"Ueber einen fundamentalen Satz aus der kinematischen Geometrie des Raumes," J. f. d. reine u. angew. Math. **90** (1881), 39-43.

On a fundamental theorem in the kinematic geometry of space

(By *C. Geiser* in Zurich)

Translated by D. H. Delphenich

In a book on descriptive geometry by **Mannheim** that appeared recently (*), the author clearly summarized the most important of the beautiful results to which he had been led in his long and successful studies of the motion of rigid bodies in space. On page 262 of it, he treated the change of position of a figure that was constrained by four conditions, and gave the theorem:

When a figure of invariable form is displaced in such a manner that four of its points remain on four given surfaces, for any position of that figure, the normals to the trajectory surfaces of all of its points will meet the same two lines.

One finds the remark there: "See Liouville's Journal of Mathematics (2) **11** (1866), in which I stated that theorem for the first time." In other, newer papers that treated kinematic geometry, the cited theorem and its numerous consequences also referred to that same source, such that it perhaps does not seem unreasonable to recall that it had already been expressed earlier. In fact, on 26 April 1855, **Steiner** presented an essay by Prof. **Schönemann** on "Die Construction von Normalen und Normalebenen gewisser krummer Flächen und Linien" that began as follows (**):

"When a solid body moves on four given surfaces with four unvarying points, in general, each point of it must move on a well-defined surface. The problem now arises of finding the normals to the surface for a well-defined point of the body by construction."

1. "If we denote the four points of the body by a, b, c, d, and the four surfaces on which those four points should move by A, B, C, D, and furthermore denote the normals to A, B, C, D that one can erect at the points a, b, c, d by α, β, γ, δ, and any point of the body by p then that will determine the normal to the surface P on which p must move. In order to do that, one lays the two lines through α, β, γ, δ that cut all four of them, and which one knows can both be real or both imaginary; those two lines shall be called

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(*) *Cours de Géométrie descriptive, comprenant les Éléments de la Géométrie cinématique*, by **A. Mannheim**, professor at l'École polytechnique. Paris, Gauthier-Villars, 1880.

(**) Berliner Monatsberichte, annual issue for 1855, page 255.

guidelines. If one now lays a straight line through the point p and both guidelines then it will be the desired normal to the surface P."

One recognizes **Mannheim**'s theorem in this statement precisely and easily convinces oneself by inspecting the original paper of the important consequences that **Schönemann** had wished to infer from this fundamental theorem.

Since the **Schönemann** paper contained only results, while **Mannheim** applied essentially geometric methods in his treatises, I would like to add a simple, analytical proof of the stated theorem in order to also make my own contribution of the discussion of it (*).

The rigid body will be referred in one case to a rectangular coordinate system X, Y, Z that is fixed in it, and in the other case, to a likewise rectangular system Ξ, H, Z in absolute space. If the coordinates of a point of the body are x, y, z in the first system and ξ, η, ζ in the second one then the following relations will exist:

$$(1)\qquad \left\{\begin{aligned} \xi &= a\,x + b\,y + c\,z + f, \\ \eta &= a'\,x + b'\,y + c'z + f', \\ \zeta &= a''x + b''y + c''z + f'', \end{aligned}\right.$$

in which:

$$(2)\qquad \left\{\begin{aligned} a^2 + b^2 + c^2 &= 1, \\ a'^2 + b'^2 + c'^2 &= 1, \\ a''^2 + b''^2 + c''^2 &= 1, \end{aligned}\right.$$

and

$$(3)\qquad \left\{\begin{aligned} a'a'' + b'b'' + c'c'' &= 0, \\ a''a + b''b + c''c &= 0, \\ aa' + bb' + cc' &= 0. \end{aligned}\right.$$

Now, if the body moves in such a way that each point of it can proceed on a surface that is associated with it then the surface element that belongs to a position ξ, η, ζ of the point x, y, z will be determined by two infinitely-close, mutually-independent positions of the same point: $\xi + d\xi$, $\eta + d\eta$, $\zeta + d\zeta$; $\xi + \delta\xi$, $\eta + \delta\eta$, $\zeta + \delta\zeta$, which will yield:

$$(4^d)\qquad \left\{\begin{aligned} d\xi &= x\,da + y\,db + z\,dc + df, \\ d\eta &= x\,da' + y\,db' + z\,dc' + df', \\ d\zeta &= x\,da'' + y\,db'' + z\,dc'' + df'', \end{aligned}\right.$$

(*) One should confer: **Ribaucour**, "Propriétés relatives aux déplacements d'un corps, assujetti à quatre conditions." Comptes rendus des séances de l'académie des sciences **76** (2 June 1873), pp. 1347.

$$(4^{\delta})\qquad \begin{cases} \delta\xi = x\;\delta a + y\;\delta b + z\,\delta c + \delta f, \\ \delta\eta = x\;\delta a' + y\;\delta b' + z\,\delta c' + \delta f', \\ \delta\zeta = x\,\delta a'' + y\,\delta b'' + z\,\delta c'' + \delta f''. \end{cases}$$

Due to (2) and (3), that gives:

$$(5)\qquad \begin{cases} a\,da + \quad b\,db + \quad c\,dc = 0, \\ a'\,da \; + \quad b'\,db + \quad c'\,dc = 0, \\ a''da'' + \; b''db'' + \; c''dc'' = 0, \end{cases}$$

$$(6)\qquad \begin{cases} a''\,da' + b''\,db' + c''\,dc' = -(a'\,da + b'\,db + c'\,dc), \\ a\,da'' + b\,db'' + c\,dc'' = -(a''da + b''db + c''dc), \\ a'\,da + b'\,db + c'\,dc = -(a\,da' + b\,db' + c\,dc'). \end{cases}$$

The quantities that are found on both sides of the equality sign in (6) shall be called *dp*, *dq*, *dr*. If one replaces the symbol *d* with δ everywhere in equations (5) and (6), which is permissible, then the quantities δp, δq, δr will also be defined.

