wedge dt$.

Calculating the reactions. – Use the operators i ( ). Form the table of the field components:

	$d\dot{\xi}$	$d\dot{\eta}$	$d\dot{\zeta}$	$d\dot{u}$	$d\dot{v}$	dp	dq	dr	$d\xi$	$d\eta$	$d\zeta$	du	dv	ω^1	ω^2	ω^3	dt
E	0	0	$-g$	0	0	0	0	0	$\dot{\xi}$	$\dot{\eta}$	$\dot{\zeta}$	$\dot{u}$	$\dot{v}$	p	q	r	1
e^1	0	0	$\frac{1}{M}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
e^2	$\frac{1}{M}$	0	0	$\frac{-1}{\mu}$	0	0	$\frac{-a}{Mk^2}$	0	0	0	0	0	0	0	0	0	0
e^3	0	$\frac{1}{M}$	0	0	$\frac{-1}{\mu}$	$\frac{a}{Mk^2}$	0	0	0	0	0	0	0	0	0	0	0
e^4	0	0	0	$\frac{1}{\mu}$	0	0	0	0	0	0	0	0	0	0	0	0	0
e^5	0	0	0	0	$\frac{1}{\mu}$	0	0	0	0	0	0	0	0	0	0	0	0

One immediately deduces the non-zero values of the i ( ) d that relate to the various constraints:

1. $$i(e^1)\,d\dot{\zeta} = \frac{1}{M}, \qquad i(\mathrm{E})\,d\dot{\zeta} = -g,$$

2. $$i(e^2)\,d(\dot{\xi} - aq - \dot{u}) = \frac{1}{M}\left(1 + \frac{a^2}{k^2}\right) + \frac{1}{\mu},$$

$$i(e^4)\,d(\dot{\xi} - aq - \dot{u}) = -\frac{1}{\mu},$$

$$i(e^4)\,d(\dot{\xi} - aq - \dot{u}) = 0,$$

3. $$i(e^3)\,d(\dot{\eta} + ap - v) = \frac{1}{M}\left(1 + \frac{a^2}{k^2}\right) + \frac{1}{\mu},$$

$$i(e^4)\,d(\dot{\eta} + ap - v) = -\frac{1}{\mu},$$

$$i(\mathrm{E})\,d(\dot{\eta} + ap - v) = 0,$$

4. $i(e^2)d(\dot{\xi}-\omega\eta)=\frac{1}{M}$, $\quad i(\mathrm{E})d(\dot{\xi}-\omega\eta)=-\omega\dot{\eta}$,

5. $i(e^3)d(\dot{\eta}+\omega\xi)=\frac{1}{M}$, $\quad i(\mathrm{E})d(\dot{\eta}+\omega\xi)=\ \omega\dot{\xi}$.

One verifies that the constraints are compatible, since the determinant is equal to $(M^3\mu^2)^{-1}$, so the values of the coefficients of the constraints that we interpret as the components of the reactions will be:

For the plane on the sphere:

$$X = M\omega\dot{\eta} = -M\ \omega^2\ \xi, \qquad Y = -M\omega\dot{\xi} = -M\ \omega^2\ \eta, \qquad N = Mg\ .$$

For the fixed plane on the moving plane (in other words, the components of the force that must be applied to the moving plane in order to realize the servo):

$$P = -\left[M+\mu\left(1+\frac{a^2}{k^2}\right)\right]\omega^2\xi, \qquad Q = -\left[M+\mu\left(1+\frac{a^2}{k^2}\right)\right]\omega^2\eta\ .$$

Equations of motion. – Form Ω_s, which is deduced from Ω by replacing $d\dot{\zeta}$ with 0 and $d\zeta$ with 0, since that constraint has zero power, replacing *dp* and *dq* with their values that one infers by differentiating the conditions of rolling without slipping, replacing ϖ^1, ϖ^2 with their values that are calculated by annulling the form Ω^1, Ω^2, resp., since those constraints have zero power, replacing $d\dot{\xi}$ and $d\dot{\eta}$ with their values that one calculates by differentiating the servo-constraints, and replacing *du* and *dv* with their values that are calculated by means of $du = P_u^4dt = \dot{u}\,dt$, $dv = P_u^5dt = \dot{v}\,dt$, resp.:

$$\dot{\xi} = \omega\,\eta, \qquad \dot{\eta} = -\,\omega\,\xi, \qquad ap = \dot{v} + \omega\,\xi,$$

$$a\,q = \omega\eta - \dot{u}, \qquad du = \dot{u}\,dt, \qquad \varpi^1 = \frac{1}{a}(\dot{v}\,dt - d\eta),$$

$$d\dot{\xi} = \omega\,d\eta, \qquad d\dot{\eta} = -\,\omega\,d\xi, \qquad a\,dp = d\dot{v} + \omega\,d\xi,$$

$$a\,dq = \omega dh - d\dot{u}, \qquad dv = \dot{v}\,dt, \qquad \varpi^2 = \frac{1}{a}(d\xi - \dot{u}\,dt),$$

so

$$\Omega_s = 2\,M\,\omega\,d\eta\wedge d\xi$$

$$+M\frac{k^2}{a^2}[(d\eta + \omega\xi\,dt)\wedge(d\dot{v} + \omega\,d\xi) + (\omega\,d\eta -)\wedge(d\xi - \omega\eta\,dt) + dr\wedge\varpi^3 + r\,dr\wedge dt]$$

$$+ M \frac{k^2}{a^2} [(\dot{v} + \omega\xi)(d\xi - \dot{u}\, dt) \wedge \varpi^3 + (\omega\eta - \dot{u})\varpi^3 \wedge (\dot{v}\, dt - d\eta)$$

$$+ r\, (\dot{v}\, dt - d\eta) \wedge (d\xi - \dot{u}\, dt)] - M\, \omega^2\, (\eta\, d\eta + \xi\, d\xi) \wedge dt\, .$$

Since the coefficients of Ω, the constraint relations, and the powers of the constraints are constant or depend upon only ξ, η, u, v, the differential equations of motion will be given by the characteristics of Ω_s :

$$\frac{\partial \Omega_s}{\partial (d\dot{u})} = - M \frac{k^2}{a^2} (d\xi - \omega\eta\, dt) = 0,$$

$$\frac{\partial \Omega_s}{\partial (d\dot{v})} = - M \frac{k^2}{a^2} (d\eta + \omega\xi\, dt) = 0,$$

$$\frac{\partial \Omega_s}{\partial (dr)} = - M \frac{k^2}{a^2} (\varpi^3 - r\, dt) = 0,$$

$$\frac{\partial \Omega_s}{\partial (d\xi)} = - 2\, M\omega\, d\eta$$

$$+ M \frac{k^2}{a^2} [- \omega (d\eta + \omega\, \xi\, dt) - (\omega\, d\eta - d\dot{u}) + (\dot{v} + \omega\xi)\, \varpi^3 - r\, (\dot{v}\, dt - d\eta)]$$

$$- M\, \omega^2 \xi\, dt = 0,$$

$$\frac{\partial \Omega_s}{\partial (d\eta)} = + 2M\omega\, d\xi$$

$$+ M \frac{k^2}{a^2} [(d\dot{v} + \omega\, d\xi) + \omega (d\xi - \omega\eta\, dt) + (\omega\xi - \dot{u})\, \varpi^3 - r\, (\dot{v}\, dt - d\eta)]$$

$$- M\, \omega^2 \xi\, dt = 0,$$

$$\frac{\partial \Omega_s}{\partial \varpi^3} = M \frac{k^2}{a^2} [- dr - (\dot{v} + \omega\xi)(d\xi - \dot{u}\, dt) + (\omega\eta - \dot{u})(\dot{v}\, dt - d\eta)] = 0\, .$$

The first two equations give:

$$d\xi - \omega\eta\, dt = 0, \qquad d\eta + \omega\xi\, dt = 0\, ,$$

and upon integrating:

$$\xi = A \cos \omega t + B \sin \omega t, \qquad \eta = - A \sin \omega t + B \cos \omega t\, .$$

One gets the last one upon taking the first two into account: $dr = 0$. Equations (4) and (5) define $\dot{u}$ and $\dot{v}$; i.e., the velocity of motion of the plane:

$$M\,\frac{k^2}{a^2}\,d\dot{u} = -M\,\omega^2\left(1+\frac{k^2}{a^2}\right)\xi\,dt\,, \qquad M\,\frac{k^2}{a^2}\,d\dot{v} = -M\,\omega^2\left(1+\frac{k^2}{a^2}\right)\eta\,dt\,.$$

Example VI. – Generalized constraint: Simplified problem of radio-guided engines. – A heavy solid body is subject to the action of a motor that exerts a known force **F** on the solid body at each instant that points along the tangent to the trajectory of the center of gravity G of solid body. The motion is supposed to be planar and to take place in the material symmetry plane of the solid body, whose existence we postulate, namely, xOz. We suppose that a mechanism directs the tangent to the trajectory of G towards a moving body M whose coordinates are functions of time t : $x\,(t)$, $z\,(t)$. Find the system of equivalent forces that are necessary for the realization of that motion.

Let:

ξ, η	be the coordinates of the center of gravity G of the body,
θ	be the angle of rotation of the solid body around its center of gravity,
α	be the angle between the tangent to the trajectory to G and the horizontal axis,
M	be the mass of the body,
$M\,k^2$	be its moment of inertia around an axis that is perpendicular to the symmetry plane zOx.

The associated exterior form for the free solid body is:

$$\Omega = M\;(d\dot{\xi}\wedge d\xi + d\dot{\zeta}\wedge d\zeta + k^2 d\dot{\theta}\wedge d\theta)\; - M\;(\dot{\xi}\,d\xi + \dot{\zeta}\,d\zeta + k^2\dot{\theta}\,d\theta)\wedge dt$$
$$+ (F\cos\,\alpha\,d\xi + F\sin\,\alpha\,d\zeta - Mg\;d\zeta)\wedge dt\,.$$

Conditions:

1. The tangent to the trajectory must point along GM :

$$a^1 = \dot{\xi}\,(\zeta - z) - \dot{\zeta}\;(\xi - x) = 0\,.$$

2. The angle of rotation of the body around itself must be equal to α, up to a constant:

$$(\xi - x)\sin\,\theta - (\zeta - z)\cos\,\theta = 0,$$

so

$$a^2 = (\dot{\xi} - \dot{x})\sin\,\theta - (\dot{\zeta} - \dot{z})\cos\,\theta + [(\xi - x)\cos\,\theta + (\zeta - z)\sin\,\theta]\,\dot{\theta} = 0.$$

Since one knows nothing *a priori* about the forces that are necessary for realizing a similar constraint, we shall take the power of the ensemble to be $P = \lambda\dot{\xi} + \mu\dot{\zeta} + \nu\dot{\theta}$, and thus get the form Ω_l. The components of the field of the form $\Omega + \Omega_l$ are:

$$\mathrm{E} = \left(\frac{1}{M}(F\cos\alpha + \lambda), \frac{1}{M}(F\sin\alpha + \mu - Mg), \frac{\nu}{Mk^2}, \dot{\xi}, \dot{\zeta}, \dot{\theta}, 1\right).$$

Write down that the forms da^1 and da^2 belong to the sub-module of the characteristics of $\Omega + \Omega_l$; i.e., that:

$$i\,(\mathrm{E})\,da^1 = 0, \quad i\,(\mathrm{E})\,da^2 = 0.$$

Upon denoting the distance GM by D and the velocity of the body by V, one must have:

$$\lambda \sin\alpha - (\mu - Mg)\cos\alpha - M\frac{V}{D}(\dot{z}\cos\alpha - \dot{x}\sin\alpha) = 0,$$

$$\lambda \sin\alpha - (\mu - Mg)\cos\alpha + D\frac{\nu}{k^2} + M(\ddot{z}\cos\alpha - \ddot{x}\sin\alpha)$$
$$+ 2M\,[(\dot{\xi} - \dot{x})\cos\alpha + (\dot{\zeta} - \dot{z})\sin\alpha]\,\dot{\alpha} = 0,$$

which is a system that shows that only ν and $\lambda \sin\alpha - \mu\cos\alpha$ are determined.

If one lets W denote the velocity of the moving M and lets φ denote the angle between the tangent to the trajectory and the horizontal then one will get:

$$\lambda \sin\alpha - \mu\cos\alpha = -Mg\cos\alpha + \frac{M}{D}VW\cos(\alpha - \varphi),$$

$$\frac{\nu}{k^2} = \frac{M}{D}g\cos\alpha - \frac{M}{D}VW\cos(\alpha - \varphi) - 2\frac{M}{D}\dot{\alpha}[V - W\cos(\alpha - \varphi)] + \frac{M}{D}(\ddot{x}\sin\alpha - \ddot{z}\cos\alpha).$$

CHAPTER FOUR

THEORY OF UNILATERAL CONSTRAINTS: CASE OF ONE CONSTRAINT

§ I. – General considerations.

For a mechanical system S that is defined by the characteristic field E for a form Ω of degree two on a differentiable manifold V_{2n+1} that is restricted by a constraint of type a = 0, λe (viz., a constraint for which one knows that the direction of the constraint force that is defined by the field e), physical reasons will impose a sign on λ that is known *a priori*. One can always arrange for the sign that is imposed upon λ to be positive by replacing e with $- e$. In all of what follows, λ and e will be assumed to satisfy that condition. Let M be a point of V_{2n+1} that belongs to the submanifold $a = 0$.

Definition. – We say that a constraint of type $a = 0$, λe has *class U* if the function $a(t)$ that is defined by the value of a along the integral curve that is tangent to the vector field E at M_0 has a well-defined sign.

One can always suppose that the sign imposed upon a (t) is the positive sign for the neighborhood on the right $t > t_0$ (t_0 is the value of t at M_0), because one can always replace a with $- a$, which is a hypothesis that we suppose to be realized in what follows.

Justification for those considerations. – The preceding conventions might seem arbitrary. We shall show how they originate in some examples.

1. A solid body S in contact with another one S', with or without friction.

We saw (Chap. II, § VI) that a similar constraint is defined by:

$$u^3 = 0, \qquad P = N\,[n^3 - f\,|\,\mathbf{V}_g\,| - \delta\,|\,\boldsymbol{\omega}_2\,| - \omega\,|\,\boldsymbol{\omega}_n\,|\,]\,.$$

The separation of the solid bodies can happen only if $u^3 > 0$; contact happens when $N > 0$.

That type of constraint is the origin of the term *unilateral constraint* that is given to constraints of class U.

2. H. Béghin's servo constraint.

A disc B that is framed by an angle β is coaxial with a disc A that is framed by an angle α. The constraint $a = \dot{\beta} - \dot{\alpha} = 0$ is realized by applying a couple whose power is $P = \lambda\dot{\beta}$ to the disc B by way of an electrical control device that constitutes an index that is invariantly coupled with B. When the index that is coupled to A just touches the index that is coupled to B, an electrical current will be established, and a motor that turns in the trigonometric sense will apply a couple whose power is $P = \lambda\dot{\beta}$. The unique sense of rotation on the motor imposes the constraint that $\lambda > 0$. The breakdown of the constraint can take place with no deterioration of the contacts only if $a = \dot{\beta} - \dot{\alpha} > 0$.

3. More generally, the zero-power constraints and the Appell constraints in which the mechanism that realizes them imposes the condition that $a > 0$ for the constraint to break down and $\lambda > 0$ for the constraint to be acceptable.

The questions that one poses are then: When one knows the initial conditions at the point M_0 of V_{2n+1}, which belongs to the submanifold $a = 0$, what motion will it produce? Do the mechanical conditions $a(t) > 0$, $\lambda > 0$ that are imposed *a priori* determine the possible motions?

§ II. – Possible motions at M_0.

Two motions are possible at M_0 :

1. The unconstrained motion (M) that is defined by the integral curve Σ that issues from M_0 and is tangent to the vector field E.

a is a numerical function on V_{2n+1} that becomes a function of t on the curve Σ. That motion will be acceptable if that function, which is zero at t_0, becomes positive, which is a hypothesis that translates into $\left(\frac{da}{dt}\right)_0 > 0$, and if that derivative is zero, it will translate into $a_0^{(n+1)} > 0$, which is a derivative that certainly exists, because $i(\mathrm{E}) \cdot da \neq 0$ (Chap. II, § III).

From Theorem II of Chapter I, § III, that condition translates into $(i(\mathrm{E}) \cdot da)_0 > 0$, or into $(i(\mathrm{E}) \cdot da)_0^{(n+1)} > 0$ if the first n derivatives are zero.

2. The constrained motion (M_l) that is defined by the integral curve Σ_l that issues from M_0, which is tangent to the vector field $\mathrm{E} + \lambda e$. That motion will be acceptable when $\lambda > 0$.

Now, λ is defined by the equation $i(\mathrm{E}) \cdot da + \lambda\, i(e) \cdot da = 0$. The constraint is assumed to be compatible (Chap. II, § IV), $i(e) \cdot da \neq 0$, so:

$$\lambda = -\frac{i(\mathrm{E})\, da}{i(e)\, da}.$$

λ is defined to be a numerical function on V_{2n+1} that will become a function of t along the integral curve Σ_l whose n^{th} derivative is:

$$\lambda^{(n)} = [i\,(\mathrm{E} + \lambda e)\,d\lambda]^{(n)}.$$

Theorem I:

If $(i\,(\mathrm{E})\cdot da)^{(n)}$ is zero for $r \leq n$ and non-zero for $r = n + 1$ at M_0 then λ and its first $(n-1)$ derivatives will be zero at M_0 and:

$$-(i\,(e)\cdot da)_0\,\lambda_0^{(n)} = (i\,(\mathrm{E})\cdot da)_0^{(n+1)}.$$

By definition:

$$\lambda\,(r) = i\,(\mathrm{E} + \lambda e)\,d\lambda^{(r-1)} = i\,(\mathrm{E})\,d\lambda^{(r-1)} + \lambda\,i\,(e)\,d\lambda^{(r-1)}.$$

$(i\,(\mathrm{E})\cdot da)_0 = 0$ implies that $\lambda_0 = 0$, so:

$$\lambda_0^{(r)} = (i\,(\mathrm{E})\,d\lambda^{(r-1)})_0 = (i(\mathrm{E})\,d\lambda)_0^{(r-1)}.$$

By recurrence, λ can be presented in the form of a product $u \cdot v$, with $u = i\,(\mathrm{E})\cdot da$, $v = \dfrac{-1}{i(e)\,da}$, λ and its first $(n-1)$ derivatives at M_0 are zero, the n^{th} of which is given by:

$$\lambda_0^{(n)} = -\frac{(i\,(\mathrm{E})\,da)_0^{(n+1)}}{(i\,(e)\cdot da)_0}.$$

Consequence: Suppose that $(i\,(\mathrm{E})\cdot da)_0^{(r)} = 0$ for $r \leq n$ and $\neq 0$ for $r = n + 1$.

1. For a motion M, the first non-zero derivative of the function $a\,(t)$ will have order $(n + 1)$ and will have the value $a_0^{(n+1)} = (i\,(\mathrm{E})\cdot da)_0^{(r)}$.

2. For a motion (M_l), λ and its first $(n-1)$ derivatives will be zero at M_0, so the n^{th} derivative of λ can be calculated from:

$$-(i\,(e)\cdot da)_0\,\lambda_0^{(n)} = (i\,(\mathrm{E})\cdot da)_0^{(n+1)}.$$

One can summarize the preceding results in the formula:

$$a_0^{(n+1)} - (i\,(e)\cdot da)_0\,\lambda_0^{(n)} = (i\,(\mathrm{E})\cdot da)_0^{(n+1)}, \tag{II.1}$$

in which one sets $\lambda = 0$ for a motion M, and $a = 0$ for a motion (M_l). It is important for the mechanical applications to remark that $(i\,(\mathrm{E})\cdot da)_0^{(n+1)}$ depends upon only the initial conditions and the forces that are applied to the system besides the ones that are necessary to realize the constraint. Formula (II.1) can also be established by means of the following theorem:

Theorem II:

If f and λ are two numerical functions on V_{2n+1} that are regular, along with their derivatives in the neighborhood of a point M, and E *and e are two given fields without singularities in the neighborhood of M then the $(n + 1)^{\text{th}}$ derivative of the function f relative to the field* E + λe *can be calculated by means of the formula:*

$$f_{\mathrm{E}+\lambda e}^{(n+1)} = (\lambda\,\varphi_1)^{(n)} + \ldots + (\lambda\,\varphi_p)^{(n-p+1)} + \ldots + \lambda\,\varphi_{n+1} + (i\,(\mathrm{E})\,df)^{(n+1)}, \tag{II.2}$$

in which $\varphi_1 = i\,(e)\cdot df$, $\varphi_p = i\,(e)\cdot d\,(i\,(\mathrm{E})\cdot df)^{(p+1)}$, and $(\lambda\,\varphi_p)^{(n-p+1)}$ denotes the $(n-p+1)^{\text{th}}$ derivative of the product $\lambda\,\varphi_p$ with respect to the field E + λe .

By definition, $f_{\mathrm{E}+\lambda e}^{1} = i\,(\mathrm{E})\cdot df + \lambda\, i\,(e)\cdot df$.

Upon setting $df = \Phi_1$, $i\,(e)\cdot df = \varphi_1$, the n^{th} derivative of $f^{(1)}$ will be written:

$$f^{(n+1)} = \Phi_1^{(n)} + (\lambda\,\varphi_1)^{(n)}. \tag{1}$$

One can apply the procedure that was employed for the function f to the function Φ:

$$\Phi_1^{(1)} = i\,(\mathrm{E})\cdot d\Phi_1 + \lambda\, i\,(e)\cdot d\Phi_1 .$$

Upon setting $i\,(\mathrm{E})\cdot d\Phi_1 = \Phi_2$, $i\,(e)\cdot d\Phi_1 = \varphi_2$, the $(n-1)^{\text{th}}$ derivative of $\Phi_1^{(1)}$ can be written:

$$\Phi_1^{(n)} = \Phi_2^{(n-1)} + (\lambda\,\varphi_2)^{(n-1)}. \tag{2}$$

One pursues the argument, and after $(n + 1)$ operations, one will get:

$$\Phi_1^{(1)} = i\,(\mathrm{E})\cdot d\,\Phi_n + \lambda\,\varphi_{n+1} . \tag{$n+1$}$$

Upon adding corresponding sides of the preceding $(n + 1)$ relations, after remarking that $i\,(\mathrm{E})\cdot d\,\Phi_n = (i\,(\mathrm{E})\cdot df)^{(n+1)}$, one will get (II.2).

Application. – If λ and its first $(n - 1)$ derivatives are zero at M_0 then formula (II.2) will give:

$$f_0^{(n+1)} - \lambda_0^{(n)}\,(i\,(e)\,df\,)_0 = (i\,(\mathrm{E})\,df\,)_0^{(n+1)} . \tag{II.3}$$

When that relation is applied to $f = a$, that will give (II.1). Formula (II.1), thus-proved, will give $a_0^{(n+1)} = (i\,(\mathrm{E})\cdot da)_0^{(n+1)}$ for a motion (M) upon setting $\lambda = 0$ and $\lambda_0^{(n)}$ for a motion (M_l) upon setting $a = 0$, and it will show, moreover, that upon varying n, λ and its first $(n-1)$ derivatives will be zero at M_0, if $(i\,(\mathrm{E})\cdot da)_0^{(n+1)}$ is the first non-zero derivative at M_0 .

Discussing the possibilities.

Theorem:

1. *If* $(i\ (e)\cdot da)_0 > 0$ *then the initial conditions will suffice to determine the ultimate motion.*

2. *If* $(i\ (e)\cdot da)_0 < 0$ *then the initial conditions can determine the ultimate motion, but that will be impossible if* $(i\ (e)\cdot da)_0 > 0$ *or* $(i\,(\mathrm{E})\cdot da)_0^{(n+1)}$*, and indeterminate when:*

$$(i\ (\mathrm{E})\cdot da)_0 > 0 \qquad \text{or} \qquad (i\,(\mathrm{E})\cdot da)_0^{(n+1)} > 0.$$

Indeed:

1. If $(i\ (\mathrm{E})\cdot da)_0 > 0$ and $(i\ (\mathrm{E})\cdot da)_0 \neq 0$ then the relation (II.1), with $n = 0$, will give for:

$$(i\ (\mathrm{E})\cdot da)_0 > 0: \quad \begin{cases} \lambda = 0, & \left(\dfrac{da}{dt}\right)_0 > 0 \quad \text{the motion } M \text{ is acceptable} \\ a = 0, & \lambda_0 \quad < 0 \quad \text{the motion } M \text{ is unacceptable} \end{cases}$$

$$(i\ (\mathrm{E})\cdot da)_0 < 0: \quad \begin{cases} \lambda = 0, & \left(\dfrac{da}{dt}\right)_0 < 0 \quad \text{the motion } M \text{ is unacceptable} \\ a = 0, & \lambda_0 \quad > 0 \quad \text{the motion } M \text{ is acceptable} \end{cases}$$

If $(i\ (\mathrm{E})\cdot da)_0 = 0$ and $(i\,(\mathrm{E})\cdot da)_0^{(n+1)} \neq 0$ then the relation (II.1), with $n = 0$, will give that (M) and (M_l) are both possible for $(i\ (\mathrm{E})\cdot da)_0 > 0$:

$$\begin{cases} \text{for} \quad \lambda = 0, & \left(\dfrac{da}{dt}\right)_0 > 0 \\ \text{for} \quad a = 0, & \lambda_0 \quad > 0. \end{cases}$$

There will then be indeterminacy.

If $(i\ (\mathrm{E})\cdot da)_0 < 0$ then (M) and (M_l) will both be impossible:

$$\begin{cases} \text{for} \quad \lambda = 0, & \left(\dfrac{da}{dt}\right)_0 < 0 \\ \text{for} \quad a = 0, & \lambda_0 \quad < 0. \end{cases}$$

If $(i\,(\mathrm{E})\cdot da)_0 = 0$ and $(i\,(\mathrm{E})\cdot da)_0^{(n+1)} \neq 0$ then one get the same results by using (II.1).

Geometric picture for that discussion. – Consider the vectors $\mathbf{E}_0$ and $\boldsymbol{e}_0$ in the tangent space to V_{2n+1} at M_0 . The submanifold $a = 0$ divides V_{2n+1} into two regions in the neighborhood of M_0 . Associate the form da with the vector $\boldsymbol{a}$ whose origin in M_0 and which points in the direction of the positive region. Associate the two covariant components of $\mathbf{E}_0$ and $\boldsymbol{e}_0$ relative to $\boldsymbol{a}$ with the vectors:

$$\boldsymbol{a} = (i\,(\mathrm{E})\cdot da)_0\,\boldsymbol{a}, \qquad \mathbf{A} = -\,(i\,(e)\cdot da)_0\,\boldsymbol{a},$$

respectively, which are collinear with $\boldsymbol{a}$.

