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# The Compton effect according to Schrödinger's theory

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The frequencies and intensities that are radiated by the Compton effect are calculated according to Schrödinger's theory. The quantum-mechanical quantities are obtained from the classical quantities as geometric means from the initial and final states of the process.

**1. Construction of the differential equation for $\psi$.** Heisenberg and Schrödinger have given methods for the determination of quantum frequencies and intensities. The Compton effect was already calculated by Dirac [1]) using the Heisenberg method. Here, the same problem shall be treated by the Schrödinger method. The Schrödinger process has the advantage that it serves as a useful mathematical tool. It is based upon the determination of a quantity $\psi$ for an isolated electron that is a function of the Cartesian space coordinates $x_1$, $x_2$, $x_3$ and time $t$. Schrödinger has presented two rules for arriving at linear, second-order, partial differential equations that $\psi$ must satisfy. Both have a certain relationship to the classical prescription by which one obtains the Hamilton-Jacobi differential equation for the action function $W$: One substitutes the derivatives of $W$ with respect to the coordinates for the corresponding impulses $p_1$, $p_2$, $p_3$ and the derivative with respect to time for $E$ in the relation $f(x, t, p, E) = 0$ that defines $E$. According to one of Schrödinger's rules [2]), instead of the derivatives, one replaces the derivatives with their symbol multiplied by $h/2\pi i$ and applies the resulting differential operator to $\psi$ (in which symmetry assumptions must be made in order to avoid indeterminacy). The classical quantum prescriptions are written:

$$p_k = \frac{\partial W}{\partial x_k}, \qquad E = -\frac{\partial E}{\partial t}; \qquad p_k = \frac{h}{2\pi i}\frac{\partial}{\partial x_k}, \qquad E = -\frac{h}{2\pi i}\frac{\partial}{\partial t}, \tag{1}$$

when one introduces the imaginary quantities:

$$x_4 = ict, \qquad p_4 = \frac{iE}{c} \tag{2}$$

in the symmetric form:

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[1]) P. A. M. Dirac, Proc. Roy. Soc. **111** (1926), 405.

[2]) E. Schrödinger, Ann. d. Phys. **79** (1926), 734.

$$p_\alpha = \frac{\partial W}{\partial x_\alpha}, \qquad\qquad p_\alpha = \frac{h}{2\pi i}\frac{\partial}{\partial x_\alpha}, \tag{1a}$$

in which here, as also in what follows, $k$ means 1, 2, 3, and $\alpha$ means 1, 2, 3, 4.

In relativistic mechanics, the defining equation for the kinetic energy reads:

$$\sum p_k^2 - \frac{E^2}{c^2} + m^2c^2 = 0. \tag{3}$$

($m$ = electron mass, $c$ = velocity of light), or, from (2):

$$\sum p_\alpha^2 + m^2c^2 = 0. \tag{3a}$$

Now, put the electron in an electromagnetic field with the vector potential components $\Phi_1$, $\Phi_2$, $\Phi_3$, and the scalar potential $\Phi_0$, between which there exists the relation:

$$\sum \frac{\partial \Phi_k}{\partial x_k} + \frac{1}{c}\frac{\partial \Phi_0}{\partial x_0} = 0, \tag{4}$$

and from which, the electric and magnetic field strengths can be calculated according to the formulas:

$$E_k = -\frac{\partial \Phi_0}{\partial x_k} - \frac{1}{c}\sum \frac{\partial \Phi_k}{\partial t}, \qquad H_1 = \frac{\partial \Phi_3}{\partial x_2} - \frac{\partial \Phi_2}{\partial x_3}, \tag{5}$$

and cyclic permutations. If we introduce:

$$\Phi_4 = i\Phi_0 \tag{6}$$

then (4) and (5), with the use of ($2_1$), assume the form:

$$\sum \frac{\partial \Phi_\alpha}{\partial x_\alpha} = 0, \tag{4a}$$

$$E_k = i\left(\frac{\partial \Phi_4}{\partial x_k} - \frac{\partial \Phi_k}{\partial x_4}\right), \qquad H_1 = \frac{\partial \Phi_3}{\partial x_2} - \frac{\partial \Phi_2}{\partial x_3}. \tag{5a}$$

These formulas show that $\Phi_\alpha$ is determined up to an additive expression of the form $\partial f / \partial t$, where $f$ satisfies the wave equation $\sum \frac{\partial^2 f}{\partial x_\alpha^2} = 0$.

If a field is present then one clarifies that energy means kinetic energy plus field energy $e\Phi_0$ ($c$ = electron charge), and then, on the grounds of invariance, impulse means kinetic impulse plus "field impulse" $e/c\ \Phi_k$. (3) and (3a) become:

$$\sum\left(p_k - \frac{e}{c}\Phi_k\right)^2 - \frac{(E-e\Phi_0)^2}{c^2} + m^2c^2 = \sum\left(p_\alpha - \frac{e}{c}\Phi_\alpha\right)^2 + m^2c^2 = 0. \tag{7}$$

From (1a), the Hamilton-Jacobi (Schrödinger, resp.) differential equation then becomes:

$$\sum\left(\frac{\partial W}{\partial x_\alpha} - \frac{e}{c}\Phi_\alpha\right)^2 + m^2c^2 = 0, \tag{8}$$

or

$$\left\{\sum\left(\frac{h}{2\pi i}\frac{\partial}{\partial x_\alpha} - \frac{e}{c}\Phi_\alpha\right)^2 + m^2c^2\right\}\psi = 0,$$

respectively, or, after carrying out the square and multiplying by $-4\pi^2/h^2$:

$$\sum\frac{\partial^2\psi}{\partial x_\alpha^2} - \frac{4\pi i}{h}\sum\Phi_\alpha\frac{\partial\psi}{\partial x_\alpha} - \frac{4\pi^2}{h^2}\left(\frac{e^2}{c^3}\sum\Phi_\alpha^2 + m^2c^2\right)\psi = 0; \tag{9}$$

the first indeterminacy that is present – viz., whether one should write $\sum\Phi_\alpha\frac{\partial\psi}{\partial x_\alpha}$ or $\sum\frac{\partial(\Phi_\alpha\psi)}{\partial x_\alpha}$ – is lifted, on the grounds of (4a). An increase in $\Phi_\alpha$ by $\partial f/\partial x_\alpha$ corresponds to an increase of $W$ by $e/c\ f$ and a multiplication of $\psi$ by $e^{\frac{2\pi i}{h}\frac{e}{c}f}$.

