"Zur Lichtfortpflanzung nach der Relativitätstheorie," Ann. d. Phys. **72** (1923), 421-456.

1. *On the propagation of light in the theory of relativity*.

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In the first part of the work it will be shown that the influence of matter on electromagnetic phenomena is equivalent to the influence of a gravitational field with potentials $k\ u_\mu\ u_\nu$ (k is the Fresnel transmission coefficient, u_μ is the four-velocity). Through this reduction to the vacuum, one immediately obtains the principle of least action and thus, in particular, the energy tensor of the electromagnetic field in ponderable bodies. On thus arrives at the tensor that was presented by M. Abraham.

In the second part, the wave equations that are valid for arbitrary linear tensors will be derived. Additional terms appear in the expressions that come from the special theory (which, by the equivalence principle, will be valid in "fictitious" gravitational fields that arise from transformations), and which include the uncontracted and contracted curvature tensors. Nevertheless, these expressions obey the rules of calculation that one uses in the special theory.

As will be shown in the third part, the field, the four-potential, and the six-potential (Hertz tensor) will therefore satisfy the generalized wave equation. The advantage of the six-potential is the fact that it satisfies no extra conditions beyond the wave equation. It has precisely the same relationship with the six-polarization that the four-potential does with the four-current.

In the fourth part, the assumptions under which one can speak of light rays in the context of geometrical optics will be specified. The world lines of the rays are the null geodetic lines in a gravitational field that includes the one that is actually present, as well as the one that corresponds to the four-velocity of the matter.

1.

Transformation of the electromagnetic equations. The equations of the electromagnetic field, which define the foundations for the study of the propagation of light, will ordinarily be written as:

$$\frac{\partial F_{ik}}{\partial x^l}+\frac{\partial F_{kl}}{\partial x^i}+\frac{\partial F_{li}}{\partial x^k}=0\ , \tag{1}$$

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}(\sqrt{g}H^{ik})\ =s^i\ , \tag{2}$$

$$H_i=\varepsilon F_i\ , \tag{3}$$

$$u_i\,F_{kl}+u_k\,F_{li}+u_l\,F_{ik}=\mu\,(u_i\,H_{kl}+u_k\,H_{li}+u_l\,H_{ik}) \tag{4}$$

(5) $$s_i + u_i\,(s^k\,u_k) = \sigma F_i\,,$$

in which we have set:

(6) $$F_i = F_{ik}\,u^k\,, \qquad H_i = H_{ik}\,u^k\,,$$

to abbreviate. u^i is the four-velocity of matter:

(7) $$u^i = \frac{dx^i}{\sqrt{-ds^2}}\,,$$

such that:

(8) $$g_{ik}\,u^i\,u^k = u^i\,u_i = -1\,.$$

(The line element has one positive and three negative dimensions.) The meaning of the remaining quantities is well-known. To these equations, one must add the inhomogeneous Einstein gravitational equations, in which the sum of the elastic and electromagnetic energy tensors is used. We will give the latter equations later on.

Equations (3) and (4) collectively represent six mutually independent equations; they thus express the fact that in a rest system (and for normal values of g_{ik}) one has $\mathfrak{D} = \varepsilon\,\mathfrak{E}$, $\mathfrak{B} = \mu\,\mathfrak{H}$. We would like to solve them for the H_{ik}. Multiplication of (4) by u^i yields, when one observes (3), (6), and (8):

$$\begin{aligned} -F_{kl} + u_k\,F_l - u_l\,F_k &= \mu\,(-H_{kl} + u_k\,H_l - u_l\,H_k) \\ &= \mu\,\{-H_{kl} + \varepsilon\,(u_k\,F_l - u_l\,F_k)\} \end{aligned}$$

or:

(9) $$\mu\,H_{ik} = F_{ik} + (\varepsilon\mu - 1)(u_i\,F_k - u_k\,F_i)\,.$$

Conversely, (3) and (4) also follow from (9). We can therefore replace (3) and (4) with (9).

The four-velocity does not appear in the differential equations (1) and (2). We would like to write the equations in such a way that the gravitational field will also be banished to the additional equations. We replace (1) and (2) with:

(1′) $$\frac{\partial F_{ik}}{\partial x^l} + \frac{\partial F_{kl}}{\partial x^i} + \frac{\partial F_{li}}{\partial x^k} = 0\,,$$

(2′) $$\frac{\partial \mathfrak{H}^{ik}}{\partial x^k} = \mathfrak{s}^i$$

and also make the substitution:

(10) $$\mathfrak{H}^{ik} = \sqrt{g}\,H^{ik}\,, \qquad \mathfrak{s}^i = \sqrt{g}\,s^i$$

in the remaining terms. The electromagnetic field is thus characterized by:

$$F_{ik}\,, \qquad \mathfrak{H}^{ik}, \tag{11}$$

in which F_{ik} is a linear tensor and $\mathfrak{H}^{ik}$ is a linear tensor density [1]). The gravitational field is characterized by:

$$g_{ik}\,, \tag{12}$$

the electrical state of the matter by:

$$\varepsilon,\ \mu,\ \sigma,\ \mathfrak{s}^i, \tag{13}$$

in which ε, μ, σ are scalars and $\mathfrak{s}^i$ is a vector density. The mechanical state of the matter is characterized by mechanical constants and the functions:

$$x^i = x^i\,(a_1,\ a_2,\ a_3,\ \tau), \tag{14}$$

that represent the world lines of the material points (which are distinguished from each other by way of a_1, a_2, a_3, while τ gives the distance along the world line). Only the derivatives of these functions appear in the electric and mechanical equations. The four-velocity is a combination of the defining objects (12) and (14):

$$u^i = \frac{\dfrac{\partial x^i}{\partial \tau}}{\sqrt{-g_{\mu\nu}\dfrac{\partial x^\mu}{\partial \tau}\dfrac{\partial x^\nu}{\partial \tau}}}\,. \tag{15}$$

For the transformation so performed and the development that results from it, it is crucial that in the "dielectric" equation (9), which is definitive for the propagation of light in transparent bodies, the quantities (12) and (15) appear only in the relation:

$$\gamma^{ik} = g^{ik} - (\varepsilon\,\mu - 1)\,u^i\,u^k\,. \tag{16}$$

In order to show this, we next write (9) in contravariant form:

$$\mu\,H^{ik} = F^{ik} + (\varepsilon\,\mu - 1)\,(u^i\,F^k - u^k\,F^i)\,. \tag{9'}$$