If $(i\,(e)\cdot da)_0 > 0$ then $\boldsymbol{a}$ and $\mathbf{A}$ have opposite directions, and α will correspond to a motion (M) or (M_l) .

If $(i\,(e)\cdot da)_0 > 0$ then $\boldsymbol{a}$ and $\mathbf{A}$ have the same direction, and if α has the opposite direction to $\boldsymbol{a}$ then that will not correspond to any motion, while if α has the same direction as $\boldsymbol{a}$ then that will correspond to two motions.

§ III. – Mechanical consequences.

The preceding study showed the important role that is played by the sign of the invariant $i\,(e)\cdot da$, whose analytical expression is:

$$i\,(e)\cdot da = \sum_{i=1}^{n}\frac{\partial a}{\partial p_i}l_i\ .$$

For a constraint of the Appell type, which includes the holonomic constraints and linearly non-holonomic constraints as special cases, as we saw in Chapter II, § V, $i\,(e)\cdot da$ is equal to the norm of e, so it is always positive. For such constraints, the initial conditions suffice to determine the ultimate motion. For the other types of constraints (in particular, for constraints with friction, such as resistance to rolling or pivoting, to which we shall return in the following paragraph), the initial conditions, combined with the mechanical conditions, might not be sufficient. In the indeterminate case, if one would like to specify the final motion then it would be necessary to add a selective condition for (M) and (M_l) to the initial conditions.

Analytical reason for the indeterminacy. – By using the geometric interpretation of the differential system for the equations of motion that it defines the curves that are tangent to a field F on a manifold V_{2n+1}, with F = E for a motion (M) and $\mathrm{F} = \mathrm{E} + \lambda e$ for a motion (M_l), the field F, thus-defined, will be discontinuous on the submanifold $a = 0$. There will then be two possible integrals that correspond to the motions (M) and (M_l) when one takes the initial point M_0 to be on the submanifold $a = 0$. The mechanical considerations that $\lambda > 0$ if $a = 0$ and $a > 0$ if $\lambda = 0$ will permit one to separate the two motions only if $i\,(e)\cdot da > 0$.

Theorem:

The appearance or disappearance of a constraint will necessarily imply a discontinuity for the derivatives of the functions that represent the motion. If one is given $2n$ *functions* f_i *on* V_{2n+1} *that are independent and have no singularity in a neighborhood of* M_0 *, which is taken on the manifold* $a = 0$*, then the motions* (M) *and* (M_l) *will be defined by the differential systems:*

$$(M) \qquad \frac{df_i}{dt} = i\,(\mathrm{E}) \cdot df_i\,, \qquad\qquad (M_l) \qquad \frac{df_i}{dt} = i\,(\mathrm{E} + \lambda\, e) \cdot df_i\,,$$

with

$$\lambda\, i\,(e) \cdot da + i\,(\mathrm{E}) \cdot da = 0\,.$$

If $\lambda_0 \neq 0$ at M_0 then one will deduce that: $\left(f_{i,\mathrm{E}+\lambda e}^{(1)}\right)_0 - \left(f_{i,\mathrm{E}}^{(1)}\right)_0 = \lambda_0\,(i\,(e) \cdot df_i)_0$, which will show the discontinuity in the first-order derivatives of the functions f_i .

If λ and its first $(n-1)$ derivatives are zero at M_0 then formula (II.3) will give:

$$\left(f_{i,\mathrm{E}+\lambda e}^{(1)}\right)_0 - \left(f_{i,\mathrm{E}}^{(1)}\right)_0 = \lambda_0^{(n)}\,(i\,(e) \cdot df_i)_0\,,$$

which shows that the derivatives of order $(n + 1)$ of the functions f_i are discontinuous.

§ IV. – Examples that show the existence of indeterminacy due to a constraint.

1. Let us return to the example of a servo constraint that was described at the beginning of this chapter.

Let I_1 denote the moment of inertia of disc A, let Γ denote the couple that is applied to it, let I_2 be the moment of inertial of disc B, and suppose that the servomotor that follows B can turn only in the negative sense.

The constraint translates into $\beta - \alpha = 0$, or into $a = \dot{\beta} - \dot{\alpha} = 0$, conforming to the viewpoint of Chapter II, I, since the power that is necessary to realize that constraint is $P = -\lambda\dot{\beta}$.

The generating form Ω of the equations of motion is:

$$\Omega = I_1\, d\dot{\alpha} \wedge d\alpha + I_2\, d\dot{\beta} \wedge d\beta - I_1\, d\dot{\alpha} \wedge dt - I_2\, d\dot{\beta} \wedge dt + \Gamma_1\, d\alpha \wedge dt - \lambda\, d\beta \wedge dt\,,$$

and thus, one has the equations:

Motion (M), $\lambda = 0$ Motion (M_l), $a = 0$

$$\begin{cases} -I_1\,d\dot{\alpha} + \Gamma_1\,dt = 0, \\ d\alpha - \dot{\alpha}\,dt = 0, \\ I_2\,d\dot{\beta} = 0, \\ d\beta - \dot{\beta}\,dt = 0, \end{cases} \qquad \begin{cases} -I_1\,d\dot{\alpha} + \Gamma_1\,dt = 0, \\ d\alpha - \dot{\alpha}\,dt = 0, \\ I_2\,d\dot{\beta} - \lambda\,dt = 0, \\ d\beta - \dot{\beta}\,dt = 0. \end{cases}$$

The components of the field E are Γ_1 / I_1, 0, $\dot{\alpha}$, $\dot{\beta}$, 1, and those of the field e are 0, $-1/I_2$, 0, 0, 0.

It will then result that $i(e) \cdot da = -1 / I_2$, $i(\mathrm{E}) \cdot da = -\Gamma_1 / I_1$. In this case, the relation (II.1) that permits one to discuss the possibilities is written:

$$\frac{da}{dt} + \frac{\lambda}{I_2} = -\frac{\Gamma_1}{I_1}.$$

Discussion:

if $\Gamma_1 > 0$:

$$\begin{cases} \dfrac{da}{dt} = -\dfrac{\Gamma_1}{I_1} < 0 \quad \text{for } \lambda = 0 \quad \text{then } (M) \text{ will be unacceptable} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{since } da/dt \text{ must be positive,} \\ \lambda = -\dfrac{\Gamma_1}{I_1} < 0 \quad \text{for } a = 0 \quad \text{then } (M_l) \text{ will be unacceptable} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{since } \lambda \text{ must be positive,} \end{cases}$$

if $\Gamma_1 < 0$:

$$\begin{cases} \dfrac{da}{dt} = -\dfrac{\Gamma_1}{I_1} > 0 \quad \text{for } \lambda = 0 \quad \text{then } (M) \text{ will be acceptable} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{since } da/dt \text{ is positive,} \\ \lambda = -\dfrac{\Gamma_1}{I_1} > 0 \quad \text{for } a = 0 \quad \text{then } (M_l) \text{ will be acceptable} \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{since the two discs will turn clockwise.} \end{cases}$$

In the case where Γ_1 is negative, there will be indeterminacy. That indeterminacy can be eliminated only by an arbitrary choice of $da / dt \neq 0$ for a motion (M) and $\lambda \neq 0$ for a motion (M_l).

2. It is clear that one can multiply the examples of the preceding type by associating an electrical contact that creates a force field with a constraint relation that is coupled with the contact.

3. Case of two solid bodies in contact with sliding friction and resistance to rolling and pivoting.

In Chapter II, § VI, we calculated $i\,(e)\cdot da$ for two solid bodies in point-like contact. The relation that permits us to discuss the possibilities has the form:

$$\left(\frac{da}{dt}\right)_0 - (i\,(e)\cdot da)_0\, N = (i\,(e)\cdot da)_0\,,$$

with

$$i\,(e)\cdot da = A + f\,(B \sin s + C \cos s) + \delta(D \cos \sigma + F \sin \sigma) + \varepsilon\,\varpi G\,.$$

If $(i\,(e)\cdot da)_0 > 0$ for $(i\,(\mathrm{E})\cdot da)_0 > 0$ then the only possibility will be $\left(\frac{da}{dt}\right)_0 = (i\,(\mathrm{E})\cdot da)_0$; i.e., the two solid bodies will separate. For $(i\,(\mathrm{E})\cdot da)_0 < 0$, $(i\,(e)\cdot da)_0\, N = (i\,(\mathrm{E})\cdot da)_0$, they will slide over each other.

If $(i\,(e)\cdot da)_0 > 0$ for $(i\,(\mathrm{E})\cdot da)_0 < 0$ then there will be two possibilities: $\left(\frac{da}{dt}\right)_0 > 0$, which is separation of the two solid bodies, and $N > 0$, which is sliding contact between the solid bodies, and for $(i\,(\mathrm{E})\cdot da)_0 < 0$, one has the impossibility of either separation or contact.

If one neglects the resistance to rolling and pivoting then the impossible case was pointed out for the first time by Painlevé and is known by the name of the *Painlevé paradox* ([20]), who likewise pointed out that one can encounter some indeterminacies in problems with friction. If one neglects sliding friction and the resistance to pivoting then L. Roy ([21]) pointed out that the resistance to rolling can likewise give rise to some indeterminacies and impossibilities.

We remark that pivoting friction or the combination of sliding friction and resistance to rolling and pivoting can give rise to the same phenomena.

Those anomalies, which do not appear in that way for the classical holonomic and linearly non-holonomic constraints, have led some to think that they are due to the imperfection in Coulomb's laws and that more precise laws would avoid them. The study that was made in this chapter shows that this not the case, since the origin of the paradoxes and the indeterminacies lives in the concept of constraint itself and is concerned with an arrangement of the vectors ***e*** (viz., the constraint field) and ***a*** (which is associated with the form da) at a point M_0 on the submanifold $a = 0$ of V_{2n+1} if $i\,(e)\cdot da$ is negative.

The case of sliding friction is therefore just the first case where those facts were confirmed historically.

([20]) Cf., PAINLEVÉ, C. R. Acad. sci. Paris **121** (1895) and *Leçons sur le frottement*, Hermann, Paris, 1895.

([21]) Cf., L. ROY, Congrés de mécanique appliquée.

CHAPTER V

THEORY OF UNILATERAL CONSTRAINTS: CASE OF SEVERAL CONSTRAINTS

§ I. – Mechanical aspect of the problem.

When one considers a mechanical system S that is restricted by p constraints of class U (Chap. IV, § I), with the initial conditions being given, it is natural to demand to know what the possible motions will be and in what cases the conditions that were imposed *a priori* for the constraints to break down and the constraint factors will be sufficient to separate the motions.

The mechanical system S will be defined by the characteristic field E of a form Ω of degree two on a differentiable manifold V_{2n+1} , and each of the p constraints will be characterized by an analytic submanifold $a^k = 0$ and a constraint field $\lambda_h\, e^h$, where λ_h is a numerical function to be determined, e^h is a known field, and the index h varies from 1 to p. Giving the initial conditions amounts to giving a point M_0 of V_{2n+1} . The entire discussion is based upon the following theorem:

Theorem I:

If f, λ_1 , …, λ_p are $(p + 1)$ *numerical functions on V_{2n+1} that are regular, along with their partial derivatives in the neighborhood of a point M, and if* E, e^1, e^2, …, e^p *are* $(p + 1)$ *given fields without singularities in the neighborhood of M then the* $(n + 1)^{\text{th}}$ *derivatives of the function f relative to the field* $\text{E} + \sum_{h=1}^{p} \lambda_h\, e^h$ *can be calculated by means of the formula:*

$$f_n^{(n+1)} = (\lambda_h\, \varphi_1^h)^{(n)} + \ldots + (\lambda_h\, \varphi_p^h)^{(n-p+1)} + \ldots + \lambda_h\, \varphi_1^h + (i\,(E)\, df)^{(n+1)}, \tag{V.1}$$

with $\varphi_1^h = i\,(e^h) \cdot df$, $\varphi_p^h = i\,(e^h) \cdot d\,(i\,(\text{E}) \cdot df)^{(p)}$ *and summation over the dummy index h in each parenthesis* $(\lambda_h\, \varphi_1^h)^{(n-p+1)}$*, while the exponent* $(n - p + 1)$ *indicates derivation with respect to the field* $\text{E} + \sum_{h=1}^{p} \lambda_h\, e^h$.

The proof is carried out as it was for Theorem II, Chap. IV. By definition:

$$f^{(1)} = i\,(\text{E})\, df + \lambda_h\, i\,(e^h) \cdot df .$$

Upon setting $i\,(\text{E}) \cdot df = \Phi_1$, $i\,(e^h) \cdot df = \varphi_1^h$, the n^{th} derivative of $f\,(1)$ will be:

(1) $$f^{(n+1)} = \Phi_1^{(n)} + (\lambda_h\, \varphi_1^h)^{(n)}.$$

One can apply the procedure that was employed with the function f to the function Φ_1.

Upon setting $i\,(\mathrm{E}) \cdot d\Phi_1 = \Phi_2$, $i\,(e^h) \cdot d\Phi_1 = \varphi_2^h$, the $(n-1)^{\text{th}}$ derivative of $\Phi_1^{(1)}$ will give:

(2) $$\Phi_1^{(n)} = \Phi_2^{(n-1)} + (\lambda_h\, \varphi_2^h)^{(n-1)}.$$

One continues the argument, and after $(n + 1)$ analogous operations, one will get:

$(n + 1)$ $$\Phi_n^{(1)} = i\,(\mathrm{E})\, d\Phi_n + \lambda_h\, \varphi_{n+1}^h.$$

Upon adding corresponding sides of the preceding $(n + 1)$ relations and remarking that $i\,(\mathrm{E}) \cdot d\Phi_n = (i\,(\mathrm{E}) \cdot df)^{(n+1)}$, one will get (V.I). One can apply (V.I) to the p functions $a^1, \ldots, a^p$ at a point M_0 , which gives:

(V.2) $$(a^k)_0^{(n+1)} - (\lambda_h\, \varphi_1^{hk})_0^{(n)} \cdots - (\lambda_k\, \varphi_1^k)_0^{(n-p+1)} \cdots - (\lambda_k\, \varphi_{n+1}^k)_0 = (i\,(\mathrm{E}) \cdot da^k)_0^{(n+1)}.$$

In particular, if $(i\,(\mathrm{E}) \cdot da^k)_0 \neq 0$ for all k from 1 to p, which means that E does not belong to any of the tangent spaces to the submanifolds $a^k = 0$, then:

(V.3) $$(a^k)_0^{(1)} - \lambda_h\, (i(e^k) \cdot da^k)_0 = (i\,(\mathrm{E}) \cdot da^k)_0 \qquad (k = 1, \ldots, p).$$

The relations (V.3), whose right-hand sides do not depend upon the various possibilities, will allow us to discuss the nature of the possible motions:

Hypothesis A: If the motion takes place with the breakdown of the p constraints then one must set $\lambda_1 = 0, \ldots, \lambda_p = 0$ in the left-hand side of (V.3), which will give:

$$(a^k)_0^{(1)} = (i\,(\mathrm{E}) \cdot da^k)_0 \qquad (k = 1, \ldots, p).$$

Hypothesis A will be acceptable if $(i\,(\mathrm{E}) \cdot da^k)_0 > 0$.

Hypothesis B: If the motion of S takes place while respecting the p constraints then one must set:

$$(a^1)_0^{(1)} = 0, \qquad \ldots, \qquad (a^p)_0^{(1)} = 0$$

in the right-hand side of (V.3), which will give the following system of p linear equations for the determination of the λ_k :

$$-\ \lambda_h (i\,(e_k) \cdot da^k)_0 = (i\,(\mathrm{E}) \cdot da^k)_0 .$$

Since one supposes that the constraints are compatible (cf., Chap. III, § II), that system will admit a solution.

Hypothesis B will be acceptable if $\lambda_1 > 0, \ldots, \lambda_p > 0$.

Hypothesis C: If the motion of *S* takes place while respecting *k* of the constraints then one can suppose that they are the first *k* of them, to fix ideas, and that the other ($p - k$) break down, so one must set:

$$(a^1)_0^{(1)} = 0, \ldots, (a^k)_0^{(1)} = 0, \qquad \lambda_{k+1} = 0, \ldots, \lambda_p = 0$$

in the left-hand side of (V.3).

One will then be led to solve a linear system in the unknowns:

$$\lambda_1, \ldots, \lambda_k, (a^{k+1})_0^{(1)}, \ldots, (a^p)_0^{(1)},$$

namely:

$$\begin{aligned}
- \lambda_1\, i(e^1) \cdot da^1 - \ldots - \lambda_k\, i(e^k) \cdot da^1 &= i(\mathrm{E}) \cdot da^1, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
- \lambda_1\, i(e^1) \cdot da^k - \ldots - \lambda_k\, i(e^k) \cdot da^k &= i(\mathrm{E}) \cdot da^k, \\
(a^{k+1})_0^{(1)} - \lambda_1\, i(e^1) \cdot da^{k+1} - \ldots - \lambda_k\, i(e^k) \cdot da^{k+1} &= i(\mathrm{E}) \cdot da^{k+1}, \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
(a^p)_0^{(1)} - \lambda_1\, i(e^1) \cdot da^p - \ldots - \lambda_k\, i(e^k) \cdot da^p &= i(\mathrm{E}) \cdot da^p,
\end{aligned}$$

whose determinant is non-zero, since the *k* constraints that are taken from the *p* constraints must be compatible.

Hypothesis C will be acceptable if:

$$\lambda_1 > 0, \ldots, \lambda_k > 0, \ (a^{k+1})_0^{(1)} > 0, \ldots, (a^p)_0^{(1)} > 0.$$

§ II. – Equivalent problem in *analysis situs*.

One can give a geometric form to the simultaneous examination of these various hypotheses that will simplify the discussion.

We remark that if a constraint of type $a = 0$, λe is given then that will mean that one knows, on the one hand, the field *e*, which consists of elements in the tangent spaces *T* to V_{2n+1} , and on the other hand, the form *da*, which consists of elements in the space T' that is dual *T*. *a* and *e* are the physical realization of the imposed constraint. The two elements *e* and *da* are independent of any coordinate system. Those two elements are associated with the invariant $i(e) \cdot da$; i.e., $(i(e) \cdot da)_0$ will have a value at a point *M* of V_{2n+1} that is independent of the choice of coordinates. We will have the following p^2 invariants at M_0 relative to the *p* fields $e^1, \ldots, e^p$ and the *p* forms $da^1, \ldots, da^p$:

$$i(e^h) \cdot da^k = r^{hk}.$$

The p elements r^{hk} are the elements of a matrix $\| r^{hk} \|$.

Now consider a p-dimensional vector space E_p , and in that space, on the one hand, p arbitrary independent vectors $\mathbf{a}^1$, $\mathbf{a}^2$, …, $\mathbf{a}^p$ that have the same origin, and on the other, p vectors $\mathbf{A}^1$, $\mathbf{A}^2$, …, $\mathbf{A}^p$ that are defined by:

$$\left\| \begin{matrix} \mathbf{A}^1 \\ \mathbf{A}^2 \\ \vdots \\ \mathbf{A}^3 \end{matrix} \right\| = \left\| -r^{hk} \right\| \times \left\| \begin{matrix} \mathbf{a}^1 \\ \mathbf{a}^2 \\ \cdots \\ \mathbf{a}^p \end{matrix} \right\| .$$

The p equations (V.3) are equivalent to the vectorial relation:

$$\text{(V.4)} \qquad \mathbf{a} + \lambda_h \, \mathbf{A}^h = \boldsymbol{\alpha} .$$

(The change of sign comes from the definition that was given to the vectors $\mathbf{A}^h$.) Hypothesis A means that one can decompose the vector $\boldsymbol{\alpha}$ in E_p into the p vectors $\mathbf{a}^1$, …, $\mathbf{a}^p$, and that its components will all be positive as a result of the fact that $\boldsymbol{\alpha}$ belongs to a well-defined region R_0 in E_p .

Hypothesis B means that one can decompose the vector $\boldsymbol{\alpha}$ in E_p into the p vectors $\mathbf{A}^1$, …, $\mathbf{A}^p$ and that its components will all be positive as a result of the fact that $\boldsymbol{\alpha}$ belongs to a well-defined region R_p in E_p .

Hypothesis C means that one can decompose $\boldsymbol{\alpha}$ into the k vectors $\mathbf{A}^1$, $\mathbf{A}^2$, …, $\mathbf{A}^k$ and the $(p - k)$ vectors $\mathbf{a}^{k+1}$, …, $\mathbf{a}^p$, and that its components will all be positive as a result of the fact that $\boldsymbol{\alpha}$ belongs to a well-defined region R_k^{p-k} in E_p .

The problem of analysis that consists of determining the possible motions of the system S restricted by p constraints of class U when one knows that initial conditions is then equivalent to a problem in *analysis situs.*

One is given a sheaf of $2p$ vectors in a Euclidian space E_p that have the same origin:

$$\begin{matrix} \mathbf{a}^1 \ldots & \mathbf{a}^k \ldots & \mathbf{a}^p \\ \mathbf{A}^1 \ldots & \mathbf{A}^k \ldots & \mathbf{A}^p . \end{matrix}$$

If one takes p of those vectors:

$$\mathbf{A}^{i_1}, \ldots, \mathbf{A}^{i_k}, \mathbf{a}^{i_{k+1}}, \ldots, \mathbf{a}^{i_p},$$

in which $(i_1 , \ldots, i_k , i_{k+1} , \ldots, i_p)$ is a permutation of the first p integers and they are chosen in such a way that two vectors will not appear in the same column of the preceding matrix, then one will get a p-hedron. One can associate a region $E_p = R_{i_1 \cdots i_k}^{i_{k+1} \cdots i_p}$ with each p-hedron that we call the *internal region* of the p-hedron, such that if $\boldsymbol{\alpha}$ is in $R_{i_1 \cdots i_k}^{i_{k+1} \cdots i_p}$ then the components of $\boldsymbol{\alpha}\,(X_{i_1}, \ldots, X_{i_{k+1}}, x_{i_{k+1}}, \ldots, x_{i_p})$ relative to that p-hedron will all be positive numbers:

$$\alpha = X_{i_1}\mathbf{A}^{i_1} + \cdots + X_{i_k}\mathbf{A}^{i_k} + x_{i_{k+1}}\mathbf{a}^{i_{k+1}} + \cdots + x_{i_p}\mathbf{a}^{i_p}.$$

The preceding decomposition of α corresponds to a motion $M_{i_1\cdots i_k}^{i_{k+1}\cdots i_p}$ of the system in which k constraints that are taken from the p persist, while $(p - k)$ others will break down. When α is given, the number of possible motions will correspond to the number of regions $R_{i_1\cdots i_k}^{i_{k+1}\cdots i_p}$ to which α can belong simultaneously.

Total number of possibilities. – There are as many regions $R_{i_1\cdots i_k}^{i_{k+1}\cdots i_p}$ that one can realize as there are combinations of k vectors $\mathbf{A}^{i_1}, \ldots, \mathbf{A}^{i_k}$ that are taken from among p of them, namely C_p^k. The total number of regions is then equal to the sum of the combinations:

$$C_p^0 + C_p^1 + \cdots + C_p^k + \cdots + C_p^p = 2^p.$$

The number of possibilities in a mechanical system that is restricted by p constraints of class U will then be 2^p.

Necessary and sufficient conditions for the uniqueness of the possibilities. – In order for a vector α to correspond to only one motion, it is necessary and sufficient that the 2^p regions $R_{i_1\cdots i_k}^{i_{k+1}\cdots i_p}$ should have no common p-dimensional domain or that they should have no common internal point. Since the vectors $\mathbf{A}^1, \ldots, \mathbf{A}^p$ are defined by starting with the vectors $\mathbf{a}^1, \ldots, \mathbf{a}^p$ by means of the matrix $\| - r^{hk} \|$, one will be led to study the conditions that the elements of that matrix must satisfy in order for the various regions $R_{i_1\cdots i_k}^{i_{k+1}\cdots i_p}$ to have no common p-dimensional domain.

Theorem II:

The necessary and sufficient condition for all of the internal regions to 2^p *p-hedra in a p-dimensional Euclidian space that are obtained by taking p vectors in the various columns of the matrix:*

$$\begin{matrix} \mathbf{a}^1 \ldots & \mathbf{a}^k \ldots & \mathbf{a}^p \\ \mathbf{A}^1 \ldots & \mathbf{A}^k \ldots & \mathbf{A}^p \end{matrix}$$

to have no common p-dimensional domain is that all of the determinants of the diagonal minors that one can extract from the square matrix $\| r^{hk} \|$ *should be positive, where the vectors* $\mathbf{A}^1, \ldots, \mathbf{A}^p$ *are defined as functions of the vectors* $\mathbf{a}^1, \ldots, \mathbf{a}^p$ *by means of the matrix* $\| - r^{hk} \|$.

Preliminary lemmas:

1. Two p-hedra that are constructed from k different vectors will have no common p-dimensional domain only if at least one of the k inequalities:

(V.5) $$x_i\, X_i < 0,\ \ldots,\ x_{i_k} X_{i_k} < 0$$

is satisfied, where $x_{i_1}, \ldots, x_{i_k}, X_{i_1}, \ldots, X_{i_k}$ denote the components of a vector of E_p with respect to the k different vectors.

Since the two p-hedra are composed of k different vectors, that amounts to saying that if one takes $\mathbf{a}^j$ from column j in order to compose the first p-hedron then one must take $\mathbf{A}^j$ from the same column in order to compose the second p-hedron.

It will then result that the same vector $\boldsymbol{\alpha}$ will have the following expressions with respect to the two bases that relate to those two p-hedra:

$$\boldsymbol{\alpha} = x_{i_1}\mathbf{a}^{i_1} + \cdots + x_{i_q}\mathbf{a}^{i_q} + X_{i_{q+1}}\mathbf{A}^{i_{q+1}} + \cdots + X_{i_k}\mathbf{A}^{i_k} + x_{i_{k+1}}\mathbf{a}^{i_{k+1}} + \cdots + x_{i_p}\mathbf{a}^{i_p},$$
$$\boldsymbol{\alpha} = X_{i_1}\mathbf{A}^{i_1} + \cdots + X_{i_q}\mathbf{A}^{i_q} + x_{i_{q+1}}\mathbf{a}^{i_{q+1}} + \cdots + x_{i_k}\mathbf{a}^{i_k} + x'_{i_{k+1}}\mathbf{a}^{i_{k+1}} + \cdots + x'_{i_p}\mathbf{a}^{i_p}.$$

To say that the internal regions to those two p-hedra have no common p-dimensional domain means that one cannot simultaneously have:

$$x_{i_1} > 0,\ \ldots,\ x_{i_q} > 0, \quad X_{i_{q+1}} > 0,\ \ldots,\ X_{i_k} > 0,$$
$$X_{i_1} > 0,\ \ldots, X_{i_q} > 0, \quad x_{i_{q+1}} > 0,\ \ldots,\ x_{i_k} > 0;$$

in other words, at least one of the k inequalities (V.5) must be true.