The differential equation (9), together with the one for the complex conjugate function $\bar{\psi}$, can be obtained from the variation of the integral:

$$\left.\begin{aligned} J &= \int H\,dx_1dx_2dx_3dx_4, \\ H &= \sum\frac{\partial\psi}{\partial x_\alpha}\frac{\partial\bar{\psi}}{\partial x_\alpha} + \frac{2\pi i}{h}\frac{e}{c}\sum\left(\bar{\psi}\frac{\partial\psi}{\partial x_\alpha} - \psi\frac{\partial\bar{\psi}}{\partial x_\alpha}\right)\Phi_\alpha + \frac{4\pi^2}{h^2}\left(\frac{e^2}{c^2}\sum\Phi_\alpha^2 + m^2c^2\right)\psi\bar{\psi}, \end{aligned}\right\} \tag{10}$$

when one treats $\psi$ and $\bar{\psi}$ as independent functions whose variations vanish on the boundary of the integration domain. This yields the generalization of the other Schrödinger rule [1]): One Hermitizes the Hamilton-Jacobi equation (8):

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[1]) E. Schrödinger, Ann. d. Phys. **79** (1926), 361.

$$\left(\frac{\partial W}{\partial x_\alpha} - \frac{e}{c}\Phi_\alpha\right)\left(\frac{\partial \bar{W}}{\partial x_\alpha} - \frac{e}{c}\Phi_\alpha\right) + m^2c^2 = 0,$$

and makes the substitution $W = h / 2\pi i \log \psi$ in it, with which, after multiplying by $4\pi^2 / h^2 \psi\bar{\psi}$, the left-hand side goes to the expression $H$ in (10). However, instead of setting $H = 0$, one sets the variation of the integral $\int H \, dx_1 \, dx_2 \, dx_3 \, dx_4$ equal to zero. In the limit $h = 0$, $W$ becomes real and (9) goes to (8).

If the potentials are time-independent then one can, in agreement with (1), make the Ansatz:

$$\psi = u \, e^{-\frac{2\pi i}{h}Et}, \tag{11}$$

with time-independent $u$. (9) and (10) then become:

$$\sum \frac{\partial^2 u}{\partial x_k^2} - \frac{4\pi i}{h}\frac{e}{c}\sum \Phi_k \frac{\partial u}{\partial x_k} - \frac{4\pi^2}{h^2}\left(\frac{e^2}{c^2}\sum \Phi_k^2 - \frac{(E - e\Phi_0)^2}{c^2} + m^2c^2\right) u\bar{u} = 0, \tag{9a}$$

$$\left.\begin{aligned} J &= \int H \, dx_1 dx_2 dx_3 dx_4, \\ H &= \sum \frac{\partial u}{\partial x_k}\frac{\partial \bar{u}}{\partial x_k} + \frac{2\pi i}{h}\frac{e}{c}\sum\left(\bar{u}\frac{\partial \psi}{\partial x_k} - u\frac{\partial \bar{\psi}}{\partial x_k}\right)\Phi_k \\ &\qquad + \frac{4\pi^2}{h^2}\left(\frac{e^2}{c^2}\sum \Phi_k^2 - \frac{(E - e\Phi_0)^2}{c^2} + m^2c^2\right) u\bar{u}. \end{aligned}\right\} \tag{10a}$$

In the case of classical mechanics, one must replace $E$ with $E + mc^2$, and go to the limit $c = \infty$; in this, $e/c \, \Phi_k$ then remains untouched, since the $c$ here arises from the fact that $e$ is thought of as measured in electromagnetic units. In this sense, one must replace $\partial / \partial t$ with $\partial / \partial t - 2\pi i / h \, mc^2$ in (9) and (10) and $(E - e\Phi_0)^2 / c^2 - m^2c^2$ with $2m(E - c\,\Phi_0)$ in (9a) and (10a). For $\Phi_k = 0$, the last two equations then take on the form of the ones that Schrödinger published [1]).

**2. Determination of the radiation from $\psi$.** Classically, one computes the radiation with the help of the motion of the electron. Starting from a complete integral of (8) with the three constants $c_k$, one obtains the motion in the state that is defined by these constants by means of the formula:

$$\frac{\partial W}{\partial c_k} = d_k, \tag{12}$$

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[1]) E. Schrödinger, Ann. d. Phys. *loc. cit.* and **79** (1926), 489.

where the $d_k$ are three more constants. When (12) is solved, it gives the coordinates as functions of time.

In quantum theory, one cannot speak of the motion in a state, since all the motions are coupled with each other. The possible radiations are the spatially-distributed currents and charges of the one system, which are derived from $\psi$ in the following way: If we multiply (9) by $\bar{\psi}$ and the complex conjugate equation that is valid for $\bar{\psi}$ by $\psi$ and subtract both equations from each other then we obtain, while observing (4a):

$$\sum \frac{\partial s_\alpha}{\partial x_\alpha} = 0, \tag{13}$$

with

$$s_\alpha = i\left(\bar{\psi}\frac{\partial \psi}{\partial x_\alpha} - \psi\frac{\partial \bar{\psi}}{\partial x_\alpha} - \frac{4\pi i}{h}\frac{e}{c}\Phi_\alpha \psi\bar{\psi}\right). \tag{14}$$

In order to go to a real representation, if we set:

$$s_k = s_k, \qquad s_4 = ic\,\rho \tag{15}$$

then (13) can be written:

$$\sum \frac{\partial s_k}{\partial x_k} + \frac{\partial \rho}{\partial t} = 0. \tag{13a}$$