The surface element that is laid through the points ξ, η, ζ; $\xi + d\xi, \eta + d\eta, \zeta + d\zeta$; $\xi + \delta\xi, \eta + \delta\eta, \zeta + \delta\zeta$ is parallel to the one that is determined by 0, 0, 0 ; $d\xi, d\eta, d\zeta$; $\delta\xi, \delta\eta, \delta\zeta$; the normal to it that is laid through ξ, η, ζ will then have the equations:

$$(7)\qquad \Xi = \xi + \rho\,(d\eta\,\delta\zeta - d\zeta\,\delta\eta),\quad \mathrm{H} = \eta + \rho\,(d\zeta\,\delta\xi - d\xi\,\delta\zeta),\quad \mathrm{Z} = \zeta + \rho\,(d\xi\,\delta\eta - d\eta\,\delta\xi).$$

$d\xi, d\eta, d\zeta$; $\delta\xi, \delta\eta, \delta\zeta$ are expressed as linear functions of *x*, *y*, *z* by means of equations (4). However, with the help of the solution of (1) for *x*, *y*, *z*, and subsequent substitution into (4), they will also be represented as linear functions of ξ, η, ζ. With consideration given to (5) and (6), one will have:

$$(8)\qquad \begin{cases} d\xi = dr\cdot\eta - dq\cdot\zeta + (f''\,dq - f'\,dr + df), \\ d\eta = dp\cdot\zeta - dq\cdot\xi + (f\;\;dr - f\;\;dp + df'), \\ d\zeta = dq\cdot\xi - dq\cdot\eta + (f'\,dp - f''\,dq + df''), \end{cases}$$

in which one only has to exchange *d* with δ in order to obtain $\delta\xi, \delta\eta, \delta\zeta$.

Equations (7) represent all of the normals that belong to the surface elements that are described by all points of the body when the *x*, *y*, *z*, or – what amounts to the same thing – the ξ, η, ζ assume all possible values. Despite the three-fold extended manifold of value systems of ξ, η, ζ, according to **Schönemann**' theorem, the totality of all normals now define only a two-fold extended manifold, and indeed, one that one calls, with **Kummer** (*), a *ray system of order and class one.*

(*) Abhandlungen der Berliner Akademie, annual issue for 1866, page 14.

In order to arrive at this result, one recalls (*) that the equations:

$$\Xi = \xi + \sigma\xi', \qquad \mathrm{H} = \eta + \sigma\eta', \qquad \mathrm{Z} = \zeta + \sigma\zeta' \tag{9}$$

represent the four-fold manifold of all lines in space for unrestricted variability of ξ, η, ζ; ξ', η', ζ', but a ray system of order and class one will also be immediately cut out of it, when one establishes two linear, homogeneous equations between the ξ', η', ζ':

$$p_1\xi' + q_1\eta' + r_1\zeta' = 0, \qquad p_2\xi' + q_2\eta' + r_2\zeta' = 0, \tag{10}$$

in which:

$$\begin{cases} p_1 = \gamma_1\eta - \beta_1\zeta + \lambda_1, & q_1 = \alpha_1\zeta - \gamma_1\xi + \mu_1, & r_1 = \beta_1\xi - \alpha_1\eta + \nu_1, \\ p_2 = \gamma_2\eta - \beta_2\zeta + \lambda_2, & q_2 = \alpha_2\zeta - \gamma_2\xi + \mu_2, & r_2 = \beta_2\xi - \alpha_2\eta + \nu_2, \end{cases} \tag{11}$$

and the α, β, γ, λ, μ, ν mean arbitrary constants. Due to (10), equations (9) can be written:

$$\Xi = \xi + \rho(q_1 r_2 - r_1 q_2), \quad \mathrm{H} = \eta + \rho(r_1 p_2 - p_1 r_2), \quad \mathrm{Z} = \zeta + \rho(p_1 q_2 - q_1 p_2), \tag{12}$$

and one will thus have the equations of a ray system that will go to equations (7) when one makes the subsequent substitutions:

$$\frac{p_1 \quad q_1 \quad r_1 \;\vert\; \alpha_1 \quad \beta_1 \quad \gamma_1 \;\vert\; \lambda_1 \qquad \mu_1 \qquad \nu_1}{d\xi \quad d\eta \quad d\zeta \;\vert\; dp \quad dq \quad dr \;\vert\; f''\,dq - f'\,dr + df \quad f\,dr - f''\,dp + df' \quad f'\,dq - f\,dr + df''} \tag{13}$$

and appends the ones that one obtains when one replaces the upper index 1 with 2 and the lower sign d with δ. The two guidelines will be found to be the two common generators of three hyperboloids whose equations emerge from:

$$q_1 r_2 - r_1 q_2 = 0, \qquad r_1 p_2 - p_1 r_2 = 0, \qquad p_1 q_2 - q_1 p_2 = 0 \tag{13}$$

by means of the substitutions (13).

Zurich, 15 April 1880.

(*) **Plücker**, *Neue Geometrie des Raumes*, Leipzig, B. G. Teubner 1868, page 62, *et seq*. The formulas that are used here are easy to verify directly, moreover.