From (10), the left-hand side is $\dfrac{\mu\,\mathfrak{H}^{ik}}{\sqrt{g}}$; the first term on the right is $g^{ir}\;g^{ks}\;F_{rs}$. Furthermore, from (6), we have:

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[1] In the terminology of H. Weyl, Raum, Zeit, Materie. 4th ed., pp. 51 and 98.

$$F^k = g^{ks} F_{sr} u^r = - g^{ks} F_{rs} u^r , \qquad\qquad F^i = g^{ir} F_{rs} u^s ,$$

such that we obtain for the right-hand side of (9′):

$$F_{rs} \{ g^{ir} g^{ks} - (\varepsilon\mu - 1)(u^i u^r g^{ks} + u^k u^s g^{ir})\}.$$

If we now add the term $(\varepsilon\mu - 1)^2 u^i u^k u^r u^s$, which vanishes due to the anti-symmetry of F_{ik} , inside the curly brackets then this bracket will become $(g^{ir} - (\varepsilon\mu - 1) u^i u^r) (g^{ks} - (\varepsilon\mu - 1) u^k u^s)$ or, according to (16), $\gamma^{ir} \gamma^{ks}$. We can therefore put (9′) into the form:

(17) $$\mu \mathfrak{H}^{ik} = \sqrt{g}\ \gamma^{ir} \gamma^{ks} F_{rs} .$$

Moreover, we introduce the quantities γ_{ik} that are reciprocal to the γ^{ik}, and which are uniquely determined by the requirement that $\gamma^{ir} \gamma_{rk} = \delta^i_k$ by way of:

(18) $$\gamma_{ik} = g_{ik} + \left(1 - \frac{1}{\varepsilon\mu}\right) u_i u_k .$$

By analogy with g the negative determinant of γ_{ik} will be denoted by γ. The ratio γ / g is an invariant (under transformation, the numerator and denominator are both multiplied by the square of the functional determinant). We may thus base our calculations on the case $u_1 = u_2 = u_3 = 0$ and obtain:

$$-\gamma = \begin{vmatrix} g_{11} & \cdots & g_{14} \\ \cdots & \cdots & \cdots \\ g_{41} & \cdots & g_{44} + \left(1 - \dfrac{1}{\varepsilon\mu}\right) u_4 u_4 \end{vmatrix} = - g - \left(1 - \frac{1}{\varepsilon\mu}\right) g\, g^{44} u_4 u_4 ,$$

if $- g\, g^{44}$ is the sub-determinant of the element (4, 4) in the determinant. From (8), one has $g^{44} u_4 u_4 = - 1$. Thus:

(19) $$\gamma = \frac{g}{\varepsilon\mu}$$

and therefore we obtain (17) in the form:

(20) $$\mathfrak{H}^{ik} = \sqrt{\frac{\varepsilon}{\mu}} \sqrt{\gamma}\ \gamma^{ir} \gamma^{ks} F_{rs} ,$$

from which our assertion is proved.

We will therefore have to introduce, in place of the line element ds with the coefficients g_{ik}, a new line element $d\sigma$ with the coefficients γ_{ik} that are given by (18). We would like to identify indices that relate to this new line element by parentheses. (Only we will write $g^{(i)(k)}$, $g_{(i)(k)}$, resp., where we have written γ^{ik}, γ_{ik}, resp., up to now.) The defining objects (11), (13), (14) naturally remain unaffected by this transformation, and we have:

$$F_{(i)(k)} = F_{ik}\,, \qquad \mathfrak{H}^{(i)(k)} = \mathfrak{H}^{ik}\,, \qquad \mathfrak{s}^{(i)} = \mathfrak{s}^{i}\,. \tag{21}$$

We will therefore omit the parentheses for these quantities. The tensors $H^{(i)(k)}$, $s^{(i)}$ that are associated with the densities $\mathfrak{H}^{(i)(k)}$, $\mathfrak{s}^{(i)}$ by (18) become, from the model of (10), and due to (19):

$$H^{(i)(k)} = \frac{\mathfrak{H}^{(i)(k)}}{\sqrt{\gamma}} = \frac{\sqrt{\varepsilon\mu}\;\mathfrak{H}^{ik}}{\sqrt{g}} = \sqrt{\varepsilon\mu}\,H^{ik}, \qquad\qquad s^{(i)} = \sqrt{\varepsilon\mu}\,s^{i}. \tag{22}$$

By our definitions, the expression $\gamma^{jr}\,\gamma^{ks}\,F_{rs}$ that appears in (20) can obviously be written $F^{(i)(k)}$, and $\dfrac{\mathfrak{H}^{ik}}{\sqrt{\gamma}}$ is $H^{(i)(k)}$. (20) then takes on the symmetric form:

$$\sqrt{\varepsilon}\,F^{(i)(k)} = \sqrt{\mu}\;\;H^{(i)(k)}, \tag{23}$$

or, when one goes over to the covariant components relative to the metric γ_{ik}:

$$\sqrt{\mu}\,F_{(i)(k)} = \sqrt{\varepsilon}\,H_{(i)(k)}\,. \tag{23′}$$

If one replaces $H^{(i)(k)}$ in (23) with $\sqrt{\varepsilon\mu}\,H^{ik}$ then one obtains the equation $F^{(i)(k)} = \mu\,H^{ik}$, and equating this with (9′) shows that the transformation formula:

$$F^{(i)(k)} = F^{ik} + (\varepsilon\mu - 1)(u^{i}\,F^{k} - u^{k}\,F^{i})\,. \tag{24}$$

is valid for $F^{(i)(k)}$. (3) and (4) go into each other when one exchanges ε and μ with their reciprocal values and likewise F with H. Under this exchange, equation (9), which is equivalent to (3) and (4), becomes:

$$\frac{F_{ik}}{\mu} = H_{ik} + \left(\frac{1}{\varepsilon\mu} - 1\right)(u_i\,H_k - u_k\,H_i)\,.$$

If one equates this with (23′), in which one observes that from (21), one has $F_{(i)(k)} = F_{ik}$, then one obtains the transformation formula:

(25) $$H_{(i)(k)} = \sqrt{\varepsilon\mu}\left[H_{ik} - \left(1 - \frac{1}{\varepsilon\mu}\right)(u_i H_k - u_k H_i)\right]$$

for $H_{(i)(k)}$.

