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On the most general problem in optics

By ANTONIO GARBASSO and GUIDO FUBINI

Translated by D. H. Delphenich

§ 1. Some simple experiments. – The media that give rise to the phenomenon of mirages are isotropic and inhomogeneous. They then have properties that depend upon the coordinates, but not the direction, of the luminous trajectory. Therefore, the rays in them appear to be curved in general, and only one of them will originate from a given point in a given direction.

However, crystals are homogeneous and anisotropic (other than the first system). All of their points are equivalent, but different directions will usually have different properties. Their trajectories are always rectilinear, and as a rule there are two of them in any direction that are characterized by different velocities.

The one category of media and the other have been the subject of infinite research. However, it does not seem that the most general problem of the propagation of light in an anisotropic and inhomogeneous medium has been discussed up to now.

Nonetheless, it is easy to experimentally obtain a body that is endowed with those two properties at the same time. A layer of gelatin that is put into contact with a solution (take zinc chloride, for example) will in fact appear almost immediately to be isotropic and inhomogeneous. However, a similar stratum that is subjected to mechanical actions will assume the optical characteristics of a crystal. If the diffusion and deformation are made to act in succession then the specimen must be anisotropic and inhomogeneous as a result.

Indeed, experiments verify that prediction.

A stratum of pure gelatin in the form of an elongated right parallelepiped is placed between the two faces of a small screw vise. The gelatin is then positioned between two Nicol prisms in such a way that the light traverses it normal to the lateral faces without compression.

If the Nicols are parallel and one turns the specimen until it reaches the position in which the colors assume their maximum vividness then the bright field will present, for example, the appearance that is illustrated in Figure 1 (see the table) (†).

If one then turns the analyzer Nicol through 90° then the colors will change, and the illuminated region will appear to be as in Figure 2.

In one case and the other, the field will then have a generally uniform color that changes only slowly in the parts of it that are closest to the faces of the vice. The same point will present complementary colors in the two tests.

(†) Translator: The figures and tables were unavailable to me at the time of translation.

If one takes another sample of gelatin of equal height and different thickness and one compresses it to the same degree then the phenomenon will not change in appearance, but the dominant colors will be modified. For example, instead of one red and one green, one might have one yellow and one blue.

Now take a third parallelepiped of gelatin whose one lateral face is put into contact with a solution of zinc chloride at some point in time and, as before, it is compressed while placing the face that is used for diffusion in contact with one face of the vice.

The colors of chromatic polarization between parallel Nicols and between crossed Nicols (Fig. 3 and 4) seem to be distributed in a way that is completely different from the preceding one. Indeed, colors change continuously in the illuminated field from one face to the other. That is precisely because the specimen has a different thickness in different regions.

As for the isochromatic lines, they are naturally composed of the lines that are perpendicular to the common direction of diffusion and deformation.

§ 1. A proposal for a theory. – When one acknowledges the experimental existence of anisotropic and inhomogeneous media, that presents an opportunity to subject them to calculation in order to make that characteristic less obvious. To that end, let x, y, z be orthogonal Cartesian coordinates and let a layer S be bounded by two planes z = const.

Now suppose that S behaves like a uniaxial crystal at any point, that is to say, the ellipsoid of elasticity is an ellipsoid of rotation throughout.

Further suppose that:

a) The polar axes of the given ellipsoids are always parallel to the z-coordinate.

b) The lengths of the individual axes (polar or not) are functions of only z.

Now assume that in order to determine the propagation of light in S, one can:

a) Consider S to be the sum of n homogeneous layers S_i that are pair-wise adjacent and bounded by planes z = const.

b) Calculate the propagation in that system of layers.

c) Pass to the limit of $n = \infty$.

Finally, agree that the incident ray is formed of a system of plane waves, and choose the y-axis to be parallel to the plane of the wave, for simplicity. The system of waves that is produced by the successive reflections and refractions will again be systems of plane waves with the plane of the wave being parallel to the y-axis.

Let φ_i denote the angle that the normal to the plane of the wave in the i^{th} layer makes with the z-axis, and let V_i denote the velocity of propagation for the corresponding system of waves in the sense that is normal to the plane of the wave.

It is well-known that with those notations, one will have:

$$\frac{\sin\varphi_i}{V_i} = h, \tag{1}$$

in which h is a constant that depends upon only the initial data.

