“Application des formes extérieures du 2e ordre à la dynamique newtonienne et relativiste,” Ann. Inst. Fourier **3** (1951), 277-285.

The application of second-order exterior forms to Newtonian and relativistic dynamics

By **François GALLISOT** (Grenoble)

Translated by D. H. Delphenich

Introduction

Mechanics is generally conceived of in terms of kinematical quantities such as velocity and acceleration and dynamical ones like force, with the differential of the velocity vector being coupled to the product of the force vector and *dt* by Newton’s principle. When one assumes that mechanical phenomena are capable of being described by differential equations, it becomes interesting to have a generating form for the differential equations that is completely invariant under the transformations of the point group that acts upon a set of 2*n* position and velocity variables.

From our viewpoint in Newtonian mechanics, a material point of mass *m* is framed by seven variables x^i, t, v^i (i varies from 1 to 3), to which one associates an exterior form that is constructed from the differentials dx^i, dt, dv^i :

$$\omega = m\,\delta_{ij}\,dv^i \wedge dx^j - m\,\delta_{ij}\,v^i\,dv^j \wedge dt + \delta_{ij}\,X^i\,dx^j \wedge dt$$

(δ_{ij} is the Kronecker symbol, and X^i are the components of the force *F* that is applied to the point), ω is invariant under the transformations of the Galilean group, and its expression will have the same form with respect to any orthonormal Galilean frame. The differential equations of motion are the associated equations to ω:

$$\frac{\partial\omega}{\partial(dx^j)} = -\,m\,dv^i + X^i\,dt = 0, \qquad \frac{\partial\omega}{\partial(dv^i)} = m\,(dx^j - v^j\,dt) = 0\,.$$

It is essential to point out that it is the associated equations to ω that couple the parameters v^i to the differentials of the parameters of position x^i and time.

We have shown ([1]) that in the context of classical mechanics for a material system with 2*n* position and velocity parameters, one can always associate a second-order Cartan

---

([1]) Cf., Communication au Congrès des Sociétés Savantes, Grenoble, 1952, in press. C. R. Acad. Sci. **234** (1952), 2148-2150.

exterior form whose associated equations define the differential equations of motion, when they are combined with the constraints of an arbitrary nature that are imposed.

I. – Establishing the dynamical equations of continuous media in Newtonian mechanics.

1. Lemma:

If one is given a second-order exterior form ω that is defined on a manifold V_{2n} then one can associate it with a bilinear form $\omega(\delta, d)$. The equations of motion will annul $\omega(\delta, d)$ for any choice of the δ.

Let:

$$\omega = k_{\alpha\beta}\, d\rho^\alpha \wedge d\rho^\beta - b_{\alpha_0}\, d\rho^\alpha \wedge dt,$$

in which $k_{\alpha\beta}$ is an antisymmetric tensor, and b_{α_0} is a vector.

One can associate ω with the bilinear form:

$$\omega(\delta, d) = \delta\rho^\alpha \wedge \frac{\partial\omega}{\partial(d\rho^\alpha)} + \delta\rho^\beta \wedge \frac{\partial\omega}{\partial(d\rho^\beta)} + \delta t \wedge \frac{\partial\omega}{\partial(dt)},$$

or

$$\omega(\delta, d) = k_{\alpha\beta}\,[\delta\rho^\alpha \wedge d\rho^\beta - \delta\rho^\beta \wedge d\rho^\alpha] - b_{\alpha_0}\,[\delta\rho^\alpha \wedge dt - \delta t \wedge d\rho^\alpha].$$

If d denotes a differential element of the integral manifold V_1 in V_{2n+1}, and the differential equations of motion are:

(1) $$\frac{\partial\omega}{\partial(d\rho^\alpha)} = 0, \qquad \frac{\partial\omega}{\partial(d\rho^\beta)} = 0, \qquad \frac{\partial\omega}{\partial(dt)} = 0$$

(which is a consequence of the preceding $2n$ equations) then one will indeed have $\omega(\delta, d) = 0$ for any δ on an integral manifold.