For the line element $d\sigma$ we generally find, from (18), that:

(26) $$\begin{cases} d\sigma^2 = g_{ik}dx^i dx^k + \left(1 - \dfrac{1}{\varepsilon\mu}\right)(u_i dx^i)^2 \\ \qquad = ds^2 + \left(1 - \dfrac{1}{\varepsilon\mu}\right)(u_i dx^i)^2. \end{cases}$$

For the world direction of matter, one has $dx^i = u^i\sqrt{-ds^2}$, and thus, due to (8), $d\sigma^2 = \dfrac{ds^2}{\varepsilon\mu}$.

One thus has:

(27) $$u^{(i)} = \frac{dx^i}{\sqrt{-d\sigma^2}} = u^i\sqrt{\varepsilon\mu}\,.$$

The covariant components $u_{(i)} = \gamma_{ir}\,u^{(r)}$ become, when one again considers (8):

(28) $$u_{(i)} = \frac{u_i}{\sqrt{\varepsilon\mu}}\,.$$

From (21_1) and (27), we have:

(29) $$F_{(i)} = F_{(i)(k)}\,u^{(k)} = F_{ik}\,u^k\sqrt{\varepsilon\mu} = F_i\,\sqrt{\varepsilon\mu}\,,$$

and, from (16):

(30) $$F^{(i)} = \gamma^{ir}\,F_{(i)} = (g^{ir} - (\varepsilon\mu - 1)\,u^i\,u^r)\,F_r\sqrt{\varepsilon\mu} = F^i\,\sqrt{\varepsilon\mu}\,,$$

so one has $F_r\,u^r = 0$. In a completely analogous way, one recognizes that:

(31) $$H^{(i)} = H^i\,, \qquad H_{(i)} = H_i\,.$$

We are now in a position to also write Ohm's law (5) for the new line element. The contravariant components of the left-hand side of (5) are, from (22_2), (27), and (28) equal to $\dfrac{1}{\sqrt{\varepsilon\mu}}(s^{(i)} + u^{(i)}(u_{(k)}\,s^{(k)}))$, and, from (30), the contravariant form of the right-hand side is $\dfrac{\sigma F^{(i)}}{\sqrt{\varepsilon\mu}}$. Thus:

(32) $$s^{(i)} + u^{(i)}(u_{(k)}\, s^{(k)}) = \sigma F^{(i)} ,$$

or, written covariantly(relative to γ_{ik}):

(32′) $$s_{(i)} + u_{(i)}(u^{(k)}\, s_{(k)}) = \sigma F_{(i)} .$$

For the $s_{(i)}$ that appear, one obtains, from (18) and (22_2):

(33) $$s_{(i)} = \gamma_{ir}\, s^{(r)} = \sqrt{\varepsilon\mu}\left[s_i + \left(1 - \frac{1}{\varepsilon\mu}\right) u_i (u^r s_r) \right].$$

In summary, we can therefore state the following theorem:

If one transforms the gravitational field g_{ik} into γ_{ik} by means of the transformation formula:

$$\gamma_{ik} = g_{ik} + \left(1 - \frac{1}{\varepsilon\mu}\right) u_i\, u_k ,$$

i.e., if one constructs the components of the differential equations of the fundamental field quantities F_{ik} , $\mathfrak{H}^{ik}$, $\mathfrak{s}^i$ relative to this metric, then the additional equations assume the form:

$$\sqrt{\varepsilon}\; F_{(i)(k)} = \sqrt{\mu}\; H_{(i)(k)} , \qquad s_{(i)} + u_{(i)}(u^{(k)}\, s_{(k)}) = \sigma F_{(i)} .$$

The four-velocity drops out of the dielectric equation, whereas Ohm's law retains its form.

For non-conducting ($\sigma = 0$), *uncharged* ($u^k\, s_k = 0$), *homogeneous* (ε, μ *constant*) *media the differential equations for the field F are identical with those of the pure vacuum with a gravitational field* γ_{ik} .

In the latter case, we can give our theorem the following two physical interpretations:

1. Electromagnetic phenomena in ponderable bodies are the same as in a vacuum that is governed by a field with the potentials $\left(1 - \frac{1}{\varepsilon\mu}\right) u_i\, u_k$ in addition to the existing gravitational field. In the special theory, by restriction to quantities of first order:

$$\gamma_{\alpha\beta} = 1 , \qquad \gamma_{\alpha 4} = -\left(1 - \frac{1}{\varepsilon\mu}\right)\frac{v_\alpha}{c} , \quad \gamma_{44} = -\frac{1}{\varepsilon\mu} ,$$

and the line element $d\sigma$ becomes:

(34) $$d\sigma^2 = dx_\alpha\, dx^\alpha - 2\left(1 - \frac{1}{\varepsilon\mu}\right) v_\alpha\, dx^\alpha\, dt - \frac{c^2}{\varepsilon\mu}\, dt^2 .$$

The velocity of light in a given direction will be determined (as we will show in part four) by $d\sigma^2 = 0$. If the light velocity and the velocity of the body are parallel to each other then the well-known formula:

$$\text{(35)} \qquad \frac{dx}{dt} = \frac{c}{\sqrt{\varepsilon\mu}} + \left(1 - \frac{1}{\varepsilon\mu}\right) v ,$$

follows from (34), or, to the same degree of precision:

$$\text{(35′)} \qquad \frac{dx}{dt} = \frac{c}{\sqrt{\varepsilon\mu} - (\varepsilon\mu - 1)\frac{v}{c}} .$$

(35′) can thus be interpreted by saying that light propagates as if it were in a medium with an index of refraction $\sqrt{\varepsilon\mu} - (\varepsilon\mu - 1)\frac{v}{c}$. Our interpretation of the transformation theorem is a generalization of this interpretation. We have, as it were, a "rest transformation" before us.

In pre-relativistic physics, the term $\left(1 - \frac{1}{\varepsilon\mu}\right) v$ in (35) was spoken of as the "dragging of the ether." We can give our theorem an analogous interpretation.