Now let $1 / a_i$, $1 / c_i$ be the semi-axes of the elastic ellipsoid in the i^{th} layer, and accordingly, let $1 / c_i$ be the semi-axis that is parallel to the z-axis.

One will have:

$$V_i = a_i \tag{2}$$

or

$$\begin{aligned} V_i^2 &= a_i^2 + (c_i^2 - a_i^2)\sin^2\varphi_i\,, \\ &= c_i^2 + (a_i^2 - c_i^2)\cos^2\varphi_i\,. \end{aligned} \tag{3}$$

$\cos\varphi_i$ is determined, up to sign, from (1). Those equations will then imply two possible systems of waves, in each of which, the displacement will have the direction cosines:

$$\begin{matrix} -\cos\varphi_i \cdot \cos\theta_i\,, & \sin\theta_i\,, & \sin\varphi_i \cdot \cos\theta_i\,, \\ \cos\varphi_i \cdot \cos\theta_i\,, & \sin\theta_i\,, & \sin\varphi_i \cdot \cos\theta_i\,, \end{matrix}$$

respectively.

It is obvious that one will set:

$$\theta_i = 0 \tag{4}$$

or

$$\theta_i = \frac{\pi}{2} \tag{5}$$

according to whether one treats ordinary or extraordinary rays, resp.

The resulting displacements will then have the components:

$$\left\{ \begin{aligned} u_i &= -A_i \cos\varphi_i \cdot \cos\theta_i \cdot \sin O_i + B_i \cos\varphi_i \cdot \cos\theta_i \cdot \sin\Omega_i\,, \\ v_i &= \qquad\quad A_i \sin\theta_i \cdot \sin O_i + \qquad\quad B_i \cos\theta_i \cdot \sin\Omega_i\,, \\ w_i &= \quad A_i \sin\varphi_i \cdot \cos\theta_i \cdot \sin O_i + B_i \sin\varphi_i \cdot \cos\theta_i \cdot \sin\Omega_i\,, \end{aligned} \right. \tag{6}$$

in which A_i and B_i are the lengths of the two partial displacements, and O_i and Ω_i are the corresponding phases, which are defined by equations of the type:

$$(7)\qquad \begin{cases} O_i = \left(x\dfrac{\sin\varphi_i}{V_i} + z\dfrac{\cos\varphi_i}{V_i} - t + D_i \right)\lambda = (hx-t)\lambda + \left(z\dfrac{\cos\varphi_i}{V_i} + D_i \right)\lambda, \\ \Omega_i = \left(x\dfrac{\sin\varphi_i}{V_i} - z\dfrac{\cos\varphi_i}{V_i} - t + \Delta_i \right)\lambda = (hx-t)\lambda + \left(z\dfrac{\cos\varphi_i}{V_i} + \Delta_i \right)\lambda. \end{cases}$$

In (7), λ is the constant 2π, the unit of time is chosen to be the period of the individual vibrations, and D_i and Δ_i are constants throughout each layer ([1]).

§ 3. Ordinary rays. – One sets:

$$\theta_i = 0, \qquad V_i = a_i\,, \qquad v_i = 0,$$

and it will result from (6) and (7) that:

$$(8)\qquad \begin{cases} u_i = L_i \cos(hx+t)\lambda + M_i \sin(hx-t)\lambda, \\ w_i = N_i \cos(hx-t)\lambda + P_i \sin(hx-t)\lambda, \end{cases}$$

with

$$(9)\qquad \begin{cases} L_i = \cos\varphi_i\,(-A_i \sin C_i - B_i \sin\Gamma_i), \\ M_i = \cos\varphi_i\,(-A_i \cos C_i + B_i \cos\Gamma_i), \\ N_i = \sin\varphi_i\,(\ \ A_i \sin C_i - B_i \sin\Gamma_i), \\ P_i = \sin\varphi_i\,(\ \ A_i \sin C_i - B_i \sin\Gamma_i), \end{cases}$$

$$(10)\qquad \begin{cases} C_i = \left(z\dfrac{\cos\varphi_i}{V_i} + D_i \right)\lambda, \\ \Gamma_i = \left(z\dfrac{\cos\varphi_i}{V_i} + \Delta_i \right)\lambda. \end{cases}$$

From the laws of reflection and refraction, the u_i and w_i must be continuous in the planes that separate the layers. If:

$$z = z_{i-1}$$

is the plane that separates the $(i-1)^{\text{th}}$ layer from the i^{th} one then one will have:

$$(u_i)_{z=z_{i-1}} - (u_{i-1})_{z=z_{i-1}} = 0,$$
$$(w_i)_{z=z_{i-1}} - (w_{i-1})_{z=z_{i-1}} = 0\,.$$

([1]) Kirchhoff, *Mathematische Optik*, 1891, page 219.