We are then justified in speaking of the $s_k$ as the components of a current density and $\rho$ as a charge density. The continuity equation (13a) then exists between these quantities, and *a priori* they do not have to satisfy any other condition in order for them to serve as the sources of an electromagnetic field in Maxwell's equations. The factor $1/i$ was added in (14) in order to make $s_k$ and $\rho$ real. One easily confirms that these quantities are independent of the aforementioned indeterminacy in the potentials $\Phi_\alpha$ . They will be obtained from the Hamilton function $H$ (10) by derivation with respect to the potentials, as is also the case in Mie's theory of matter [1]). One has:

$$s_\alpha = -\frac{h}{2\pi}\frac{e}{c}\frac{\partial H}{\partial \Phi_\alpha}. \tag{16}$$

The field that is generated by the density is given by the retarded potentials:

$$\Phi_\alpha = \frac{1}{c}\int \frac{[s_\alpha]}{R}dx, \qquad dx = dx_1\,dx_2\,dx_3 \tag{17}$$

by means of formula (5a). $R$ is the distance from the volume element $dx$ to the origin and square bracket shall indicate that $t$ is set equal to the value $t - R/c$. The radiation is equal to the radiation that originates at the electric center of mass for the charges. This center of mass is defined by:

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[1]) Cf., e.g., M. v. Laue, *Relativitätstheorie II,* eq. (271).

$$e\,X_k = \int x_k\,\rho\,dx, \qquad e = \int \rho\,dx, \tag{18}$$

which one can summarize as:

$$e\,X_\alpha = \int x_\alpha\,\rho\,dx. \tag{18a}$$

From the continuity equation (13a), when the current vanishes on the boundary of the space to a sufficient degree, it then follows that:

$$0 = \sum_k \int \frac{\partial s_k}{\partial x_k}\,dx = -\int \frac{\partial \rho}{\partial t}\,dx,$$

$$0 = \sum_r \int \frac{\partial (x_k s_r)}{\partial x_r}\,dx = -\int x_k \frac{\partial \rho}{\partial t}\,dx + \int s_k\,dx.$$

The first equation says that the total charge is constant in time, as it must be, and the second one, that the velocity of the center of charge is given by:

$$e\,\frac{dX_k}{dt} = \int s_k\,dx, \tag{19}$$

or, together with the last equation (18):

$$e\,\frac{dX_\alpha}{dt} = \int s_\alpha\,dx \;{}^{1}). \tag{19a}$$

In order for the field to be the classical one for $h = 0$ (i.e., the correspondence principle), (18) must go to the totality of all possible classical motions for $h = 0$ ²). In particular, the total charge must be equal to the charge of the electron, as we have already suggested by the notation.

We next assume that, for natural boundary conditions, equation (9) possesses a sequence of discrete solutions $\psi_1$, $\psi_2$, …, which we summarize by the sum:

$$\psi = \sum_l z_l \psi_l\,. \tag{20}$$

The (real) constants $z_l$ are definitive of the weight of the state $l$. The densities (14) become:

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¹) Editor's remark. One can, with E. Madelung (Naturwiss. **14** (1926), 1004), regard the current as electricity moving with the velocity $\mathfrak{u} = \mathfrak{s}/\rho$ ($\mathfrak{s} = s_1, s_2, s_3$). Its mass density is then $m\sigma = m\rho/e$. $X_k$ and $dX_k / dt$ are then the position and velocity of the center of mass, resp. – By neglecting the magnetic field and relativity, (14) yields $\mathfrak{s} = 1/i \cdot (\bar{\psi}\,\mathrm{grad}\,\psi - \psi\,\mathrm{grad}\,\bar{\psi}) = 2\psi\bar{\psi}\,\mathfrak{a}''$, (with Madelung's notation), $\rho = \frac{4\pi m}{h}\psi\bar{\psi}$, such that $\mathfrak{u} = \frac{h}{2\pi m}\,\mathfrak{a}''$, as with Madelung.

²) In this determination, the possibility of additional terms that vanish for $h = 0$ still exists. (Cf., rem. I, pp. 12).

$$s_\alpha = \sum_{lm} z_l z_m s_\alpha^{(lm)}\,, \tag{21}$$

with

$$s_\alpha^{(lm)} = i\left(\bar{\psi}_m \frac{\partial \psi_l}{\partial x_\alpha} - \psi_l \frac{\partial \bar{\psi}_m}{\partial x_\alpha} - \frac{4\pi i}{h}\frac{e}{c}\Phi_\alpha \psi_l \bar{\psi}_m\right). \tag{21a}$$

The $s_\alpha^{(lm)}$ define the elements of a Hermitian matrix, so they can be derived from a Hermitian matrix $H^{(lm)}$ in the manner of (16), which arises from the $H$ in (10) in such a way that one replaces $\psi$ and $\psi_l$ with $\bar{\psi}$ and $\bar{\psi}_m$, resp. According to (18), (19), and (21), the motion will be represented by:

$$X_k = \sum_{lm} z_l z_m X_k^{(lm)}\,, \qquad \frac{dX_k}{dt} = \sum_{lm} z_l z_m \frac{dX_k^{(lm)}}{dt}\,, \tag{22}$$

with

$$cX_k^{(lm)} = \int x_k\, \rho^{(lm)}\, dx, \qquad c\frac{dX_k^{(lm)}}{dt} = \int s_k^{(lm)} dx\,. \tag{22a}$$

The $X_k^{(lm)}$ are the Heisenberg matrices, in the event that the functions $\psi_l$ are suitably normalized. In the case of (11), its Schrödinger representation follows from (22a) [1]).

If the index $l$ is capable of taking on continuous values then integrals appear in place of the sums.