2. We decompose the world displacement PQ into the co-moving component PP' that is parallel to the four-velocity:

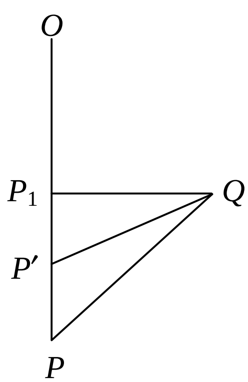

Fig. 1

$$\text{(36)} \qquad -\left(1 - \frac{1}{\sqrt{\varepsilon\mu}}\right) u^i\, u_r\, dx^r ,$$

and the ether component $P'Q$:

$$d\xi^i = dx^i + \left(1 - \frac{1}{\sqrt{\varepsilon\mu}}\right) u^i\, u_r\, dx^r .$$

For the distance $P'Q$ relative to *the actual existing metric* this yields:

$$g_{ik}\, d\xi^i\, d\xi^k = g_{ik}\, dx^i\, dx^k + \left(1 - \frac{1}{\sqrt{\varepsilon\mu}}\right) (u_r\, dx^r)^2 .$$

Hence, from (26), it is equal to the distance PQ according to the metric γ_{ik}. If PQ is a light ray in the body then one has $d\sigma^2 = 0$, from which it follows that $g_{ik}\, d\xi^i\, d\xi^k = 0$, i.e., $P'Q$ is a light ray in vacuo (for the actually existing gravitational field g_{ik}). The two differ by the co-moving term (36). Let P_1 be the orthogonal projection of Q in the direction u, such that $PP' = \left(1 - \frac{1}{\sqrt{\varepsilon\mu}}\right) PP_1$. The analogues in the Hertzian theory were the decomposition PP_1, P_1Q, which would correspond to an infinitely large vacuum velocity of light in the rest system (in which P_1 and Q are equal), which is entirely consistent with the Galileian principle of relativity that this theory was founded on. One can thus think of the factor $\left(1 - \frac{1}{\sqrt{\varepsilon\mu}}\right)$ as the "relativistic" drag coefficient. [1])

From the comparison with the vacuum field equations, we can immediately derive the *principle of least action* and thus, from the results of the general theory of relativity, the *energy tensor*. We omit Ohm's law and regard $\mathfrak{s}^i$ as a property that is inherent to the material, like ε and μ [cf., (13)]. From now on, we then have the additional equation (20), which we can also write as:

(20′) $$\mathfrak{H}^{ik} = \sqrt{\frac{\varepsilon}{\mu}} \sqrt{\gamma}\, F^{(i)(k)},$$

from the postulates that we encountered. (20) ((20′), resp.) differs from the additional equation to (1′) and (2′) that occurs in vacuo for the gravitational field γ_{ik} only by the fact that the factor $\sqrt{\frac{\varepsilon}{\mu}}\sqrt{\gamma}$ stands in place of $\sqrt{\gamma}$. Thus, by this substitution we obtain, with no further assumptions, the electrical action density $L^{(\varepsilon)}$ of the action principle for the vacuum (given the gravitational field γ_{ik}):

(37) $$\delta_\varphi W^{(\varepsilon)} = \delta_\varphi \int L\, dS = 0, \qquad (dS = dx^1\, dx^2\, dx^3\, dx^4),$$

(38) $$L^{(\varepsilon)} = \tfrac{1}{4}\sqrt{\gamma} F_{rs} F^{(r)(s)} - \mathfrak{s}^i \varphi_i,$$

in which the four-potential φ_i is connected with the field F by the equation:

(39) $$F_{ik} = \frac{\partial \varphi_k}{\partial x_i} - \frac{\partial \varphi_i}{\partial x_k}.$$

The electrical action density in the absence of matter is:

(40) $$L^{(\varepsilon)} = \frac{1}{4}\sqrt{\frac{\varepsilon}{\mu}} \sqrt{\gamma} F_{rs} F^{(r)(s)} - \mathfrak{s}^i \varphi_i,$$

[1] The first interpretation is: if one would like to equate the circumstances for moving bodies with those at rest, which is obviously appropriate to the spirit of relativity theory, then, instead of the ether being associated with a property of matter, namely, a velocity, on the contrary, matter is associated with a gravitational field, hence, a property of the ether.

or, from (20′):

$$L^{(\varepsilon)} = \tfrac{1}{4} F_{rs} \mathfrak{H}^{rs} - \mathfrak{s}^i \varphi_i \,, \tag{40′}$$

if only the four-potential is to be varied (which we intended by the use of the notation δ_φ). These variations shall vanish at the boundary of the region over which the integral in (37) is taken. [1])

In order to obtain the equations for the material we have add the mechanical action $W^{(m)}$ to the electrical one $W^{(\varepsilon)}$. Only the functions (14) are to be varied in the variational equation:

$$\delta_m W^{(\varepsilon)} + \delta_m W^{(m)} = 0 \,. \tag{41}$$

(This is intended by the notation δ_m, and we let $\delta_m = 0$ on the boundary.) Thus the four-velocity (15) experiences the local change:

$$\delta_m u^i = \frac{\partial\, \delta x^i}{\partial x^r} u^r - \frac{\partial u^i}{\partial x^r} \delta u^r + u^i u_\mu \frac{\partial\, \delta x^\mu}{\partial x^r} u^r + \tfrac{1}{2} u^i \frac{\partial\, g_{\mu\nu}}{\partial x^r} u^\mu u^\nu \delta x^r \,. \tag{42}$$

(Naturally, this quantity is a vector. In order for that to also be the case in the present representation, one must only replace the ordinary derivatives with the covariant derivatives, which will be clarified in part two [2]).) The deformation of the material, which is characterized by δx^i, will also carry along with it the electrical corpuscles that are included in it. One easily obtains the variation [3]) of ε and μ (bound electrons) and $\mathfrak{s}^i$ (free electrons) that is induced by the variation δx^i from the transformation character of these quantities (ε, μ are scalars, $\mathfrak{s}^i$ is a contravariant tensor density):

(43) [4])

$$\delta_m\, \varepsilon = -\frac{\partial \varepsilon}{\partial x^r} \delta x^r \,, \qquad \delta_m\, \mu = -\frac{\partial \mu}{\partial x^r} \delta x^r \,.$$

$$\delta_m\, \mathfrak{s}^i = \mathfrak{s}^r \frac{\partial\, \delta x^i}{\partial x^r} - \frac{\partial}{\partial x^r} (\mathfrak{s}^i \delta x^r) = \frac{\partial}{\partial x^r} (\delta x^i \mathfrak{s}^r - \delta x^r \mathfrak{s}^i) - \frac{\partial\, \mathfrak{s}^r}{\partial x^r} \delta x^r \,. \tag{44}$$

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[1]) The principle (37), (40′) has been presented by E. Henschke, Berlin diss. 1912. Ann d. Phys. **42**, pp. 887, 1913.