In words, the increments that the u and w are subjected to when one takes:

$$\varphi_i\,,\ D_i\,,\ \Delta_i\,,\ \text{and}\ a_i\,,$$

instead of:

$$\varphi_{i-1}\,,\quad D_{i-1}\,,\quad \Delta_{i-1}\,,\ \text{and}\ a_{i-1}\,,$$

resp., and *does not vary the* z must be zero.

However, since that must happen for any value of x, the L_i, M_i, N_i, and P_i will enjoy the same property. In particular, one will have:

$$(L_i)_{z=z_{i-1}} - (L_{i-1})_{z=z_{i-1}} = 0,$$

and therefore:

$$(L_i)_{z=z_i} - (L_{i-1})_{z=z_{i-1}} = (L_i)_{z=z_i} - (L_i)_{z=z_{i-1}}\,,$$

or, upon neglecting the higher-order infinitesimals with respect to the thicknesses of the individual layers:

$$\begin{aligned}\frac{(L_i)_{z=z_i} - (L_{i-1})_{z=z_{i-1}}}{z_i - z_{i-1}} &= \frac{(L_i)_{z=z_i} - (L_i)_{z=z_{i-1}}}{z_i - z_{i-1}} \\ &= \cos\varphi_i\,(-A_i\cos C_i - B_i\cos\Gamma_i)\,\frac{\cos\varphi_i}{V_i}\,\lambda \\ &= -\frac{\cos^2\varphi_i}{V_i\sin\varphi_i}\,\lambda P_i\,.\end{aligned}$$

When one passes to the limit, the L, M, N, and P will become functions of only z, and one will get:

$$\begin{aligned}\frac{dL}{dz} &= -\frac{\cos^2\varphi_i}{V_i\sin\varphi_i}\,\lambda P\,, \\ &= -\frac{1-a^2h^2}{a^2h}\,\lambda P\,,\end{aligned} \tag{11}$$

and similarly:

$$\frac{dP}{dz} = h\,\lambda\,L\,, \tag{11′}$$

$$\frac{dP}{dz} = \frac{1-a^2h^2}{a^2h}\,\lambda N\,, \tag{12}$$

$$\frac{dN}{dz} = -\,h\,\lambda\,M\,. \tag{12′}$$

However, it follows directly from (11) and (11′), (12) and (12′) that the N and P must satisfy:

$$\frac{d^2\psi}{dz^2} = -\lambda^2 \frac{1-a^2h^2}{a^2}\psi \,. \tag{13}$$

The study of ordinary rays is then reduced to the integration of (13).

If ψ_1 and ψ_2 denote two independent integrals of that equation and α_1, α_2, β_1, β_2 and denote four constants then one can certainly set:

$$N = \alpha_1 \psi_1 + \alpha_2 \psi_2 \,,$$

$$P = \beta_1 \psi_1 + \beta_2 \psi_2 \,,$$

$$L = \frac{1}{h\lambda}\left(\beta_1 \frac{d\psi_1}{dz} + \beta_2 \frac{d\psi_2}{dz}\right),$$

$$M = -\frac{1}{h\lambda}\left(\alpha_1 \frac{d\psi_1}{dz} + \alpha_2 \frac{d\psi_2}{dz}\right),$$

and therefore:

$$\text{(I)} \qquad \begin{cases} u = \dfrac{1}{h\lambda}[(\beta_1 \psi_1' + \beta_2 \psi_2')\cos(hx-t)\lambda - (\alpha_1 \psi_1' + \alpha_2 \psi_2')\sin(hx-t)\lambda], \\ w = (\alpha_1 \psi_1 + \alpha_2 \psi_2)\cos(hx-t)\lambda - (\beta_1 \psi_1' + \beta_2 \psi_2')\sin(hx-t)\lambda. \end{cases}$$

In the present case, those equations define the system of waves that is the simplest generalization of the systems of ordinary plane waves in homogeneous media.

