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OPTICS.

Purely geometric proof of the fundamental principle of the theory of caustics

By M. GERGONNE.

Translated by: D. H. Delphenich

On page 14 of the present volume, we expressed the view that we could provide our readers with a proof of the fundamental principle of the theory of surface caustics for refraction that is just as simple as the one that was given by Dupin for caustic surfaces under reflection. An article by Timmermans, a professor of mathematics at the royal college of Ghent, that was inserted into the *Correspondance mathématique et physique* of the kingdom of the Netherlands (tome I, no. 6, page 336), which is a collection that is not widely distributed in France, put me into a position of achieving that goal beyond my hopes. It is true that the author falls a little short, and his proof relates to only plane curves. However, there is little to be done in order to extend it to curved surfaces, and, at the same time, to give it all of the development that is seems to lack. That is then the objective of what we propose to do here.

Let two arbitrary curved surfaces be situated in whatever way that one desires with respect to each other, but absolutely fixed in space. Imagine two concentric spheres that are moving and have variable radii, but in such a manner that their radii always preserve a constant ratio between them. Furthermore, suppose that these spheres move and vary in size in space in such a manner that they are constantly tangent to the two surfaces in question, respectively. Their common center will generate a third surface that one must relate to the other two.

Suppose, to begin with, that the two given surfaces are planar surfaces that we represent by p and p', respectively. It is easy to see that the third one P will also be a planar surface that passes through the intersection of the first two. Indeed, let there be two spheres of arbitrary locations and sizes, let M be their common center, and let m and m' be the points of contact with the two planes p and p', respectively, in such a way that Mm and Mm' are radii of these two spheres, radii whose ratio is assumed to be constant. Let a third plane P be laid through that center M and the common section of the two planes p and p', and choose the point M_1 on it arbitrarily. Drop perpendiculars M_1m_1 and M_1m_1' from that point onto the planes p and p', resp.; these perpendiculars will be parallel to Mm and Mm', respectively. Upon then letting I denote the point where the line MM_1 meets the common section of the three planes, one will have:

$$\frac{Mm}{M_1m_1} = \frac{IM}{IM_1}, \qquad \frac{Mm'}{M_1m_1'} = \frac{IM}{IM_1},$$

and, in turn:

$$\frac{Mm}{M_1m_1} = \frac{Mm'}{M_1m_1'}, \qquad \text{or furthermore} \qquad \frac{M_1m_1}{M_1m_1'} = \frac{Mm}{Mm'}.$$

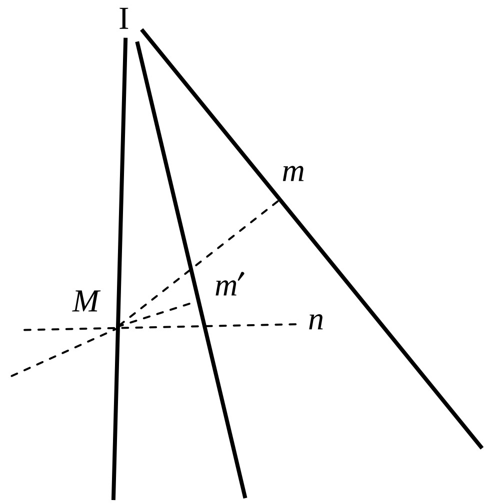
Therefore, one describes two concentric spheres at the point M_1, which is taken to be the common center, that have the radii M_1m_1 and M_1m_1'. These spheres will be tangent to the two planes p and p', respectively, and their radii will have a given constant ratio. That will be one of the locations of our two spheres whose locations and sizes vary in space. One thus sees that all of the points M_1 of the plane P will be centers of such systems of spheres, and it is, moreover, easy to see that they will be the only ones in space.

Now, suppose that the two given fixed surfaces are arbitrary; denote them by s and s', and let S be the unknown surface that is the locus of the centers of the spheres. Let M be one of the locations of the common center on that surface. For that location, let m and m' be the points of contact of these two spheres with the surfaces s and s', respectively. For an infinitely small change in the position of the common center, and as a result, in the size of the spheres, one can replace the two surfaces s and s' with their tangent planes p and p' at m and m', resp., and then, from what was proved above, the common center M can be assumed to move on a plane P that passes through the intersection of the other two. That plane P is then the tangent plane at M to the surface S that is described by the common center. Therefore, in all of the positions of the common center of the two spheres, the planes P, p, p' that are tangent to the surfaces S, s, s' at the points M, m, m' will intersect along the same line, which varies in position like the point M.

Imagine a plane Π through the point M that is perpendicular to that line, so the line will be reciprocally perpendicular to it. The planes P, p, p', in which that line is likewise found to be contained, will thus also be perpendicular to the plane Π. It will then result from this that the radii Mm and Mm', which are perpendicular to the planes p and p', respectively, and consequently normal to the surfaces s and s', resp., will be in that plane, and that the same thing will be true for the perpendicular that is drawn through the point M to the plane P that is normal to the surface S at that point.