2. Two p-hedra that have only one pair of differing basis vectors will have no common p-dimensional domain only if the quotient of the two diagonal determinants that are constructed from the elements of the matrix $\| -r^{hk} \|$ is negative.

If the two non-common vectors are $\mathbf{a}^k$ and $\mathbf{A}^k$ then $\boldsymbol{\alpha}$ will have the following expressions in the bases that relate to the two p-hedra:

$$\boldsymbol{\alpha} = X_1\,\mathbf{A}^1 + \ldots + X_{k-1}\,\mathbf{A}^{k-1} + x_k\,\mathbf{a}^k + x_{k+1}\,\mathbf{a}^{k+1} + \ldots + x_p\,\mathbf{a}^p,$$
$$\boldsymbol{\alpha} = X'_1\mathbf{A}^1 + \ldots + X'_{k-1}\mathbf{A}^{k-1} + X'_k\,\mathbf{A}^k + x'_{k+1}\mathbf{a}^{k+1} + \ldots + x'_p\,\mathbf{a}^p.$$

Now, if $\xi_1, \ldots, \xi_k, \xi_{k+1}, \ldots, \xi_p$ are the components of $\mathbf{a}^k$ in the basis $\mathbf{A}^1, \ldots, \mathbf{A}^k, \mathbf{a}^{k+1}, \ldots, \mathbf{a}^p$ then:

$$\mathbf{a}^k = \xi_1\,\mathbf{A}^1 + \ldots + \xi_k\,\mathbf{A}^k + \xi_{k+1}\,\mathbf{a}^{k+1} + \ldots + \xi_p\,\mathbf{a}^p.$$

Upon replacing $\mathbf{a}^k$ with its value in the first expression for $\boldsymbol{\alpha}$ and identifying the coefficients of $\mathbf{A}^k$, one will get:

$$X_k = \xi_k\, x_k\,.$$

The equivalent condition to $x_k\,X_k < 0$ is then $\xi_k < 0$. ξ_k is expressed by means of the minor determinants that are extracted from the matrix $\| -r^{hk} \|$. In order to do that, it will suffice to express the vectors $\mathbf{a}^1, \ldots, \mathbf{a}^k$ as functions of the vectors $\mathbf{A}^1, \ldots, \mathbf{A}^k, \mathbf{a}^{k+1}, \ldots, \mathbf{a}^p$; i.e., to solve the system:

$$- r^{11}\,\mathbf{a}^1 - \ldots - r^{1k}\,\mathbf{a}^k = \mathbf{A}^1 + r^{1k+1}\,\mathbf{a}^{k+1} + \ldots + r^{1p}\,\mathbf{a}^p,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$- r^{k1}\,\mathbf{a}^1 - \ldots - r^{kk}\,\mathbf{a}^k = \mathbf{A}^1 + r^{kk+1}\,\mathbf{a}^{k+1} + \ldots + r^{kp}\,\mathbf{a}^p,$$

so $\xi_k = \dfrac{\Delta_{12\cdots k-1}}{\Delta_{12\cdots k}}$, with:

$$\Delta_{12\ldots k-1} = \begin{vmatrix} -r^{11} & \cdots & -r^{1k-1} \\ \vdots & & \vdots \\ -r^{k-11} & \cdots & -r^{k-1k-1} \end{vmatrix}, \qquad \Delta_{12\ldots k} = \begin{vmatrix} -r^{11} & \cdots & -r^{1k} \\ \vdots & & \vdots \\ -r^{k1} & \cdots & -r^{kk} \end{vmatrix}.$$

$\Delta_{12\ldots k-1}$ and $\Delta_{12\ldots k}$ are two diagonal minor determinants that are extracted from the matrix $\| - r^{hk} \|$. The necessary and sufficient condition for the internal regions to the two *p*-hedra that are constructed from the two series of vectors:

$$\mathbf{A}^1, \ldots, \mathbf{A}^{k-1}, \mathbf{a}^k, \mathbf{a}^{k+1}, \ldots, \mathbf{a}^p,$$
$$\mathbf{A}^1, \ldots, \mathbf{A}^{k-1}, \mathbf{A}^k, \mathbf{a}^{k+1}, \ldots, \mathbf{a}^p$$

should have no common *p*-dimensional domain is that $\dfrac{\Delta_{12\cdots k-1}}{\Delta_{12\cdots k}} < 0$.

3. All of the possible pairs of two *p*-hedra that have only one pair of differing basis vectors will have no common *p*-dimensional domain if all of the diagonal minors of the matrix $\| r^{kk} \|$ are positive.

Let us apply Lemma to the following various pairs of *p*-hedra:

1. The regions internal to the two *p*-hedra that are defined by the two series of vectors:

$$\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^k, \ldots, \mathbf{a}^p,$$
$$\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{A}^k, \ldots, \mathbf{a}^p$$

will have no common *p*-dimensional domain if $- r^{kk} < 0$, namely, $r^{kk} > 0$.

Consequence: The diagonal elements of the matrix $\| r^{kk} \|$ must all be positive.

2. The regions internal to the two *p*-hedra that are defined by the two systems of vectors:

$$\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^g,\ \mathbf{A}^k,\ \mathbf{a}^j, \mathbf{a}^k, \ldots, \mathbf{a}^p,$$
$$\mathbf{a}^1, \mathbf{a}^2, \ldots, \mathbf{a}^g, \mathbf{A}^k, \mathbf{A}^j, \mathbf{a}^k, \ldots, \mathbf{a}^p$$

will have no common *p*-dimensional domain if $\dfrac{-r^{kk}}{\begin{vmatrix} -r^{kk} & -r^{kj} \\ -r^{jk} & -r^{jj} \end{vmatrix}} < 0$.

Upon taking the preceding condition into account, one must have:

$$\begin{vmatrix} r^{kk} & r^{kj} \\ r^{jk} & r^{jj} \end{vmatrix} > 0.$$

Consequence: The second-order diagonal minors of the matrix $\| r^{kk} \|$ must be positive.

3. One can follow the argument by recurrence. The internal regions to the two p-hedra that are defined by the two systems of vectors:

$$\begin{aligned} &\mathbf{a}^1, \mathbf{a}^2, \dots, \mathbf{a}^k, \mathbf{A}^\lambda, \dots, \mathbf{A}^\mu, \mathbf{a}^\nu, \mathbf{a}^{\nu+1}, \dots, \mathbf{a}^p, \\ &\mathbf{a}^1, \mathbf{a}^2, \dots, \mathbf{a}^k, \mathbf{A}^\lambda, \dots, \mathbf{A}^\mu, \mathbf{a}^\nu, \mathbf{a}^{\nu+1}, \dots, \mathbf{a}^p \end{aligned}$$

will have no common p-dimensional domain if:

$$\frac{\begin{vmatrix} -r^{\lambda\lambda} & \cdots & -r^{\lambda\mu} \\ -r^{\mu\lambda} & \cdots & -r^{\mu\mu} \end{vmatrix}}{\begin{vmatrix} -r^{\lambda\lambda} & \cdots & -r^{\lambda\nu} \\ -r^{\nu\lambda} & \cdots & -r^{\nu\nu} \end{vmatrix}} < 0 \qquad \text{or} \qquad \frac{(-1)^{k+1}}{(-1)^k} \frac{\begin{vmatrix} r^{\lambda\lambda} & \cdots & r^{\lambda\mu} \\ r^{\mu\lambda} & \cdots & r^{\mu\mu} \end{vmatrix}}{\begin{vmatrix} r^{\lambda\lambda} & \cdots & r^{\lambda\nu} \\ r^{\nu\lambda} & \cdots & r^{\nu\nu} \end{vmatrix}} < 0 .$$

By hypothesis, the determinant in the numerator is positive, so since the determinant in the denominator has a different parity, one must have:

$$\begin{vmatrix} r^{\lambda\lambda} & \cdots & r^{\lambda\nu} \\ r^{\nu\lambda} & \cdots & r^{\nu\nu} \end{vmatrix} > 0 .$$

Proof of Theorem II:

1. First, imagine the case of two p-hedra that are defined by the following two sequences of vectors:

$$\begin{aligned} &\mathbf{a}^1, \ \mathbf{a}^2, \dots, \mathbf{a}^k, \ \mathbf{a}^{k+1}, \dots, \mathbf{a}^p, \\ &\mathbf{A}^1, \mathbf{A}^2, \dots, \mathbf{A}^k, \mathbf{a}^{k+1}, \dots, \mathbf{a}^p. \end{aligned}$$

The same vector α will have the following expression in the two bases that relate to the two p-hedra:

$$\begin{aligned} \alpha &= x_1 \mathbf{a}^1 + \dots + x_k \mathbf{a}^k + x_{k+1} \mathbf{a}^{k+1} + \dots + x_p \mathbf{a}^p, \\ \alpha &= X_1 \mathbf{A}^1 + \dots + X_k \mathbf{A}^k + x'_{k+1} \mathbf{a}^{k+1} + \dots + x'_p \mathbf{a}^p . \end{aligned}$$

We shall show that at least one of the k inequalities (V.6) is satisfied when all of the diagonal minors of the matrix $\| r^{kk} \|$ are positive:

(V.6) $$x_1 X_1 < 0, \qquad \dots, \qquad x_k X_k < 0.$$

The equations:

$$\mathbf{A}^1 = - r^{11}\,\mathbf{a}^1 - \ldots - r^{1k}\,\mathbf{a}^k - r^{1k+1}\,\mathbf{a}^{k+1} - \ldots - r^{1p}\,\mathbf{a}^p,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\mathbf{A}^k = - r^{k1}\,\mathbf{a}^1 - \ldots - r^{kk}\,\mathbf{a}^k - r^{kk+1}\,\mathbf{a}^{k+1} - \ldots - r^{kp}\,\mathbf{a}^p$$

shows that one can pass from the $X_1, \ldots, X_k$ to the $x_1, \ldots, x_k$ by the substitution:

$$\text{(V.7)} \qquad \left\{ \begin{aligned} -x_1 &= r^{11} X_1 + \cdots + r^{k1} X_k, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ -x_k &= r^{1k} X_1 + \cdots + r^{kk} X_k, \end{aligned} \right.$$

and since $\Delta_k = \det \begin{vmatrix} r^{11} & \cdots & r^{1k} \\ r^{k1} & \cdots & r^{kk} \end{vmatrix} \neq 0$ (> 0), it will admit an inverse (V.8):

$$\text{(V.8)} \qquad \left\{ \begin{aligned} -X_1 &= \frac{R_1^1}{\Delta_k} x_1 + \cdots + \frac{R_1^k}{\Delta_k} x_k, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ -X_k &= \frac{R_k^1}{\Delta_k} x_1 + \cdots + \frac{R_k^k}{\Delta_k} x_k. \end{aligned} \right.$$

If one of the inequalities $x_1 < 0, \ldots, x_k < 0$ was always satisfied when α is in the domain D ($X_1 > 0, \ldots, X_k > 0$), while one of the inequalities (V.6) was always true, then the proposition would not be proved.

Therefore, suppose that none of the inequalities $x_1 < 0, \ldots, x_k < 0$ is always satisfied when α is in D. The variables $x_1, \ldots, x_k$ cannot be always positive when α is in D, because, in particular, for $X_1 = 0, \ldots, X_j \neq 0, \ldots, X_k = 0$, the equations (V.7) will give $x_j = -\dfrac{r^{jj}}{\Delta_k} X_j$; since r^{jj} and Δ_k (diagonal minor) are positive, x_j will be negative. The variables $x_1, \ldots, x_k$ will then be annulled when α traverses D. In particular, for $x_1 = 0, x_2 = 0, \ldots, x_k \neq 0$, equations (V.8) will give the variables $X_1, \ldots, X_k$ as:

$$-X_1 = \frac{R_1^j}{\Delta_k} x_j, \ \ldots, \ -X_j = \frac{R_j^j}{\Delta_k} x_j, \ \ldots, \ -X_k = \frac{R_k^j}{\Delta_k} x_j.$$

From Lemma 3, Δ_k and R_j^j diagonal minors are positive. x_j is negative, because X_j is positive in D, so it will result that all of the coefficients $R_1^j, \ldots, R_k^j$ are positive. If the argument is repeated for all values of j from 1 to k, then all of the coefficients of the equations (V.8) will be positive.

When α is in D, since the first of equations (V.8) are all negative, the same thing must be true for the right-hand sides, so at least one of the variables $x_1, \ldots, x_k$ will be negative. Thus, at least one of the inequalities (V.6) is satisfied.

2. Now imagine the general case. Let two arbitrary p-hedra be constructed from k different vectors. The same vector $\boldsymbol{\alpha}$ in E_p will have the following expressions in the two bases relative to those two p-hedra:

$$\boldsymbol{\alpha} = x_{i_1}\mathbf{a}^{i_1} + \cdots + x_{i_q}\mathbf{a}^{i_q} + X_{i_{q+1}}\mathbf{A}^{i_{q+1}} + \cdots + X_{i_k}\mathbf{A}^{i_k} + x_{i_{k+1}}\mathbf{a}^{i_{k+1}} + \cdots x_{i_p}\mathbf{a}^{i_p},$$
$$\boldsymbol{\alpha} = X_{i_1}\mathbf{A}^{i_1} + \cdots + X_{i_q}\mathbf{A}^{i_q} + x_{i_{q+1}}\mathbf{a}^{i_{q+1}} + \cdots + x_{i_k}\mathbf{a}^{i_k} + x'_{i_{k+1}}\mathbf{a}^{i_{k+1}} + \cdots x'_{i_p}\mathbf{a}^{i_p}.$$

In order to find the substitution that permits one to pass from the:

$$x_{i_1}, \ldots, x_{i_q}, X_{i_{q+1}}, \ldots, X_{i_k} \qquad \text{to the} \qquad X_{i_1}, \ldots, X_{i_q}, x_{i_{q+1}}, \ldots, x_{i_k},$$

one evaluates the vectors of the first basis as functions of those of the second one. The defining relations of the vectors:

$$\mathbf{A}^1, \ldots, \mathbf{A}^k, \ldots, \mathbf{A}^p$$

imply the following system in the unknowns:

$$-\mathbf{a}^{i_1}, \ldots, -\mathbf{a}^{i_q}, -\mathbf{A}^{i_{q+1}}, \ldots, -\mathbf{A}^{i_k},$$

namely:

$$\begin{cases} -r^{i_1 i_1}\mathbf{a}^{i_1} - \cdots - r^{i_1 i_q}\mathbf{a}^{i_q} = r^{i_1 i_{q+1}}\mathbf{a}^{i_{q+1}} + \cdots + r^{i_1 i_p}\mathbf{a}^{i_p} + \mathbf{A}^{i_1}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ -r^{i_q i_1}\mathbf{a}^{i_1} - \cdots - r^{i_q i_q}\mathbf{a}^{i_q} = r^{i_q i_{q+1}}\mathbf{a}^{i_{q+1}} + \cdots + r^{i_q i_p}\mathbf{a}^{i_p} + \mathbf{A}^{i_q}, \\ -r^{i_{q+1} i_1}\mathbf{a}^{i_1} - \cdots - r^{i_{q+1} i_q}\mathbf{a}^{i_q} - \mathbf{A}^{i_{q+1}} = r^{i_{q+1} i_{q+1}}\mathbf{a}^{i_{q+1}} + \cdots + r^{i_{q+1} i_p}\mathbf{a}^{i_p}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ -r^{i_k i_1}\mathbf{a}^{i_1} - \cdots - r^{i_k i_q}\mathbf{a}^{i_q} - \mathbf{A}^{i_k} = r^{i_k i_{q+1}}\mathbf{a}^{i_{q+1}} + \cdots + r^{i_k i_p}\mathbf{a}^{i_p}. \end{cases}$$

We remark that the determinant of that system $\Delta_{i_1 \cdots i_q}$ is a diagonal minor of the matrix $\| r^{kk} \|$. The calculation of $-\mathbf{a}^{i_1}, \ldots, -\mathbf{a}^{i_{q+1}}, -\mathbf{A}^{i_{q+1}}, \ldots, -\mathbf{A}^{i_k}$ shows that the coefficients $R^{i_1}_{i_1}, \ldots, R^{i_q}_{i_q}, R^{i_{q+1}}_{i_{q+1}}, \ldots, R^{i_k}_{i_k}$ are diagonal minors of $\| r^{kk} \|$:

$$-\mathbf{a}^{i_1} = \frac{R^{i_1}_{i_1}}{\Delta_{i_1,-i_q}}\mathbf{A}^{i_1} + \cdots + \frac{R^{i_1}_{i_q}}{\Delta_{i_1,-i_q}}\mathbf{A}^{i_q} + \frac{R^{i_1}_{i_{q+1}}}{\Delta_{i_1 i_q}}\mathbf{a}^{i_{q+1}} + \cdots + \frac{R^{i_1}_{i_k}}{\Delta_{i_1 i_q}}\mathbf{a}^{i_k} + \cdots + \frac{R^{i_1}_{i_p}}{\Delta_{i_1,-i_q}}\mathbf{a}^{i_p},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$

$$-\mathbf{a}^{i_q} = \frac{R^{i_q}_{i_1}}{\Delta_{i_1,-i_q}}\mathbf{A}^{i_1} + \cdots + \frac{R^{i_q}_{i_q}}{\Delta_{i_1,-i_q}}\mathbf{A}^{i_q} + \frac{R^{i_{q+1}}_{i_{q+1}}}{\Delta_{i_1 i_q}}\mathbf{a}^{i_{q+1}} + \cdots + \frac{R^{i_{q+1}}_{i_k}}{\Delta_{i_1 i_q}}\mathbf{a}^{i_k} + \cdots + \frac{R^{i_q}_{i_p}}{\Delta_{i_1,-i_q}}\mathbf{a}^{i_p},$$

$$-\mathbf{A}^{i_{q+1}} = \frac{R^{i_{q+1}}_{i_1}}{\Delta_{i_1,-i_q}}\mathbf{A}^{i_1} + \cdots + \frac{R^{i_{q+1}}_{i_q}}{\Delta_{i_1,-i_q}}\mathbf{A}^{i_q} + \frac{R^{i_{q+1}}_{i_{q+1}}}{\Delta_{i_1 i_q}}\mathbf{a}^{i_{q+1}} + \cdots + \frac{R^{i_{q+1}}_{i_k}}{\Delta_{i_1 i_q}}\mathbf{a}^{i_k} + \cdots + \frac{R^{i_{q+1}}_{i_p}}{\Delta_{i_1,-i_q}}\mathbf{a}^{i_p},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$-\mathbf{A}^{i_k} = \frac{R_{i_1}^{i_k}}{\Delta_{i_1,-i_q}}\mathbf{A}^{i_1} + \cdots + \frac{R_{i_q}^{i_k}}{\Delta_{i_1,-i_q}}\mathbf{A}^{i_q} + \frac{R_{i_{q+1}}^{i_k}}{\Delta_{i_1 i_q}}\mathbf{a}^{i_{q+1}} + \cdots + \frac{R_{i_k}^{i_k}}{\Delta_{i_1 i_q}}\mathbf{a}^{i_k} + \cdots + \frac{R_{i_p}^{i_k}}{\Delta_{i_1 i_q}}\mathbf{a}^{i_p}.$$

One deduces the following formulas from these relations by the substitution (V.9):

$$\text{(V.9)}\qquad \left\{\begin{aligned}
-X_{i_1} &= \frac{R_{i_1}^{i_1}}{\Delta_{i_1 i_q}}x_{i_1} + \cdots + \frac{R_{i_q}^{i_q}}{\Delta_{i_1 i_q}}x_{i_q} + \frac{R_{i_1}^{i_{q+1}}}{\Delta_{i_1 i_q}}X_{i_{q+1}} + \cdots + \frac{R_{i_1}^{i_k}}{\Delta_{i_1 i_q}}X_{i_k},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
-X_{i_q} &= \frac{R_{i_q}^{i_1}}{\Delta_{i_1 i_q}}x_{i_1} + \cdots + \frac{R_{i_q}^{i_q}}{\Delta_{i_1 i_q}}x_{i_q} + \frac{R_{i_q}^{i_{q+1}}}{\Delta_{i_1 i_q}}X_{i_{q+1}} + \cdots + \frac{R_{i_q}^{i_k}}{\Delta_{i_1 i_q}}X_{i_k},\\
-x_{i_{q+1}} &= \frac{R_{i_1}^{i_1}}{\Delta_{i_1 i_q}}x_{i_1} + \cdots + \frac{R_{i_{q+1}}^{i_q}}{\Delta_{i_1 i_q}}x_{i_q} + \frac{R_{i_{q+1}}^{i_{q+1}}}{\Delta_{i_1 i_q}}X_{i_{q+1}} + \cdots + \frac{R_{i_{q+1}}^{i_k}}{\Delta_{i_1 i_q}}X_{i_k},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
-x_{i_k} &= \frac{R_{i_k}^{i_1}}{\Delta_{i_1 i_q}}x_{i_1} + \cdots + \frac{R_{i_k}^{i_q}}{\Delta_{i_1 i_q}}x_{i_q} + \frac{R_{i_k}^{i_{q+1}}}{\Delta_{i_1 i_q}}X_{i_{q+1}} + \cdots + \frac{R_{i_k}^{i_k}}{\Delta_{i_1 i_q}}X_{i_k}.
\end{aligned}\right.$$

Conversely, evaluate the vectors of the second basis as functions of those of the first one. One then deduces, by a calculation that is analogous to the one in formulas (V.10), that:

$$\text{(V.10)}\qquad \left\{\begin{aligned}
-x_{i_1} &= \frac{\rho_{i_1}^{i_1}}{\Delta}X_{i_1} + \cdots + \frac{\rho_{i_1}^{i_q}}{\Delta}X_{i_q} + \frac{\rho_{i_1}^{i_{q+1}}}{\Delta}x_{i_{q+1}} + \cdots + \frac{\rho_{i_1}^{i_k}}{\Delta}x_{i_k},\\
-x_{i_q} &= \frac{\rho_{i_q}^{i_1}}{\Delta}X_{i_1} + \cdots + \frac{\rho_{i_q}^{i_q}}{\Delta}X_{i_q} + \frac{\rho_{i_1}^{i_{q+1}}}{\Delta}x_{i_{q+1}} + \cdots + \frac{\rho_{i_k}^{i_k}}{\Delta}x_{i_k},\\
-X_{i_{q+1}} &= \frac{\rho_{i_{q+1}}^{i_1}}{\Delta}X_{i_1} + \cdots + \frac{\rho_{i_{q+1}}^{i_q}}{\Delta}X_{i_q} + \frac{\rho_{i_{q+1}}^{i_{q+1}}}{\Delta}x_{i_{q+1}} + \cdots + \frac{\rho_{i_{q+1}}^{i_k}}{\Delta}x_{i_k},\\
-X_{i_k} &= \frac{\rho_{i_k}^{i_1}}{\Delta}X_{i_1} + \cdots + \frac{\rho_{i_k}^{i_q}}{\Delta}X_{i_q} + \frac{\rho_{i_k}^{i_{q+1}}}{\Delta}x_{i_{q+1}} + \cdots + \frac{\rho_{i_k}^{i_k}}{\Delta}x_{i_k}.
\end{aligned}\right.$$

Once one has remarked that the diagonal elements in formulas (V.9) and (V.10) are composed of quotients of diagonal minors of the matrix $\| r^{kk} \|$ (so they will all be positive), it will suffice to recall, point-by-point, the argument that was made in the preceding case in order to show that the one of the variables $x_{i_1}, \ldots, x_{i_q}, X_{i_{q+1}}, \ldots X_{i_k}$ will be negative in the domain D ($X_{i_1} > 0, \ldots, X_{i_q} > 0, x_{i_{q+1}} > 0, \ldots, x_{i_k} > 0$), which will imply that the internal regions to the two p-hedra have no common p-dimensional domain.

Theorem III:

The p-dimensional space E_p is completely covered by the 2^p p-hedra that have no p-dimensional domain in common.

Let α be a vector in E_p that is determined by its components $a^1, \dots, a^p$ with respect to the basis that is composed of the vectors $\mathbf{a}^1, \mathbf{a}^2, \dots, \mathbf{a}^p$. Since α is arbitrary in E_p, the components will have arbitrary signs that define a sequence of p – and + signs that will be well-defined when one takes the vectors in the order $\mathbf{a}^1, \dots, \mathbf{a}^p$. The decomposition of α into vectors of the corresponding bases relative to each p-hedron:

$$\mathbf{a}^{i_1}, \dots, \mathbf{a}^{i_q}, \qquad \mathbf{A}^{i_{q+1}}, \dots, \mathbf{A}^{i_k}, \ \mathbf{a}^{i_{k+1}}, \dots, \mathbf{a}^{i_p}$$

will generate a coordinate system whose sequence of signs will be well-defined when one arranges the indices of the vector in the natural order (viz., 1 to p), which is an operation that is always possible, since one takes one and only one vector from each column of the matrix:

$$\begin{matrix} \mathbf{a}^1, & \mathbf{a}^2, & \dots, & \mathbf{a}^{p-1}, & \mathbf{a}^p \\ \mathbf{A}^1, & \mathbf{A}^2, & \dots, & \mathbf{A}^{p-1}, & \mathbf{A}^p. \end{matrix}$$

The sequence of signs that relate to the two p-hedra are different, or else their internal regions would not be distinct. (It will suffice to change the orientation of the vectors for which the signs are all negative in order to obtain a contradiction.) Now, there are 2^p sequences of different signs and 2^p regions: Therefore, any sequence of p signs corresponds to a region and conversely. In particular, there will exist a sequence of p signs that are all positive. It will then result that the space E_p is, in fact, covered by the set of internal regions to the 2^p p-hedra and that α is in a well-defined region.