§ 4. Extraordinary rays. – One sets:

$$\theta_i = \frac{\pi}{2}, \qquad V_i^2 = c_i^2 + (a_i^2 - c_i^2)\cos^2\varphi_i \,, \qquad u_i \,, \qquad w_i = 0 \,,$$

and it will result from (6) and (7) that:

$$v_i = Q_i \cos(hx-t)\,\lambda + R_i\,(hx-t)\,\lambda \,, \tag{14}$$

with

$$\begin{cases} Q_i = A_i \sin C_i - B_i \sin \Gamma_i \,, \\ R_i = A_i \cos C_i + B_i \cos \Gamma_i \,. \end{cases} \tag{15}$$

It follows that:

$$a_i^2 \frac{\partial v_i}{\partial z} = \lambda\, a_i^2 \frac{\cos\varphi_i}{V_i} [E_i \cos (hx - t)\, \lambda + F_i \sin (hx - t)\, \lambda], \tag{16}$$

in which one sets:

$$\begin{cases} E_i = A_i \cos C_i - B_i \cos \Gamma_i , \\ F_i = -A_i \sin C_i \ - B_i \sin \Gamma_i , \end{cases} \tag{17}$$

for ease of writing.

Upon recalling ([1]) that v_i and $a_i^2 \dfrac{\partial v_i}{\partial z}$ must be continuous under the passage from one layer to the next, one will find that the same thing must happen for Q_i , R_i , $\lambda\, a_i^2 \dfrac{\cos\varphi_i}{V_i} E_i$, $\lambda\, a_i^2 \dfrac{\cos\varphi_i}{V_i} F_i$. If one then applies the preceding methods and passes to the limit then the result will be:

$$\frac{dQ}{dz} = \frac{\cos\varphi}{V} \lambda E , \tag{18}$$

$$\frac{d}{dz}\left(a^2 \ \frac{\cos\varphi}{V} E \right) = -a^2 \frac{\cos^2\varphi}{V^2} \lambda Q , \tag{18′}$$

$$\frac{dQ}{dz} = \frac{\cos\varphi}{V} \lambda F , \tag{19}$$

$$\frac{d}{dz}\left(a^2 \ \frac{\cos\varphi}{V} F \right) = -a^2 \frac{\cos^2\varphi}{V^2} \lambda R . \tag{19′}$$

However, it follows immediately from (18) and (18′), (19) and (19′) that the Q and R must satisfy:

$$\frac{d}{dz}\left(a^2 \frac{d\chi}{dz} \right) + a^2 \frac{\cos^2\varphi}{V^2} \lambda^2 \chi = 0,$$

or, from (1) and (3):

$$\frac{d}{dz}\left(a^2 \frac{d\chi}{dz} \right) + (1 - c^2 h^2) \lambda^2 \chi = 0 , \tag{20}$$

which plays the same role for extraordinary rays that (13) does in the case of ordinary rays.

If χ_1 and χ_2 denote two independent integrals of (20) and γ_1 , γ_2 , δ_1 , and δ_2 denote four constants then one can set:

$$Q = \gamma_1\, \chi_1 + \gamma_2\, \chi_2 ,$$

$$R = \delta_1\, \chi_1 + \delta_2\, \chi_2 ,$$

([1]) KIRCHHOFF, *loc. cit.*, page 231.

and therefore:

$$v = (\gamma_1\, \chi_1 + \gamma_2\, \chi_2) \cos (hx - t)\, \lambda + (\delta_1\, \chi_1 + \delta_2\, \chi_2) \sin (hx - t)\, \lambda\,. \tag{II}$$

§ 5. Theoretical considerations. – Among the many forms that one can give to the equations of optics, and which are all equivalent for homogeneous media, it is noteworthy that one of them continues to persist for inhomogeneous media, as well.

The form to which we allude that is valid for homogeneous and inhomogeneous bodies (at least when the ellipsoid of elasticity at any point is one of rotation with its axis parallel to the *z*-axis) is the following one:

$$\left\{\begin{aligned} -\frac{\partial^2 u}{\partial t^2} &= \frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}, \\ -\frac{\partial^2 v}{\partial t^2} &= \frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}, \\ -\frac{\partial^2 w}{\partial t^2} &= \frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}, \end{aligned}\right. \tag{21}$$

$$\left\{\begin{aligned} X &= a^2 \left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} \right), \\ Y &= a^2 \left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} \right), \\ Z &= a^2 \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} \right). \end{aligned}\right. \tag{22}$$

It is, in fact, easy to verify that the *u*, *v*, and *w* that are defined by our formulas (I) and (II) will satisfy (21) and (22), while they will not satisfy the other systems, which are equivalent to them only in the case of homogeneous media.