**3. Application to the Compton effect.** The primary radiation will be described by a plane, linearly-polarized wave with a direction $n_1$, $n_2$, $n_3$, and an oscillation number $\nu$. Its potentials are:

$$\Phi_\alpha = a_\alpha \cos \varphi, \qquad a_4 = i\, a_0\,, \tag{23}$$

with a phase:

$$\varphi = \frac{2\pi\nu}{c}\left(\sum n_k x_k - ct\right) = \sum l_\alpha a_\alpha = lx, \tag{24}$$

if one sets

$$l_k = \frac{2\pi\nu}{c} n_k\,, \qquad l_4 = i\, l_0 = i\frac{2\pi\nu}{c}\,, \tag{25}$$

and sums of the form $\sum f_\alpha g_\alpha$ are written $fg$, to abbreviate. The relation $\sum n_k^2 = 1$ and condition (4a) yield:

$$l^2 = 0, \qquad al = 0. \tag{26}$$

From (5a), the field is:

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[1]) E. Schrödinger, Ann. d. Phys. **79** (1926), 734.

$$\left.\begin{aligned} E_k &= i(a_k l_4 - a_4 l_k)\sin\varphi = \frac{2\pi\nu}{c}(a_0 n_k - a_k)\sin\varphi, \\ H_1 &= (a_2 l_3 - a_3 l_2)\sin\varphi, \quad \text{and cyclic permutations.} \end{aligned}\right\} \tag{27}$$

The electric vector lies in the plane through the vector $a_k$ and the wave normal that is perpendicular to it, while the magnetic vector is perpendicular to that plane. Both of them have a magnitude that is equal to $\frac{2\pi\nu}{c}\sqrt{aa}\,\sin\varphi$.

With the values (23) for the $\Phi_\alpha$, while neglecting the $a_\alpha^2$, the differential equations (8) and (9) read:

$$\sum\left(\frac{\partial W}{\partial x_\alpha}\right)^2 - 2\left(b\frac{\partial W}{\partial x}\right)\cos\varphi + m^2c^2 = 0,$$

$$\sum\frac{\partial^2\psi}{\partial x_\alpha^2} - \frac{4\pi i}{h}\left(b\frac{\partial\psi}{\partial x}\right)\cos\varphi - \frac{4\pi^2}{h^2}m^2c^2\psi = 0,$$

with

$$b_\alpha = \frac{e}{c}\,a_\alpha. \tag{28}$$

They are solved by:

$$W = px + \frac{pb}{pl}\sin\varphi, \qquad \psi = e^{\frac{2\pi t}{h}W}, \tag{29}$$

if the relation (3a) exists between the integration constants $p_\alpha$ (which likewise implies their meaning), as one easily confirms by observing (26) [1]).

We next determine the classical motion from (12). We take $p_k = c_k$ for the independent integration constants, such that, from (3):

$$\frac{\partial E}{\partial p_k} = \frac{c^2 p_k}{E}. \tag{30}$$

Formula (12) yields, when one goes to a real representation by means of (2), (23), (28), and (25):

$$x_k = \frac{c^2 p_k}{E}t + \frac{c}{E(pl)}\left[\frac{pb}{pl}\left(l_k\frac{E}{c} - l_0 p_k\right) - \left(b_k\frac{E}{c} - b_0 p_k\right)\right]\sin\varphi + d_k. \tag{31}$$

From (24), and in our approximation (except for a constant), the phase is set to:

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[1]) The relation (29) between $\psi$ and $W$, which was true only for small $h$ up to now, is also true here rigorously, when one does not neglect $b^2$, which adds the term $-\frac{b^2}{8pl}(2\varphi + \sin 2\varphi)$.

$$\varphi = 2\pi\nu \left( \sum n_k \frac{cp_k}{E} - 1 \right) t = \frac{c^2}{E} (pl)\, t. \tag{32}$$

The velocities are then:

$$\frac{dx_k}{dt} = \frac{c^2 p_k}{E} t + \frac{c^3}{E^2} \left[ \frac{pb}{pl} \left( l_k \frac{E}{c} - l_0 p_k \right) - \left( b_k \frac{E}{c} - b_0 p_k \right) \right] \cos \varphi. \tag{33}$$

The motion then consists in a uniform, rectilinear motion with the velocity $v$ ($v_k = c^2 p_k / E$), over which a harmonic oscillation is overlaid with the frequency:

$$\nu_0 = \nu \left( 1 - \sum n_k \frac{cp_k}{E} \right) = \nu \left( 1 - \frac{v}{c} \cos \vartheta \right), \tag{34}$$

where $\vartheta$ is the angle between the direction of the velocity and the wave normal.

The laws of quantum motion and radiation are deduced from the knowledge of the densities $s_\alpha$. For the sake of normalization, we multiply the solution $\psi$ of (29) by a (real) function $C(p_1, p_2, p_3)$ of the constants $p_k$, and using the template (20), we define the total solution:

$$\psi = \int z(p)\, C(p)\, e^{\frac{2\pi i}{h} W}\, dp, \qquad dp = dp_1\, dp_2\, dp_3\,, \tag{35}$$

where the integral is extended over all of $p$-space. Analogous to the energy-impulse vector of the electron, we introduce the corresponding quantity for the primary light quantum:

$$\pi_\alpha = \frac{h}{2\pi} l_\alpha, \text{ i.e.,} \qquad \pi_k = \frac{h\nu}{c}\, n_k\,, \quad \pi_4 = i\, \frac{\varepsilon}{c} = i \frac{h\nu}{c}. \tag{36) [from (25)}$$

The de Broglie phases for the electron and light quantum are then:

$$f = \frac{2\pi}{h} (p\, x), \qquad \varphi = \frac{2\pi}{h} (\pi\, x). \tag{37}$$