Remarks 1:

1. The preceding result can be further expressed as: For any one-dimensional manifold γ in the $(2n + 1)$-dimensional space, in order for the manifold V_2 that is generated by the integral manifolds V_1 that are based upon γ, one will have that the integral $\int_{V_2} \omega(\delta, d) = 0$.

2. Since the form $\omega(\delta, d)$ is zero on any manifold V_2 that is generated by the integral manifolds V_1 that are based upon γ, $\omega(\delta, d)$ will be a linear form in the first integrals of the differential system (1).

If one takes $\delta = d$ then since $\omega = \frac{1}{2}(d, d)$, one can deduce that the exterior form ω is expressed solely in terms of the differentials of the first integrals of the system (1).

$$\omega = k_{ij}\ dc^i \wedge dc^j, \qquad \text{the } k_{ij} \text{ are functions of } c^i,\ c^j, \text{ and a variable } t.$$

3. $d\omega = 0$, where ω is expressed in terms of the differentials of the first integrals:

$$\omega = k_{ij}\ dc^i \wedge dc^j .$$

Since ω is a closed form, there will locally exist a $\overset{*}{\omega}$ such that $d\overset{*}{\omega} = \omega$. The form $\overset{*}{\omega}$ is likewise expressed in terms of the differentials of the first integrals of the system; $\overset{*}{\omega}$ is a linear integral invariant of the system ([1]) in sense of E. Cartan.

2. Equations of continuous media. – Consider a three-dimensional medium that is referred to a curvilinear coordinate system q^i (i varies from 1 to 3). Let V be a finite volume, and let ΔV a volume element; let ρ be the density of matter at a point, and let ΔS be a surface element on the boundary of V.

Associate the volume V with the form:

$$\Omega\ (\delta, d) = \int_V \rho\, \omega_C\ \Delta V + \int_V \omega_{f_V}\ \Delta V + \int_{FV} \omega_{f_S}\ \Delta S\ ,$$

with

(2) $$\omega_C = \delta p_i \wedge dq^i - \delta q^i \wedge dp_i - \delta T \wedge dt,$$

which is the kinetic part of ω in Hamiltonian form.

(3) $$\omega_{f_V} = \sum_{i=1}^{3} Q_i\, \delta q^i \wedge dt$$

is the dynamical part of ω that corresponds to the volume forces, and:

(4) $$\omega_{f_S}\ \Delta S \sum_{i,j=1}^{3} -\Delta S_j T\ \delta^{ij} q_i \wedge dt$$

is the dynamical part of ω that corresponds to the surface forces.

Transform the integral $\int_{FV} \omega_{f_S}\, \Delta S$ by using Stokes's formula, and let D denote the symbol of absolute derivation.

$$\int_{FV} \omega_{f_S}\, \Delta S = dt \wedge \int_{FV} T^{ij}\, \delta q_i\, \Delta S_j = dt \wedge \int_V \left[\frac{\mathrm{D}(T^{ij})}{\mathrm{D}q^j} \delta q_i + T^{ij} \frac{\mathrm{D}(\delta q_i)}{\mathrm{D}q^j} \right] \Delta V .$$

The components δq^i enter into (2) and (3) in a contravariant form. One can put them into covariant form by means of the contravariant metric tensor g^{ij}, $\delta q^i = g^{ij}\, \delta q_j$:

$$\omega_C = \delta p_i \wedge \left(dq^i - \frac{\partial T}{\partial p_i} dt \right) - g^{ij} \left(dp_j - \frac{\partial T}{\partial p^j} dt \right) \wedge \delta q_i ,$$

$$\omega_{f_V} = \sum_{i=1}^{3} Q_i\, \delta q^i \wedge dt ,$$

upon introducing the contravariant components of the volume forces, so:

$$\begin{aligned} \Omega = &- \int_V \left[\rho\, g^{ij} \left(dp_j - \frac{\partial T}{\partial p^j} dt \right) - Q^i dt + \frac{\mathrm{D}(T^{ij})}{\mathrm{D}q^j} dt \right] \wedge \delta q_i\, \Delta V \\ &+ \int_V \rho\, \delta p_i \wedge \left[dp^i - \frac{\partial T}{\partial p^i} dt \right] \Delta V + dt \wedge \int_V T^{ij} \frac{\mathrm{D}(T^{ij})}{\mathrm{D}q^j} \Delta V . \end{aligned} \tag{5}$$

