"Sur le réduction à un principe variationnel des équations du movement d'un fluide visqueux incompressible, Ann. Inst. Fourier **1** (1949), 157-162.

On the reduction of the equations of motion of a viscous, incompressible fluid to a variational principle

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1. – One knows that it is not possible to carry out the reductions of the equations of motion of an incompressible viscous fluid that is not moving slowly in the general case by considering the velocity field at an instant ([1]).

It seems interesting to see whether such a principle can be obtained, not by imagining the fluid at one instant, but by following a certain mass $\mathcal{M}$ in its motion.

Having defined a set $\mathcal{E}$ of virtual motions of $\mathcal{M}$ between two instants t_0 and t_1, one will be tempted to construct a function $\mathcal{L}$ that is defined on $\mathcal{E}$ and is such that $\int_{t_0}^{t_1} \mathcal{L}\, dt$ is stationary for any element of $\mathcal{E}$ that verifies the indefinite equations of motion.

One likewise shows along that path that one will arrive at a negative result for the general case, at least if one is limited to a certain class of functions $\mathcal{L}$.

Furthermore, the method that is followed will lead us to express the Navier equations for a viscous incompressible fluid in terms of the Lagrange variables and point to a rapid method for obtaining those equations.

2. – Let a real motion of a mass $\mathcal{M}$ of the fluid between the moments t_0 and t_1 correspond to certain initial conditions and limits, and let it be under the action of external forces that depend upon a potential. Let $\mathcal{D}_t$ denote the domain that is occupied at the instant t, let Σ_t be its boundary (one can suppress the index t), while $\mathcal{D}_0$ and Σ_0 denote $\mathcal{D}_{t_0}$ and Σ_{t_0}. In particular, Σ might or might not be the wall of a deformable vessel with a constant volume that contains the fluid. Suppose that the motion of Σ is given. Consider the set of virtual motions of $\mathcal{M}$ between t_0 and t_1 that are:

1. Compatible with the incompressibility condition.

([1]) See **H. VILLAT**, *Leçons sur les fluides visqueux*, pp. 103.

2. Continuous, and if Σ is a wall that bounds the fluid then they will adhere to that wall.

3. Such that the positions and velocities of the fluid elements t_0 and t_1 are the same as they would be in the real motion.

More precisely, when space is referred to rectangular axes $Ox_1x_2x_3$, $\mathcal{E}$ will be composed of continuous vectorial functions:

$$OM = f(P, t) \; (P(a_1, a_2, a_3) \in \mathcal{D}_0 + \Sigma_0 \, ; \, M(x_1, x_2, x_3) \in \mathcal{D} + \mathcal{S} \, ; \, t \in [t_0, t_1]) \tag{1}$$

that satisfy:

a)
$$\Delta \equiv \frac{D(x_1, x_2, x_3)}{D(a_1, a_2, a_3)} = 1 \, .$$

(1) *b*) f reduces to a given function of P and t for:

$$P \in \Sigma_0 \, .$$

c) f and $\partial f / \partial t$ reduce to given functions of P for $t = t_0$ and $t = t_1$.

Let $V(u_1, u_2, u_3)$ denote the vector $\partial f / \partial t$. The x_i are the Euler variables, and the a_i are the Lagrange variables.

3. – By analogy with Hamilton's principle, one lets $\mathcal{T} + \mathcal{U}$ enter in place of $\mathcal{L}$, in which:

$$\mathcal{T} = \tfrac{1}{2} \int_{\mathcal{D}_0} \rho V^2 \, d\tau_0$$

(ρ = density, which is a constant), and:

$$\mathcal{U} = \tfrac{1}{2} \int_{\mathcal{D}_0} \rho U \, d\tau_0$$

(U is a given function of M that is a potential for the external force).

One knows the role that is played in the case of slow motions by the dissipation function ψ that is defined in terms of Euler variables by:

$$\psi(V) = \sum_i \left(\frac{\partial u_i}{\partial x_i} \right)^2 + \tfrac{1}{2} \sum_{i \neq j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^2 . \tag{3}$$

The integral $\mu\int_{\mathcal{D}} 2\psi\, d\tau$ (in which μ is the coefficient of viscosity) represents the heat that is dissipated per unit time and at the instant t by the effect of viscosity. One introduces the quantity:

(4) $$Q(t) = \int_{t_0}^{t_1} dt\, \mu \int_{\mathcal{D}} 2\psi\, d\tau$$

into $\mathcal{L}$, which represents the heat that is dissipated in the mass $\mathcal{M}$ between t_0 and t_1, and which is homogeneous in the expression $\mathcal{T} + \mathcal{U}$.

Finally, since the functions x_i (a_1, a_2, a_3, t) are coupled by (2.a), one can introduce the expression Δ into $\mathcal{L}$ by means of a multiplier λ that is an undetermined function of the x_i.