According to (29), the phase $2\pi / h\, W$ will then be:

$$\frac{2\pi}{h} W = f + k \sin \varphi, \tag{38}$$

with

$$k = \frac{pb}{p\pi}. \tag{38a}$$

Moreover, we construct the $s_\alpha$ of (14) with (35). From (37), (38), and (38a), one has:

$$\psi\bar{\psi} = \int e^{\frac{2\pi i}{h} \delta W} z(p)\, z(p')\, C(p)\, C(p')\, dp\, dp',$$

$$\psi \frac{\partial \psi}{\partial x_\alpha} = \frac{2\pi i}{h} \int (p_\alpha + k\, \pi_\alpha \cos \varphi)\; e^{\frac{2\pi i}{h}\delta W}\; z(p)\, z(p')\, C(p)\, C(p')\, dp\, dp',$$

in which $\delta F(p)$ means the difference $F(p) - F(p')$. If one takes the complex conjugate of the last expression, when one simultaneously exchanges primed and unprimed quantities (which is allowed, since it does not affect the notation of the integration variables), then what one gets is:

$$\psi \frac{\partial \bar{\psi}}{\partial x_\alpha} = -\frac{2\pi i}{h} \int (\, p'_\alpha + k'\, \pi_\alpha \cos \varphi)\; e^{\frac{2\pi i}{h}\delta W}\; z(p)\, z(p')\, C(p)\, C(p')\, dp\, dp'.$$

Therefore, we have everything all at once that it takes to be able to define the $s_\alpha$ of (14). When one considers (23) and (28), one finds that:

$$s_\alpha = \frac{2\pi}{h} \int \{\sigma p_\alpha + (\pi_\alpha\, \sigma k - 2b_\alpha) \cos \varphi\}\; e^{\frac{2\pi i}{h}\delta W}\; z(p)\, z(p')\, C(p)\, C(p')\, dp\, dp',$$

where $\sigma F(p)$ means the sum $F(p) + F(p')$. In our approximation, here, from (38), $e^{\frac{2\pi i}{h}\delta W}$ must be replaced with $e^{i\delta f}(1 + i\sigma k \sin \varphi)$, such that the curly bracket, when multiplied by $e^{\frac{2\pi i}{h}\delta W}$, equals:

$$\{\sigma p_\alpha\, e^{i\delta f} + i\delta k\; \sigma p_\alpha \sin \varphi + (\pi_\alpha\, \sigma k - 2b_\alpha) \cos \varphi\}\; e^{i\delta f},$$

in which one can also introduce:

$$\mathfrak{R}\{\sigma p_\alpha\, e^{i\delta f} + (\delta k\; \sigma p_\alpha + \pi_\alpha\, \sigma k - 2b_\alpha) \cos \varphi\}\; e^{i\,(\delta f + \varphi)}$$

[$\mathfrak{R}$ = real part] [1]). One confirms this, when one switches $i$ with $-i$ and the primed with the unprimed quantities and takes the arithmetic mean of both integrals in:

$$s_\alpha = \frac{2\pi}{h} \mathfrak{R} \int \{\sigma p_\alpha + T_\alpha\, e^{i\,(\delta f + \varphi)})\; z(p)\, z(p')\, C(p)\, C(p')\, dp\, dp', \tag{39}$$

$$T_\alpha = \delta k\; \sigma p_\alpha + \pi_\alpha\, \sigma k - 2b_\alpha\,. \tag{40}$$

One can therefore write the corresponding cosine in (39), instead of the $e$-functions.

In order to determine the functions $C$, we compare the "quantum motion" (19) with the classical motion (33) [1]). We thus have to integrate (39) over the space of all $x_k$. The integral over the $p'_k$ and $x_k$ that thus arises can be put into the form:

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[1]) The quantities with the index $\alpha = 4$ are thus to be considered as real. Their imaginary values are first introduced in the construction of their real parts.

$$\int F(P,P')\, e^{\frac{2\pi i}{h}\sum x_k (P_k - P'_k)} dP'dx,$$

which, from the Fourier integral theorem, is equal to:

$$h^2\, F(P, P').$$

Thus, with $P_k = p_k$, $P'_k = p'_k$, one has:

$$\left.\begin{aligned} \delta f &= \frac{2\pi}{h}\sum x_k (P_k - P'_k) - \frac{2\pi}{h}(E - E')t, \\ \int \sigma p_\alpha e^{i\delta f} z(p')C(p')\, dp'dx &= 2h^2 p_\alpha z(p)C(p), \end{aligned}\right\} \tag{41}$$

since $p_4 = p'_4$ – i.e., $E = E'$ – follows from $p_k = p'_k$. With $P_k = p_k + \pi_k$, $P'_k = p'_k$, one has:

$$\left.\begin{aligned} \delta f + \varphi &= \frac{2\pi}{h}\sum x_k (P_k - P'_k) = \frac{2\pi}{h}(E + \varepsilon - E')t, \\ \int T_\alpha e^{i(\delta f + \varphi)} z(p')C(p')\, dp'dx &= h^2 T_\alpha e^{-2\pi i v^* t} z(p)C(p), \end{aligned}\right\} \tag{42}$$

$$v^* = \frac{E + \varepsilon - E'}{h}, \tag{42a}$$

in which one sets $p'_k = p_k + \pi_k$. If one introduces:

$$\pi^*_k = 0, \qquad \pi^*_4 = i\,\frac{hv^*}{c} \tag{43}$$

then one can write this condition, together with (42a), in the symmetric form:

$$p_\alpha + \pi_\alpha = p'_\alpha + \pi'_\alpha. \tag{44}$$

Therefore, according to (39), (41), and (42), (19a) reads:

$$e\,\frac{dX_\alpha}{dt} = 4\pi h^2 \int p_\alpha z^2(p)\, C^2(p)\, dp + 2\pi h^2 \int T_\alpha z(p)\, z(p')\, C(p)\, C(p') \cos 2\pi v^* t\, dp. \tag{45}$$

If one then sets:

$$C^2 = \frac{ec^2}{4\pi h^2 E} \tag{46}$$

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[1]) The coordinates $X_k$ in (18) are obtained from (19) by integrating over time. The integration constants play a role in the determination of the radiation.

and sets $z^2 dp$ equal to the weight of the state $p$ – i.e., the relative number of electrons in this state – then the parts in (33) and (45) that originate in the uniform motion come into agreement for $\alpha = k$ [1]). For $\alpha = 4$, the second integral in (45) must vanish, due to the temporal constancy of the total charge $e$. This yields the relation:

$$\int z^2\, dp = 1 \tag{47}$$

for the weight, as it must be.