In the expression for Ω, we have not taken the internal forces in the volume V into account, because those forces are completely unknown. From the postulate of classical mechanics that the power generated by the internal forces is zero for a velocity field that is a field of moments, which is equivalent to postulating that the work done by internal forces is zero, if δq denotes a displacement in the three-dimensional space then we shall consider δq to be something that corresponds to the displacements in what follows. One has the Killing equations:

$$\frac{\mathrm{D}(\delta q_i)}{\mathrm{D}q^j} + \frac{\mathrm{D}(\delta q_j)}{\mathrm{D}q^i} = 0. \tag{6}$$

For each point of the medium, if the δq satisfy equations (6) then the d that correspond to the differential equations of motion for that point will be such that Ω = 0.

a) Upon choosing the δq to be the translations that are solutions to the equations:

$$\frac{\mathrm{D}(\delta q_i)}{\mathrm{D}q^j} = 0, \tag{7}$$

when Ω is in the form (5), it will reduce to:

$\Omega\,(\delta, d)$

$$= - \int_V \left\{ \left[\rho\, g^{ij} \left(dp_j - \frac{\partial T}{\partial p^j} dt \right) - Q^i dt + \frac{\mathrm{D}(T^{ij})}{\mathrm{D}q^j} dt \right] \wedge \delta q_i - \rho\, \delta p_i \wedge \left(dq^i - \frac{\partial T}{\partial p_i} dt \right) \right\} \Delta V \, .$$

The integral on the right-hand side must be zero for any choice of continuous functions that are solutions to (7) and any δp_i that are arbitrary continuous functions, so the following six equations will result:

(8)
$$\begin{cases} \rho\, g^{ij} \left(dp_j - \dfrac{\partial T}{\partial q^j} dt \right) - Q^i dt + \dfrac{\mathrm{D}(T^{ij})}{\mathrm{D}q^j} dt = 0, \\ dq^i - \dfrac{\partial T}{\partial p_i} dt = 0, \end{cases}$$

which are reducible to the three classical second-order equations:

(9)
$$\rho\, \gamma^i = Q^{\,i} - \frac{\mathrm{D}(T^{ij})}{\mathrm{D}q^j}, \qquad \gamma^i = \ddot{q}^{\,i} + \Gamma^i_{jk}\, \dot{q}^{\,j}\, \dot{q}^{\,k},$$

in which Γ^i_{jk} is the Christoffel symbol of the second kind.

b) Upon choosing the δq to be arbitrary displacements and taking equations (8) into account, Ω will reduce to:

$$\Omega\,(\delta, d) = dt \wedge \int_V T^{ij} \frac{\mathrm{D}(\delta q_i)}{\mathrm{D}q^j} \Delta V = \frac{dt}{2} \wedge \int_V \left(T^{ij} \frac{\mathrm{D}(\delta q_i)}{\mathrm{D}q^j} + T^{\,ji} \frac{\mathrm{D}(\delta q_j)}{\mathrm{D}q^i} \right) \Delta V \, ,$$

or, upon taking (6) into account:

$$\Omega\,(\delta, d) = dt \wedge \int_V T^{ij} \frac{\mathrm{D}(\delta q_i)}{\mathrm{D}q^j} \Delta V = \frac{dt}{2} \int_V (T^{ij} - T^{\,ji}) \frac{\mathrm{D}(\delta q_i)}{\mathrm{D}q^j} \Delta V \, .$$

Since Ω is zero for a choice of continuous functions that are solutions to (6) and $\dfrac{\mathrm{D}(\delta q_i)}{\mathrm{D}q^j} \neq 0$, and the preceding integral is zero, one must have $T^{\,ij} = T^{\,ji}$, which shows that the constraint tensor is symmetric.