Number of conditions that the elements of the matrix $\| r^{hk} \|$ satisfy in the case of reduction. – When one writes out that all of the diagonal minors of the matrix $\| r^{hk} \|$ are positive, one will get $(2^p - 1)$ conditions because:

$$C_p^1 + C_p^2 + \cdots + C_p^k + \cdots + C_p^p = 2^p - 1.$$

These $(2^p - 1)$ conditions bear upon the p^2 elements of the matrix. One can mention two simple cases where that number of conditions can be reduced. In order to do that, consider a vector α of E_p whose coordinates with respect to the two bases:

$$\mathbf{a}^1, \mathbf{a}^2, \dots, \mathbf{a}^p \quad \text{and} \quad \mathbf{A}^1, \mathbf{A}^2, \dots, \mathbf{A}^p$$

we denote by x_i and X_i, respectively, so:

$$\alpha = \sum_{i=1}^{p} x_i \, \mathbf{a}^i, \qquad \alpha = \sum_{i=1}^{p} X_i \, \mathbf{A}^i.$$

The fact that the internal regions to the two p-hedra that are defined by those two sequences of vectors are distinct translates into one of the inequalities:

$$x_1 X_1 < 0, \ldots, x_k X_k < 0, \ldots, x_p X_p < 0 .$$

Then consider the bilinear form $\psi(x, X) = \sum_{i=1}^{p} x_i X^i$, which generates the quadratic form $\psi(X) = \frac{1}{2}(r^{hk} + r^{kh}) X^h X^k$, since $\| x \| = \| - r^{hk} \|^* \| X \|$, where $\| - r^{hk} \|^*$ denotes the transpose of $\| - r^{hk} \|$.

One will then have the identity $\sum_{i=1}^{p} x_i X^i \equiv - \psi(X)$.

It will then result that if the form $\psi(X)$ is always positive-definite in E_p then one of the inequalities $x_1 X_1 < 0, \ldots, x_p X_p < 0$ will surely be satisfied. Since the fact that the form is always positive-definite in E_p is independent of the chosen basis, one can take the bases to be two systems arbitrary p-hedra, which shows that the internal regions to those two p-hedra have no common p-dimensional domain. There are then two simple cases:

First case: The quadratic form $\psi = \frac{1}{2}(r^{hk} + r^{kh}) X^h X^k$ is positive-definite.

Second case: The matrix $\| r^{hk} \|$ is symmetric, and the associated form ψ is positive-definite.

In these two cases, the $(2^p - 1)$ conditions reduce to p conditions that one can obtain by expressing the idea that a nested chain of minors of $\| r^{hk} \|$ are all positive.

§ III. – Study of the case $(i(\mathrm{E}) \cdot da^k) = 0$.

In all of the foregoing, we have not envisioned the case in which the vector α is found in the internal domain of one of the 2^p p-hedra, since we supposed in Paragraph I that $(i(\mathrm{E}) \cdot da^k)_0 \neq 0$ for all k from 1 to p. It is easy to see what happens when α is, more generally, in a k-dimensional domain ($k < p$) that is common to 2^k p-hedra.

First, imagine the special case where $\alpha = 0$ in E_p ; i.e., that $(i(\mathrm{E}) \cdot da^k)_0 = 0$ for all k from 1 to p. Formula (V.2) shows that if the $(i(\mathrm{E}) \cdot da^k)_0^{(n+1)}$ are the first non-zero derivatives for all k from 1 to p then, on the one hand, for an order less than n, the λ_k will be zero at M_0 , as well as all of their derivatives up to order $(n - 1)$ inclusive, and on the other hand, the first non-zero derivatives of the a^k will have order $(n + 1)$. It will then result that:

$$(a^k)_0^{(n+1)} - (\lambda_k)^{(k)} (i(e^h) \cdot da^k)_0 = (i(\mathrm{E}) \cdot da^k)_0^{(n+1)}$$

or

$$(a^k)_0^{(n+1)} - r^{hk} (\lambda_k)_0^{(n)} = (i(\mathrm{E}) \cdot da^k)_0^{(n+1)} .$$

Upon interpreting that relation in the vector space E_p , one will see that the discussion of the possibilities will be accomplished by playing the same game with the vectors $\mathbf{a}^1$, …, $\mathbf{a}^p$, $\mathbf{A}^1$, …, $\mathbf{A}^p$ as before. One determines $(a^k)_0^{(n+1)} \cdot (\lambda_h)_0^{(n)}$ by means of the vector $\boldsymbol{\alpha}$ that is defined by its components $(i(\mathrm{E}) \cdot da^k)_0^{(n+1)}$.

General case. – If $\boldsymbol{\alpha}$ belongs to the domain that is common to 2^k p-hedra ($k < p$), which means that $(i(\mathrm{E}) \cdot da^k)_0 = 0$ for k values of h, then it will suffice to decompose $\boldsymbol{\alpha}$ into $\boldsymbol{\alpha}_k$, which belongs to E_k , namely, the space common to the 2^k p-hedra, and $\mathbf{C}_{\alpha k}$, which belongs to the space complementary to E_k with respect to E_p .

$$\boldsymbol{\alpha}_k = X_{i_1}\mathbf{A}^{i_1} + \cdots + X_{i_q}\mathbf{A}^{i_q} + x_{i_{q+1}}\mathbf{a}^{i_{q+1}} + \cdots + x_{i_k}\mathbf{a}^{i_k} ,$$

$$\mathbf{C}_{\alpha k} = x_{i_{k+1}}\mathbf{a}^{i_{k+1}} + \cdots + x_{i_p}\mathbf{a}^{i_p} .$$

Since:

$$X_{i_1} = 0,\ \ldots,\ X_{i_q} = 0,\ x_{i_{q+1}} = 0,\ \ldots,\ x_{i_k} = 0,\ \boldsymbol{\alpha}_k = 0,$$

by hypothesis, one comes back to the preceding case. It will suffice to take the components of $\boldsymbol{\alpha}_k$ to be the first derivatives of $(i\ (\mathrm{E}) \cdot da^k)^{(n+1)}$ that are not all zero for h from i_1 to i_k . If certain derivatives of order $(n + 1)$ are simultaneously zero then one considers a subspace of E_k .

§ IV. – Mechanical interpretation of the preceding results.

The following theorem results from the preceding study:

Theorem IV:

When a mechanical system S is restricted by p constraints of class U, namely, $a^k (p_i , q^i, t) = 0$, $\Omega = \lambda_k\, l_i^k dq^i$, $a^k > 0$, $\lambda_k > 0$, *if the initial conditions satisfy the equations* $a^k(p_i^0, q_0^i, t^0) = 0$ *then there will be* 2^p *possible situations. The necessary and sufficient condition for the uniqueness of the motions is that all of the diagonal minors that one can extract from the product matrix* $A \cdot L$ *must be positive:*

$$\left(A = \left\| \frac{\partial a^k}{\partial p_i} \right\|, \quad L = \| l_i^h \| \right).$$

Application to Appell constraints. – The matrix $\| r^{hk} \|$ is symmetric for Appell constraints, which have zero power and include the holonomic and non-holonomic constraints as special cases (cf., Chapter II, § V) . Indeed, for such constraints, $l_i^h =$

$\partial a^h / \partial \dot{q}^i$. Since that matrix is the product of two matrices A and L, with $A = \left\| \frac{\partial a^h}{\partial p_i} \right\|$, $L = \left\| l_i^h \right\|$, it will result from the fact that:

$$\frac{\partial a^h}{\partial p_i} = \frac{\partial a^h}{\partial \dot{q}^j} \cdot \frac{\partial \dot{q}^j}{\partial p_i} = l_j^k g^{ji}$$

that in the case of Appell constraints $A = L^* \cdot G$, in which L^* denotes the transpose of the matrix L, and G denotes the symmetric matrix $\| g^{ji} \|$.

The relation $\| g^{hk} \| = A \cdot L = L^* \cdot G$. L shows that $\| r^{hk} \|$ is symmetric. The following theorem will then result from Theorem IV and the second case of reduction under the conditions that were pointed out at the end of Paragraph III:

Theorem V:

For a mechanical system S that is restricted by p constraints that are, on the one hand, of class U, and on the other hand, of Appell type, the initial conditions will be sufficient to determine the final motion.

Other types of constraints. – For the other types of constraints, the initial conditions might not be sufficient to determine the final motion if the diagonal minors of the matrix $\| r^{hk} \|$ are not all positive. In particular, that is what happens in systems of solid bodies in contact with friction when the sliding velocities at the various points of contact are not zero. In the indeterminate case, it is necessary to append some conditions that are equivalent to $da^h / dt \neq 0$, $\lambda_h \neq 0$ to the initial conditions in order to specify the final motion.

§ V. – Constraints for which the expression for power can depend upon the existence or breakdown of other constraints.

In the foregoing, we were occupied with constraints of the type $a^h = 0$, $\lambda_k e^k$, where the field e^k is known *a priori*, and consequently those constraint will be independent of the existence or breakdown of the other constraints. The discussion of the possibilities can be made in the case of solid bodies in contact with friction where the sliding velocity is zero at the initial incident, and that will lead one to envision some constraints whose expression for the power can vary according to the existence or breakdown of some other constraints, and consequently, their constraint fields e^k might depend upon other constraints. In order to simplify things, imagine a solid body with a permanent symmetry plane that is in contact with the line of intersection of that symmetry plane with a fixed perpendicular plane. If u and w are the components of the velocity of the point of contact of the solid body with the plane then:

a) Contact that involves rolling without slipping translates into two zero-power constraints $w = 0$, $P_2 = \mu\, w$; $u = 0$, $P_1 = \lambda\, u$.

b) The cessation of contact translates into $dw / dt > 0$, $\mu = 0$, and it will also imply that $\lambda = 0$. The power of the constraint $u = 0$ will then depend upon the realization or breakdown of the constraint *w*.

c) The appearance of sliding ($w_0 = 0$, $u_0 = 0$, $dw / dt = 0$, $du / dt \neq 0$) will modify the power P_2, which will become $P_2 = \mu\,(w + \varepsilon f\, u)$, with $\varepsilon\,(du / dt) < 0$, $\varepsilon = \pm 1$. The power P_2 will then depend upon whether du / dt is positive, negative, or zero; i.e., upon the breakdown or realization of the constraint *u*.

For a set of solid bodies that depend upon $2n$ position and velocity parameters and have *p* mutual contacts, the examination of the possible situations at the instant t_0 in the case where the sliding velocities of the *p* contacts are zero at t_0 can be accomplished in the following way, although it does not admit a geometric form that is as simple as the one that was envisioned above: Having chosen *p* moving tri-rectangular trihedra $P_h x^h y^h z^h$ whose summits are P_h , respectively, which are the h^{th} contact points, and the axis $P_h\, z^h$ is oriented along the common normal, it would be advantageous to begin to write the system of equations that will permit one to determine the rolling without slipping and the cessation of contact.

The condition of rolling without slipping at the h^{th} contact translates into:

$$\begin{array}{lll} u^h = 0, & v_h = 0, & w = 0, \\ P = X^h\, u^h, & P = Y^h\, v^h, & P = Z^h\, w^h . \end{array}$$

Rolling without slipping at the *p* contact points constitutes $3p$ zero-power constraints. From our theory, we must suppose that they are compatible, which will always permit one to choose the parameters to be the $3p$ quantities u^h, v^h, w^h. The conclusions of § IV tell us that there will exist a quadratic form with $3p$ components for the reactions and that $3p$ of the equations can always be put into the form:

$$\text{(I)} \qquad \left\{ \begin{aligned} \frac{du^h}{dt} - \frac{\partial\varphi}{\partial X^h} &= \alpha^h, \\ \frac{dv^h}{dt} - \frac{\partial\varphi}{\partial Y^h} &= \beta^h, \\ \frac{dw^h}{dt} - \frac{\partial\varphi}{\partial Z^h} &= \gamma^h, \end{aligned} \right. \qquad (h = 1, \ldots, p),$$

whose right-hand sides are independent of the possibilities.

These equations immediately indicate the contacts that break down $dw^h / dt > 0$ and the acceptable rolling without slipping by writing that $\frac{du^h}{dt} = \frac{dv^h}{dt} = \frac{dw^h}{dt} = 0$, $z^h > 0$, $(X^h)^2 + (Y^h)^2 < (f_h\, Z^h)^2$, where f_h is the coefficient of friction at the h^{th} contact.

The appearance of slipping at the p contacts. – In order to envision the hypothesis that slipping appears everywhere, it will suffice to replace $\frac{du^h}{dt}$ with $\frac{d\rho^h}{dt}\cos\sigma_h$, $\frac{dv^h}{dt}$ with $\frac{d\rho^h}{dt}\sin\sigma_h$, $\frac{dw^h}{dt}$ with 0, X^h with $-f_k Z^k \cos\sigma_h$, and Y^h with $-f_k Z^k \sin\sigma_h$, where ρ^h and σ_h are the polar coordinates of the sliding velocity at the h^{th} contact with respect to the axes $P_k x^k y^h$. One will then obtain the system:

$$
\text{(II)} \quad \left\{
\begin{array}{ll}
1. & \dfrac{d\rho^h}{dt} - \dfrac{\partial\varphi}{\partial X^h}\cos\sigma_h - \dfrac{\partial\varphi}{\partial Y^h}\sin\sigma_h = \alpha^h\cos\sigma_h + \beta^h\sin\sigma_h, \\
2. & -\dfrac{\partial\varphi}{\partial X^h}\cos\sigma_h - \dfrac{\partial\varphi}{\partial Y^h}\sin\sigma_h = \alpha^h\cos\sigma_h + \beta^h\sin\sigma_h, \\
3. & -\dfrac{\partial\varphi}{\partial Z^h} = \gamma^h.
\end{array}
\right.
$$

Equations (II.3) determine the Z^h as functions of the σ^h. Equations (II.2) determine the σ_h by means of a system of trigonometric equations.

Equations (II.1) determine the $d\rho^h / dt$. The values that were found will be acceptable when $Z^k > 0$, $d\rho^h / dt > 0$.

Remark. – Once the angles σ_h have been determined by calculation, one can discuss the signs of $Z^k > 0$, $d\rho^h / dt > 0$ by using a vector representation that is analogous to the one that was used in § II of this chapter, since the directions of the constraint fields will be known under those conditions.

Case of slipping appearing at k contacts and rolling without slipping at the (p – k) **other contacts.** – One can suppose that the slipping appears at the first k contacts. One will deduce the following system from (I) by a method that is similar to the preceding one:

$$
\text{(III)} \quad \left\{
\begin{array}{lll}
1. & \dfrac{d\rho^h}{dt} - \dfrac{\partial\varphi}{\partial X^h}\cos\sigma_h - \dfrac{\partial\varphi}{\partial Y^h}\sin\sigma_h = \alpha^h\cos\sigma_h + \beta^h\sin\sigma_h, & \\
2. & -\dfrac{\partial\varphi}{\partial X^h}\sin\sigma_h - \dfrac{\partial\varphi}{\partial Y^h}\cos\sigma_h = \alpha^h\sin\sigma_h - \beta^h\cos\sigma_h, & h = 1 \text{ to } k, \\
3. & -\dfrac{\partial\varphi}{\partial Z^h} = \gamma^h, & \\
4. & -\dfrac{\partial\varphi}{\partial X^h} = \alpha^h, \quad -\dfrac{\partial\varphi}{\partial Y^h} = \beta^h, \quad -\dfrac{\partial\varphi}{\partial Z^h} = \gamma^h, & h = k \text{ to } 3p - 3k.
\end{array}
\right.
$$

The reactions, which are $3p - 3k - k$ in number, are determined as functions of the σ_h by the last $3p - 3k - k$ equations. The k equations (III.2) determine the σ_h. The $d\rho^h / dt$ are determined by the k equations (III.1). One can then envision the $2p$ conditions for validity: $d\rho^h / dt > 0$, $Z^h > 0$, $(X^h)^2 + (Y^h)^2 < (Z^h)^2$ (h varies from $k + 1$ to p in the last ones).

We shall confine ourselves to these generalities, since our objective is to show that our method will permit us to organize a very delicate discussion.

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CHAPTER SIX

STUDY OF THE DIFFERENTIAL EQUATIONS OF MECHANICS WHEN CONSIDERED TO BE CHARACTERISTICS OF A FORM OF DEGREE 2 THAT IS DEFINED ON A DIFFERENTIABLE MANIFOLD V_{2n+1}.

In this chapter, we propose to show how the viewpoint that consists of envisioning the system Σ of differential equations of motion of a mechanical system as the characteristics of a form Ω of degree two that is defined on an indefinitely-differentiable manifold V_{2n+1} will permit one to known the structure of the sub-module of functions that are solutions to Σ.

The practical applications are easily deduced from that, and in particular, in the integrable case.

In order to carry out that study, it is advantageous to envision the differential, infinitesimal transformation, and anti-derivation operators from a viewpoint that was pointed out by H. Cartan ([22]). The following paragraph is abstracted from his presentation.

§ I. – Definition and properties of operators.

Graded algebras. – If A is an associative algebra over a commutative ring K with unity then a *graded structure* is defined when one is given some homogeneous vector subspaces A^p of degree ($p \geq 0$) such that the vector space A is the direct sum of the A^p and the product of an element of A^p and an element of A^q will be an element of A^{p+q}.

Endomorphism of degree r. – An endomorphism ([23]) of the vector structure on A is said to have *degree r* when it maps A^p into A^{p+r} for each p.

Products and compositions of endomorphisms. – If λ and μ are two endomorphisms of degrees r and r', respectively, then one calls the product of the two endomorphisms, when taken in the order λ, μ, the *endomorphism* $\mu\lambda$ *of degree* $r + r'$. (The operation is performed from right to left.)

That product is not generally commutative. Considering the two endomorphisms $\mu\lambda$ and $\lambda\mu$ will lead to two new endomorphisms:

1. The symmetric composition $\lambda\mu + \mu\lambda$.

2. The antisymmetric composition $\lambda\mu - \mu\lambda = [\lambda, \mu]$, which is called the *bracket* of the endomorphisms λ, μ.

([22]) Cf., H. Cartan, *Colloque de topologique de Bruxelles 1950,* Masson, Paris.
([23]) Cf., N. Bourbaki, *Algébre,* Chap. I, § **4**, pp. 49, Paris, Hermann, 1942.

Special endomorphisms:

1. *Derivation.* – One calls any endomorphism θ of A of even degree that enjoys the property:

$$\theta(a \cdot b) = \theta(a) \cdot b + a \cdot \theta(b)$$

with respect to the multiplication in A a *derivation*. If A possesses a unity element I then $\theta(I) = 0$.

2. *Anti-derivation.* – One calls any endomorphism δ of A of odd degree that possesses the property:

$$\delta(a \cdot b) = \delta(a) \cdot b + (-1)^p\, a \cdot \delta(b)$$

with respect to the multiplication in A for $a \in A^p$, $b \in A^q$ an *anti-derivation.*

In addition, $\delta(I) = 0$.

Composition of derivation and anti-derivation. – One verifies that a consequence of the preceding two definitions is that:

1. The bracket of two derivations $[\theta_1, \theta_2]$ is a derivation.

2. The bracket $[\theta, \delta]$ of a derivation and an anti-derivation is a derivation.

3. The symmetric composition $\delta_1\delta_2 + \delta_2\delta_1$ of two anti-derivations is an anti-derivation.

4. The square of an anti-derivation is a derivation.

Differential and exterior algebra:

Definition: One calls the anti-derivation of degree + 1 that possesses the property that $d \cdot d = 0$, moreover, the *differential* d.

There exists a graded algebra relative to the anti-derivation d that is called the *exterior algebra* whose unitary module A^1 is composed of the differentials of the functions of the ring K, where A^0 is identified with the ring of operators K, and the functions are imagined to be differential forms of degree 0.

H. Cartan's operators $i\,(x)$ **and** $\theta\,(x)$ **on a differentiable manifold. –** In the applications to indefinitely-differentiable manifolds that we have in mind, the tangent vector fields constitute a module T over the ring K of indefinitely-differentiable numerical functions. T', which is the dual to T, is the module of differential forms of degree 1. $A\,(T)$, which is the exterior algebra over the module T', is the exterior algebra of differential forms of all degrees.

Operator i (x). – Since the algebra A (T') is generated, in the multiplicative sense, by its elements of degree 0 and 1, an anti-derivation will be determined when it is known on the subspaces A^0 and A^1.

Any $x \in T$ defines an anti-derivation of degree – 1 of the algebra A (T') that is called the *interior product by* x, which is zero on A^0, and on A^1, it reduces to the scalar product that defines the duality between T and T'.

Remark. – The operator i (x) has square zero, because $i(x) \cdot i(x)$ a zero derivation on A^0 and A^1, so it will be zero everywhere.

Operator θ (x). – Any $x \in T$ defines a derivation of degree 0 that is composed symmetrically of the anti-derivations d and i (x) :

(VI.1) $$\theta(x) = i(x) \cdot d + d \cdot i(x) .$$

Operator ε (x). – Any $x \in T$ defines an endomorphism of degree 0 that is the bracket of the anti-derivations d and i (x):

(VI.2) $$\varepsilon(x) = [d, i(x)] = d \cdot i(x) - i(x) \cdot d .$$

Remarks:

1. If $\Omega \in A(T')$ then $\theta(x)\,\Omega$ will be the infinitesimal transformation of Ω by $\theta(x)$; hence, the name "infinitesimal transformation" that was given to the operator $\theta(x)$.

2. The operators θ and d commute.

3. The operators ε and d anti-commute.

4. n successive applications of $\theta(x)$ will give:

$$\theta^{(n)}(x) = (i(x) \cdot d)^{(n)} + (d \cdot i(x))^{(n)}.$$

In particular, when one applies it to a function f in the ring K, the preceding formula will give:

$$\theta^{(n)}(x)\, f = (i(x)\, df)^{(n)} .$$

Endomorphisms that relate to two fields. – There are some endomorphisms that correspond to two tangent vector fields x, y that are derivations and anti-derivations, and we shall specify the ones that will be useful to us.

1. The product of the endomorphism i (y) i (x) is an endomorphism of degree – 2. As a result of the associativity of the exterior product:

$$i(y)\, i(x) = i(y \wedge x) .$$

The preceding endomorphism maps a form of degree two into A^0. If $\Omega = k_{ij}\, dx^i \wedge dx^j$ (k_{ij} is an antisymmetric tensor) then:

$$i\,(y \wedge x)\,\Omega = k_{ij}\,(x^i\, y^j - y^i\, x^j) = \sum_{i,j=0}^{2n} k_{ij} \begin{vmatrix} x^i & x^j \\ y^i & y^j \end{vmatrix}.$$

That is the left interior product of $y \wedge x$ and Ω.

2. The bracket of a derivation $\theta(x)$ of degree 0 and an anti-derivation $i\,(y)$ of degree – 1 is an anti-derivation of degree – 1, so one will be led to consider the anti-derivation that relates to a field that is composed of x, y and denoted $[x, y]$, namely, the *Lie bracket*. That notation is justified because that anti-derivation will be zero on A^0, and on A^1, it will reduce to $i\,([x, y])\, df$ relative to the element df, so one will have the formula:

$$[\theta(x), i\,(y)] = \theta(x)\, i\,(y) - i\,(y)\, \theta(x) = i\,([x, y])\,. \tag{VI.3}$$

3. Consider the two endomorphisms $\theta(x) \cdot i\,(y)$ and $i\,(y) \cdot \theta(x)$. Their difference will give a new endomorphism:

$$\begin{aligned} \theta(x)\, i\,(y) &= i\,(x)\, di\,(y) + di\,(x) \cdot i\,(y), \\ i\,(x)\, \theta(y) &= i\,(x)\, i\,(y)\, d + i\,(x) \cdot d \cdot i\,(y), \\ \theta(x)\, i\,(y) - i\,(x)\, \theta(y) &= di\,(x \wedge y) - i\,(x \wedge y)\, d\,. \end{aligned}$$

Consider the endomorphism $\varepsilon(x \wedge y)$ that relates to the composed element $x \wedge y$:

$$\theta(x)\, i\,(y) - i\,(x)\, \theta(y) = \varepsilon(x \wedge y) = di\,(x \wedge y) - i\,(x \wedge y)\, d\,. \tag{VI.4}$$

4. When one permutes x and y in (VI.3), one will get:

$$[\theta(y), i\,(x)] = \theta(x)\, i\,(y) - i\,(x)\, \theta(y) = i\,([y, x])\,. \tag{VI.3′}$$

Upon subtracting (VI.3′) from (VI.4) and taking into account the fact that $[y, x] = -[x, y]$, one will get:

$$\theta(x)\, i\,(y) - \theta(y)\, i\,(x) = \varepsilon(x \wedge y) + i\,([x, y])\,. \tag{VI.5}$$

5. The bracket of two derivations $\theta(x)$, $\theta(y)$ of degree 0 is a derivation of degree 0, so the corresponding derivation will be the derivation with respect to the Lie bracket $[x, y]$.

An application of (VI.1) will give:

$$\begin{aligned} \theta(x)\, \theta(y) &= \theta(x)\,[i\,(y)\, d + di\,(y)], \\ \theta(y)\, \theta(x) &= \theta(y)\,[i\,(x)\, d + di\,(x)]\,. \end{aligned}$$

Upon taking into account the fact that the operators θ and d commute:

$$[\theta(x),\ \theta(y)] = \theta(x)\ \theta(y) - \theta(y)\ \theta(x)$$
$$= [\theta(x)\ i\,(y) - \theta(y)\ i\,(x)] \cdot d + d \cdot [\theta(x)\ i\,(y) - \theta(y)\ i\,(x)].$$

Formula (VI.5) permits us to transform the right-hand side into:

$$[\theta(x),\ \theta(y)] = [\varepsilon(x \wedge y) + i\,([x, y])]\ d + d\ [\varepsilon(x \wedge y) + i\,([x, y])]\ .$$

Now, $\varepsilon(x \wedge y)\ d + d\varepsilon(x \wedge y) = 0$, as a consequence of (VI.4) or the anti-commutativity of the operators ε and d. The right-hand side then reduces to $i\,([x, y])\ d + di\,([x, y])$, and from (VI.1), that is nothing but $\theta([x, y])$, so:

$$[\theta(x),\ \theta(y)] = \theta([x, y])\ . \tag{VI.6}$$

The operator θ then operates upon not only T' and $A\,(T')$, but also on T, and the field that is the transform of y by $\theta(x)$ will be denoted by $[x, y] = \theta(x) \cdot y$.

§ II. – Study of the system of differential equations Σ that is characteristic to Ω.

Take the form $\Omega = k_{ij}\ dx^i \wedge dx^j$, where i and j vary from 0 to $2n$, and k_{ij} are the components of an antisymmetric tensor, and are functions in the ring K. We denote the system of characteristics of Ω by Σ .

Definition. – One calls the element E of T such that:

$$i\,(\mathrm{E})\ \Omega = 0$$

the *characteristic field associated with* Ω.

By hypothesis, the form Ω will have rank $2n$ (i.e., $\Omega^n \neq 0$). The field E is then defined by the preceding equation only up to a numerical function as a factor, since its $(2n + 1)$ components are proportional to the $(2n + 1)$ determinants that are the minors of order $2n$ that are extracted from the matrix:

$$\left\| \begin{matrix} 0 & k_{12} & \cdots & k_{1,2n} & k_{10} \\ k_{21} & 0 & \cdots & k_{2,2n} & k_{20} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ k_{2n,1} & \cdots & \cdots & 0 & k_{2n,0} \end{matrix} \right\| .$$

It is advantageous to choose the proportionality factor such that $i\,(\mathrm{E})\ dt = 1$, as we did in Chapter I, § III.