We would like to show that from (46) the oscillatory parts also come into agreement when $h = 0$. From (44), it follows that:

$$p'^2 = p^2 + 2p\pi + \pi^2 - 2p\pi^* - 2\pi\,\pi^* + \pi^{*2},$$

or since, from (3a), $p'^2 = p^2 = -\,m^2c^2$, and from (26) and (36), $\pi^2 = 0$, one has:

$$p\pi = p\pi^* + \pi\,\pi^* - \frac{\pi^{*2}}{2}\,. \tag{48}$$

From this, when one goes to the real representation using (2), (36), and (43), it follows that:

$$v^* = \frac{v\left(1 - \sum \frac{cn_k\, p_k}{E}\right)}{1 + \frac{2hv - hv^*}{2E}}\,. \tag{48a}$$

Thus, $v^*$ agrees with $v_b$ in (34) for $h = 0$. From (44), the $T_\alpha$ in (40) becomes:

$$T_\alpha = 2k\,\pi_\alpha + \delta k(2p_\alpha - \pi_\alpha^*) - 2b_\alpha\,. \tag{49}$$

If we multiply (44) by $b_\alpha$ and sum over $\alpha$ then we obtain: $p'b = pb - \pi^* b$, due to the relation $\pi\, b = 0$ that follows from ($26_2$), in conjunction with (28) and (36). Analogously, upon multiplying (44) by $\pi_\alpha$ we obtain $p'\pi = p\pi - \pi\pi^*$, due to the relation $\pi^2 = 0$ that we already employed. Thus, from (38a), one has:

$$\delta k = \frac{pb}{p\pi} - \frac{pb - \pi^* b}{p\pi - \pi\pi^*} = \frac{p\pi \cdot \pi^* b - pb \cdot \pi\pi^*}{p\pi(p\pi - \pi\pi^*)}\,,$$

or, from (48):

$$\delta k = \frac{\pi^* b - k \cdot \pi\pi^*}{\pi^* p - \frac{\pi^{*2}}{2}}\,. \tag{50}$$

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[1]) It is very plausible to assume that the uniform, rectilinear motion coincides classically and quantum-theoretically, such that the additional terms in remark 2 on pp. 6 drop out.

Equations (48), (49), and (50) follow from (44) and are valid independently of the special values (43) for $\pi_\alpha^2$. For these values, from (50), one gets:

$$\delta k = \frac{b_4 - k\pi_4}{p_4 - \frac{\pi_4^*}{2}}. \tag{50a}$$

With the abbreviation $\mathfrak{p}_\alpha = p_\alpha - \pi_\alpha^2 / 2$, it then arises from (49) that:

$$T_\alpha = \frac{2}{\mathfrak{p}_4}[k\,(p_\alpha \mathfrak{p}_4 - \pi_4\, \mathfrak{p}_\alpha) - (b_\alpha \mathfrak{p}_4 - b_4\, \mathfrak{p}_\alpha)\}. \tag{51}$$

From this, it then follows that $T_4 = 0$. As we have already concluded above, the oscillatory part in (45) then vanishes for $\alpha = 4$. For $\alpha = k$, from (36), (38a), and (43), and with $\mathfrak{p}_4 = i\, \mathfrak{E}/c = \frac{i}{c}\left(E - \frac{h\nu^*}{2}\right)$, one has:

$$T_k = \frac{2c}{\mathfrak{E}}\left\{\frac{pb}{pl}\left(l_k \frac{\mathfrak{E}}{c} - l_0 p_k\right) - \left(b_k \frac{\mathfrak{E}}{c} - b_0 p_k\right)\right\}. \tag{51a}$$

The oscillatory part of $dX_k / dt$ in (45) then reads, with the use of (46):

$$\int \frac{c^3 z(p) z(p')}{\mathfrak{E}\sqrt{EE'}}\left\{\frac{pb}{pl}\left(l_k \frac{\mathfrak{E}}{c} - l_0 p_k\right) - \left(b_k \frac{\mathfrak{E}}{c} - b_0 p_k\right)\right\} \cos 2\pi\nu^* t\, dp.$$

For $h = 0$, one has $\mathfrak{E} = E = E'$, $p_k = p_k'$, such that this expression agrees with the oscillatory part in (33), since, as we found above, $z^2\, dp$ is the weight of the state $p$.

For the determination of the frequencies and intensities, we must further substitute (39) into (17). We can then restrict ourselves to the oscillatory part, since obviously the uniform, rectilinear motion does not contribute to the radiation. In the usual approximation, for the distant reference point in $e^{i\,(\delta f + \varphi)}$ we replace the $R$ in $t - R/c$ with $r - \sum \xi_k x_k$, where $r$ is the distance from the reference point to a mean position in the charge domain and the $\xi_k$ are the direction cosines of $r$ (observation direction). We simply replace the $R$ in the denominator of (17) with $r$. With:

$$P_k = p_k + \pi_k - \frac{E + \varepsilon}{c}\xi_k, \qquad P_k' = p_k' - \frac{E'}{c}\xi_k, \qquad \nu^* = \frac{E + \varepsilon - E'}{h}, \tag{52}$$

one gets:

$$[\delta f + \varphi] = \frac{2\pi}{h}\sum (P_k - P_k')x_k - 2\pi\nu^*\left(t - \frac{r}{c}\right). \tag{52a}$$

From (17) and (39), when one replaces $C$ with its value (46), the radiation potential then becomes:

$$\Phi_\alpha = \frac{ec}{2h^3 r}\Re\int \frac{T_\alpha z(p) z(p')}{\sqrt{EE'}}\, e^{\frac{2\pi i}{h}\sum (P_k - P'_k)x_k - 2\pi i \nu^*(t - r/c)}\, dp\, dp' dx\,. \tag{53}$$