Remarks 1:

1. Upon taking the continuity equation into account, one can put the nine equations into the form that was pointed out by Lichnerowicz ([2]), which will then lead to the equations of the relativistic mechanics of continuous media:

$$\frac{\partial(\rho v^i)}{\partial t}+\frac{\mathrm{D}(\rho v^i v^j + T^{ij})}{\mathrm{D}q^j} = Q^i .$$

2. In order to establish the Killing equations (6), it will suffice to write down the idea that the absolute differential of the square of the element of length *ds* is zero under a point transformation.

II. – Mechanical equations of a point in special relativity.

Special relativity is based upon the following postulates:

1) The velocity an electromagnetic wave is a constant *c* with respect to any frame *R* on the space-time manifold V_4 .

Upon assuming that V_4 can be referred to some Galilean coordinates *x*, *y*, *z*, *t*, which are formed by a tri-rectangular trihedron that is associated with a time variable, that postulate will imply the existence of the fundamental metric invariant ([3]):

(1) $$ds^2 = c^2\, dt^2 - dx^2 - dy^2 - dz^2,$$

in which *ds* denotes the arc length of the trajectory at the point in V_4 .

2) A point is endowed with a rest energy e_0 , which an essentially-positive number.

3) A force **F** that acts upon a material point will be defined with respect to the frame *R* by the contravariant components of a quadri-vector F^i. For **F** = 0, the trajectories are geodesics of V_4 that are defined by the equations that are associated with an exterior form of degree 2.

Axiomatically, associate the point *M* with the second-order exterior form ω that is constructed from the differentials of the four parameters x^i (*i* varies from 1 to 4, with $ct = x^4$) and the differentials of the four parameters u^i that are coupled with the relation:

(2) $$(u^4)^2 - \sum_{i=1}^{3}(u^i)^2 = 1,$$

([2]) Cf., Lichnérowicz, *Éléments de calcul tensoriel*, Armand Colin, no. 259, pp. 157.
([3]) Cf., E. Borel, *Introduction géométrique à quelques théories physique*, Paris, 1914, pp. 8.

(3) $\omega =$

$$e_0\left[c\,du^4\wedge dt-\sum_{i=1}^{3}du^i\wedge dx^i-u^4du^4\wedge ds+\sum_{i=1}^{3}u^i\,du^i\wedge ds\right]+\left[F^4dt-\sum_{i=1}^{3}F^idx^i\right]\wedge ds\,,$$

or

(4) $$\omega = e_0\, k_{ij}\,[du^{\,i}\wedge dx^{\,j} - u^i\, du^{\,j}\wedge ds] + k_{ij}\, F^{\,i}\, dx^{\,j}\wedge ds,$$

with

$$k_{ij} = 0, \qquad \text{if } i \neq j, \qquad \begin{array}{ll} k_{ij} = +1 & \text{for} \quad i = j = 4, \\ k_{ij} = -1 & \text{for} \quad i = j = 1, 2, 3. \end{array}$$

Theorem:

The form ω is invariant under the transformations of the Lorentz group that are applied simultaneously to the variables x^i, u^i and leave the two quadratic forms 1 *and* 2 *invariant.*

Let two Galilean frames be given on V_4 to which one associates the two systems of eight variables:

1. The frame x^i, u^i .
2. The frame ξ^σ, α^σ.

One passes from the first one to the second by means of the formulas:

$$\xi^\sigma = a_i^\sigma x^i + b^\sigma\,, \qquad \alpha^\sigma = a_i^\sigma u^i\,,$$

so

$$d\xi^\sigma = a_i^\sigma dx^i\,, \qquad d\alpha^\sigma = a_i^\sigma du^i\,.$$

The invariance of the quadratic form in (2), $k_{ij}\; u^{\,i}\; u^{\,j} = k_{\rho\sigma}\;\alpha^\rho\,\alpha^\sigma$ implies the following properties for the matrix $A = \|\, a_i^\sigma \,\|$:

$$k_{\rho\sigma}a_i^\rho a_j^\sigma = k_{ij} = \begin{cases} \;\;0 & \text{for} \quad i \neq j, \\ +1 & \text{for} \quad i = j = 4, \\ -1 & \text{for} \quad i = j = 1, 2, 3. \end{cases}$$