Let I be the point where the intersection of the three planes P, p, p' is cut by the plane Π, and consider what happens in the latter plane. One has:

$$\sin nMm = \sin mIM = \frac{Mm}{IM},$$
$$\sin nMm' = \sin m'IM = \frac{Mm}{IM}.$$



Therefore:

$$\frac{\sin nMm}{\sin nMm'} = \frac{Mm}{Mm'}.$$

Due to the fact that the right-hand side of this equation is assumed to be constant for all sizes and locations of the system of two spheres, the left-hand side must be constant, as well. The characteristic property of the surface that it is the locus of the centers of two spheres can thus be stated as follows:

If two concentric spheres move in space with variable radii (although the radii still have a constant ratio, moreover) in such a manner that that they are constantly tangent to two arbitrary, but given, fixed surfaces, respectively, then the locus of their common center will be a third surface such that if one draws normals to the other surfaces from any of its points then these normals will be in the same plane as the normal that is drawn from the same point to the surface that is the locus of the centers. In addition, the sines of the angles that are defined by the first two normals with it will have a constant ratio that is equal to that of the radii of the two spheres (*).

Therefore, if one supposes that the surface S is the separating surface between two media for which the sines of incidence and refraction have the same constant ratio as the radii of the two spheres and that the incident rays are all normal to the surface s then the refracted rays will all be normal to the surface s'. One then has this theorem:

Imagine two homogeneous media with unequal refringent powers that are separated from each other by a surface of an arbitrary nature, and rays of light that penetrate from one of these media into the other. If the incident rays are directed in space in such a manner that they can traverse the same surface orthogonally then the refracted rays will also be directed in space in such a manner that they will traverse a common surface orthogonally, and conversely. In addition, each orthogonal trajectory surface of incident

(*) The editors of the *Correspondence* advised us in a note that Timmermans was in possession of this theorem before he was made aware of the article on page 345 of our volume XV.

rays will always provoke an orthogonal trajectory surface of refracted rays such that at any point of the separating surface between the two media where one draws normals to these other two surfaces the lengths of these respective normals will have a constant ratio that is equal to that of the sine of incidence to the sine of refraction.

It has already been observed several times before, and notably on page 14 of the present volume, that reflection is only a special case of refraction, namely, the one in which the sines of incidence and refraction differ only by their signs. Therefore, the foregoing will implicitly contain the entire theory of caustic surfaces for reflection.

It was observed on page 15 that the theory of planar caustics, either by refraction or by reflection, is only a special case of that of caustic surface. Therefore, what little that we have just said implicitly contains the entire theory of planar caustics and caustic surfaces, whether by refraction or by reflection.

Let us take a look back at this point. We shall refer our thinking to the point of departure of the geometers in the theory that we will consider and rapidly survey the space that one traverses. In 1682, Tschirnhausen was the first to comment on the planar caustic that is formed by parallel rays that are reflected in the circle and propose to look for the equation. This problem, which is only a diversion today, was quite difficult back then. He gave a solution that was falsely attributed to Cassini, Mariotte, and de la Hire, who were commissioners of the royal academy of science in Paris. That effort was unsuccessful in attracting geometers to that sort of curves, which one soon perceives might give the true key to all of the mysteries of optics. Bernoulli, l'Hôpital, Carré, and some others have, in due course, made it the special object of their research and gave general methods for obtaining the equation of an arbitrary planar caustic, either by reflection or by refraction.

In 1810, Malus was the first to consider the general theory of caustic surfaces and found some beautiful theorems. However, some errors in the calculations, which is an almost inevitable result of a very complicated analysis, led him to deny these theorems the generality that they actually deserve. In 1822, Dupin recalled Malus's theory, gave it the complement that it was lacking, and in 1823, in an analysis that is likewise quite complicated (*Annales*, t. XIV, page. 129), we deduce *the possibility of replacing the effect of an arbitrary number of refractions and reflections on the rays that are originally normal to the same, arbitrary surface with either one refraction or one reflection.*

Some research that related to some special cases of reflection and refraction (*Annales*, t. V, page 283, t. XI, page 229, and t. XIV, page 1) led us in 1815 to suspect that most often *the especially complicated caustics might very well be only developments of other curves that are really quite simple*. In 1825, Sturm, by characterizing the curve whose caustic relative to a circle is a developable (*Annales*, t. XV, page 205), gave new weight to that conjecture. Almost at the same time, Quetelet published some elegant theorems in planar caustics, in general (*Mémoires de l'Académie royale des Sciences* de Bruxelles, t. III, page 89), where those of Sturm related to only special cases, moreover. After having proved these theorems by analysis (t. XV, page 345), we extended them to caustic surfaces on the first page of the present volume, along with Sarrus, almost at the same time as us, or rather, we gave a simple and general theorem that contains all of the theory of caustics and surface caustics, whether by refraction or by reflection. All that remains to be desired is a simple proof of that theorem, and now Timmermans has produced one that has reached such a stage that it can be introduced into one's education at even the most elementary level, and one can only be surprised that in the interval of nearly a century and a half so many geometers have worked so hard and made so many calculations in order to finally arrive at a result that was, so to speak, right under their noses. Except for the applications, which always offer practical difficulties, that theory can presently be regarded as complete. However, one must pass through various detours in order to reach that point, because in all situations in which there are both more general and simpler things at the same time, it is the ordinary that last presents itself to one's thinking. Many other theories are also associated with an apparent perfection of the collective efforts of geometers, and they have proved useful to science only by directing one's meditations towards a very important objective. At the point to which we have arrived today, we have, indeed, much less of a need to create new theories as a need to reduce the theories that are already known to their simplest forms, if one might be permitted to say that.