[Set:

$$(hx - t)\, \lambda = \mu\,,$$

and meanwhile when one substitutes the values (I) and (II) in the first of (21), it will result that:

$$\begin{aligned} -\frac{\partial^2 u}{\partial t^2} - \frac{\partial Y}{\partial z} + \frac{\partial Z}{\partial y} = \frac{\cos\mu}{h\lambda} \Big[&(\beta_1 \psi_1' + \beta_2 \psi_2')\lambda^2 - (\beta_1 \psi_1' + \beta_2 \psi_2')\, a^2 h^2 \lambda^2 + (\beta_1 \psi_1'''' + \beta_2 \psi_2'''') a^2 - (\beta_1 \psi_1 + \beta_2 \psi_2)\, h^2 \lambda^2 \frac{da^2}{dz} \\ &+ (\beta_1 \psi_1'' + \beta_2 \psi_2'') \frac{da^2}{dz} \Big] \\ - \frac{\sin\mu}{h\lambda} \Big[&(\alpha_1 \psi_1' + \alpha_2 \psi_2')\lambda^2 - (\alpha_1 \psi_1' + \alpha_2 \psi_2')\, a^2 h^2 \lambda^2 + (\alpha_1 \psi_1'''' + \alpha_2 \psi_2'''') a^2 - (\alpha_1 \psi_1 + \alpha_2 \psi_2)\, h^2 \lambda^2 \frac{da^2}{dz} \end{aligned}$$

$$+\ (\alpha_1\,\psi_1'' + \alpha_2\,\psi_2'')\frac{da^2}{dz}\Bigg]$$

$$= \frac{\cos\mu}{h\lambda}\left\{\beta_1\frac{d}{dz}[a^2\psi_1'' + (1-a^2h^2)\lambda^2\psi_1] + \beta_2\frac{d}{dz}[a^2\psi_2'' + (1-a^2h^2)\lambda^2\psi_2]\right\}$$

$$-\ \frac{\sin\mu}{h\lambda}\left\{\alpha_1\frac{d}{dz}[a^2\psi_1'' + (1-a^2h^2)\lambda^2\psi_1] + \alpha_2\frac{d}{dz}[a^2\psi_2'' + (1-a^2h^2)\lambda^2\psi_2]\right\} = 0,$$

by virtue of equations (13). In these formulas, one sets da^2 / dz in place of $d\,(a)^2 / dz$.]

Apart from any mechanical or electromagnetic interpretation, one should then see that the form described is the *most natural and general* form that one can give to the equations of optics. Moreover, the result of our study that was obtained assumes *only* the theory of optics in homogeneous media and then proceeds along a *purely mathematical* path that is not perhaps devoid of any interest, in its own right.

§ 6. Fermat's principle. – The equation $\dfrac{\sin\varphi_i}{V_i} = h$ permits one to determine the form of rays in the media that were just studied. Using the well-known formulas of optics, which couple the direction cosines of a ray to the cosines of the normal to the plane wave, one easily finds that equations (1) are also equivalent in the present case to the ones that *when rays traverse the individual layers, they will follow the path that takes the least time to traverse it*.

[For example, let us study extraordinary rays, and first suppose that the medium is homogeneous. With the notations from what follows, the direction cosines of the ray will be proportional to:

$$c^2 \sin\varphi,\ 0, \qquad a^2\cos\varphi\,.$$

If we denote them by ξ, η, and ζ, resp., then we will have:

$$\begin{cases} \xi = \dfrac{c^2\sin\varphi}{\sqrt{a^4\cos^2\varphi + c^4\sin^2\varphi}}, \\ \eta = 0, \\ \zeta = \dfrac{a^2\sin\varphi}{\sqrt{a^4\cos^2\varphi + c^4\sin^2\varphi}}. \end{cases}$$

