"Über ein neues allgemeines Grundsetz der Mechanik," J. reine angew. Math. **4** (1829), 232-235.

On a new general foundation for mechanics

(By Herrn Hofrat and Prof. Dr. **Gauss** in Göttingen)

Translated by D. H. Delphenich

As is known, the principle of virtual velocities converts all of statics into a mathematical problem, and **d'Alembert**'s principle for dynamics reduces that study, in turn, to statics. Therefore, it is in the nature of things that no one has given any new basic principle for the study of motion and equilibrium that would already include both of the latter principle, and from which they could be derived. In the meantime, however, due to that situation, a new principle would not seem to be worthless. It would always remain interesting and instructive to abstract a new and preferable viewpoint for the laws of nature such that one would easily solve this or that problem from it, or that it would add a special reasonableness to that problem. The great geometer who built the structure of mechanics on the basis of the principle of virtual velocities so brilliantly, did not reject the raising of **Maupertuis**'s principle of least work to a state of greater determinacy and generality, which is a principle to which one can appeal with great advantage from time to time (*).

The peculiar character of the principle of virtual velocities consists of the fact that it is a general formula for solving all static problems and a paradigm for all other principles, without taking the credit for them so directly that it would already recommend itself as something plausible, as long as only expresses it.

In that regard, the principle that I have proposed here seems to have one advantage. However, it also has a second one, namely, that it encompasses the law of motion and rest in completely the same way and in greatest generality. Therefore, it would very much be in order that from the gradual development of science and the teaching of individuals, simple things should come before complicated things and specialized things should come before the generalities. Nonetheless, once one has reached the higher standpoint, the spirit will demand the converse process, whereby all of statics appears to be only an entirely special case of mechanics. Even the aforementioned geometer seem to place some value upon that notion when he regarded one of the advantages of the principle of least action as being that it included both equilibrium and motion at the same

---

(*) Remark by the author: However, permit me to point out here that I did was not satisfied by the way that another great geometer attempted to prove **Huyghens** law for the extraordinary refraction of light in crystals with double refraction by means of the principle of least action. In fact, the admissibility of that basic law depends essentially upon the conservation of *vis viva*, which would merely constrain the positions and speeds of moving points without having any influence on the direction of motion, which was, however, assumed in the aforementioned attempt. It seems to me that in systems of emanations, all endeavors to link the phenomena of double refraction with the general laws of dynamics must remain fruitless as long as one considers light particles to be merely points.

time when one expresses them in such a way that the *vis viva* is smallest for both of them, which is a remark that seems to be more clever than true, however, since the minimum in both cases occupies an entirely different place in both of them.

The new principle is the following one:

The motion of a system of material points that are always coupled to each other in some way, and whose motion is, at the same time, always subject to external constraints, agrees with the free motion at each moment to the greatest possible extent or with the least possible constraint when one considers a measure of the constraint that the system suffers at each point in time to be the sum of the products of the squares of the deviations of each point from the free motion of its mass.

Let $m, m', m'', \ldots$ be the masses of the points. Let $a, a', a'', \ldots$, be their positions at time t, and let $b, b', b'', \ldots$, be the positions that they would assume after the infinitely-small time interval dt as a result of the forces that act upon them during that time and the speeds and directions that would be attained at tome t, in the event that they were all completely free. The actual positions $c, c', c'', \ldots$ will then be the ones that are compatible with all of the conditions on the system and for which $m\,(b\,c)^2 + m'(b'c')^2 + m''(b''c'')^2 + \ldots$ is a minimum.

Equilibrium is obviously only a special case of the general law, and the condition for:

$$m\,(b\,c)^2 + m'(b'c')^2 + m''(b''c'')^2 + \ldots$$

itself to be a minimum, or the persistence of the system in the rest state, is that the free motion of the individual points should lie closer than any of the other possible ones that might emerge.

The derivation of our principle from the two that were cited above comes about easily in the following way:

The force that acts upon the material point m is obviously composed, first of all, of the force that is coupled with the speed and direction at time t that takes a to c in the time dt and a second one that would lead from rest to c by way of cb in the same time if one considered the point to be free. The same thing will be true for the other points. From **d'Alembert**'s principle, the points $m, m', m'', \ldots$ must then be equilibrium under the effect of only the second forces along $cb, c'b', c''b'', \ldots$ at the positions $c, c', c'', \ldots$ due to the constraints on the system.

From the principle of virtual velocities, equilibrium would demand that the sum of the products of each of the three factors (namely, each of the masses $m, m', m'', \ldots$ the lines $cb, c'b', c''b'', \ldots$, and any others that would project onto the latter, resp. due to the possible motions of that point that are compatible with the constraints on the system) would always have to equal zero, as one ordinarily expresses (*), or rather, more

(*) Remark by the author: The usual expression always assumes constraints that make the opposite of any possible motion equally possible, such as, e.g., that a point should remain on a well-defined surface, that the distance between two points should be unvarying, and the like. However, that is an unnecessary, and not always appropriate, restriction on nature. The outer surface of an impermeable body does not require that a material point that is found upon it must remain upon it, but merely prohibits it from appearing on the other side. For a tensed, inextensible, but flexible, string between two points, only an correctly, by saying that each sum can never be positive. Therefore, if γ, γ', γ'', … are positions that are different from c, c', c'', …, but still compatible with the constraints on the system, and θ, θ', θ'', … are the angles that $c\,\gamma$, $c'\gamma'$, $c''\gamma''$, … make with cb, $c'b'$, $c''b''$, …, then $\sum m \cdot cb \cdot c\gamma \cdot \cos\theta$ will always be either 0 or negative. Now, since:

$$\gamma b^2 = c\,b^2 + c\,\gamma^2 - 2\,cb \cdot c\gamma \cdot \cos\theta,$$

it will then be clear that:

$$\sum m \cdot \gamma b^2 - \sum m \cdot c\,b^2 = \sum m \cdot c\,\gamma^2 - 2 \sum m \cdot cb \cdot c\gamma \cdot \cos\theta$$

will always be positive as a result, so $\sum m \cdot \gamma b^2$ will always be greater than $\sum m \cdot c\,b^2$; i.e., that will be a minimum. Q. E. D.

It is very remarkable that although free motions cannot exist constraints are imposed, by their very nature, they will be modified in the same way by the method of least squares, which relates to quantities that are necessarily coupled with each other by dependencies, as the calculating mathematician will confirm by experience. That analogy can be pursued even further, although I presently do not intend to do so.

---

increase in the distance between two points is impossible, but not a decrease, etc. Why then would we not choose to express the law of virtual velocities in such a way that it encompasses *all* cases right from the beginning?