Theorem I:

Any $x \in T$ corresponds to a Pfaff form π that is zero along the integral curves of Σ*: Conversely, any Pfaff form that is zero along the integral curves of* Σ *corresponds to an $x \in T$, modulo* E .

$$i(x)\,\Omega = \pi,$$

$$i(x)\,i(\mathrm{E})\,\Omega = -\,i(\mathrm{E})\,i(x)\,\Omega,$$

since

$$i(\mathrm{E})\,\Omega = 0, \qquad i(\mathrm{E})\,\pi = 0,$$

which shows that π belongs to a sub-module of characteristic forms of Ω*, so π will be zero on the integral curves of* Σ *.*

Conversely, if a Pfaff form is zero on the integral curves of Σ then $i(\mathrm{E})\,\pi = 0$. Since the form π belongs to a sub-module of characteristic forms, there will exist an x modulo E such that $i(x)\,\Omega = \pi$.

First integral of Σ. – From now on, we shall call any closed Pfaff form that is zero on the integral curves of Σ a *first integral* of Σ.

That definition is justified because such a form is not always the differential of a function in the ring K that is defined on V_{2n+1} , but it is only a function that is defined on a neighborhood of V_{2n+1} . In other words, any point of V_{2n+1} belongs to a neighborhood U such that the restriction of the form to U is the differential of a function that is defined on U.

Any closed form $\bar{\pi}$ of degree 1 that is zero on the integral curves of Σ will correspond to an $\bar{x} \in T$ modulo E whose $(2n + 1)$ components are $(2n + 1)$ functions that are solutions of a system of linear first-order partial differential equations:

(IV.7) $$d(i(\bar{x})\,\Omega) = 0.$$

Infinitesimal transformations of Σ :

Definition. – An arbitrary element ω^p of degree p in $A(T')$ is said to *admit* an infinitesimal transformation that is generated by $\mathrm{X} \in T$ if the operator $\theta(\mathrm{X})$ transforms ω^p to zero in the space of forms when it is applied to it:

$$\theta(\mathrm{X})\,\omega^p = 0 .$$

In particular, if Ω has degree two then it will admit the infinitesimal transformation that is defined by $\theta(\mathrm{X})$ then the same thing will be true for the differential system Σ . We remark that the $(2n + 1)$ components of the field X that is the generator of the infinitesimal transformations for Ω are solutions to a first-order system of partial differential equations (VI.8):

(VI.8) $$\theta(\mathrm{X})\,\Omega = i\,(\mathrm{X})\,d\Omega + di\,(\mathrm{X})\,\Omega = 0.$$

A. – STUDY OF THE CASE $d\Omega = 0$.

A comparison of (VI.7) and (VI.8) will show that if $d\Omega = 0$ then the systems of partial differential equations that define the infinitesimal transformations and the fields x that are generators of first integral will be the same. Hence, one has the theorem:

Theorem II:

Any X *that is the generator of an infinitesimal transformation will correspond to a first integral, and conversely.*

Example. – Let $\Omega = dp_i \wedge dq^i - dH \wedge dt$. We look for the condition for Ω to admit the infinitesimal transformation that is defined by the operator $\theta(t)$:

$$\theta(t)\,\Omega = d\left(dH - \frac{\partial H}{\partial t}dt\right) = 0, \qquad d\left(\frac{\partial H}{\partial t}dt\right) = 0,$$

which shows that $\partial H / \partial t$ must be a function of only the variable t, namely, $V'(t)$. That infinitesimal transformation corresponds to a first integral:

$$H - V(t) = h.$$

It is the Painlevé integral $T_2 - T_0 - U - V(t) = h$.

In the particular case of $\partial H / \partial t \equiv 0$, it will be the classical *vis viva* integral.

Consequence of the preceding theorem. – The sub-module of the tangent vector fields to the manifold V_{2n+1} that are generators of infinitesimal transformations for Σ corresponds to the sub-module of first integrals by duality with respect to Ω.

Let X, Y be two fields that are generators of infinitesimal transformations for Ω :

$$\theta(\mathrm{X})\,\Omega = 0 = di\,(\mathrm{X})\,\Omega, \quad i\,(\mathrm{X})\,\Omega = \pi,$$
$$\theta(\mathrm{Y})\,\Omega = 0 = di\,(\mathrm{Y})\,\Omega, \quad i\,(\mathrm{Y})\,\Omega = \sigma.$$

If $[\mathrm{X}, \mathrm{Y}] \neq 0$ then the Lie bracket of the two fields will correspond to an infinitesimal transformation $\theta([\mathrm{X}, \mathrm{Y}])$, and as a result, a new first integral:

$$\theta([\mathrm{X}, \mathrm{Y}])\,\Omega = di\,([\mathrm{X}, \mathrm{Y}])\,\Omega = 0.$$

One obtains its expression by remarking that under the present hypotheses, formula (VI.3) will give:

$$i\,([\mathrm{X}, \mathrm{Y}])\,\Omega = \theta(\mathrm{X})\,i\,(\mathrm{Y})\,\Omega - i\,(\mathrm{Y})\,\theta(\mathrm{X})\,\Omega = \theta(\mathrm{X})\,\sigma = i\,(\mathrm{X})\,d\sigma + di\,(\mathrm{X})\,\sigma$$

$$= di\ (\mathrm{X} \wedge \mathrm{Y})\ \Omega\ .$$

The new first integral is essentially the differential of a function in the ring K whose expression is:

$$\sum_{i,j=0}^{2n} k_{ij} \begin{vmatrix} Y^i & Y^j \\ X^i & X^j \end{vmatrix};$$

i.e., the interior product on the left of i (X ∧ Y) and Ω .

In particular, if $\Omega = dp_i \wedge dq^i - dH \wedge dt$, and if the forms π and σ are the differentials of the functions f and g in the ring K then if X is a solution $i\ (\mathrm{X})\ \Omega = df$ that is defined modulo $\mathrm{E} = \left(-\frac{\partial H}{\partial q^i}, \frac{\partial H}{\partial p_i}, 1\right)$, then one will have $\mathrm{X} = \left(\frac{\partial f}{\partial q^i}, -\frac{\partial f}{\partial p_i}, 0\right)$, so:

$$i\ (\mathrm{X} \wedge \mathrm{Y})\ \Omega = \sum_{i=1}^{n} \begin{vmatrix} \dfrac{\partial g}{\partial p_i} & \dfrac{\partial g}{\partial q^i} \\ \dfrac{\partial f}{\partial p_i} & \dfrac{\partial f}{\partial q^i} \end{vmatrix}.$$

One recognizes the Poisson bracket ($f\ g$) in the right-hand side.

Corollary:

If the fields X *and* Y, *whose bracket* [X, Y] *is non-zero, correspond to the first integrals df and dg, respectively, then the field* [X, Y] *will correspond to a new first integral that is the differential of the Poisson bracket. The bracket* [[X, Y], Z] *will correspond to the bracket* ((f, g), h). *The bracket identity corresponds to the Poisson identity.*

Fields in involution. – If [X, Y] = 0 then we will say that the two fields X and Y are *in involution.* The two corresponding first integrals are said to be *in involution.*

Theorem III:

Any field X *that is the generator of an infinitesimal transformation for* Ω *is in involution with the characteristic field* E = 0.

We remark that θ (E) Ω = 0, as a consequence of the fact that $d\Omega = 0$ and the definition of E, namely, i (E) Ω = 0 . It will then result that any element that is coupled to Ω will be mapped to zero in the corresponding set by θ (E) . In particular, the field X that is the generator of an infinitesimal transformation will be mapped to zero in the tangent space:

$$\theta\,(\mathrm{E})\ \mathrm{X} = [\mathrm{E}, \mathrm{X}] = 0.$$

Case of *n* known fields in involution. – Suppose that one knows n fields $X_1, X_2, \ldots, X_n$ that are pair-wise in involution and are generators of infinitesimal transformations of Ω. They will correspond to n closed Pfaff forms $\pi_1, \ldots, \pi_n$ that are first integrals of Σ :

$$i(X_i)\,\Omega = \pi_i\,, \qquad\qquad i(X_j)\,\pi_i = 0\,.$$

Since the n fields X_i are supposed to be linearly independent, the integrals π_i will be independent. The form $\pi = \pi_1 \wedge \ldots \wedge \pi_n$ of degree n will be non-zero.

1. Ω belongs to the sub-module of n forms $\pi_1, \ldots, \pi_n$.

If q is an integer then:

$$i(X_i)\,\Omega^q = q\,\pi_i \cdot \Omega^{q-1},$$

$$\begin{aligned} i(X_j \wedge X_i)\,\Omega^q &= q\,i(X_j)\,\pi_i \wedge \Omega^{q-1} + (-1)\,q\,(q-1)\,\pi_i \wedge \pi_i \wedge \Omega^{q-2} \\ &= (-1)\,q\,(q-1)\,\pi_i \wedge \pi_i \wedge \Omega^{q-2}\,, \end{aligned}$$

so for $q > n$:

$$\begin{aligned} i(X_1 \ldots X_n)\,\Omega^q &= (-1)^{n-1}\,q \ldots (q-n-1)\,\pi_1 \wedge \ldots \wedge \pi_n \wedge \Omega^{q-n} \\ &= (-1)^{n-1}\,q \ldots (q-n-1)\,\pi \wedge \Omega^{q-n}. \end{aligned}$$

Upon taking $q = n + 1$, since Ω has rank $2n$, $\Omega^{n+1} = 0$, so $\pi \wedge \Omega = 0$, which is an equation that will show that Ω belongs to the sub-module of n forms π_i . One can then write:

$$\Omega = \sum_{j=1}^{n} \pi_j \wedge \omega^j\,.$$

2. The n forms ω^j are closed modulo the n integrals π_j .

$d\Omega = \sum_{j=1}^{n} \pi_j \wedge d\omega^j$, because the $d\pi_j$ are zero. Since $d\,\Omega$ is zero:

$$\sum_{j=1}^{n} \pi_j \wedge d\omega^j = 0.$$

Upon multiplying the two sides by $\pi_1 \wedge \ldots \wedge \hat{\pi}_j \wedge \ldots \wedge \pi_n$, where the ^ sign that is placed above π_j means that the term in question does not appear in the product, one will get:

$$\pi \wedge d\omega^j = 0,$$

which is an equation that shows that $d\omega^j$ belongs to the sub-module of n forms $\pi_1, \ldots, \pi_n$; in other words, the n forms ω^j are closed on the submanifolds that are defined by the n first integrals $\pi_1 = 0, \ldots, \pi_n = 0$.

Theorem IV:

If one knows n fields that are pair-wise in involution and are generators of infinitesimal transformations of Ω *then the system* Σ *will be integrable by quadratures.*

That theorem is only a translation of the Liouville-Cartan theorem into the language of manifolds.

Remark. – The forms π_1, π_n , ω^1, ω^n are exact differentials only locally or on a neighborhood of V_{2n+1} .

For such a neighborhood, one can calculate the integrals and put them into the form $r_j = r_j(p_i, q^i, t)$, $s^j = s^j(p_i , q^i, t)$. With that reservation, Ω can be expressed by means of just the differentials of the first integrals:

$$\Omega = dr_j \wedge ds^j .$$

Ω can be put into that form in an infinitude of ways by a transformation of the infinite symplectic group that acts upon the set $r_1 , \ldots, r_n , s^1, \ldots, s^n$. The n functions $r_1 , \ldots, r_n$, which are pair-wise in involution, constitute the generic elements of a sub-ring.

Application. Search for the integrable cases of the characteristic system Σ of a form Ω of type $\Omega = dp_i \wedge dq^i - dH \wedge dt$. – If one knows how to integrate Σ then one will know a representation for the first integrals r_j = const., s^j = const. Ω, whose exterior derivative is zero, can then be put into the form $\Omega = dr_j \wedge ds^j$.

There are 2^n ways of considering Ω to be the exterior differential of a Pfaff form:

$$\omega = \sum_{i=1}^{h} p_i \, dq^i - \sum_{i=1}^{h} q^{n-h+i} dp_{n-k+i} - H\, dt . \tag{VI.9}$$

Consider the form $\Omega^* = d(\omega + s^j dr_j)$ that is defined on a manifold V_{4n+1} ([24]). The form Ω^* is zero, by construction, on the submanifolds $r^j(p_i , q^i, t) = r^j$, $s^j(p_i , q^i, t) = s^j$, so $\omega + s^j dr_j$ will be a closed form, and when one writes that it is equal to the differential of a function $f(q^i, p_{n+h-i} , r_j , t)$, in which the indices i and j vary from 1 to h and 1 to n, respectively, one will get:

$$p_i = \frac{\partial f}{\partial q^i}, \qquad q^{n-h+i} = -\frac{\partial f}{\partial p_{n-h+i}}, \qquad H = -\frac{\partial f}{\partial t}, \qquad s^j = \frac{\partial f}{\partial r_j} .$$

When one is given $f(q^i, p_{n+h-i} , r_j, t)$ such that the determinant $\left| \dfrac{\partial^2 f}{\partial q^i \, \partial r_j}, \cdots, \dfrac{\partial^2 f}{\partial p_{n-h+i} \, \partial r_j} \right|$ is non-zero, H can be obtained by eliminating the r_j from the equations:

([24]) V_{4n+1} is the topological product of V_{2n+1} with a symplectic manifold Sp_{2n} .

$$p_i = \frac{\partial f}{\partial q^i}, \qquad q^{n-h+i} = -\frac{\partial f}{\partial p_{n-h+i}}, \qquad H = -\frac{\partial f}{\partial t}.$$

The first integrals r_j are defined by the n implicit equations:

$$p_i = \frac{\partial f}{\partial q^i}, \qquad q^{n-h+i} = -\frac{\partial f}{\partial p_{n-h+i}}.$$

The first integrals s^j = const. are defined modulo r_i = const. by $s^j = \partial f / \partial r_j$.

Remarks:

1. The n functions $r^j (p_i , q^i, t)$ are obviously in involution since $\Omega = dr_i \wedge ds^i$.

2. Giving the form df on the submanifolds:

$$p_i = \frac{\partial f}{\partial q^i}, \qquad q^{n-h+i} = -\frac{\partial f}{\partial p_{n-h+i}}, \qquad H = -\frac{\partial f}{\partial t}, \qquad s^j = \frac{\partial f}{\partial r_j}$$

of V_{4n+1} is then a practical means for obtaining a set of n functions in involution.

3. Two closed forms ω, ω' of type (VI.9) will generate different integrability cases only if the local submanifolds $r^j (p_i , q^i, t)$ = const. that correspond to those integrability cases are different. Hence, if one writes out that the form $\omega = p_i\, dq^i - H\, dt + s^j\, dr_j$ is closed on the submanifolds p_i as functions of only (q^i, r_i) then one will get the same case upon writing that the form:

$$\omega' = - q^i\, dp_i - H\, dt + s^j\, dr_j$$

is closed on the submanifolds q^i that are functions of only (r_i , p_i) .

4. In order for Ω to not be the sum of p forms, it is necessary and sufficient that f must not be the sum of p functions f_k that each depend upon a different set of variables r_j .

Case of mechanics. – H must be a positive-define quadratic form at p_i : $H = A^{ij} p_i p_j + A^i p_i + B$, where the A^{ij}, A^i, B are functions of $(n + 1)$ variables q^i, t .

That fact leads us to the preliminary question of what the canonical changes of variables will be that preserve the degree of the variables p_i . If a change of variables for which the homologue of the set of p is the set of r preserves the degree of a relation in the p then it will be linear with respect to the r, and conversely. The form $\Omega = dp_i \wedge dq^i - dH \wedge dt$ will become $\Omega = dr_j \wedge ds^j - dK \wedge dt$. The form $dp_i \wedge dq^i - dH \wedge dt - dr_j \wedge ds^j + dK \wedge dt$, which is defined on the a manifold V_{4n+1} , will be zero on the submanifolds $p_i = p_i (r_j , s^j, t)$, $q^i = q^i (r_j , s^j, t)$, so the form $q^i\, dp_i + r_i\, ds^i + H\, dt - K\, dt$ will be the differential of a function that is linear in the p_i , namely, $p_i\, \varphi^i (s^j, t) + \varphi^0 (s^j, t)$, so:

$$q^i = \varphi^i(s^j, t), \qquad r_j = p_i \frac{\partial\varphi^i}{\partial s^j}, \quad H - K = p_i \frac{\partial\varphi^i}{\partial t} + \frac{\partial\varphi^0}{\partial t}.$$

The equations $q^i = \varphi^i(s^j, t)$ show that the desired canonical transformations are nothing but the transformations of the point-like pseudo-group Q that acts upon the variables (q^i, t).

It will then result that from the viewpoint that we have chosen, two functions H must be considered to be distinct only if they are not deduced from each other by a point-like transformation Q.

The pseudo-group Q likewise transforms any integral that is algebraic in the p_i into an algebraic integral of the same nature. If the n functions r_j, which are pair-wise in involution, are algebraic with respect to the p_i then since the r_j are defined by means of the implicit equations $p_i = \partial f / \partial q^i$, the partial derivatives $\partial f / \partial q^i$ must be functions that are algebraic with respect to r_j. One will then obtain certain integrable cases simply by choosing the form df to have coefficients that are algebraic in the r_j.

Remark. – It is quite certain that in the practical applications of that method, one will be limited by the fact that one does not know how to solve the algebraic equations. One can sometimes get around that difficulty by using one of $2n$ forms (VI.9) that are given by the integral invariant ω.

EXAMPLE I. – *n integrals that are linear in p_i.* – Any integral p_i that has the form $\alpha^i p_i + \alpha^0$ = const., where the α are functions of $(q^1, \dots, q^n, t)$, can be reduced to the form $\bar{p}_n + \bar{\alpha}^0$ = const. by a transformation Q.

$\alpha^i p_i + \alpha^0$ will become $\alpha^i \bar{p}_j \dfrac{\partial\bar{q}^j}{\partial q^i} + \alpha^0$ = const. by a transformation Q. One can satisfy that relation by choosing the $\bar{q}^j$ to be functions of $q^1, \dots, q^n, t$ such that:

$$\alpha^i \frac{\partial\bar{q}^j}{\partial q^i} = 0 \qquad \text{for } j \text{ that varies from 1 to } n-1, \qquad \alpha^i \frac{\partial\bar{q}^h}{\partial q^i} = 1.$$

$\bar{q}^1, \dots, \bar{q}^{n-1}$ will then be $(n - 1)$ distinct integrals of the system (t plays the role of a parameter):

$$\frac{dq^1}{\alpha^1} = \dots = \frac{dq^n}{\alpha^n}.$$

$\bar{q}^n$ is a solution to the system:

$$\frac{dq^1}{\alpha^1} = \dots = \frac{dq^n}{\alpha^n} = \frac{d\bar{q}^n}{1}$$

that is obtained by means of one quadrature by taking the preceding $(n - 1)$ integrals into account.

An integral that is linear with respect to the p_i can be put into the form:

$$p_n + a_0 = r_n$$

by a transformation Q, so the function $f(q^i, r_j, t)$ will have the form $f = r_n\, q^n + g_n + \varphi_n$, in which g_n is a function that is independent of r_n, and φ_n is a function that is independent of the q^n. If there are n integrals that are linear with respect to the p_i after the successive Q transformations then the function f can be put into the form:

$$f = \sum_{i=1}^{n} r_i\, q^i + g + \varphi,$$

in which g is a function of $(q^1, \ldots, q^n, t)$ that is independent of $r_1, \ldots, r_n$, and φ is a function of $t, r_1, \ldots, r_n$.

$H = -\frac{\partial g}{\partial t} - \frac{\partial \varphi}{\partial t}$ must be quadratic with respect to the p_i, since one passes from the p_i to the r_i by the linear substitution:

$$p_i = \frac{\partial g}{\partial q_i} + r_i,$$

so $-\partial\varphi / \partial t$ must be a form that is quadratic with respect to the $r_1, \ldots, r_n$:

$$-\frac{\partial \varphi}{\partial t} = \tfrac{1}{2}(A^{ij}\, r_i\, r_j + A^i\, r_i + A^{00}), \qquad \varphi = -\tfrac{1}{2}\int (A^{ij} r_i\, r_j + A^i r_i + A^{00})\, dt.$$

The reduced form of H by means of Q is then:

$$2H = A^{ij}\left(p_i - \frac{\partial g}{\partial q^i}\right)\left(p_j - \frac{\partial g}{\partial q^j}\right) + A^0\left(p_i - \frac{\partial g}{\partial q^i}\right) + A^{00} - 2\frac{\partial g}{\partial t},$$

in which the A^{ij}, A^i, A^{00} are functions of only the variable t, and g is an arbitrary function of $q^1, \ldots, q^n, t$. Hence, one has the theorem:

Theorem:

The necessary and sufficient condition for there to be n linear integrals in involution is that H should be reducible to the form above by a transformation of the point-like pseudo-group.

II. $(n-1)$ **integrals that are linear in** p_i **, while the** n^{th} **one is quadratic in the** p_i . – From the foregoing case, the presence of $(n-1)$ linear integrals will lead one to put f into the form:

$$f = \sum_{j=1}^{n-1} r_j\, q^j + g + \varphi$$

by successive transformations Q, in which g is a function of the $q^1, \ldots, q^n, t$ that is independent of the $r_1, \ldots, r_n$, and φ is a function of the $r_1, \ldots, r_n$ that is independent of the $q^1, \ldots, q^n$.

$\left(p_n - \frac{\partial g}{\partial q^n}\right) = \frac{\partial \varphi}{\partial q^n}$, where $\frac{\partial \varphi}{\partial q^n}$ is a function of r_n ; since the integral r_n must be quadratic, r_n will have the form:

$$r_n = A^{nn}\left(p_n - \frac{\partial g}{\partial q^n}\right)^2 + 2B\left(p_n - \frac{\partial g}{\partial q^n}\right) + C,$$

which is an expression in which B and C are functions that are linear and quadratic, respectively, with respect to the r_i (j varies from 1 to n – 1). One then deduces that:

$$\frac{\partial \varphi}{\partial q^n} = \frac{-B + \sqrt{B^2 - A^{nn}(C - r_n)}}{A^{nn}}.$$

H must be quadratic, so since $H + \frac{\partial g}{\partial t} = -\frac{\partial \varphi}{\partial t}$, $\frac{\partial \varphi}{\partial t}$ must become a quadratic form of rank n with respect to the p_i when one replaces the $r_1, \ldots, r_n$ with their values. That quadratic form can always be written in the form:

$$\frac{\partial \varphi}{\partial t} = \lambda\left(p_n - \frac{\partial g}{\partial q^n}\right)^2 + 2\mu\left(p_n - \frac{\partial g}{\partial q^n}\right) + \nu,$$

which is an expression in which μ and ν are linear and quadratic, respectively, with respect to the p_j (j varies from 1 to n – 1). If one replaces the $p_1 - \frac{\partial g}{\partial q^1}, \ldots, p_n - \frac{\partial g}{\partial q^n}$ as functions of the $r_1, \ldots, r_n$ then $\frac{\partial \varphi}{\partial t}$ will depend linearly upon the radical $\sqrt{B^2 - A^{nn}(C - r_n)}$:

$$\frac{\partial \varphi}{\partial t} = D - \frac{E}{A^{nn}}\sqrt{B^2 - A^{nn}(C - r_n)}.$$

The expression $\frac{\partial \varphi}{\partial q^n} dq^n + \frac{\partial \varphi}{\partial t}$ must be an exact differential for any r, so the expression $\sqrt{B^2 - A^{nn}(C - r_n)}\,(dq^n - E\,dt)\,(1 / A^{nn})$ must be itself an exact differential. A^{nn} and E depend upon only q^n and t ($r_1, \ldots, r_n$ enter into E only as parameters), so if one considers an integrating factor Φ for $dq^n - E\,dt$ then the particular transformation $\bar{q}^n = \bar{q}^n(q^n, t)$ will show that:

$$\frac{1}{\Phi A^{nn}}\sqrt{B^2 - A^{nn}(C - r_n)}\, d\bar{q}^n$$

can be an exact differential only if the functions that enter under the radical depend upon only $\bar{q}^n$.

Consequence. – One can always reduce to the case:

$$\frac{\partial\varphi}{\partial q^n} = \frac{-\left(\sum_{i=1}^{n-1} A^{ni} r_i + A^n\right) + \sqrt{\left(\sum_{i=1}^{n-1} A^{ni} r_i + A^n\right)^2 - A^{nn}\left(\sum_{i,j=1}^{n-1} A^{ij} r_i\, r_j + 2\sum_{i=1}^{n-1} A^i r_i + A^{00} - r_n\right)}}{A^{nn}}$$

by a transformation Q of the point-like pseudo-group, in which the A depend upon only the variable q^n, and:

$$\frac{\partial\varphi}{\partial t} = -\tfrac{1}{2}\left(\sum_{i,j=1}^{n-1} a^{ij} r_i\, r_j + 2\sum_{i=1}^{n-1} a^i r_i + a^{00} + a^n r_n\right),$$

in which the a depend upon only t.

One then deduces H and the quadratic integral r_n :

$$\begin{aligned} 2H = {} & \sum_{i,j=1}^{n-1} a^{ij}\left(p_i - \frac{\partial g}{\partial q^i}\right)\left(p_j - \frac{\partial g}{\partial q^j}\right) + \sum_{i=1}^{n-1} a^i\left(p_j - \frac{\partial g}{\partial q^i}\right) + a^{00} \\ & + a^n\left[\sum_{i,j=1}^{n-1} A^{ij}\left(p_i - \frac{\partial g}{\partial q^i}\right)\left(p_j - \frac{\partial g}{\partial q^j}\right) + 2A^i\left(p_i - \frac{\partial g}{\partial q^i}\right) + A^{00}\right] - 2\frac{\partial g}{\partial t}, \end{aligned}$$

$$r_n = \sum_{i,j=1}^{n-1} A^{ij}\left(p_i - \frac{\partial g}{\partial q^i}\right)\left(p_j - \frac{\partial g}{\partial q^j}\right) + 2A^i\left(p_i - \frac{\partial g}{\partial q^i}\right) + A^{00} .$$

Special case: $g \equiv 0$, the $a \equiv 0$, except for $a^n = 1$, and the $A^i \equiv 0$. One then gets Delassus's integrable case.