[2]) One has: $\delta_m u^i = u^r D_r \delta x^i - \delta x^r D_r u^i + u^i u_\mu\, u^r D_r \delta x^\mu$.

[3]) H. Weyl., loc. cit., pp. 212.

[4]) We thus omit the deformation state of the material from the variability of ε and μ.

From the second form of $\delta_m\, \mathfrak{s}^i$, one sees that $\delta_m\, \mathfrak{s}^i$ is a vector density (cf., eq. (98) and (99)). If we assume the conservation law for electricity $\frac{\partial \mathfrak{s}^r}{\partial x^r} = 0$ that is given by (2′) the (44) simplifies to:

$$\delta_m\, \mathfrak{s}^i = \frac{\partial}{\partial x^r}(\delta x^i\, \mathfrak{s}^r - \delta x^r\, \mathfrak{s}^i)\,. \tag{44′}$$

The vector u^i will therefore not be affected by the variation δx^i because the metric remains fixed, but the distance between two material spacetime points generally varies. (Cf., footnote 1, pp. 12.)

The action density $L^{(m)}$ that is associated with $W^{(m)}$ is dependent upon the first derivative of the functions (14) and the g_{ik}. [1])

The energy theorem follows from the electrical equations (37) and the mechanical equations (41). One shows this without making use of the explicit expression for $L^{(e)}$ and $L^{(m)}$, as is well-known [2]), by considering a variation for which all quantities, including the gravitational field, are affected. Then, due to the invariance of $W^{(e)}$ and $W^{(m)}$ it follows identically:

$$\delta_\varphi\, W^{(e)} + \delta_m\, W^{(e)} + \delta_g\, W^{(e)} = 0\,, \tag{45}$$

$$\delta_m\, W^{(m)} + \delta_g\, W^{(m)} = 0\,, \tag{46}$$

in which δ_g refers to the change in the g_{ik}. We set:

$$\delta_g\, L^{(e)} = \tfrac{1}{2}\mathfrak{T}_{ik}\, \delta g^{ik}\,, \qquad \delta_g\, L^{(m)} = \tfrac{1}{2}\mathfrak{M}_{ik}\, \delta g^{ik}\,; \tag{47}$$

$\mathfrak{T}_{ik}$ and $\mathfrak{M}_{ik}$ are the electrical and mechanical energy tensor densities. In δ_g, one must replace δg^{ik} with:

$$\delta g^{ik} = g^{ir}\frac{\partial\, \delta x^k}{\partial x^r} + g^{kr}\frac{\partial\, \delta x^i}{\partial x^r} - \frac{\partial g^{ik}}{\partial x^r}\delta x^r\,. \tag{48}$$

(One recognizes the tensor character of δg^{ik} when we replace the ordinary derivatives with covariant ones, as above.) Integration by parts yields:

$$\delta_g\, W^{(e)} = -\int\left(\frac{\partial \mathfrak{T}_i^r}{\partial x^r} + \tfrac{1}{2}\mathfrak{T}_{\alpha\beta}\frac{\partial g^{\alpha\beta}}{\partial x^i}\right)\delta x^i\, dS \tag{49}$$

and an analogous formula (49′) for $\delta_g\, W^{(m)}$. From (45), (46), (49), and (49′), it follows that:

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[1]) G. Herglots, Ann. d. Phys., **36**, pp. 493, 1911; G. Nordström. Versl. Amst. **25**, pp. 836, 1916.

[2]) H. Weyl, loc. cit, § 28.

(50) $$\delta_\varphi W^{(e)} + \delta_m W^{(e)} + \delta_m W^{(m)} = \int \left(\frac{\partial \mathfrak{S}_i^{\ r}}{\partial x^r} + \tfrac{1}{2} \mathfrak{S}_{\alpha\beta} \frac{\partial g^{\alpha\beta}}{\partial x^i} \right) \delta x^i dS ,$$

in which $\mathfrak{S}_{ik}$ is the sum of the two energy tensor densities. Equations (37) and (41) thus have, in fact, the energy-impulse theorem:

(60) $$\frac{\partial \mathfrak{S}_i^{\ r}}{\partial x^r} + \tfrac{1}{2} \mathfrak{S}_{\alpha\beta} \frac{\partial g^{\alpha\beta}}{\partial x^i} = 0$$

as a consequence.

We can easily derive the electrical energy tensor that comes from (47_1) on the basis of our analogous theorem for the vacuum. In vacuo, for the gravitational field γ_{ik} one has:

(61) $$\delta_\gamma L^{(e)} = \tfrac{1}{2} \mathfrak{T}_{(i)(k)}\, \delta\gamma^{ik} ,$$

(62) $$\mathfrak{T}_{(i)}^{\ (k)} = \sqrt{\gamma}\, (F_{ir}\, F^{(k)(r)} - F_{rs}\, F^{(r)(s)}\, \delta_i^k).$$

Here, we must once again simply replace $\sqrt{\gamma}$ with $\sqrt{\frac{\varepsilon}{\mu}}\sqrt{\gamma}$ and obtain, when we again consider (20′):

(63) $$\mathfrak{T}_{(i)}^{\ (k)} = F_{ir}\, \mathfrak{H}^{kr} - \tfrac{1}{4} F_{rs}\, \mathfrak{H}^{rs}\, \delta_i^k .$$

In order to define the relationship between $\mathfrak{T}_i^k$ and $\mathfrak{T}_{(i)}^{\ (k)}$, we must express $\delta\gamma^{ik}$ in terms of δg^{ik}. From (15), one has:

(64) [1]) $$\delta_g u^i = \tfrac{1}{2} u^i u^\mu u^\nu\, \delta g_{\mu\nu} = -\tfrac{1}{2}\, u^i u_\mu u_\nu\, \delta g^{\mu\nu} ,$$

and thus:

(65) $$\delta\gamma^{ik} = \delta\{g^{ik} - (\varepsilon\mu - 1) u^i u^k\} = \delta g^{ik} - (\varepsilon\mu - 1) u^i u^k u_\mu u_\nu \delta g^{\mu\nu} .$$

---

[1]) If one replaces δg^{ik} with the value (48) here then one has:

$$\delta_g u^i = -u^i u_\mu \frac{\partial\, \delta x^\mu}{\partial x^r}\, u^r + \tfrac{1}{2}\, u^i \frac{\partial\, g^{\mu\nu}}{\partial x^r}\, u_\mu u_\nu\, \delta u^r = -u^i u_\mu \frac{\partial\, \delta x^\mu}{\partial x^r}\, u^r - \tfrac{1}{2}\, u^i \frac{\partial\, g_{\mu\nu}}{\partial x^r}\, u^\mu u^\nu\, \delta u^r .$$

Thus, the last term in (42) will compensate:

(42′) $$\delta u^i = \delta_m u^i + \delta_g u^i = \frac{\partial\, \delta x^\mu}{\partial x^r}\, u^r - \frac{\partial\, u^i}{\partial x^r}\, \delta x^r .$$

In fact, this is the local variation of u^i due solely to the shift δx^i.

If we substitute this in (61) and equate the result with (47_1) then we find that:

$$\mathfrak{T}_{ik} = \mathfrak{T}_{(i)(k)} + (\varepsilon\mu - 1)\, u_i\, u_k\, \mathfrak{T}_{(\mu)(\nu)}\, u^\mu\, u^\nu\,. \tag{66}$$

This yields the mixed components when one multiplies the left-hand side and the second term on the right by g^{kl} and the first term on the right by $\gamma^{kl} + (\varepsilon\mu - 1)\, u^i\, u^k$:

$$\mathfrak{T}_i^{\ l} = \mathfrak{T}_{(i)}^{\ (l)} + (\varepsilon\mu - 1)(\mathfrak{T}_{(i)(k)}\, u^k + u_i \mathfrak{T}_{(\mu)(\nu)}\, u^\mu\, u^\nu)\, u^l\,. \tag{66′}$$

In the rest system (and for normal values of the g_{ik}) the factor of $(\varepsilon\mu - 1)\, u^l$ has the values $\mathfrak{T}_{(\alpha)(4)}$, 0 ($\alpha$ = 1, 2, 3). Thus:

$$\mathfrak{T}_i^{\ l} = \mathfrak{T}_{(i)}^{\ (l)} + (\varepsilon\mu - 1)(\mathfrak{T}_{ik}\, u^k + u_i \mathfrak{T}_{\mu\nu}\, u^\mu\, u^\nu)\, u^l\,. \tag{67}$$

and thus, according to (63):

$$T_i^{\ k} = F_{ir}\, H^{kr} - \tfrac{1}{4} E_{ir}\, H^{kr}\, \delta_k^{\ l} - (\varepsilon\mu - 1)\, \Omega_i\, u^k\,, \tag{67′}$$

in which we have introduced the "rest ray":

$$\Omega_i = -\,(T_r^{\ i}\, u^r + u^i\, T_{\mu\nu}\, u^\mu\, u^\nu). \tag{68}$$

If $\mathfrak{S}^\alpha$ is the energy current (α = 1, 2, 3), then one has $T_4^{\ \alpha} = -\dfrac{\mathfrak{S}^\alpha}{c}$, and in the rest system:

$$\Omega^\alpha = \frac{\mathfrak{S}^\alpha}{c}\,, \qquad \Omega^4 = 0 \qquad (\alpha = 1, 2, 3). \tag{68′}$$

If one substitutes (67′) into (68) then one finds, because $\Omega^r\, u_r = 0$:

$$\Omega^i = F_l\, H^{il} - F_l\, H^l\, u^i = u_k\, F_l\, (H^{ik}\, u^l + H^{kl}\, u^i + H^{li}\, u^k)\,. \tag{69}$$

The *energy tensor of M. Abraham* is determined by means of formulas (67′) and (69). [1])

[1] W. Pauli, Jr., Enz. d. math. Wiss. v. **19**, formula (303). From (50) and (60), one also has the variational principle:

$$\delta_\varphi\, W^{(l)} + \delta_m\, W^{(l)} + \delta_m\, W^{(l)} = 0\,, \qquad (?) \tag{A}$$

in which δ_φ means that the four-potential is varied. The field F_{ik} will then be varied, as well, *but not the field* H_{ik}, because the u^i experiences the variation (42), and not the one in (42′), footnote 1, pp. 12 (if the metric field remains unchanged). I. Ishiwara, Ann. d. Phys. **42**, pp. 986, 1913, started with the principle (A), but he concluded (loc. cit., equation (15a)) that H will be affected by the variation. The same thing would then follow for u^i, i.e., one bases the variation δu^i on the assumption (42′), which *does not agree with* (42) for constant g_{ik}. However, if one substitutes the variation (42′) in the identity (45) for $\delta_m\, W^{(l)}$ then one must vary the g_{ik} in $\delta_m\, W^{(l)}$ only to the extent that they do not occur in the u^i. One then has $\delta\gamma^{ik}$ =

2.

The wave expressions. For the sake of simplicity, in what follows we will occupy ourselves with propagation of light in (uncharged) homogeneous isolated bodies. For these, as we will see, the equations of the vacuum for the gravitational field γ_{ik} are valid. For the time being, we will also omit the parentheses around the indices (and also write g_{ik} for γ_{ik}).