It will result from this that:

$$\begin{cases} \sin^2\varphi = \dfrac{a^4\xi^2}{a^4\xi^2 + c^4\zeta^2}, \\ \cos^2\varphi = \dfrac{c^4\zeta^2}{a^4\xi^2 + c^4\zeta^2}, \end{cases}$$

and since:

$$V^2 = a^2\cos\varphi + c^2\sin\varphi\,,$$

one will have directly that:

$$(\alpha) \qquad \frac{\sin^2\varphi}{V^2} = \frac{a^2\xi^2}{a^2c^2\xi^2 + c^4\zeta^2}.$$

Now consider an inhomogeneous medium that is composed of two homogeneous layers separated by the plane $z = Z$. Let O_1 be a point of the first layer, let O_2 be a point of the second one, and the light that goes from O_1 to O_2 will pierce the plane $z = Z$ at I. Draw perpendiculars to that plane from O_1 and O_2, and let P_1 and P_2 be their feet.

Set:

$$O_1 P_1 = e_1, \qquad O_2 P_2 = c_2,$$
$$P_1 P_2 = d, \qquad P_1 I = x,$$

and one will meanwhile have:

$$O_1 I = \sqrt{e_1^2 + x^2},$$
$$O_2 I = \sqrt{e_2^2 + (d-x)^2}.$$

It still remains for us to calculate the velocity $\mathfrak{D}$ in the direction of the rays. However, it is known (KIRCHHOFF, *loc. cit.*, page 208, *et seq.*) that in general one has:

$$\mathfrak{D}^2 = \frac{(a^2+c^2)V^2 - a^2c^2}{V^2} = \frac{a^2c^2}{a^2\xi^2 + c^2\zeta^2},$$

and therefore, for the first medium:

$$\mathfrak{D}_1 = \sqrt{\frac{a_1^2\,c_1^2}{a_1^2\,\dfrac{x^2}{e_1^2+x^2} + c_1^2\,\dfrac{e_1^2}{e_1^2+x^2}}},$$

and for the second one:

$$\mathfrak{D}_2 = \sqrt{\frac{a_2^2\,c_2^2}{a_2^2\,\dfrac{(d-x)^2}{e_2^2+(d-x)^2} + c_2^2\,\dfrac{e_2^2}{e_1^2+(d-x)^2}}},$$

in which the subscripts 1 and 2 distinguish the quantities that relate to one or the other layer.

The equation:

$$\frac{\partial}{\partial x}\left(\frac{O_1 I}{\mathfrak{D}_1} + \frac{O_2 I}{\mathfrak{D}_2}\right) = 0$$

will then take the form:

$$\frac{a_1 x}{\sqrt{a_1^2\,c_1^2\,x^2 + c_2^4\,e_1^2}} = \frac{a_2(d-x)}{\sqrt{a_2^2\,c_2^2\,(d-x)^2 + c_2^4\,e_2^2}},$$

which corresponds to (1), from (α).]

It will then follow that even in our anisotropic and inhomogeneous medium, the equations of the rays will be obtained from:

$$\partial \int \frac{ds}{\mathfrak{D}} = 0, \tag{23}$$

in which *ds* is, as usual, the line element, and $\mathfrak{D}$ is the velocity of propagation of a system of plane waves in the direction of the ray.

In other words: *Fermat's principle continues to be valid for the more general media that we have considered*.

[Equation (23) can be easily put into a more explicit form. Recall the value of $\mathfrak{D}$ that is given by:

$$\mathfrak{D}^2 = \frac{a^2 c^2}{a^2 \xi^2 + c^2 \zeta^2},$$

and one will indeed have:

$$\frac{ds}{\mathfrak{D}} = \frac{\sqrt{a^2 \xi^2 + c^2 \zeta^2}}{a\,c} ds$$

$$= \frac{\sqrt{a^2 (dx^2 + dy^2) + c^2 dz^2}}{a\,c},$$

and therefore, in place of (23), one will have:

$$\partial \int \frac{\sqrt{a^2 (dx^2 + dy^2) + c^2 dz^2}}{a\,c} = 0 .$$

In words: The rays in our inhomogeneous and anisotropic media are the geodetics of the metric that is defined by the line element:

$$ds^2 = \frac{a^2 (dx^2 + dy^2) + c^2 dz^2}{a^2 c^2} .$$

Optics in our media then gives us a method that (one can say) *serves, in a certain regard, to experimentally realize a geometry that is even more general than ordinary Euclidian or non-Euclidian geometry.*]