One can summarize the foregoing in the theorem:

Theorem:

The necessary condition for the system Σ to have a quadratic first integral and $(n - 1)$ *integrals that are linear with respect to the* p_i *is that H must be reducible to the form above by the point-like pseudo-group Q.*

III. h **integrals that are quadratic in** p_i **and** $(n - h)$ **that are linear in** p_i . – The $(n - h)$ linear integrals can always be reduced to the form $p_i - \partial g / \partial q^i = r_i$ ($i = 1$ to $n - h$) by a transformation Q, and df can be written:

$$df = \sum_{i=1}^{n-1} r_i\, dq^i + dg\,(q^1, \ldots, q^n, t) + d\varphi,$$

in which $r_{n-h+1}, \ldots, r_n$ denote the h quadratic integrals, we remark that if the α are constants then $\sum_{n-h+1=1}^{n} \alpha^k r_k$ will also be a quadratic integral, so φ can be only a function of the forms that are linear in $r_{n-h+1}, \ldots, r_n$ whose coefficients are functions of the variables $q^{n-h+1}, \ldots, q^n, t$.

Now, the n^{th} quadratic integral can always be put into the form:

$$r_n = A^{nn} p_n^2 + 2B\, p_n + C,$$

in which B and C depend upon $p_{n-1}, \ldots, p_{n-h+1}$. It will then result that r_n must enter linearly under a radical of second degree. Since φ is a function of forms that are linear in the $r_{n-h+1}, \ldots, r_n$, the same thing must be true for the other ones. One can reduce to the case in which that radical depends upon only one variable by a transformation of the point-like group Q. One will then be led to the following expression for df:

$$df = \sum_{i=1}^{n-h} r_i\, dq^i + \sqrt{2} \sum_{j,k=n-h+1}^{n} \sqrt{\varphi_k^j\, r_j + \varphi_k}\; dq^k + \left(\sum_{i=1}^{n-h} \tfrac{1}{2} a^{ij} r_i\, r_j + \sum_{i=1}^{n} a^i r_i + a^0 \right) dt + dg\,,$$

in which the φ_k^j are functions of only the q^k, the φ_k are forms that are quadratic in r_j (j = 1 to $n - h$) with coefficients that are functions of only the q^k, and the a depend upon only t. The corresponding function H will then be defined by:

$$0 = \begin{vmatrix} a^{n-h+1} & \cdots & a^n & -\left[H + \dfrac{\partial g}{\partial t} + a^0 + \displaystyle\sum_{i,j,=1}^{n-h} \tfrac{1}{2} a^{ij} \left(p_i - \dfrac{\partial g}{\partial q^i} \right)\left(p_j - \dfrac{\partial g}{\partial q^j} \right) + a^i \left(p_i - \dfrac{\partial g}{\partial q^i} \right) \right] \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_k^{n-h+1} & \cdots & \varphi_k^n & \tfrac{1}{2}\left(p_k - \dfrac{\partial g}{\partial q^k} \right)^2 - \varphi_k \\ \cdots & \cdots & \cdots & \cdots \\ \varphi_n^{n-h+1} & \cdots & \varphi_n^n & \tfrac{1}{2}\left(p_n - \dfrac{\partial g}{\partial q^n} \right)^2 - \varphi_n \end{vmatrix}.$$

Special cases:

1. n quadratic integrals. One gets the generalized Staeckel case by taking the $a^{ij} \equiv 0$, and the φ_k to be a function of the variable q^k.

2. One gets the ordinary Staeckel case by taking $g \equiv 0$ and all of the a to be zero, except for $a^n = 1$.

3. Liouville's integrable case. This is simpler to establish directly by taking:

$$df = \sqrt{2}\sum_{i=1}^{n-1}\sqrt{B_i(r_i - C_i + A_i\,r_n}\,dq^i + \sqrt{2}\sqrt{B_i\sum_{i=1}^{n-1}(-r_i) - C_n + A_n\,r_n}\,dq^n - r_n\,dt\,.$$

One then deduces that:

$$\tfrac{1}{2}p_i^2 = B_i\,(r_i - C_i + A_i\,r_n), \qquad \tfrac{1}{2}p_n^2 = B_n\left(\sum_{i=1}^{n-1}(-r_i) - C_n + A_n\,r_n\right), \qquad H = r_n\,,$$

so $2H = \dfrac{1}{\sum_{i=1}^{n} A_i}\left[\sum_{i=1}^{n}\left(\dfrac{p_i^2}{B_i} + C_i\right)\right]$, or in classical form:

$$2H = \sum_{i=1}^{n} A_i \sum_{i=1}^{n} B_i(\dot{q}^i)^2 + \frac{\sum_{i=1}^{n} C_i(q^i)}{\sum_{i=1}^{n} A_i(q^i)}\,.$$

IV. n **integrals that are linear with respect to the** q^i. – If there exist n integrals that are linear with respect to the q^i then the function f must have the form $f = r_j\,\varphi^j\,(p_1, \dots, p_n, t) + \psi\,(p_1, \dots, p_n, t)$, because $q^i = r_j\,\varphi^j / \partial p_i$. The simplest sub-ring of n functions in involution is composed of the n functions $s^j = \partial f / \partial r_j = \varphi^j(p_1, \dots, p_n, t)$. A canonical transformation that preserves the degree of the q^i will lead us to the case where:

$$f = r_i\,\varphi^i\,(p_i\,, t) + \psi(p_1, \dots, p_n, t),$$

in which the functions φ^i depend upon only the variables p_i and t, since they are functions that are independent of the r_i. Eliminating the r_i from the $(n-1)$ equations:

$$q^i = r_j\frac{\partial\varphi^i}{\partial p_j} + \frac{\partial\psi}{\partial p_i}\,, \qquad H = r_j\frac{\partial\varphi^i}{\partial t} + \frac{\partial\psi}{\partial t}$$

will give:

$$H = q^i\,\frac{\dfrac{\partial\varphi^j}{\partial t}}{\dfrac{\partial\varphi^i}{\partial p_j}} + \frac{\partial\psi}{\partial t} - \frac{\dfrac{\partial\varphi^j}{\partial t}}{\dfrac{\partial\varphi^i}{\partial p_j}}\cdot\frac{\partial\psi}{\partial p_i}\,.$$

That function must be quadratic:

1.
$$\frac{\dfrac{\partial \varphi^j}{\partial t}}{\dfrac{\partial \varphi^i}{\partial p_j}} = a_i(t)\, p_i^2 + 2b_i(t)\, p_i + c_i(t)\,,$$

which shows that φ^i is a function of the solution R_i of the Riccati equation $dp_i / dt = a_i(t)\, p_i^2 + 2\, b_i(t)\, p_i + c_i(t)$.

2.
$$\frac{\partial \psi}{\partial t} - \frac{\partial \psi}{\partial p_i}[\, a_i(t)\, p_i^2 + 2\, b_i(t)\, p_i + c_i(t)] = a^{ij} p_i\, p_j + 2\, a^i\, p_i + a^{00},$$

in which the a are functions of t, ψ is a function of $R_1, \ldots, R_n$ and t, which one determines by quadratures.

In summary, one constitutes a sub-ring of n functions in involution by giving the solutions to n Riccati equations.

A variant of the process consists of giving the sub-ring by means of n functions $s^1, \ldots, s^n$.

Examples:

1.
$$s^n = \frac{t}{2} + \int \frac{dp_n}{p_n^2 + \alpha}, \qquad s^i = \frac{(p_n^2+\alpha)^{\beta_i}}{p_i} + \gamma^i \int (p_n^2+\alpha)^{\beta_i - 1} dp_n\,,$$

$$f = 2r_n\left(\frac{t}{2} + \int \frac{dp_n}{p_n^2+\alpha}\right) + \sum_{i=1}^{n-1} r_i \left[\frac{(p_n^2+\alpha)^{\beta_i}}{p_i} + \gamma^i \int (p_n^2+\alpha)^{\beta_i-1} dp_n\right],$$

$$q^i = -\, r_i\, \frac{(p_n^2+\alpha)^{\beta_i}}{p_i}\,,$$

$$q_n = +\,2r_n \frac{1}{p_n^2+\alpha} + \sum_{i=1}^{n-1} r_i \left[\frac{2\beta_i\, p_n (p_n^2+\alpha)^{\beta_i-1}}{p_i} + \gamma^i (p_n^2+\alpha)^{\beta_i-1}\right], \qquad H = r_n\,,$$

so

$$2H = \sum_{i=1}^{n-1} \gamma^i\, p_i^2\, q^i + 2\sum_{i=1}^{n-1} \beta_i\, p_i\, p_n\, q^i + (p_n^2+\alpha) q^n\,.$$

2.
$$s^n = \frac{t}{2} + \int \frac{dp_n}{p_n^2+\alpha}, \qquad s^i = p_i (p_n^2+\alpha)^{\beta_i}\,,$$

$$f = 2r_n \int \frac{dp_n}{p_n^2+\alpha} + r_n\, t + \sum_{i=1}^{n-1} r_i\, p_i (p_n^2+\alpha)^{\beta_i} + \psi\,,$$

$$q^i = r_i\,(p_n^2 + \alpha)^{\beta_i} + \frac{\partial \psi}{\partial p_i},$$

$$q^n = 2r_n\,\frac{1}{p_n^2 + \alpha} + \frac{\partial \psi}{\partial p_n} + 2\sum_{i=1}^{n-1} r_i\; p_i\; \beta_i\; p_n\,(p_n^2 + \alpha)^{\beta_i - 1}.$$

One determines ψ by means of the condition:

$$-(p_n^2 + \alpha)\frac{\partial \psi}{\partial p_n} + 2\,p_i\;\beta_i\;p_n\,\frac{\partial \psi}{\partial p_i} = \sum_{i=1}^{n-1} \gamma^i\,p_i^2\,,$$

namely:

$$\sum_{i=1}^{n-1} \gamma^i\,p_i^2\,(p_n^2 + \alpha)^{2\beta_i} \int \frac{dp_n}{(p_n^2 + \alpha)^{2\beta_i + 1}} = \psi,$$

so:

$$2H = \sum_{i=1}^{n-1} \gamma^i\,p_i^2 - 2\sum_{i=1}^{n-1} \beta_i\;p_i\;p_n q^i + q^n\,(p_n^2 + \alpha)\,.$$

Special case: $n = 2$, $\alpha = 0$. One finds spiral surfaces whose geodesics are determined by quadratures (cf., Darboux, tome III, pp. 81).

B. – STUDY OF THE CASE $d\,\Omega \neq 0$.

Theorem V:

If $\mathrm{X} \in T$ and $\overline{x} \in T$ *correspond to an infinitesimal transformation and a first integral of* Σ, *respectively, then the Lie bracket* $[\mathrm{X},\ \overline{x}\,] \neq 0$ *will correspond to a new first integral of* Σ *whose expression will be* $i\,(\mathrm{X} \wedge \overline{x}\,)\,\Omega$.

The formula (VI.3) gives $i\,[\mathrm{X},\ \overline{x}\,]\,\Omega = \theta(\mathrm{X})\ i(\overline{x})\,\Omega - i(\overline{x})\ \theta(\mathrm{X})\,\Omega$.

The hypotheses $\theta(\mathrm{X})\,\Omega = 0$, $i(\overline{x})\,\Omega = \overline{\pi}$, $d\overline{\pi} = 0$ implies that:

$$i\,[\mathrm{X},\ \overline{x}\,]\,\Omega = di\,(\mathrm{X} \wedge \overline{x}\,)\,\Omega.$$

If the Lie bracket $[\mathrm{X},\ \overline{x}\,]$ is a non-zero element of T then when the operator $i\,[\mathrm{X},\ \overline{x}\,]$ is applied to Ω, it will then correspond to the differential of a function in the ring K, which will be a function whose expression is $i\,(\mathrm{X} \wedge \overline{x}\,)\,\Omega$. Since the differential of that function is zero along the integral curves of Σ, it will be a new first integral of Σ . Its calculation will be immediate:

$$i\,(\mathrm{X} \wedge \overline{x}\,)\,\Omega = k_{ij}\begin{vmatrix} \overline{x}^i & \overline{x}^j \\ X^i & X^j \end{vmatrix}.$$

The study of the system Σ is carried out in conformity with Lie's ideas by means of one's knowledge of certain infinitesimal transformations.

Definition. – We say that r elements $X^1, X^2, \ldots, X^r \in T$, which are generators of infinitesimal transformations of Ω define a complete system if:

$$[X^\rho, X^\sigma] = \gamma_\tau^{\rho\sigma} X^\tau, \qquad \rho, \sigma, \tau \in (1 \text{ to } r),$$

where the $\gamma_\tau^{\rho\sigma}$ are functions in the ring K.

Lemma:

There is an infinitude of ways of choosing $2n$ *elements* $x^1, \ldots, x^{2n}$ *that belong to* T *and are such* $i(X^i)\, i(x^j)\, \Omega = i(X^i \wedge x^j)\, \Omega$ *is equal to* 1 *for* $j = i$, *and* 0 for $j \neq i$.

If the X^i are known then the components of x^j (j fixed) will be solutions of the r linear equations ($r < 2n + 1$):

$$i(X^i \wedge x^j)\, \Omega = 0 \qquad \text{for} \qquad j \neq i, \quad i(X^i \wedge x^j)\, \Omega = 1 \qquad \text{for } j \neq i\,.$$

One can benefit from the arbitrariness to choose solutions for the x^j that are as simple as possible.

Let $\pi^i = i(x^i)\, \Omega$, where the index i varies from 1 to $2n$, so when the corresponding $2n$ Pfaff forms are equated to 0, that will constitute one of the expressions for a system of differential equations Σ.

Theorem VI:

The system of $(2n - r)$ *Pfaff forms* $\pi^{r+1}, \ldots, \pi^{2n}$ *is completely integrable, so the* r *forms* $\pi^1, \ldots, \pi^r$ *are invariant forms.*

We apply formula (VI.5) to a form π^i by taking x, y to be two fields X^ρ, X^σ:

$$\theta(X^\rho)\, i(X^\sigma)\, \pi^i - \theta(X^\sigma)\, i(X^\rho)\, \pi^i = \varepsilon(X^\rho \wedge X^\sigma)\, \pi^i + i\,[X^\rho, X^\sigma]\, \pi^i. \tag{VI.10}$$

From the way that the forms π^i were chosen, the left-hand side is always zero.

1. *For* $i > r$:

$$i(X^\sigma)\, \pi^i = 0, \qquad i(X^\rho)\, \pi^i = 0, \qquad i(X^\rho \wedge X^\sigma)\, \pi^i = \gamma_\tau^{\rho\sigma} \cdot i(X^\tau)\, \pi^i = 0,$$

$$\varepsilon(X^\rho \wedge X^\sigma)\, \pi^i = -\, i(X^\rho \wedge X^\sigma)\, d\pi^i.$$

The vanishing of the left-hand side of (VI.10) will imply that:

$$i\,(X^\rho \wedge X^\sigma)\,d\pi^i = 0.$$

The $x^1, \ldots, x^{2n}$ form an ordinary differential system that is then completely integrable, so the $d\pi^i$ will belong to the sub-module of the π^i. Since the c^i_{jk} are functions of the ring K, one can then write:

$$d\pi^i = c^i_{jk}\,(\pi^j \wedge \pi^k)\,.$$

Let us study what the hypotheses on the c^i_{jk} will imply:

$$i\,(X^\rho \wedge X^\sigma)\,d\pi^i = c^i_{jk}\,i\,(X^\rho \wedge X^\sigma)\cdot(\pi^j \wedge \pi^k)\,,$$

$$0 = c^i_{jk}\,i\,(X^\rho)\,i\,(X^\sigma)\cdot(i\,(x^j)\,\Omega) \wedge (i\,(x^j)\,\Omega)\,.$$

Since $i\,(X^\sigma)\cdot i\,(x^j)\,\Omega$ is non-zero only for $j = \sigma$, it will then result that the preceding equality will reduce to:

$$c^i_{jk}\,i\,(X^\rho)\,i\,(X^\sigma)\,\Omega = 0\,.$$

Since $i\,(X^\rho)\cdot i\,(x^k)\,\Omega$ is non-zero only for $k = \rho$, one must have $c^i_{\sigma\rho} = 0$; i.e., $c^i_{jk} = 0$ for $j \leq r$, $k \leq r$.

Those conditions imply that $d\pi^{r+\alpha}$ belongs to sub-module of forms $\pi^{r+\alpha}$; in other words, the system of $(2n - r)$ forms $\pi^{r+\alpha}$ is completely integrable.

2. *For* $i \leq r$:

$$i\,(X^\tau)\,\pi^i = 0, \text{ if } \quad i \neq \tau, \qquad i\,(X^\tau)\,\pi^i = 1, \text{ if } \quad i = \tau\,.$$

The vanishing of the left-hand side of (VI.10) gives:

$$0 = i\,(X^\rho \wedge X^\sigma)\,d\pi^\tau + \gamma^{\rho\sigma}_\tau\,.$$

Upon taking into account the fact that $d\pi^i = c^i_{jk}\,\pi^j \wedge \pi^k$:

$$i\,(X^\rho)\,i\,(X^\sigma)\,c^i_{jk}\,\pi^j \wedge \pi^k = c^i_{jk}\;i\,(X^\rho)\,i\,(X^\sigma)\,i\,(x^j)\,\Omega \wedge i\,(x^k)\,\Omega\,.$$

Since $i\,(X^\sigma)\,i\,(x^j)\,\Omega$ is non-zero only for $j = \sigma$, its value will then be 1.
Since $i\,(X^\rho)\,i\,(x^k)$ is non-zero only for $k = \rho$, its value will then be 1.

$$\gamma^{\rho\sigma}_\tau = c^\tau_{\sigma\rho} = -\,c^\tau_{\rho\sigma}\,.$$

In addition, the c^i_{jk} will generally be non-zero functions for $\tau = i \leq r$ and j and $k > r$.

Let us show that the r forms $\pi^1, \ldots, \pi^r$ are invariant forms:

$$d\pi^i = c^i_{jk}\, \pi^j \wedge \pi^k \qquad \text{for} \qquad (i = 1 \text{ to } r,\, j,\, k = 1 \text{ to } 2n).$$

Since the $2n$ forms $\pi^1, \ldots, \pi^{2n}$ are expressed linearly as functions of the $2n$ differentials of the first integrals of Σ, if one performs that substitution on the left-hand side then upon introducing conveniently-chosen functions of the first integrals:

$$d\pi^i = \sum k^i_{\alpha\alpha^*}\, dc^\alpha \wedge dc^{\alpha*} \qquad \text{with} \qquad \alpha^* = \alpha + n\,,$$

in which the $k^i_{\alpha\alpha^*}$ are functions of the first integral and one of the variables (t, for example). If u denotes an arbitrary first integral or the variable t then upon taking the exterior derivative of the two sides of the last equations, one will have:

$$d\,(d\pi^i) = 0 = \sum \frac{\partial k^i_{\alpha\alpha^*}}{\partial u}\, du \wedge dc^\alpha \wedge dc^{\alpha*}\,.$$

Since there is just the one term $du \wedge dc^\alpha \wedge dc^{\alpha*}$, $\partial k^i_{\alpha\alpha^*} / \partial u = 0$, and $k^i_{\alpha\alpha^*}$ can only be a function of the two integrals c^α and $c^{\alpha*}$. One can reduce $d\pi^i$ to the form:

$$d\pi^i = \sum_{\alpha=1}^{n} d\bar{c}^\alpha \wedge d\bar{c}^{\alpha*}$$

by a change of first integrals $c^\alpha = c^\alpha(c^\alpha, \bar{c}^{\alpha*})$.

It will then result that $\pi^i = \bar{c}^\alpha d\bar{c}^{\alpha*}$. Since π^i is expressed in terms of only first integrals and their differentials, it will indeed be an invariant for the system Σ.

Practical consequences for the integration of Σ. – Having chosen the $2n$ forms π^i in conformity with the Lemma, one can begin the search for the solutions to the completely-integrable system $\pi^{r+1}, \ldots, \pi^{2n}$. When a solution to that system is known, one will be reduced to the integration of a differential system of r forms $\pi^1, \ldots, \pi^r$ for which one knows r invariant forms.

The search for a solution to the completely-integrable system $\pi^{r+1}, \ldots, \pi^{2n}$ is simplified by the following remarks:

1) If one knows that there exist p first integrals then one can benefit from the arbitrariness that exists in the choice of the x^i to make those first integrals belong to the sub-module of $\pi^{r+1}, \ldots, \pi^{2n}$.

2) If the mechanical system is restricted by q constraints:

$$a^k = a^k(p_i\,,\, q^i,\, t) = 0, \quad \Omega_d = L_i\, dq^i \wedge dt\,,$$

which are compatible with the infinitesimal transformations, then one chooses some of the x^i in such a way that the da^k will belong to the sub-module of $(2n - r)$ forms $\pi^{r+1}, \ldots, \pi^{2n}$.

One can then reduce the order of the completely-integrable system $\pi^{r+1}, \ldots, \pi^{2n}$ by $(p + q)$ units.

More particularly, if $2n - r = p + q$ and if the r forms $\pi^1, \ldots, \pi^r$ are closed modulo the forms $\pi^{r+1}, \ldots, \pi^{2n}$ then the integration will be accomplished by quadratures.

§ III. – Applying the preceding methods.

I. – Heavy point moving with friction on an inclined plane. Recall the notations of Chapter I, § **1**, and set $\Omega = \omega / m$, so:

$$\Omega = dv \wedge (\cos\alpha\, dx + \sin\alpha\, dy) + v\, d\alpha \wedge (-\sin\alpha\, dx + \cos\alpha\, dy)$$
$$- v\, dv \wedge dt + g \sin i\, dx \wedge dt - fg \cos i\, (\cos\alpha\, dx + \sin\alpha\, dy) \wedge dt\,.$$

One immediately knows three fields X, Y, T that generate infinitesimal transformations for Ω, and whose components are given in the table:

	v	α	x	y	t
X	0	0	1	0	0
Y	0	0	0	1	0
T	0	0	0	0	1

One then has:

$$i\,(\mathrm{X})\,\Omega = - dv \cos\alpha + v \sin\alpha\, d\alpha + g \sin i\, dt - f\, g \cos i \cos\alpha\, dt,$$
$$i\,(\mathrm{Y})\,\Omega = - dv \sin\alpha - v \cos\alpha\, d\alpha - f\, g \cos i \sin\alpha\, dt,$$
$$i\,(\mathrm{T})\,\Omega = v\, dv - g \sin i\, dx + f\, g \cos i \cos\alpha\, dx + f\, g \cos i \sin\alpha\, dy.$$

Let us determine four forms π_x, π_y, π_t, π by means of four fields x, y, t, u, conforming to the lemma. The components of the field x satisfy the equations:

$$i\,(x)\, i\,(\mathrm{X})\,\Omega = -1, \qquad i\,(x)\, i\,(\mathrm{Y}) = 0, \qquad i\,(x)\, i\,(\mathrm{T}) = 0.$$

One can take the components of x to be:

$$\cos a, \quad -\frac{\sin\alpha}{v}, \qquad 0, \qquad -\frac{v \cot\alpha}{fg \cos i}, \qquad 0,$$

so

$$i\,(x)\,\Omega = \pi_x = dx + \frac{1}{fg \cos i}\left(v\, dv \cos\alpha + \frac{v^2 \cos^2\alpha}{\sin\alpha} \cdot d\alpha\right).$$

Upon doing the same thing with y and t, one will get:

$$\text{the components of} \quad y \quad \text{are} \quad \left(\sin\alpha, \quad \frac{\cos\alpha}{v}, \quad 0 \quad -\frac{v}{fg\cos i}, \quad 0\right),$$

so

$$i\,(y)\,\Omega = \pi_y = dy + \frac{1}{fg\cos i}\left(v\,dv\sin\alpha + v^2\cos\alpha\cdot d\alpha\right),$$

and

$$\text{the components of} \quad t \quad \text{are} \quad \left(0, \quad 0, \quad \frac{1}{g\sin i}, \quad -\frac{\cot\alpha}{g\sin i}, \quad 0\right),$$

so

$$i\,(t)\,\Omega = \pi_t = dt + \frac{1}{g\sin i}\frac{v\,d\alpha}{\sin\alpha}.$$

The field u such that $i\,(u)\;i\,(\mathrm{X}) = 0$, $i\,(u)\;i\,(\mathrm{Y}) = 0$, $i\,(u)\;i\,(\mathrm{T}) = 0$ will have the components $\left(0, \quad 0, \quad 0, \quad \frac{f\cot i}{v\sin\alpha}, \quad \frac{1}{v\sin\alpha} - \frac{f\cot i\cos\alpha}{v\sin\alpha}\right)$, so:

$$i\,(u)\,\Omega = \pi = \frac{\cos\alpha\,d\alpha}{\sin\alpha} + \frac{dv}{v} - f\cot i\frac{d\alpha}{\sin\alpha}.$$

The completely-integrable system reduces to the single equation $\pi = 0$. It is a closed form, so it can be integrated by quadrature: $v = \dfrac{c}{\sin\alpha}\left(\tan\dfrac{\alpha}{2}\right)^{f\cot i}$. The three π_x, π_y, π_t are closed modulo π. The integration is accomplished by quadratures.

II. Motion of an electrified particle of charge e and gravitating mass m in an electric field E and a magnetic field H.

We suppose that the electric field is derived from a force function, and when it is added to the force function from which weight is derived, it will be denoted by f. The conditions that the field II must satisfy will be specified later.