For optical questions, in place of the field equations one employs the wave equations that are derived from them. In the special theory of relativity, the wavelike propagation of a quantity A is given by an equation of the form:

$$\Box A = \frac{\partial^2 A}{\partial x_1^2} + \frac{\partial^2 A}{\partial x_2^2} + \frac{\partial^2 A}{\partial x_3^2} - \frac{\partial^2 A}{\partial x_4^2} = 0 .$$

This sum of second differential quotients is the (four-dimensional) Laplace operator applied to A. Next, we would like to present the corresponding operators for the general theory. This is easily achieved when we replace the ordinary differentiation with so-called covariant differentiation. [1])

If we let p be a scalar, φ_i be a vector, and T_{ik} be a tensor then the covariant differential quotients look like:

(70) $$D_i\, p = p_i = \frac{\partial p}{\partial x_i},$$

(71) $$D_k\, \varphi_i = \varphi_{ik} = \frac{\partial \varphi_i}{\partial x^k} - \begin{Bmatrix} i\,k \\ l \end{Bmatrix} \varphi_l ,$$

(72) $$D_l\, T_{ik} = T_{ikl} = \frac{\partial T_{ik}}{\partial x_l} - \begin{Bmatrix} i\,k \\ m \end{Bmatrix} T_{mk} - \begin{Bmatrix} k\,l \\ m \end{Bmatrix} T_{im} ,$$

and for a general tensor:

(73) $$D_m T_{i_1 i_2 \cdots i_s} = T_{i_1 i_2 \cdots i_s , m} = \frac{\partial T_{i_1 i_2 \cdots i_s}}{\partial x^m} - \begin{Bmatrix} i_1\, m \\ n \end{Bmatrix} T_{n\, i_2 \cdots i_s \cdots} - \begin{Bmatrix} i_s\, m \\ n \end{Bmatrix} T_{i_1 \cdots i_{s-1}\, n} .$$

The fact that we denoted differentiation by an index is appropriate due to the following rules that we define:

δg^{ik}, and, from (61), one arrives at the tensor density (63), which leads to the energy tensor that was given by Minkowski (W. Pauli, loc. cit., formula (301)), as this, in fact, also emerges in the explicit calculations of I. Ishiwara.

[1]) This comes from E. B. Christoffel. The given rules 1 to 4 were presented by G. Ricci and T. Levi-Civita, Math. Ann. **54**, pp. 135, 1901. For the proof of the tensor character of the covariant derivatives, cf., e.g., M. v. Laue, Relativitätsprinzip II, § 19.

Rule 1. The contravariant derivative will be obtained from the covariant one by the usual transition from covariant to contravariant components; i.e.:

(74) $$D^m T_{i_1 i_2 \cdots i_s} = T_{i_1 i_2 \cdots i_s}{}^{m} = g^{mr} T_{i_1 i_2 \cdots i_s r} .$$

Rule 2. One goes from the derivative of covariant components to that of contravariant ones in the same way; i.e.:

(75) $$D_m T_{i_1 i_2 \cdots i_s}{}^{k_1 \cdots k_t}{}_{l_1 \cdots l_n} = g^{k_1 r_1} g^{k_2 r_2} \cdots g^{k_t r_t} D_m T_{i_1 i_2 \cdots i_s\, r_1 \cdots r_t\, l_1 \cdots l_n} .$$

On the basis of these two rules, one can therefore raise or lower corresponding indices in an equation that includes these general derivatives, regardless of whether the index refers to a component or a derivative.

With the help of the identity:

(76) $$\frac{\partial g^{ik}}{\partial x^l} = -g^{nk} \begin{Bmatrix} n\, l \\ i \end{Bmatrix} - g^{in} \begin{Bmatrix} n\, l \\ k \end{Bmatrix},$$

one can treat the corresponding mixed components on the right-hand side of (75) in the same way as the ones on the left. For example, from (72) and (75), one has:

$$D_l\, T^i{}_k = g^{is}\, T_{skl} = g^{is} \frac{\partial T_{sk}}{\partial x^l} - g^{is} \begin{Bmatrix} s\, l \\ m \end{Bmatrix} T_{mk} - g^{is} \begin{Bmatrix} k\, l \\ m \end{Bmatrix} T_{sm}$$

and for the first term on the right, one can, from (76), write:

$$\frac{\partial}{\partial x^l}(g^{is} T_{sk}) - \frac{\partial g^{is}}{\partial x^l} T_{sk} = \frac{\partial T^i{}_k}{\partial x^l} + g^{ns} \begin{Bmatrix} n\, l \\ i \end{Bmatrix} T_{sk} + g^{in} \begin{Bmatrix} n\, l \\ s \end{Bmatrix} T_{sk} ,$$

such that the formula:

$$D_l\, T^i{}_k = \frac{\partial T^i{}_k}{\partial x^l} + \begin{Bmatrix} n\, l \\ i \end{Bmatrix} T^n{}_k - \begin{Bmatrix} k\, l \\ n \end{Bmatrix} T^i{}_n$$

results. In general, one has:

$$(77)\qquad \begin{cases} D_m T_{i_1\cdots i_s}{}^{k_1\cdots k_t}{}_{l_1\cdots l_n} = \dfrac{\partial}{\partial x^m} T_{i_1\cdots i_s}{}^{k_1\cdots k_t}{}_{l_1\cdots l_n} \\ \quad -\sum_{\lambda=1}^{s} \begin{Bmatrix} i_\lambda\ m \\ p \end{Bmatrix} T_{i_1\cdots i_{\lambda-1}\,p\,i_{\lambda+1}\cdots i_s}{}^{k_1\cdots k_t}{}_{l_1\cdots l_n} \\ \quad +\sum_{\lambda=1}^{t} \begin{Bmatrix} p\ m \\ k_\lambda \end{Bmatrix} T_{i_1\cdots i_s}{}^{k_1\cdots k_{\lambda-1}\,p\,k_{\lambda+1}\cdots k_t}{}_{l_1\cdots l_n} \\ \quad -\sum_{\lambda=1}^{n} \begin{Bmatrix} l_\lambda\ m \\ p \end{Bmatrix} T_{i_1\cdots i_s}{}^{k_1\cdots k_t}{}_{l_1\cdots l_{\lambda-1}\,p\,l_{\lambda+1}\cdots l_n} \end{cases}$$

Rule 3. The rules of ordinary differentiation are valid for the differentiation of sums and products.

This follows from the fact that in a geodetic coordinate system – in which the derivative of the g_{ik} vanishes, and therefore the three-index symbols, as well – the covariant derivatives coincide with the ordinary ones.