One will then take $\mathcal{L}$ to have the form:

(5) $$\mathcal{L} = \int_{t_0}^{t_1}\left(\mathcal{T} + \mathcal{U} + \alpha\, Q + \int_{\mathcal{D}_0} \lambda\, \Delta\, d\tau_0\right) dt$$

(α is a numerical factor) or:

(6) $$\mathcal{L} = \int_{\mathcal{D}_0} d\tau_0 \int_{t_0}^{t_1} dt\left(\tfrac{1}{2}\rho V^2 + \rho U + 2\alpha\,\mu \int_{t_9}^{t_1} \psi\, dt\right).$$

4. – The function ψ that is defined by (3) in terms of Euler variables is supposed to be expressed in terms of Lagrange variables in that formula (6). In order to make that change of variables, one remarks that if g is a function of the x_i or the a_i [which are coupled by (1)] then one will have:

$$\frac{\partial g}{\partial x_1} = \frac{D(g, x_2, x_3)}{D(x_1, x_2, x_3)} = \frac{D(g, x_2, x_3)}{D(a_1, a_2, a_3)} \qquad \text{(by virtue of the fact that } \frac{D(x_1, x_2, x_3)}{D(a_1, a_2, a_3)} = 1\text{)},$$

and the expressions that are obtained for:

$$\frac{\partial g}{\partial x_2} \quad \text{and} \quad \frac{\partial g}{\partial x_3}$$

by circular permutation.

Hence:

(7) $$\Psi = A_1^2 + A_2^2 + A_3^2 + \tfrac{1}{2}(B_1^2 + B_2^2 + B_3^2),$$

with:

$$A_1 = \frac{\partial u_1}{\partial x_1} = \frac{D(\partial x_1/\partial t, x_2, x_3)}{D(a_1, a_2, a_3)},$$

(8)

$$B_1 = \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \frac{D(x_1, x_2, \partial x_2 / \partial t)}{D(a_1, a_2, a_3)} + \frac{D(x_1, \partial x_3 / \partial t, x_3)}{D(a_1, a_2, a_3)},$$

and the analogous expression for A_2, B_2, A_3, B_3 that are obtained by permutation.

Formulas (7) and (8) show that Ψ is presented in terms of Lagrange variables as a function of the $\frac{\partial x_i}{\partial a_j}$ and the $\frac{\partial^2 x_i}{\partial t\,\partial a_j}$. It will then introduce the following expressions into the calculation of the variation:

$$X_i = \sum_j \frac{\partial}{\partial a_j} \frac{\partial \Psi}{\partial\left(\dfrac{\partial x_i}{\partial a_j}\right)}, \qquad Y_i = \sum_j \frac{\partial}{\partial a_j} \frac{\partial \Psi}{\partial\left(\dfrac{\partial^2 x_i}{\partial t\,\partial a_j}\right)}.$$

Let us make the calculation of Y_1 explicit. We have:

$$\frac{\partial \Psi}{\partial\left(\dfrac{\partial^2 x_i}{\partial t\,\partial a_j}\right)} = 2A_1 \frac{D(x_2, x_3)}{D(a_2, a_3)} + B_2 \frac{D(x_1, x_2)}{D(a_2, a_3)} - B_3 \frac{D(x_1, x_3)}{D(a_2, a_3)},$$

and the two equalities that are deduced from them by circular permutations of the a_i on both sides.

One will then find that:

$$\begin{aligned} Y_1 &= 2\frac{D(x_1, x_2, x_3)}{D(a_1, a_2, a_3)} + \frac{D(B_2, x_1, x_2)}{D(a_1, a_2, a_3)} - \frac{D(B_3, x_1, x_3)}{D(a_1, a_2, a_3)} \\ &= 2\frac{\partial A_1}{\partial x_1} + \frac{\partial B_2}{\partial x_3} + \frac{\partial B_3}{\partial x_2}, \end{aligned}$$

or when one recalls (8) and the $\sum \frac{\partial u_i}{\partial x_i}$ that translate the incompressibility condition:

(10)

$$Y_1 = \Delta u_1$$

(viz., the Laplacian with respect to the x_i).

One can likewise say that:

(11)

$$Y_i = \Delta u_i \,.$$

The calculation of the X_i presents no difficulty, but it leads to some very lengthy expressions. For example, without specifying the Jacobian explicitly:

(12) $$X_1 = 2\frac{D(A_2,u_3)}{D(x_1,x_2)} + 2\frac{D(A_1,u_2)}{D(x_1,x_2)} + \frac{D(B_1,u_3)}{D(x_1,x_2)} + \frac{D(B_1,u_2)}{D(x_1,x_3)} + \frac{D(B_2,u_2)}{D(x_1,x_3)} + \frac{D(B_3,u_1)}{D(x_1,x_2)}.$$

5. – We now pass on to the calculation of the variation $\delta\mathcal{L}$ that is due to a variation $\delta f\,(\delta x_1, \delta x_2, \delta x_3)$. That variation δf verifies the boundary conditions:

(13) $$\delta f \equiv 0 \quad \text{and} \quad \frac{\partial}{\partial t}\delta f \equiv 0 \qquad \text{for } t = t_0 \text{ and } t = t_1\,; \quad \delta f \equiv 0 \ \text{ for } P \in \Sigma_0\,.$$