If one refers the Euclidian space in which the particle moves to three coordinate axes and x^1, x^2, x^3, $\dot{x}^1$, $\dot{x}^2$, $\dot{x}^3$ denote the components of the velocity of the particle, while H^1, H^2, H^3 are the components of the field H, then the form Ω that is a associated with the particle will be:

$$\Omega = \sum_{j=1}^{3} d\dot{x}^j \wedge dx^j - \sum_{j=1}^{3} \dot{x}^j dx^j \wedge dt + \frac{\partial f}{\partial x^j} dx^j \wedge dt$$

$$+\frac{e}{m}[(\dot{x}^2H^3 - \dot{x}^3H^2)\,dx^1 + (\dot{x}^3H^1 - \dot{x}^1H^3)\,dx^2 + (\dot{x}^1H^2 - \dot{x}^2H^1)\,dx^3].$$

The presence of the magnetic field implies that $d\,\Omega \neq 0$. Since the force that is due to the magnetic field is orthogonal to the displacement and the electric and gravitational fields are derived from force functions, one can define the *vis viva* integral. It corresponds to a field x in the tangent space to the manifold V_7 whose components are:

$$-\frac{\partial f}{\partial x^1},\ -\frac{\partial f}{\partial x^2},\ -\frac{\partial f}{\partial x^3},\ -\dot{x}^1,\quad -\dot{x}^2,\quad -\dot{x}^3,\quad 0,$$

in which the variables are assumed to be arranged in the order $\dot{x}^1$, $\dot{x}^2$, $\dot{x}^3$, x^1, x^2, x^3, t :

$$i\,(x)\,\Omega = \tfrac{1}{2}dv^2 - df\,.$$

In what follows, we suppose that the magnetic field revolves around the vertical axis Oz and that it components in cylindrico-polar coordinates are $U = 0$, $V = V'(\rho)$, $W = 0$, in which ρ denotes the distance to the Oz axis. In cylindrico-polar coordinates ρ, α, z, the expression for Ω in canonical form is:

$$\begin{aligned}\Omega &= dp \wedge dr + dq \wedge da + dr \wedge da \\ &-\left(p\,dp+\frac{1}{\rho^2}q\,dq+r\,dr-\frac{q^2}{\rho^3}d\rho\right)\wedge dt+\left(\frac{\partial f}{\partial\rho}d\rho+\frac{\partial f}{\partial\alpha}d\alpha+\frac{\partial f}{\partial z}dz\right)\wedge dt \\ &+\frac{e}{m}\left[\left(\frac{W}{\rho}q-V'r\right)d\rho+(U\rho r-W\rho p)d\alpha+\left(pV'-\frac{U}{\rho}q\right)dz\right]\wedge dt\,.\end{aligned}$$

Let us specify the form of the function f by the existence of certain infinitesimal transformations:

$$\mathrm{Z} = (0, 0, 0, 0, 0, 1, 0),\qquad \mathrm{A} = (0, 0, 0, 0, 1, 0, 0),\qquad \mathrm{T} = (0, 0, 0, 0, 0, 0, 1),$$

so

$$\theta(\mathrm{Z})\,\Omega = 0,\qquad \theta(\mathrm{A})\,\Omega = 0,\qquad \theta(\mathrm{T})\,\Omega = 0\,.$$

Apply the formula (VI.1), while remarking that $d\,\Omega$ can be put into the form:

$$d\,\Omega = [-\,V'dr \wedge dr \wedge dt + d\,(p\,V') \wedge dz \wedge dt]\,.$$

$\theta(\mathrm{A})\,\Omega = d\left[-dq+\frac{\partial f}{\partial\alpha}dt\right] = 0$, which is a condition that will be satisfied when $\partial f / \partial\alpha$ is a function of t.

We now suppose that $\partial f / \partial\alpha \equiv 0$, so:

$$\begin{aligned}\theta(\mathrm{Z})\,\Omega = 0 = i\,(\mathrm{Z})\,d\,\Omega + d\,(i\,(\mathrm{Z})\,\Omega) &= -\frac{e}{m}d\,(p\,V') \wedge dt + d\left[-dr+\frac{\partial f}{\partial z}dt+\frac{e}{m}(pV')dt\right] \\ &= d\left(\frac{\partial f}{\partial z}dt\right),\end{aligned}$$

which is a condition that is satisfied when $\partial f / \partial z$ is a function t, so it will be realized, in particular, when $\partial f / \partial z = -\,g$ (g is the intensity of the field of gravity).

If the preceding conditions are supposed to be realized then that will imply, in addition, the existence of two first integrals. Ω is written:

$$\Omega = dp \wedge d\rho + dq \wedge d\alpha + dr \wedge dz - \left(p\,dp + \frac{1}{\rho^2} q\,dq + r\,dr - \frac{q^2}{\rho^3} d\rho \right) \wedge dt$$
$$+ \left(\frac{\partial f}{\partial \rho} d\rho - g\,dz \right) \wedge dt + \frac{e}{m} [- V'r\,d\rho + p\,V\,dz] \wedge dt,$$

$$i\,(\mathrm{A})\,\Omega = dq\,.$$

If P denotes the field whose components are $\left(-\frac{e}{m} V', 0, 0, 0, 0, -1, 0 \right)$ then:

$$i\,(\mathrm{P})\,\Omega = -\frac{e}{m} V'(r)\,d\rho + dr + g\,dt\,.$$

Upon applying the general theory, one will know three invariant forms π^1, π^2, π^3, and three forms that define a completely-integrable system π^4, π^5, π^6. Since one knows three first integrals, one can choose three fields x that will make them the three forms π^4, π^5, π^6. An easy calculation will give:

$$\pi^1 = d\alpha - \frac{q}{p\rho^2} d\rho, \qquad \pi^4 = dq,$$

$$\pi^2 = dz + \frac{r}{p} d\rho, \qquad \pi^5 = dr - \frac{e}{m} V'(r)\,d\rho + g\,dt,$$

$$\pi^3 = -\,dt + \frac{1}{p} d\rho, \qquad \pi^5 = d\left(p^2 + \frac{q^2}{\rho^2} + r^2 - 2\,f(\rho) + 2g\,z \right).$$

When the first three forms are equated to zero, they will constitute a differential system that is not generally integrable by quadratures, modulo the last three. A special of that is: When one suppresses gravity $g = 0$, π^1, π^2, π^3 will be closed modulo π^4, π^5, π^6, and the system will be integrable by quadratures.

III. – Motion of a solid body on a plane in the case where there is a permanent symmetry plane.

From the theory that we developed, we must consider the solid body to be free. Its position is defined by three parameters: ξ and η are the coordinates of its center of gravity G, and α is the angle between Gy and Gx_1, where Gy is the normal to the plane, and Gx_1 is a half-line that is invariably fixed in the solid body. The three velocity parameters are $\dot{\xi}$, $\dot{\eta}$, $\dot{\alpha}$. The external forces other than the reaction of the plane have the general resultant X, Y, and a resultant moment with respect to G of Γ. Let M be the mass of the solid body, and let k be its radius of gyration with respect to G. The exterior form that is associated with the free solid body is:

$$\Omega_l = M(d\dot{\xi}\wedge d\xi + d\dot{\eta}\wedge d\eta + k^2 d\dot{\alpha}\wedge d\alpha) - M(\dot{\xi}\,d\dot{\xi} + \dot{\eta}\,d\dot{\eta} + k^2\dot{\alpha}\,d\alpha)\wedge dt$$
$$+ [X\,d\xi + Y\,d\eta + \Gamma\,d\alpha]\wedge dt\,.$$

The associated field E has the components:

$$\frac{X}{M},\ \frac{Y}{M},\ \frac{Z}{M},\ \dot{\xi},\ \dot{\eta},\ \dot{\alpha},\ 1.$$

Study of the constraint: Let P be the point of contact of the solid body with the plane whose coordinates with respect to the axes Gx, Gy that are parallel to the plane and normal to it, resp., are a and b, respectively. a and b are functions of α that one determines by considering the profile of the body, which is defined by the envelope of its tangents. A tangent to the profile has the equation:

$$x_1 \cos\varphi + y_1 \sin\varphi - p(\varphi) = 0$$

with respect to the axes Gx_1, Gy_1, which are fixed in the body, and a normal will have the equation:

$$-x_1 \sin\varphi + y_1 \cos\varphi - p'(\varphi) = 0\,,$$

and on the other hand:

$$\alpha + \varphi = \pi\,.$$

It will then result that the expressions for a and b as functions of a are:

$$a = p'(\pi - \alpha), \qquad b = -p(\pi - \alpha)\,.$$

The velocity of the point P, which is the point of contact of the solid body with the plane, will have components $\dot{\xi} - b\,\dot{\alpha}$, $\dot{\eta} + a\,\dot{\alpha}$ with respect to the fixed axes.

The constraint equation that expresses the condition of contact with the plane is:

$$l = \dot{\eta} + a\,\dot{\alpha} = 0.$$

The power delivered by the forces that are necessary to realize that constraint will depend upon certain hypotheses:

A. Perfectly-smooth plane. – That is a zero-power constraint:

$$P = N(\dot{\eta} + a\,\dot{\alpha}), \qquad N > 0\,.$$

The Ω_d that must be added to Ω_l in order to get Ω is:

$$\Omega_d = N\,(d\eta + a\,d\alpha) \wedge dt\,.$$

The constraint field e will have the components:

$$\left(0, \frac{1}{M}, \frac{a}{Mk^2}, 0, 0, 0, 0\right).$$

B. Rough plane with a coefficient of friction f, resistance to rolling parameter δ, and sliding and rolling of the body $\dot{\xi} - b\,\dot{\alpha} \neq 0$:

$$P = N[\dot{\eta} + a\,\dot{\alpha} + \varepsilon\, f\,(\dot{\xi} - b\,\dot{\alpha}) + \varepsilon_1 \delta\dot{\alpha}]\,, \qquad N > 0,$$

with

$$\varepsilon\,(\dot{\xi} - b\,\dot{\alpha}) < 0, \qquad \varepsilon_1\,\dot{\alpha} < 0, \qquad \varepsilon = \pm\,1, \qquad \varepsilon_1 = \pm\,1\,.$$

The form Ω_d that must be added to Ω_l in order to get Ω is:

$$\Omega_d = N\,[d\eta + a\,d\alpha + \varepsilon\,f\,(d\xi - b\,d\alpha) + \varepsilon_1\,\delta\,d\alpha] \wedge dt\,.$$

The constraint field e will have the components:

$$\left(\frac{\varepsilon f}{M}, \frac{1}{M}, \frac{a - \varepsilon\, f\, b + \varepsilon_1 \delta}{Mk^2}, 0, 0, 0, 0\right).$$

C. Body rolling without slipping on a rough plane:

$$P = T(\dot{\xi} - b\,\dot{\alpha}) + N(\dot{\eta} + a\,\dot{\alpha} + \varepsilon_1\,\delta\,\dot{\alpha})\,, \qquad N > 0,$$

with two constraint equations $m = \dot{\xi} - b\,\dot{\alpha} = 0$; $l = \dot{\eta} + a\,\dot{\alpha} = 0$, and the two inequalities $\varepsilon_1\dot{\alpha} < 0$, $|\,T\,| < f\,N$.

The form Ω_d that must be added to Ω_l in order to get Ω is:

$$\Omega_d = [T\,(d\xi - b\,d\alpha) + N\,(d\eta + a\,d\alpha + \varepsilon_1\,\delta\,d\alpha)] \wedge dt\,.$$

The two constraint fields have the components:

$$e^2 = \left(\frac{1}{M}, 0, \frac{-b}{Mk^2}, 0, 0, 0, 0\right),$$

$$e^1 = \left(0, \frac{1}{M}, \frac{a + \varepsilon_1 \delta}{Mk^2}, 0, 0, 0, 0\right).$$

Reaction of the plane. – One of the principle advantages of the use of H. Cartan's operators i (x) is that one can determine the reactions without having to previously solve the equations of motion. In the two cases A and B, the component N of the reaction is given by the equation $N\, i\, (e)\, dl + i\, (\mathrm{E})\, dl = 0$ when one replaces e with its values in each case:

$$dl = d\dot{\eta} + a\, d\dot{\alpha} + \dot{a}\,\dot{\alpha}\, d\alpha\,,$$

$$i\,(\mathrm{E})\; dl = \frac{Y}{M} + \frac{a\,\Gamma}{Mk^2} + \dot{a}\,\dot{\alpha}^2\,.$$

Case A:

$$i\,(e)\; dl = \frac{1}{M} + \frac{a^2}{Mk^2}\,, \qquad\qquad N = -\,\frac{k^2(Y + M\dot{a}\,\dot{\alpha}^2) + a\,\Gamma}{k^2 + a^2}\,,$$

which is an acceptable value when $Y + M\dot{a}\,\dot{\alpha}^2 + a\,\Gamma / k^2 < 0$.

Case B: $i\,(e)\; dl = \dfrac{1}{M} + \dfrac{a(a - \varepsilon\, f\, b + \varepsilon_1 \delta)}{Mk^2}$. That quantity must be non-zero in order for the constraint to be compatible. Hence, the value of N will be acceptable when it is positive.

$$N = -\,\frac{k^2(Y + M\dot{a}\dot{\alpha}^2) + a\,\Gamma}{k^2 + a^2 - \varepsilon\, f\; ba + \varepsilon_1\, a\, \delta}\,.$$

If $k^2 + a^2 - \varepsilon\, f\; ba + \varepsilon_1\, a\,\delta < 0$ and $k^2(Y + M\dot{a}\dot{\alpha}^2) + a\Gamma > 0$ then there will be two possible situations: viz., slipping and cessation of contact, since $dl\, /\, dt = i\, (\mathrm{E})\, dl > 0$. For $k^2(Y + M\dot{a}\dot{\alpha}^2) + a\Gamma < 0$, there is an impossible situation that one can interpret as a tangential shock ([25]).

Case C: The components T and N of the reaction of the plane are solutions to the system:

(1)
$$\begin{cases} N\, i\,(e_1)\, dl + T\, i\,(e_2)\, dl = -\,i\,(\mathrm{E})\, dl, \\ N\, i\,(e_1)\, dm + T\, i\,(e_2)\, dm = -\,i\,(\mathrm{E})\, dm. \end{cases}$$

Once one has determined the direction fields e_1 and e_2 above, since dl = $d\dot{\eta} + a\, d\dot{\alpha} + \dot{a}\,\dot{\alpha}\, d\alpha$, dm = $d\dot{\xi} - b\, d\dot{\alpha} - b\,\dot{\alpha}\, d\alpha$, one will have:

$$i\,(e_1)\; dl = \frac{1}{M} + \frac{a\,(a + \varepsilon_1 \delta)}{M\, k^2}\,, \quad i\,(e_2)\; dl = \frac{-a\, b}{M\, k^2}\,,$$

([25]) Cf., E. DELASSUS and J. PÉRÈS, Nouvelles Annales de Mathématiques (5) **2** (1923), 383-391.

$$i\,(\mathrm{E})\;dl = \frac{Y}{M} + \frac{a\,\Gamma}{M\,k^2} + \dot{a}\,\dot{\alpha}^2,$$

$$i\,(e_1)\;dm = -\frac{b\,(a+\varepsilon_1\delta)}{M\,k^2}, \qquad i\,(e_2)\;dm = \frac{1}{M} + \frac{b^2}{M\,k^2},$$

$$i\,(\mathrm{E})\;dm = \frac{X}{M} - \frac{b\,\Gamma}{M\,k^2} - b\,\dot{\alpha}^2.$$

The constraints will be compatible only if $k^2 + a^2 + b^2 - \varepsilon_1\, a\;\delta \neq 0$, so:

$$T = -\frac{(k^2 + a^2 + a\,\varepsilon_1\delta)(X - M\,\dot{b}\,\dot{\alpha}^2) + b(a+\varepsilon_1\delta)(Y + M\,\dot{b}\,\dot{\alpha}^2) - b\,\Gamma}{k^2 + a^2 + b^2 - a\,\varepsilon_1\delta},$$

$$N = -\frac{a\,b\,(X - M\,\dot{b}\,\dot{\alpha}^2) + (Y + M\,\dot{b}\,\dot{\alpha}^2)(k^2 + b^2) + a\,\Gamma}{k^2 + a^2 + b^2 - a\,\varepsilon_1\delta}.$$

The hypothesis C is acceptable only if $N > 0$ and $|\,T\,| < f\,N$.

Determining the various possibilities of escape, rolling without slipping, and slipping can be accomplished by interpreting the system (1) geometrically. Trace out two rectangular axes $O\dot{l}$, $O\dot{m}$ in a plane and consider the vectors **n** and **t** whose coordinates are $(-\,i\,(e_1)\;dl,\,-\,i\,(e_1)\;dm)$, $(-\,i\,(e_2)\;dl,\,-\,i\,(e_2)\;dm)$, resp. Similarly, trace out the half-lines D and D', $T = \pm\, f\,N$, $N > 0$. Since the left-hand sides of the system (1) are independent of the possible situations, one can interpret i (E) dl, i (E) dm as the coordinates of a point A in the plane. Draw through A, on the one hand, the parallel to **t** that cuts the support to **n** at N and cuts D and D' at α and α', resp., and on the other hand, the parallel to $O\dot{m}$ that cuts $O\dot{l}$ at $\dot{l}$. One can deduce the possible situations that can exist from the signs of N and $\dot{l}$, the position of A with respect to $\alpha\alpha'$, and the sign of the product $T\,dm\,/\,dt$; recall that $dl\,/\,dt = i$ (E) dl and $dm\,/\,dt = i$ (E) dm (cf., Chap. I, § III):

Escape: i (E) $dl > 0$

Rolling without slipping: $N > 0$, A is interior to $\alpha\alpha'$.

The onset of slipping $N > 0$, $T\,dm\,/\,dt < 0$, A external to $\alpha\alpha'$.

Note that the presence of the resistance to rolling parameter will exhibit the cases of indeterminacy and impossibility. Hence, for $a > 0$, $b > 0$, $\varepsilon_1 = -\,1$, $\delta > (k^2 + a^2)\,/\,a$, the vectors **l** and **n** have the arrangement in the adjoining figure (†). Those situations will then result for suitable values of i(E) dl and i (E) dm .

(†) Translator: That figure was not available to me at the time of translation.

Study of the differential equations of motion. – A study of the differential equations of motion can be carried out when one knows that infinitesimal transformations that the generating form Ω admits. The components of the reaction are determined as functions of the external forces and velocities, so it will suffice to express the idea that Ω_l (which corresponds to the free solid body) admits the aforementioned transformations, as well as the differential form of the constraint equation, which is $dl = d\dot{\eta} + a\,d\dot{\alpha} + \dot{a}\,\dot{\alpha}\,d\alpha$.

Hence, impose the infinitesimal transformations on the system that are generated by the fields:

$$\mathrm{T} = (0, 0, 0, 0, 0, 0, 1)\,,$$
$$\Xi = (0, 0, 0, 1, 0, 0, 0)\,,$$
$$\mathrm{H} = (0, 0, 0, 0, 1, 0, 0)\,.$$

One immediately verifies that the form dl admits them. Now, express the idea that Ω_l admits them, while supposing that the system of external forces X, Y, Γ depends upon only the variables ξ, η, α, t :

$$d\,\Omega_l = \left(\frac{\partial X}{\partial \alpha}d\alpha + \frac{\partial X}{\partial \eta}d\eta\right)\wedge d\xi \wedge dt + \left(\frac{\partial Y}{\partial \xi}d\xi + \frac{\partial Y}{\partial \alpha}d\alpha\right)\wedge d\eta \wedge dt$$
$$+ \left(\frac{\partial Z}{\partial \xi}d\xi + \frac{\partial \Gamma}{\partial \eta}d\eta\right)\wedge d\alpha \wedge dt\,.$$

1. $\theta(\mathrm{T})\,\Omega_l = i\,(\mathrm{T})\,d\,\Omega_l + d\,(i\,(\mathrm{T})\,\Omega_l)$

$$= \left(\frac{\partial X}{\partial \alpha}d\alpha + \frac{\partial X}{\partial \eta}d\eta\right)\wedge d\xi + \left(\frac{\partial Y}{\partial \xi}d\xi + \frac{\partial Y}{\partial \alpha}d\alpha\right)\wedge d\eta$$
$$+ \left(\frac{\partial Z}{\partial \xi}d\xi + \frac{\partial \Gamma}{\partial \eta}d\eta\right)\wedge d\alpha - d\,(X\,d\xi + Y\,d\eta + \Gamma\,d\alpha) = 0,$$

hence:

$$-\frac{\partial X}{\partial t}\,d\xi \wedge dt - \frac{\partial Y}{\partial t}d\eta \wedge dt - \frac{\partial \Gamma}{\partial t}\,d\alpha \wedge dt = 0,$$

which implies that:

$$\frac{\partial X}{\partial t} \equiv 0, \qquad \frac{\partial Y}{\partial t} \equiv 0, \qquad \frac{\partial \Gamma}{\partial t} \equiv 0\,.$$

2. $\theta(\mathrm{T})\,\Omega_l = i\,(\mathrm{T})\,d\,\Omega_l + d\,(i\,(\mathrm{T})\,\Omega_l)$

$$= \left(\frac{\partial X}{\partial \alpha}d\alpha + \frac{\partial X}{\partial \eta}d\eta - \frac{\partial Y}{\partial \xi}d\eta - \frac{\partial \Gamma}{\partial \xi}d\alpha\right)\wedge dt + d\,(X\,dt)$$

$$= \frac{\partial X}{\partial \xi}d\xi \wedge dt + \frac{\partial Y}{\partial \xi}d\eta \wedge dt - \frac{\partial \Gamma}{\partial \xi}d\alpha \wedge dt = 0.$$

$$\frac{\partial X}{\partial \xi} \equiv 0, \qquad \frac{\partial Y}{\partial \xi} \equiv 0, \qquad \frac{\partial \Gamma}{\partial \xi} \equiv 0 .$$

3. $\theta(\mathrm{H})\,\Omega_l = 0$ will give:

$$\frac{\partial X}{\partial \eta} \equiv 0, \qquad \frac{\partial Y}{\partial \eta} \equiv 0, \qquad \frac{\partial \Gamma}{\partial \eta} \equiv 0$$

by a similar calculation.

Consequently, X, Y, Γ are functions of only the variable α, which are conditions that we shall suppose to be realized in all of what follows.

Study of case *A*: **motion on a perfectly-smooth plane.** – Since the constraint is holonomic and independent of time, one can construct a reduced form Ω_s (cf., Chap. II, § VII, remark 3) that is obtained by taking the constraint into account:

$$\Omega_s = M\, d\dot{\xi} \wedge d\xi + M(k^2 + a^2)\, d\dot{\alpha} \wedge d\alpha - M\,[\dot{\xi}\, d\dot{\xi} + (k^2 + a^2)\,\dot{\alpha}\, d\dot{\alpha} + a\dot{\alpha}\dot{\alpha}^2\, d\alpha] \wedge dt$$
$$+ [X\, dx + (\Gamma - a\, Y)\, da] \wedge dt .$$

Conforming to the lemma in the case $d\,\Omega \neq 0$, we determine four fields ξ, t, u, v whose components are given by the following table:

	$d\dot{\xi}$	$d\dot{\alpha}$	$d\xi$	$d\alpha$	dt
ξ	$\frac{1}{M}$	$-\frac{\dot{\xi}}{M(k^2+a^2)\dot{\alpha}}$	0	0	0
t	0	$-\frac{1}{M(k^2+a^2)\dot{\alpha}}$	0	0	0
u	0	$\frac{Ma\,d\alpha^2 - \Gamma + aY}{M(k^2+a^2)\dot{\alpha}}$	0	$-\frac{\dot{\alpha}}{M}$	0
v	0	$-\frac{X}{M(k^2+a^2)\dot{\alpha}}$	$-\frac{1}{M}$	0	0

One deduces the following invariant forms from that:

$$\pi^1 = i(\xi)\,\Omega_s = d\xi - \frac{\dot{\xi}}{\dot{\alpha}}\, d\alpha, \qquad i(t)\,\Omega_s = dt - \frac{d\alpha}{\dot{\alpha}},$$

for which the system of completely-integrable forms is:

$$\pi^3 = i\,(u)\,\Omega_s = (k^2 + a^2)\ \dot\alpha\, d\dot\alpha + a\dot a\,\dot\alpha^2\, d\alpha - \frac{\Gamma - aY}{M}\, d\alpha\,,$$

$$\pi^4 = i\,(v)\,\Omega_s = d\dot\xi - \frac{X}{M\dot\alpha}\, d\alpha\,.$$

π^3 is an exact differential, so upon integrating, one will get:

$$\dot\alpha^2(k^2 + a^2) - \frac{2}{M}\int(\Gamma - aY)\, d\alpha = \text{const.}$$

π^4 is closed, modulo π^3, while π^1 and π^2 are forms that are closed modulo π^3, π^4. The system is then integrable by quadratures.

Study of case *B*: **solid body sliding on a rough plane.** – There is no appreciable advantage to using a reduced form, since a convenient choice of fields ξ, t, u, v will permit one to easily construct differential forms that do not depend upon the normal component to the reaction of the plane. We then take (†):

$$\begin{aligned}\Omega = {} & M\, d\dot\xi \wedge d\xi + M\, d\dot\eta \wedge d\eta + Mk^2\, d\dot\alpha \wedge d\alpha - M\,(\dot\xi\, d\xi + \dot\eta\, d\eta + k^2\,\dot\alpha\, d\alpha) \wedge dt \\ & + (X\, d\xi + Y\, d\eta + \Gamma\, d\alpha) \wedge dt\ \pm (?)\ N\,[d\eta + a\, d\alpha + \varepsilon f\,(d\xi - b\, d\alpha) + \varepsilon_1\,\delta\, d\alpha] \wedge dt\end{aligned}$$

Always while applying the lemma to the case $d\,\Omega \neq 0$, we determine six fields ξ, η, t, u, v, w, whose components are given by the following table:

	$d\dot\xi$	$d\dot\eta$	$d\dot\alpha$	$d\xi$	$d\eta$	$d\alpha$	dt
ξ	$\frac{1}{M}$	0	$-\frac{\dot\xi}{Mk^2\dot\alpha}$	0	0	0	0
η	0	$\frac{1}{M}$	$-\frac{\dot\eta}{Mk^2\dot\alpha}$	0	0	0	0
t	0	0	$-\frac{1}{Mk^2\dot\alpha}$	0	0	0	0
u	0	0	$\frac{(a - \varepsilon fb + \varepsilon_1\delta)Y - \Gamma}{Mk^2}$	0	$(a - \varepsilon f\, b + \varepsilon_1\,\delta)\,\dot\alpha$	$-\dot\alpha$	0

(†) Translator: The sign on the last term was not printed in the original text.

v	-1	εf	$-\dfrac{X-\varepsilon f Y}{Mk^2\dot\alpha}$	0	0	0	0
w	0	0	$\dfrac{a\dot\alpha}{Mk^2}$	0	$-\dfrac{1}{M}$	$-\dfrac{a}{Mk^2}$	0

The following invariant forms:

$$\pi^1 = i\,(\xi)\,\Omega = d\xi - \frac{\dot\xi}{\dot\alpha}\,d\alpha, \quad \pi^2 = i\,(\eta)\,\Omega = d\eta - \frac{\dot\eta}{\dot\alpha}\,d\alpha, \quad \pi^3 = i\,(t)\,\Omega = dy - \frac{1}{\dot\alpha}\,d\alpha$$

will imply the completely-integrable system of forms:

$$\pi^4 = i\,(u)\,\Omega = [(a - \varepsilon f\, b + \varepsilon_1\,\delta)\,Y - \Gamma]\,d\alpha - M\,(a - \varepsilon f\, b + \varepsilon_1\,\delta)\,\dot\alpha\,d\dot\eta + M\,k^2\dot\alpha\,d\dot\alpha,$$

$$\pi^5 = i\,(v)\,\Omega = M\,d\,(\dot\xi - \varepsilon f\dot\eta) - (X - \varepsilon f\, Y)\,\frac{d\alpha}{\dot\alpha},$$

$$\pi^6 = i\,(w)\,\Omega = d\dot\eta + a\,d\dot\alpha + \dot a\,\dot\alpha\,d\alpha\,.$$

For the calculation of $i\,(w)\,\Omega$, one takes into account the value of N that was calculated before. The form π^6 is nothing but the differential of the constraint. The form π^4, which takes the constraint into account, is a differential equation that is linear $\dot\alpha^2$, and which one can integrate by quadratures.

$$\tfrac{1}{2}M\,[k^2 + a^2 - \varepsilon f\, b\, a + \varepsilon_1\delta\, a] + M\dot a\,(a - \varepsilon f\, b + \varepsilon_1\delta)\,\dot\alpha^2\,d\alpha + [(a - \varepsilon f\, b + \varepsilon_1\delta)\,Y - \Gamma] = 0.$$

The form π^6 is closed modulo π^4. The forms π^1 and π^3 are closed modulo π^4 and π^5. The system can then be integrated by quadratures.