Hence:

$$k_{\rho\sigma}\, d\alpha^\rho\wedge d\xi^\sigma = k_{\rho\sigma}a_i^\rho a_j^\sigma du^i\wedge dx^j = k_{ij}\, du^{\,i}\wedge dx^{\,j},$$

$$k_{\rho\sigma}\,\alpha^\rho\, d\alpha^\sigma = k_{\rho\sigma}a_i^\rho a_j^\sigma u^i dx^j = k_{ij}\, u^{\,i}\, dx^{\,j},$$

$$k_{\rho\sigma}\,\Phi^{\rho}\,d\xi^{\sigma} = k_{\rho\sigma}a_i^{\rho}a_j^{\sigma}F^i dx^j = k_{ij}\,F^i\,du^j\,,$$

which gives the invariant of ω, when written in the form (4).

Equations of motion. – These are the equations that are associated with ω, which are written in the theoretical form:

$$(5)\qquad \begin{cases} \dfrac{\partial\omega}{\partial(du^4)} = e_0(c\,dt - u^4 ds), & \dfrac{\partial\omega}{\partial(dt)} = (-e_0\,du^4 + F^4 ds)\,c, \\ \dfrac{\partial\omega}{\partial(du^i)} = e_0(-dx^i + u^i ds), & \dfrac{\partial\omega}{\partial(dx^i)} = e_0\,du^i - F^i ds = 0. \end{cases}$$

Usual equations. – When one has the applications in mind, it is convenient to introduce quantities that are accessible to measurement with respect to the usual trihedron x, y, z.

The usual velocity of the point will be the vector **v** whose components are $v^i = dx^i / dt$, namely, the ratios of the differentials dx^i and dt ; one has :

$$ds^2 = c^2\,dt^2 - \sum (dx^i)^2 = c^2\,dt^2\,(1 - \beta^2),$$

upon setting:

$$\beta = \frac{v}{c}, \qquad v = \sqrt{\sum_{i=1}^{3}\left(\frac{dx^i}{dt}\right)^2}\,.$$

Upon replacing ds with $dt\,c\sqrt{1-\beta^2}$ in the equations (5), one will get the usual equations:

$$(6)\quad \begin{cases} v^i = \dfrac{dx^i}{dt} = c\,\dfrac{u^i}{u^4}, \\ u^4 = \dfrac{1}{\sqrt{1-\beta^2}}, \end{cases} \qquad (6')\quad \begin{cases} \dfrac{d}{dt}\left(\dfrac{e_0 v^i}{c\sqrt{1-\beta^2}}\right) = F^i c\sqrt{1-\beta^2}, \\ \dfrac{d}{dt}\left(\dfrac{e_0}{\sqrt{1-\beta^2}}\right) = F^4 c\sqrt{1-\beta^2}, \end{cases}$$

where the last four of them can be interpreted vectorially in the space V_4 by means of the energy-impulse vector $\mathbf{p} = (e_0 / c)\,\mathbf{u}$:

$$\frac{d}{dt}(\mathbf{p}) = \mathbf{f} \qquad \text{with} \qquad \mathbf{f} = \mathbf{F}\sqrt{1-\beta^2}\,.$$

Remarks:

1. If one wishes to obtain the equations in three-dimensional space of:

$$(u^4)^2 - \sum (u^i)^2 = 1$$

then one will deduce that:

$$c\, du^4 = c\frac{u^i}{u^4}du^i = v^i\, du^i,$$

with:

$$u^i = \frac{v^i}{c\sqrt{1-\beta^2}}, \qquad ds = c\sqrt{1-\beta^2}\, dt\,,$$

and the expression (3) for ω will become:

$$\omega = e_0 k_{ij} v^i d\left(\frac{v^j}{c\sqrt{1-\beta^2}}\right)\wedge dt - e_0 k_{ij} d\left(\frac{v^j}{c\sqrt{1-\beta^2}}\right)\wedge dx^j - k_{ij}F^i dx^j \wedge c\sqrt{1-\beta^2}dt\,, \tag{7}$$

which gives the first three equations of (6) and (6′).