If $\mathcal{L}$ is given by (5) then one will have:

(14) $$\delta\mathcal{L} \equiv \int_{\mathcal{D}_0} d\tau_0 \int_{t_0}^{t_1} dt \sum_i \Bigg\{ \rho\frac{\partial x_i}{\partial t}\,\delta\!\left(\frac{\partial x_i}{\partial t}\right) + \rho\frac{\partial U}{\partial x_i}\,\delta x_i + \frac{\partial\lambda}{\partial x_i}\,\delta x_i + 2\alpha\,\mu\int_{t_0}^{t_1} \sum_j \left(\frac{\partial\Psi}{\partial\left(\dfrac{\partial x_i}{\partial a_j}\right)}\,\delta\!\left(\frac{\partial x_i}{\partial a_j}\right) + \frac{\partial\Psi}{\partial\left(\dfrac{\partial^2 x_i}{\partial t\,\partial a_j}\right)}\,\delta\!\left(\frac{\partial^2 x_i}{\partial t\,\partial a_j}\right) \right) dt \Bigg\},$$

or after some integrations by parts and taking the relations (13) into account:

(15) $$\delta\mathcal{L} \equiv \int_{\mathcal{D}_0} d\tau_0 \int_{t_0}^{t_1} dt \sum_i \left\{ \left(-\rho\frac{\partial^2 x_i}{\partial t^2} + \rho\frac{\partial U}{\partial x_i} + \frac{\partial\lambda}{\partial x_i} + 2\alpha\,\mu Y_i \right)\delta x_i + \int_{t_0}^{t} 2\alpha\mu\left(-\frac{\partial Y_i}{\partial t} + X_i \right)\delta x_i\,dt \right\}.$$

Upon recalling the expression (11) for Y_i in terms of Euler variables, one will confirm that when one makes $\lambda = -p$, $\alpha = 1/2$, the first parenthesis will represent the component along Ox_i of the left-hand side of the indefinite equation of motion of a viscous, incompressible fluid:

$$-\rho\,\gamma + \operatorname{grad}(\rho\,U - p) + \mu\,\Delta V = 0\,.$$

Thus, for an element of $\mathcal{E}$ that verifies the indefinite equation of motion, $\delta\mathcal{L}$ will reduce to:

(16) $$\delta\mathcal{L} \equiv \int_{\mathcal{D}_0} d\tau_0 \int_{t_0}^{t_1} dt \int_{t_0}^{t} \sum_i \mu\left(-\frac{\partial Y_i}{\partial t} + X_i \right)\delta x_i\,dt\,,$$

in which the variations δx_i are annulled for $P \in \Sigma_0$, and for $t = t_0$ or $t = t_1$ and are coupled by:

(17) $$\delta \frac{D(x_1, x_2, x_3)}{D(a_1, a_2, a_3)} = \sum_i \frac{\partial}{\partial x_i} \delta x_i \equiv 0 .$$

When one takes that relation (17) into account, along with the expressions (11) and (12) for the X_i and Y_i, one can see that the form (16) for $\delta \mathcal{L}$ shows that it must not be zero in the general case.

Therefore, one verifies that $\delta \mathcal{L}$ is not annulled in the following example: the real, non-permanent, planar motions with rectilinear trajectories that are given by:

(18) $$u_1 = e^{x_2 + t}, \qquad u_2 = 0 \quad \text{or} \quad x_1 = a_1 + e^{a_2 + t} - e^{a_2}, \qquad x_2 = a_2 .$$

(The indefinite equations of motion are satisfied if $\mu = \rho$ and $p - \rho U =$ constant.)

Suppose that $\mathcal{D}_0$ is defined by: $0 \leq a_1 \leq 1$, $0 \leq a_2 \leq 1$, $t_0 = 0$ and $t_1 = 1$.

Take the variation $\delta f(\delta x_1, \delta x_2)$, with:

(19) $$\delta x_1 = t(1-t) \frac{D(x_1, \Phi^2)}{D(a_1, a_2)}, \qquad \delta x_2 = -\, t(1-t) \frac{D(\Phi^2, x_2)}{D(a_1, a_2)},$$

with

$$\Phi = a_1 \, a_2 \, (1 - a_1)(1 - a_2) ,$$

which satisfies (13) and (17).

When one takes the form of the u_i into account, the only term that will enter into the integral (16) is $-\dfrac{\partial Y_1}{\partial t} \delta x_1 = -\dfrac{\partial u_1}{\partial t} \cdot \delta x_1$, and one is assured that this integral is non-zero.

Finally, one can annul $\delta \mathcal{L}$ for only certain special motions. For example, if the X_i and $\dfrac{\partial Y_1}{\partial t}$ are identically zero, which will be the case for Poiseuille motion, then:

$$u_1 \equiv u_1(x_1, x_2), \qquad u_2 \equiv u_3 \equiv 0 .$$

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