Study of the case C : **rolling without slipping. –** We distinguish two cases according to whether one does or does not neglect the resistance to rolling couple.

a) $\delta = 0$: The two constraints then have zero power and are independent of time. As we showed in Chapter III, § IV, one can then construct a reduced form by taking the two constraints $\dot\xi - b\,\dot\alpha = 0$, $\dot\eta + a\,\dot\alpha = 0$ into account:

$$\Omega_r = M\,(b^2 + a^2 + k^2)\,d\dot\alpha \wedge d\alpha - M\,[(b^2 + a^2 + k^2)\,\dot\alpha\,d\dot\alpha + (a\dot a + b\dot b)\dot\alpha^2 d\alpha] \wedge dt]$$
$$+ (X\,b - Y\,a + \Gamma)\,d\alpha \wedge dt\,.$$

Since $d\,\Omega_r = 0$, the infinitesimal transformation that is defined by the operator $\theta\,(\mathrm{T})$ will correspond to the first integral:

$$\tfrac{1}{2} M (b^2 + a^2 + k^2)\,\dot{\alpha}^2 - \int (X\, b - Y\, a + \Gamma)\; d\alpha = \text{const.}$$

Since the time t is defined as a function of a by means of the closed form modulo the preceding one:

$$i\,(t)\,\Omega_r = dt - \frac{1}{\dot{\alpha}}\, d\alpha\,.$$

b) $\delta \neq 0$: The constraint $\dot{\xi} - b\dot{\alpha} = 0$ has zero power and is independent of time, while the other one $\dot{\eta} + a\dot{\alpha} = 0$ is not. As a consequence, we shall carry out only the partial reduction that corresponds to the former:

$$\Omega_r = M\,(b^2 + k^2)\; d\dot{\alpha} \wedge d\alpha + M\, d\dot{\eta} \wedge d\eta - M\,[(b^2 + k^2)\;\dot{\alpha}\, d\dot{\alpha} + b\dot{b}\,\dot{\alpha}^2\, d\alpha + \dot{\eta}\, d\dot{\eta}] \wedge dt$$
$$+ (X\, b + \Gamma)\; d\alpha \wedge dt + Y\, d\eta \wedge dt + N\,(d\eta + a\; d\alpha + \varepsilon_1\;\delta\, d\alpha) \wedge dt\,.$$

One determines the fields η, t, u, w, while always employing the same principle, and by way of the operator i ( ), those fields generate:

1. The invariant forms π^1, π^2 :

$$\pi^1 = i\,(\eta)\,\Omega_r = d\eta - \frac{\dot{\eta}}{\dot{\alpha}}\, d\alpha\,, \qquad\qquad \pi^2 = i\,(t)\,\Omega_r = dt - \frac{1}{\dot{\alpha}}\, d\alpha\,.$$

2. The forms π^3, π^4, constitute a completely-integrable system:

$$\pi^3 = i\,(u)\,\Omega_r = M\,(b^2 + k^2)\;\dot{\alpha}\, d\dot{\alpha} + M\, b\dot{b}\,\dot{\alpha}^2 d\alpha - M\,(a + \varepsilon_1\delta)\,\dot{\alpha}\, d\dot{\eta}$$
$$+ [X\, b + \Gamma - (a + \varepsilon_1\;\delta)\; Y]\; da$$
$$\pi^4 = i\,(w)\,\Omega_r = d(\dot{\eta} + a\dot{\alpha})\,,$$

π^3 takes into account the latter, which is the differential of the constraint relation, is a differential equation that is linear in $\dot{\alpha}^2$, and which can be integrated by quadratures:

$$\tfrac{1}{2} M\,(b^2 + k^2 + a^2 + a\;\varepsilon_1\;\delta)\;\; d\dot{\alpha}^2 + M\,(b\dot{b} + a\dot{a} + \dot{a}\,\varepsilon_1\delta)\,\dot{\alpha}^2 d\alpha - [Xb + \Gamma - (a + \varepsilon_1\;\delta)\; Y]\; d\alpha$$
$$= 0\,.$$

π^2, which is closed modulo π^3, defines the time t as a function of α by means of a quadrature.

IV. – A heavy, homogeneous solid of revolution of mass M and center of gravity G is extended by a rod that points along its axis of revolution Gz . That solid body is in contact with a fixed horizontal plane $Ox_1\, y_1$, so the rod Gz slides without friction in an orientable slot whose axis is the fixed normal Oz_1 to the plane $Ox_1\, y_1$. We shall study the rolling without slipping of the solid body on the plane, and then its slipping, while the coefficient of friction f of the solid body on the plane is constant.

The meridian of the surface of revolution that bounds the solid body is defined to be the envelope of the tangent planes to the surface. With respect to a system of axes that are fixed in the body, if λ is the polar angle of Gz between the normal GH to the tangent plane then one will have:

$$z \cos \lambda - v \sin \lambda - b(\lambda) = 0,$$
$$z \sin \lambda + v \cos \lambda - b'(\lambda) = 0 .$$

θ is the angle (Gz_1, Gz), so $\theta + \lambda = \pi$, P is the contact point of the body and the plane, and the components of GP with respect to the axes Gv_1 (horizontal to the plane Oz_1G) and Gz_1 (viz., ascending vertical) are $b'(\pi - \theta)$, $-b(\pi - \theta)$.

Let Mc^2, Ma^2 be the moments of inertia of the solid body with respect Gz and a fixed perpendicular axis, respectively. Let ρ, α, ζ be the cylindrico-polar coordinates of G. Let ψ, θ, φ be the classical Euler angles, which are the rotation parameters of the body around G.

When one uses the trihedron $Guvz$ (Gu horizontal), which moves in the body and in space, the associated exterior form to the free solid will be obtained either by applying the formula that was established in Chapter I, § II.*c* with $\Omega^1 = \omega^1 = d\theta$, $\Omega^2 = \omega^2 = \sin\theta\, d\psi$, $\Omega^3 = \omega^3 \cot \theta = \cos \theta\, d\psi$ for the motion around G or directly as the exterior derivative of the Pfaff form that generates the integral invariant:

$$\dot{\rho}\, d\rho + u\, d\alpha + \dot{\zeta}\, d\zeta + a^2 p\, \omega^1 + a^2 q\, \omega^2 + c^2 r\, \omega^3 - H\, dt,$$

with

$$\omega^1 = d\theta, \qquad \omega^2 = \sin \theta\, d\psi, \qquad \omega^3 = d\varphi = \cos \theta\, d\psi, \quad u = \rho^2 \dot{\alpha}, \; p, q, r,$$

which are components of the absolute rotation velocity of the body along the moving axes $Guvw$:

$$2H = \dot{\rho}^2 + \frac{u^2}{\rho^2} + \dot{\zeta}^2 + a^2 (p^2 + q^2) + c^2 r^2 + g\, \zeta .$$

One will then obtain:

$$\frac{\Omega}{M} = d\dot{\rho} \wedge d\rho + du \wedge d\alpha + d\zeta \wedge d\zeta - \left(\dot{\rho}\, d\dot{\rho} + \dot{\zeta}\, d\dot{\zeta} + \frac{u}{\rho^2} du - \frac{u^2}{\rho^3} d\rho \right) \wedge dt$$

$$- g\, d\zeta \wedge dt + a^2\, dp \wedge \omega^1 + a^2\, dq \wedge \omega^2 + c^2\, dr \wedge \omega^3 - c^2 r\, \omega^1 \wedge \omega^2 - a^2 r \cot q\, \omega^2 \wedge \omega^1$$

$$- [a^2 (p\, dp + q\, dq) + c^2 r\, dr] \wedge dt .$$

The characteristic field E for the free solid body has the components:

$$\frac{u^2}{\rho^3}, \; 0, \; -g, \quad q^2 \cot \theta - \frac{c^2}{a^2} r\, q,$$

$$\frac{c^2}{a^2} r\, q - pq \cot\, \theta, 0,\ \dot{\rho}\,,\ \frac{u}{\rho^2}\,,\ \dot{\zeta}\,, p, q, r, 1,$$

while the differentials of the variables are arranged in the sequence: $d\dot{\rho}$, du, $d\dot{\zeta}$, dp, dq, dr, dr, $d\alpha$, $d\zeta$, ω^1, ω^2, ω^3, dt .

A. Study of a solid body rolling without slipping.

1) **Constraints.** – The rod Gz always remains in the plane OzG, which translates into the holonomic constraint $\alpha = \psi + \pi / 2$. From our point of view, it is a zero-power constraint:

$$a_1 = \frac{u}{\rho^2} - \frac{\sin\theta}{q} = 0, \qquad P^1 = \frac{L}{M}(\dot{\alpha} - \dot{\psi})\,,$$

so the form is:

$$\Omega^1 = \frac{L}{M}\left(d\alpha - \frac{\omega^2}{\sin\theta}\right) \wedge dt\,.$$

Rolling without slipping of the body on the plane translates into three zero-power constraints. Upon writing out that the components of $\mathbf{V}_p$ are zero with respect to the trihedron Gu, Gv, Gy, one will get:

$$a^2 = -\frac{u}{\rho} - (b' \sin\,\theta + b \cos\,\theta)\, q + (b \sin\,\theta - b' \cos\,\theta)\, r = 0,$$

$$P^2 = \frac{X}{M}[-\rho\dot{\alpha} - (b' \sin\,\theta + b \cos\,\theta)\, q + (b \sin\,\theta - b' \cos\,\theta)\, r],$$

$$\Omega^2 = \frac{X}{M}[-\rho\, d\alpha - (b' \sin\,\theta + b \cos\,\theta)\, \omega^2 + (b \sin\,\theta - b' \cos\,\theta)\, \omega^3] \wedge dt\,,$$

$$a^3 = \dot{\rho} + bp = 0, \qquad P^3 = \frac{Y}{M}(\dot{\rho} + bp) = 0, \qquad \Omega^3 = \frac{Y}{M}(dr + b\, d\theta),$$

$$a^4 = \dot{\zeta} + b'p = 0, \qquad P^4 = \frac{Z}{M}(\dot{\zeta} + bp) = 0, \qquad \Omega^4 = \frac{Z}{M}(d\zeta + b'\, d\theta)\,.$$

The first six non-zero components of the constraint fields e^1, e^2, e^3, e^4 are given in the following table (the constraint fields have only n components that are not all zero in Hamiltonian coordinates).

	$d\dot{\rho}$	du	$d\zeta$	dp	dq	dr
e^1	0	1	0	0	$-\dfrac{1}{a^2\sin\theta}$	0
e^2	0	$-\rho$	0	0	$-\dfrac{b\cos\theta+b'\sin\theta}{a^2}$	$\dfrac{b\sin\theta-b'\cos\theta}{c^2}$
e^3	1	0	0	$\dfrac{b}{a^2}$	0	0
e^4	0	0	1	$\dfrac{b'}{a^2}$	0	0

The compatibility condition on the set of constraints is deduced from the determinant of the symmetric square matrix $\| i\,(e^h)\, da^k \|$:

$$\left\| \begin{matrix} \dfrac{1}{\rho^2}+\dfrac{1}{a^2\sin^2\theta} & -\dfrac{1}{\rho}+\dfrac{b'\sin\theta+b\cos\theta}{a^2\sin^2\theta} & 0 & 0 \\ -\dfrac{1}{\rho}+\dfrac{b'\sin\theta+b\cos\theta}{a^2\sin^2\theta} & 1+\dfrac{(b\cos\theta+b'\sin\theta)^2}{a^2}+\dfrac{(b\sin\theta-b'\cos\theta)^2}{c^2} & 0 & 0 \\ 0 & 0 & 1+\dfrac{b^2}{a^2} & \dfrac{bb'}{a^2} \\ 0 & 0 & \dfrac{bb'}{a^2} & 1+\dfrac{b'}{a^2} \end{matrix} \right\|,$$

namely:

$$\left(1+\frac{b^2}{a^2}+\frac{b'^2}{a^2}\right)\left[\left(\frac{1}{a\sin\theta}+\frac{b\cos\theta+b'\sin\theta}{a\rho}\right)^2+\frac{(b\sin\theta-b'\cos\theta)^2}{c^2}\left(\frac{1}{\rho^2}+\frac{1}{a^2\sin^2\theta}\right)\right]\neq 0.$$

The values of the reactions are obtained by calculating the i (E) da^k (k = 1 to 4):

$$i\,(\mathrm{E})\, da^1 = -\frac{c^2}{a^2\sin\theta}rp+\frac{2pq\cos\theta}{\sin^2\theta}-\frac{2u\dot{\rho}}{\rho^3},$$

$$i\,(E)\;da^2 = (b\cos\theta + b'\sin\theta)\left(pq\cot\theta - \frac{c^2}{a^2}rp\right) + \frac{u\dot\rho}{\rho^2} + (b + b'')(q\sin\theta + r\cos\theta)\,p,$$

$$i\,(E)\;da^3 = \frac{u^2}{\rho^2} + \left(q^2\cot\theta - \frac{c^2}{a^2}r\,p\right) - b'p^2,$$

$$i\,(E)\;da^4 = -\,g + b'\left(q^2\cot\theta - \frac{c^2}{a^2}r\,q\right) - b''\,p^2,$$

so

$$\left(1 + \frac{b^2}{a^2} + \frac{b'^2}{a^2}\right)\frac{Y}{M} = -\left(1 + \frac{b'^2}{a^2}\right)\frac{u^2}{\rho^2} - \frac{bb'}{a^2}g + b'p^2\left(1 + \frac{b'^2}{a^2} - \frac{bb''}{a^2}\right) + bq\left(\frac{c^2}{a^2}r - q\cot\theta\right),$$

$$\left(1 + \frac{b^2}{a^2} + \frac{b'^2}{a^2}\right)\frac{Z}{M} = -\frac{bb'}{a^2}\frac{u^2}{\rho^3} + \left(1 + \frac{b^2}{a^2}\right)g + p^2\left(b'' + b\frac{bb'' - b'^2}{a^2}\right) + b'q\left(\frac{c^2}{a^2}r - q\cot\theta\right),$$

$$\Delta\frac{X}{M} = \left(\frac{1}{\rho} - \frac{b\cos\theta + b'\sin\theta}{a^2\sin\theta}\right)\left(\frac{c^2 rp}{a^2\sin\theta} - \frac{2pq\cot\theta}{\sin\theta} + \frac{2u\dot\rho}{\rho^3}\right)$$

$$+\left(\frac{1}{\rho^2}\frac{1}{a^2\sin^2\theta}\right)\left[(b\cos\theta + b'\sin\theta)\left(\frac{c^2 rp}{a^2} - pq\cot\theta - \frac{u\dot\rho}{\rho^2}\right) - (b + b'')\,p(q\sin\theta + r\cos\theta)\right],$$

$$\Delta\frac{L}{M}$$

$$= \left(-\frac{1}{\rho} + \frac{b\cos\theta + b'\sin\theta}{a^2\sin\theta}\right)\left[\frac{u\dot\rho}{\rho^2} + (b + b'')\,p\,(q\sin\theta + r\cos\theta) + (b\cos\theta + b'\sin\theta)\left(pq\cot\theta - \frac{c^2 rp}{a^2}\right)\right]$$

$$+\left[1 + \frac{(b\cos\theta + b'\sin\theta)^2}{a^2} + \frac{(b\sin\theta - b'\cos\theta)^2}{c^2}\right]\left(\frac{c^2 rp}{a^2\sin\theta} - \frac{2pq\cot\theta}{\sin\theta} + \frac{2u\dot\rho}{\rho^3}\right),$$

in which Δ denotes:

$$\left(\frac{1}{a\sin\theta} + \frac{b\cos\theta + b'\sin\theta}{a\rho}\right) + \frac{(b\sin\theta - b'\cos\theta)^2}{c^2}\left(\frac{1}{\rho^2} + \frac{1}{a^2\sin\theta}\right).$$

2. Differential equations of motion. – The compatible constraints give:

$$u = \frac{\rho^2 q}{\sin\theta}, \qquad r = q\,\frac{\dfrac{\rho}{\sin\theta} + b\cos\theta + b'\sin\theta}{b\sin\theta - b'\cos\theta}, \qquad \dot\rho = -\,bp, \qquad \dot\zeta = -\,bp\,.$$

($b \sin \theta - b' \cos \theta \neq 0$, since the envelope does not reduce to a point.)

Since they have zero power, one can replace $d\alpha$ with $\varpi^2 / \sin \theta$, ϖ^2 with $\dfrac{\dfrac{\rho}{\sin\theta} + b\cos\theta + b'\sin\theta}{b\sin\theta - b'\cos\theta}$, $d\rho$ with $-b\, d\theta$, and $d\zeta$ with $-b'\, d\theta$ in Ω in order to obtain a reduced form. However, $d\rho + b\, d\theta$ is an exact differential, where ρ is a function of θ, namely, $\rho = B(\theta) + C$ (C is a constant). Since α, φ, ζ do not enter into Ω, one can obtain a reduced form Ω_s / M that is defined on a family of manifolds V_5 [submanifolds of V_{13} that are defined by the constraints and the family $\rho = B(\theta) + C$] to be the exterior differential of the Pfaff form:

$$(b^2 + b'^2 + a^2)\, p\, d\theta + \left[\frac{(B+C)^2}{\sin^2\theta} + a^2 + c^2\left(\frac{\dfrac{B+C}{\sin\theta} + b\cos\theta + b'\sin\theta}{b\sin\theta - b'\cos\theta}\right)^2\right] q\varpi^2 - H_1 dt\,,$$

with

$$2H_1 = (b^2 + b'^2 + a^2)\, p^2 + \left[\frac{(B+C)^2}{\sin^2\theta} + a^2 + c^2\left(\frac{B+C+\sin\theta(b\cos\theta + b'\sin\theta)}{\sin\theta\,(b\sin\theta - b'\cos\theta)}\right)^2\right] q - 2bg\,.$$

Since b, b', B are functions of θ, $q = \dot\psi \sin\theta$, $\varpi^2 = \sin\theta\, d\psi$, one can perform the changes of the velocity parameters:

$$\bar p = (b^2 + b'^2 + a^2)\, p,$$

$$\bar q = \left[(B+C)^2 + a^2\sin^2\theta + c^2\left(\frac{B+C+\sin\theta(b\cos\theta + b'\sin\theta)}{b\sin\theta - b'\cos\theta}\right)^2\right]\frac{q}{\sin\theta},$$

$$\frac{\Omega_s}{M} = d(\bar p\, d\theta + \bar q\, d\psi - \bar H_1 dt)\,,$$

with

$$2\bar H_1 = \frac{1}{b^2 + b'^2 + a^2}\,\bar p^2 + \frac{\bar q^2}{(B+C)^2 + a^2\sin^2\theta + c^2\left(\dfrac{B+C+\sin\theta(b\cos\theta + b'\sin\theta)}{b\sin\theta - b'\cos\theta}\right)^2} - 2bg\,.$$

Since $d\left(\dfrac{\Omega_s}{M}\right)$ is zero, any infinitesimal transformation will correspond to a first integral, so the infinitesimal transformations:

$$t = (0, 0, 0, 0, 1) \qquad \text{and} \qquad \Psi = (0, 0, 0, 1, 0)$$

will correspond to the *vis viva* integral and the linear integral $\bar{q}$ = const.; integration can be achieved by quadratures.

B. Study of the solid body slipping on the plane.

1. Constraints. – A rod sliding without friction in a slot is always characterized by:

$$a^1 = \frac{u}{\rho^2} - \frac{q}{\sin\theta} = 0, \qquad \Omega^1 = \frac{L}{M}\left(d\alpha - \frac{\omega^2}{\sin\theta}\right) \wedge dt\,.$$

Contact between the solid body and the plane is characterized by $a^2 = \zeta + b'\,p = 0$:

$$\Omega^2 = \frac{N}{M}\{dz + f\cos\sigma[\rho\,d\alpha + (b'\sin\theta + b\cos\theta)\,\omega^2 + (b'\cos\theta - b\sin\theta)\,\omega^3]$$
$$- f\sin\sigma(d\rho + b\,\omega^1)\} \wedge dt\,,$$

in which σ is the angle between the velocity of sliding $\mathbf{V}_g$ and Gu, and λ is the magnitude of $\mathbf{V}_g$:

$$\frac{\dot{\rho} + bp}{\sin\sigma} = \frac{-u/\rho - b'(r\cos\theta + q\sin\theta) + b(r\sin\theta - q\cos\theta)}{\cos\sigma} = \lambda\,.$$

The constraint fields have the components:

$$e^1 = (0, 1, 0, 0, -\frac{1}{a^2\sin\theta}, 0),$$

$$e^2 = (-f\sin\sigma, \rho f\cos\sigma, 1, -\frac{bf\sin\sigma}{a^2}, \frac{f\cos\sigma}{a^2}(b\cos\theta + b'\sin\theta),$$
$$\frac{f\cos\sigma}{c^2}(b'\cos\theta - b\sin\theta)),$$

so

$$i(e^1)\,da^1 = \frac{1}{\rho^2} + \frac{1}{a^2\sin^2\theta}, \qquad i(e^2)\,da^1 = \frac{f\cos\sigma}{\rho} - \frac{f\cos\sigma(b\cos\theta + b'\sin\theta)}{a^2\sin\theta},$$

$$i(e^1)\,da^2 = 0, \qquad i(e^2)\,da^2 = 1 - f\,\frac{bb'}{a^2}\sin\sigma.$$

The constraints are compatible if:

$$\left(\frac{1}{\rho^2} + \frac{1}{a^2\sin^2\theta}\right)\left(1 - f\,\frac{bb'}{a^2}\sin\sigma\right) \neq 0\,.$$

The constraint factors are deduced from i (E) da^1, i (E) da^2 by calculation:

$$N = \frac{g + b'' p^2 + b'\left(\frac{c^2}{a^2} rq - q^2 \cot\theta\right)}{1 - f\frac{bb'}{a^2}\sin\sigma},$$

$$L\left(\frac{1}{\rho^2} + \frac{1}{a^2\sin^2\theta}\right) = \frac{2u\dot\rho}{\rho^3} + \frac{c^2 rp}{a^2\sin\theta} - 2pq\frac{\cos\theta}{\sin^2\theta}$$
$$+ \frac{f\cos\sigma}{1 - f\frac{bb'}{a^2}\sin\sigma}\left[g + b'' p^2 + b'\left(\frac{c^2}{a^2} rq - q^2\cot\theta\right)\right].$$

2. Differential equations of motion. – The first holonomic constraint permits one to eliminate u and $d\alpha$, $u = \rho^2\dot\psi$, $d\alpha = d\psi$.

The introduction of the variables λ and σ leads one to perform the change of variables:

$$\dot\rho = \lambda\sin\sigma - bp, \qquad r = \frac{\lambda\cos\sigma + \rho\dot\psi + \sin\theta\,(b'\sin\theta + b\cos\theta)\psi}{b\sin\theta - b'\cos\theta}.$$

The second constraint permits one to eliminate $\dot\zeta$ and $d\zeta$, $\dot\zeta = -b'\,p$:

$$d\zeta = f\sin\sigma\,(d\rho + b\,d\theta) - f\cos\sigma\,[(\rho + b')\,d\psi + (b'\cos\theta - b\sin\theta)\,d\varphi + (\dot\zeta - f\lambda)\,dt].$$

The reduced form Ω_s / M depends upon the nine differentials of the variables λ, σ, ρ, $\dot\psi$, ψ, φ, q, p, t, but it does not depend upon the variables t, ψ, φ :

$$\begin{aligned}\frac{\Omega_s}{M} = d\,\{&(\lambda\sin\sigma - bp)\,d\rho + \rho^2\,\dot\psi\,d\psi - b'\,p\,f\sin\sigma\,(d\rho + b\,d\theta)\\
&+ b'\,p\,f\cos\sigma\,[(\rho + b')\,d\psi + (b'\cos\theta - b\sin\theta)\,d\varphi]\\
&+ b'\,p\,(b\,p + f\lambda)\,dt + a^2\,p\,d\theta + a^2\,\dot\psi\,\sin^2\theta\,d\psi\\
&+ c^2\frac{\lambda\cos\sigma + \rho\dot\psi + \sin\theta\,(b\cos\theta + b'\sin\theta)}{b\sin\theta - b'\cos\theta}(d\varphi + \cos\theta\,d\psi) - T\,dt\}\\
&+ g\,\{f\sin\sigma\,(d\rho + b\,d\theta) - f\cos\sigma\,[(\rho + b')\,d\psi + (b'\cos\theta - b\sin\theta)\,d\varphi)]\}\wedge dt,\end{aligned}$$

with

$$2T = (\lambda\sin\sigma - b\,p)^2 + \rho^2\dot\psi^2 + b^2\,p^2 + a^2\,(p^2 + y^2\sin^2\theta +$$
$$c^2\left(\frac{\lambda\cos\sigma + \rho\dot\psi + \sin\theta(b\cos\theta + b'\sin\theta)\,\dot\psi}{b\sin\theta - b'\cos\theta}\right)^2.$$

One knows only three obvious infinitesimal transformations for Ω_s / M, which are generated by the fields:

$$t = (0, 0, 0, \ldots, 1), \qquad \varphi = (0, 0, \ldots, 0, 0, 1, 0), \qquad \Psi = (0, 0, \ldots, 0, 1, 0, 0),$$

because it will suffice to remark that if an antisymmetric tensor $k_{\alpha\beta}$ does not depend upon one of the variables ρ^{2n} then the form $\Omega = k_{\alpha\beta}\, d\rho^{\alpha} \wedge d\rho^{\beta}$ that is defined on V_{2n+1} will admit the infinitesimal transformation (0, 0, …, 1). The solution to the problem depends upon integrating a system of five completely integrable Pfaff forms in $d\lambda$, $d\sigma$, $d\dot{\lambda}$, $d\rho$, $d\theta$, which one does not know in finite terms.