Here, we introduce the quantities $P_k$ and $P'_k$ as integration variables. The functional determinant $|\,\partial P / \partial p\,|$ of the $P_k$ with respect to the $p_k$ is orthogonal invariant. One can therefore rotate the axis-cross such that one has $p_2 = p_3 = 0$. While observing (30), one then has:

$$\frac{\partial P_1}{\partial p_1} = 1 - \frac{cp_1}{E}\,\xi_1\,, \qquad \frac{\partial P_2}{\partial p_2} = \frac{\partial P_3}{\partial p_3} = 1,$$

while all other elements of the determinant vanish. One then finds, when one likewise once more goes to a general position for the axis-cross:

$$\Delta = \left|\frac{\partial P}{\partial p}\right| = 1 - \sum \frac{cp_k}{E}\,\xi_k = 1 - \frac{v}{c}\cos\psi, \tag{54}$$

where $\psi$ is the angle between the velocity and the observation direction. $\Delta$ is then the well-known Doppler factor. The determinant $|\,\partial P' / \partial p'\,|$ is obtained from (54) when one puts the primed quantities in place of the unprimed ones. The invariance of the weight $z^2\, dp$ requires that:

$$z^2(p) = Z^2(P)\,\Delta(p), \qquad z^2(p') = Z'^2(P')\,\Delta(p'), \tag{55}$$

where $Z^2$ ($Z'^2$, resp.) is the weighting function for taking the variables $P$ ($P'$, resp.) as the basis. (53) now takes the form:

$$\Phi_\alpha = \frac{ec}{2h^3 r}\Re\int \frac{T_\alpha Z(P) Z'(P')}{\sqrt{E\,\Delta E'\,\Delta'}}\, e^{\frac{2\pi i}{h}\sum (P_k - P'_k)x_k - 2\pi i \nu^*(t - r/c)}\, dP\, dP'\, dz \tag{53a}$$

($\Delta' = \Delta(p')$). If we apply Fourier's integral theorem then we find that:

$$\Phi_\alpha = \frac{ec}{2r}\int \frac{T_\alpha Z(P) Z'(P')}{\sqrt{E\,\Delta E'\,\Delta'}} \cos 2\pi\nu^*\left(t - \frac{r}{c}\right) dP\,, \tag{56}$$

where we have substituted $P' = P$. Then, since $Z^2(P)\, dP$ and $Z'^2(P)\, dP$ are the weights of the two state domains, which combine with each other, an individual "transition" is associated with the radiation potential [1]):

$$\Phi_\alpha = \frac{ec}{2r}\frac{T_\alpha}{\sqrt{E\,\Delta E'\,\Delta'}} \cos\varphi^*\,. \tag{56a}$$

---

[1]) Cf., the representation (22). $z_l^2$ corresponds to $Z^2(P)\, dP$ and $z_m^2$ to $Z'^2(P)\, dP$, and therefore $z_l\, z_m$ corresponds to $Z(P)\, Z'(P)\, dP$.

If we introduce the scattered quantum:

$$\pi_k^* = \frac{h\nu^*}{c}\,\xi_k\,, \qquad \pi_4^* = i\,\frac{\varepsilon^*}{h} = i\,\frac{h\nu^*}{c} \tag{57}$$

then the relation $P = P'$, together with the last equation in (52), again assumes the form (44):

$$p_\alpha + \pi_\alpha = p'_\alpha + \pi'_\alpha\,. \tag{58}$$

These are the conservation laws for energy and impulse, which is the point from which the light quantum theory of Compton-Debye starts. Furthermore, since $\pi^{*2} = 0$, equation (48) reduces to:

$$p\pi = p\pi^* + \pi\pi^*. \tag{59}$$

From this, when one goes to the real representation using (2), (36), and (57), it then follows that:

$$\nu^* = \frac{\nu\left(\frac{E}{c} - \sum p_k n_k\right)}{\frac{E}{c} - \sum p_k \xi_k + \frac{h\nu}{c}(1 - \sum n_k \xi_k)}, \tag{59a}$$

or, when one introduce the angle $\Theta$ between the primary and the secondary rays, along with the previously-introduced angles $\vartheta$, $\psi$:

$$\nu^* = \frac{\nu\left(1 - \frac{v}{c}\cos\vartheta\right)}{1 - \frac{v}{c}\cos\psi + \frac{h\nu}{c}(1 - \cos\Theta)} = \frac{\nu_b}{\Delta + \frac{h\nu}{E}(1 - \cos\Theta)}\,; \tag{59b}$$

the last is true because of (34) and (54) [1]). One obtains the classical frequency from this for $h = 0$:

$$\nu_{\text{cl}} = \frac{\nu_b}{\Delta}\,; \tag{59c}$$

i.e., the frequency of motion $\nu_b$ divided by the Doppler factor $\Delta$, as it must be. If one multiplies (58) by $\pi_\alpha^*$ and sums over $\alpha$ then since $\pi^{*2} = 0$, one obtains: $p'\pi^* = p\pi^* + \pi\pi^*$, which, upon comparison with (59), yields:

$$p\pi = p'\pi^*, \tag{60}$$

or, from (2), (36), and (57), when it is written in real form:

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[1]) De Broglie, Ann. d. Phys. **3** (1925), 22.

$$\frac{h\nu}{c}\left(\sum p_k n_k - \frac{E}{c}\right) = \frac{h\nu^*}{c}\left(\sum p'_k \xi_k - \frac{E'}{c}\right),$$

or finally, with the relations (34) and (54):

$$\nu^* = \frac{E\nu_b}{E'\Delta'} = \frac{E\Delta}{E'\Delta'}\nu_{\text{cl}}\,; \tag{61}$$

the last one is true because of (59c). If one switches the primed and unprimed quantities in (58) and simultaneously switches $h$ with $-h$ then they go to each other. By means of this exchange, from (61), one gets:

$$\nu^* = \frac{E'\nu'_b}{E\Delta} = \frac{E'\Delta'}{E\Delta}\nu'_{\text{cl}}\,. \tag{61a}$$