It results from (7) that if $\beta \to 0$ then after one divides by – c, ω will have the following limit:

$$-\frac{1}{c}\lim_{\beta\to 0}\omega = \frac{e_0}{c^2}k_{ij}dv^i \wedge dx^j - \frac{e_0}{c^2}k_{ij}\, V^i dv^i + k_{ij}\, F^i du^i \wedge dt\,, \tag{8}$$

which is the generating exterior form for the differential equations for a point in Newtonian mechanics. That will give one the relativistic idea of considering the quantity $\dfrac{e_0}{c^2\sqrt{1-\beta^2}} = m$ to be the mass of a moving body and $m_0 = e_0 / c^2$ to be the rest mass.

2. *Composition of collinear velocities in relativistic mechanics.* – In our way of looking at things, the velocity with respect to a frame R is defined to be the ratio of the differentials dx^i to the differential dt : $v^i = dx^i / dt$. In order to establish the rule of composition for velocities, consider two frames (x^i), (ξ^σ) that coincide for $x^i = 0$, and then slide the axes $\Theta\xi$ and Ox along each other, while the axes $\Theta\eta$ and Oy, $\Theta\zeta$ and Oz remain parallel, respectively. One will then pass from the first one to the second one by means of the formulas:

$$\| \xi^\sigma \| = \left\| a_i^\sigma \right\| \times \left\| x^i \right\| \qquad \text{with the matrix:} \qquad \left\| a_i^\sigma \right\| = \begin{Vmatrix} a_1^1 & 0 & 0 & a_4^1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ a_1^4 & 0 & 0 & a_4^4 \end{Vmatrix}. \tag{9}$$

Hence, one passes from the system of eight variables (x^i, u^i) to the system of eight variables (ξ^σ, α^σ) by way of formulas (9) and:

$$\| \alpha^\sigma \| = \left\| a_i^\sigma \right\| \times \left\| u^i \right\|. \tag{10}$$

The invariance of the quadratic form:

$$(\alpha^4)^2 - \sum_{i=1}^{3} (\alpha^i)^2 = (u^4)^2 - \sum_{i=1}^{3} (u^i)^2$$

implies the following relations for the coefficients α_i^σ:

$$(\alpha_4^4)^2 - (\alpha_4^1)^2 = 1, \qquad (\alpha_1^4)^2 - (\alpha_1^1)^2 = 1, \qquad -\alpha_1^1\,\alpha_4^1 + \alpha_1^4\,\alpha_4^4 = 0,$$

so

$$\alpha_1^1 = \alpha_4^4 = \cosh\varphi, \qquad \alpha_4^1 = \alpha_1^4 = \sinh\varphi.$$

It will then result that the formulas for the change of frame are:

$$(11) \quad \begin{cases} \xi = x\cosh\varphi + c\,t\sinh\varphi, \\ c\,\tau = x\sinh\varphi + ct\cosh\varphi, \end{cases} \qquad (11') \quad \begin{cases} \alpha^1 = u^1\cosh\varphi + u^4\sinh\varphi, \\ \alpha^4 = u^1\sinh\varphi + u^4\cosh\varphi. \end{cases}$$

From formulas (6), the velocity of the point with respect to the first frame is $V = c\,u^1 / u^4$, and with respect to the second one, it is $W = c\ \alpha^1 / \alpha^4$.

One deduces from (11′) that:

$$W = \frac{c\,\alpha^1}{\alpha^4} = \frac{c\,u'\sinh\varphi + c\,u^4\cosh\varphi}{u^1\sinh\varphi + u^4\cosh\varphi} = \frac{c\dfrac{u^1}{u^4} + c\tanh\varphi}{1 + \dfrac{c\tanh\varphi}{c^2} : c\dfrac{u^1}{u^4}},$$

from which, it follows that:

$$(12) \qquad W = \frac{V + V_0}{1 + \dfrac{V_0}{c^2}},$$

upon setting $V_0 = c\tanh\varphi$, which is the velocity of the first frame with respect to the second one. That will imply the mechanical significance of the parameter φ, and the law of composition of collinear velocities in relativistic mechanics.

(Arrived at the Annales on 28 June 1952)

(Arrived at the Annales on 28 June 1952)

____