From (61) and (61a), it then follows that:

$$\nu^* = \sqrt{\nu_{\text{cl}}\nu'_{\text{cl}}} \tag{62}$$

and

$$E\,\Delta\,E'\,\Delta' = \frac{(E\Delta)^2\nu_{\text{cl}}}{\nu^*} = \frac{(E'\Delta')^2\nu'_{\text{cl}}}{\nu^*}\,. \tag{63}$$

We now turn to the calculation of the intensity from (56a). (56a) represents (when one ignores the weakening factor $1/r$) the potential of a plane wave of direction $\xi_1$, $\xi_2$, $\xi_3$, and frequency $\nu^*$, such that in analogy with (25) and (36) the phase can be written:

$$\varphi^* = l^* x = \frac{2\pi}{h}\cdot\pi^* x, \tag{64}$$

with

$$l^*_k = \frac{2\pi\nu^*}{c}\xi_k\,, \qquad l^*_4 = i\frac{2\pi\nu^*}{c}\,, \qquad \pi^*_\alpha = \frac{h}{2\pi}l^*_\alpha\,. \tag{64a}$$

From this, it then follows that in (56a), one can write for $T_\alpha$:

$$T_\alpha = 2(k\,\pi_\alpha + \delta k\,p_\alpha - b_\alpha), \tag{65}$$

instead of (49); the term $-\,\delta k\pi^*_\alpha$ obviously gives rise to an additional term of the form $\partial f / \partial x_\alpha$. It is easy to see that the $T_\alpha$ in (65), when expressed in terms of the unprimed or primed quantities, is independent of $h$ (when one ignores an additional term of the form $\partial f / \partial x_\alpha$). First of all, from (38a) and (36), $k\pi_\alpha$ has the property that it is independent of $h$, since $h$ cancels in the numerator and denominator. Furthermore, from (50), since $\pi^{*2} = 0$, one has:

$$\delta k = \frac{\pi^* b - k \cdot \pi\pi^*}{\pi^* p}. \tag{66}$$

Here, $h\nu^*/c$ cancels in the numerator and denominator. Since, from (40), $T_\alpha$ remains unchanged when one switches the unprimed and primed quantities and $h$ with $-h$ [in which, as we remarked above, the relations (58) remain unchanged], from (49) and (66), one has:

$$T_\alpha = 2k'\pi_\alpha + \delta k(2p'_\alpha + \pi^*_\alpha) - 2b_\alpha, \tag{65a}$$

with

$$\delta k = \frac{\pi^* b - k' \cdot \pi\pi^*}{\pi^* p'}, \tag{66a}$$

or, when one again drops the additional term $\delta k \pi^*_\alpha$:

$$T_\alpha = 2(k'\pi_\alpha + \delta k\, p'_\alpha - b_\alpha). \tag{65b}$$

With the use of (63), (56a) reads:

$$\Phi_\alpha = \sqrt{\nu^*}\, \zeta_\alpha \cos \varphi^*, \tag{67}$$

in which we have set:

$$\zeta_\alpha = \frac{ec}{2r} \frac{T_\alpha}{E\Delta\sqrt{\nu_{\text{cl}}}}. \tag{68}$$

The $\zeta_\alpha$ are independent of $h$ and, from what we said about $T$ and also from (63), they have the property that:

$$\zeta_\alpha = \zeta'_\alpha. \tag{68a}$$

The field strengths follow from (67) according to the pattern (27). One simply has to replace $\varphi$ with $\varphi^*$, $\nu$ with $\nu^*$, and the $a_\alpha$ with $\sqrt{\nu^* \zeta_\alpha}$ in (27). The electric and magnetic amplitudes will then be of the form:

$$A^* = (\nu^*)^{3/2} \zeta, \tag{69}$$

where $\zeta$ again has the property (68a). If one goes to the limit $h = 0$ then it follows that $\zeta$ is independent of $h$:

$$A^* = \left(\frac{\nu^*}{\nu_{\text{cl}}}\right)^{3/2} A_{\text{cl}}. \tag{69a}$$

If one replaces the unprimed quantities in (69a) with primed ones then, since $\zeta = \zeta'$, it becomes:

$$A'_{\text{cl}} = \nu'^{3/2}_{\text{cl}} \zeta. \tag{69b}$$

When (69) and (69b) are divided by each other, this gives:

$$A^* = \left(\frac{\nu^*}{\nu'_{\mathrm{cl}}}\right)^{3/2} A'_{\mathrm{cl}} \ . \tag{70a}$$

For the intensities, it results from (70) and (70a) that:

$$I^* = \left(\frac{\nu^*}{\nu_{\mathrm{cl}}}\right)^{3} I_{\mathrm{cl}} \ , \tag{71}$$

$$I^* = \left(\frac{\nu^*}{\nu'_{\mathrm{cl}}}\right)^{3} I'_{\mathrm{cl}} \ . \tag{71a}$$

Multiplying (70) by (70a) and (71) by (71a) gives, when one observes (62):

$$A^* = \sqrt{A_{\mathrm{cl}} A'_{\mathrm{cl}}} \tag{72}$$

and

$$I^* = \sqrt{I_{\mathrm{cl}} I'_{\mathrm{cl}}} \ . \tag{73}$$

With this, we have the result:

*The quantum frequencies and intensities of the Compton effect are equal to the geometric means of the corresponding classical quantities in the initial and final states of the process.*

For the case of the electron that is initially at rest, relation (62) was derived by Breit [1]) and relation (71) was derived by Breit [1]) from correspondence considerations and by Dirac (*loc. cit.*) using Heisenberg's theory.

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[1]) G. Breit, Phys. Rev. **27** (1926), 362.