Rule 4. Changing the order of differentiation is generally forbidden, since otherwise, the result is:

$$(78)\qquad p_{ik} - p_{ki} = 0\,,$$

$$(79)\qquad \varphi_{ikl} - \varphi_{ilk} = R^h{}_{ikl}\,\varphi_h\,,$$

$$(80)\qquad T_{iklm} - T_{ikml} = R^h{}_{ikl}\,T_{hk} + R^h{}_{klm}\,T_{ih}\,,$$

and generally:

$$(81)\qquad T_{i_1\cdots i_s lm} - T_{i_1\cdots i_s ml} = R^h{}_{i_1 lm} T_{h i_2\cdots i_s} + \cdots + R^h{}_{i_s lm} T_{i_1\cdots i_{s-1} h}\,,$$

in which:

$$(82)\qquad R^i{}_{klm} = \frac{\partial}{\partial x^l}\begin{Bmatrix} k\ m \\ i \end{Bmatrix} - \frac{\partial}{\partial x^m}\begin{Bmatrix} k\ l \\ i \end{Bmatrix} + \begin{Bmatrix} l\ n \\ i \end{Bmatrix}\begin{Bmatrix} k\ n \\ n \end{Bmatrix} - \begin{Bmatrix} m\ n \\ i \end{Bmatrix}\begin{Bmatrix} k\ l \\ n \end{Bmatrix}$$

is the Riemann-Christoffel curvature tensor. One easily confirms (81) in a geodetic coordinate system. In such a system, one has:

$$T_{i_1\cdots i_s lm} = \frac{\partial^2 T_{i_1\cdots i_s}}{\partial x^l \partial x^m} - \sum_{\lambda=1}^{s} T_{i_1\cdots i_{\lambda-1} n i_{\lambda+1}\cdots i_s}\,\frac{\partial}{\partial x^m}\begin{Bmatrix} i_\lambda\ l \\ n \end{Bmatrix},$$

$$T_{i_1\cdots i_s lm} - T_{i_1\cdots i_s ml} = \sum_{\lambda=1}^{s} T_{i_1\cdots i_{\lambda-1} n i_{\lambda+1}\cdots i_s}\left(\frac{\partial}{\partial x^l}\begin{Bmatrix} i_\lambda\ m \\ n \end{Bmatrix} - \frac{\partial}{\partial x^m}\begin{Bmatrix} i_\lambda\ l \\ n \end{Bmatrix}\right).$$

Furthermore, we set:

$$\left\{\begin{matrix}(i)\,(k)\\(l)\end{matrix}\right\} = \left\{\begin{matrix}i\;k\\l\end{matrix}\right\} + p^l_{ik}\,, \tag{83}$$

in which, according to the abbreviation that was presented in part one, the three-index symbols with the bracketed indices are constructed out of the γ_{ik} in precisely the same way that the ordinary symbols are constructed from the g_{ik}. If one substitutes, e.g., (83) in (72) then one immediately finds that:

$$D_{(l)}\,T_{ik} = D_l\,T_{ik} - p^m_{il}\,T_{mk} - p^m_{kl}\,T_{im}\,, \tag{84}$$

in which one recognizes that the p are tensors. By going to a geodetic system, one confirms the following:

$$p^l_{ik} = \gamma^{rl}\,((k\,u_k\,u_r)_i + (k\,u_r\,u_i)_k - (k\,u_i\,u_k)_l), \qquad k = 1 - \frac{1}{\varepsilon\mu}\,. \tag{85}$$

In the same way:

$$R^{(i)}{}_{(k)(l)(m)} = R^i{}_{klm} + D_l\,p^i_{km} - D_m\,p^i_{kl} + p^i_{ln}\,p^n_{kn} - p^i_{mn}\,p^n_{kl}\,. \tag{86}$$

Rule 5. One obtains the covariant derivatives that come from the γ_{ik} from the ones that come from the g_{ik} by replacing the ordinary derivatives with the covariant ones (that come from g_{ik}) and the three-index symbols with the quantities p. The curvature tensor $R^{(i)}{}_{(k)(l)(m)}$ is obtained from $R^i{}_{klm}$ by adding a term that one arrives at by the same substitution.

With these preparations, we are in a position to write down the Laplacian expressions. They are:

$$\Box\,p = p^k{}_k\,, \tag{87}$$

$$\Box_i\,\varphi = \varphi_i{}^k{}_k\,, \tag{88}$$

$$\Box_{ik}\,T = T_{ik}{}^l{}_l\,, \quad \text{etc.} \tag{89}$$

One sees that these expressions have, at the same time, the same tensor character as the quantities to which the operator is applied. From (70) and (71), only the first derivatives of the g_{ik} appear for a scalar p. Thus, the wave expressions for a scalar in general relativity will also agree with the Laplacian expression. If we employ the general notation W for a wave expression and introduce the notations:

$$\mathrm{Grad}_i\,p = \frac{\partial p}{\partial x^i}\,, \tag{90}$$

$$\mathrm{Div}\,\varphi = \varphi^k{}_k\,, \tag{91}$$

then we have, for a scalar:

(92) $$\text{Div Grad}\, p = W\, p\,.$$

The expressions (83), (84), etc., involve the second derivatives of g_{ik}. They will therefore represent wave expressions only in the context of the special theory of relativity or gravitational fields with vanishing curvature tensor; in the general case, further terms will appear in which this tensor figures. We would like to determine these extra terms.

We arrive at them when we seek to generalize relation (92) to tensors. In the special theory, this generalization is well-known. For a vector one has the formula [1])

(93) $$\text{Div Grad}\, \varphi = \text{Grad Div}\, \varphi - W\, \varphi\,,$$

in which the wave expression W is identical with $\square$. The operators Rot and Div also exist in the general theory. If we let p, φ_i, F_{ik}, S_{ikl}, L_{iklm}, etc., denote *linear* tensors of rank 0 (scalar), 1, 2, 3, 4, etc., then the rotations are defined by:

(94) $$\text{Rot}_i\, p = \frac{\partial p}{\partial x^i}\,,$$

(95) $$\text{Rot}_{ik}\, \varphi = \frac{\partial \varphi_k}{\partial x^i} - \frac{\partial \varphi_i}{\partial x^k}\,,$$

(96) $$\text{Rot}_{ikl}\, F = \frac{\partial F_{ik}}{\partial x^l} + \frac{\partial F_{kl}}{\partial x^i} + \frac{\partial F_{il}}{\partial x^k}\,,$$

(97) $$\text{Rot}_{iklm}\, S = \frac{\partial S_{klm}}{\partial x^i} - \frac{\partial S_{lmi}}{\partial x^k} + \frac{\partial S_{mik}}{\partial x^l} - \frac{\partial S_{ikl}}{\partial x^m}\,, \text{ etc.,}$$

and the divergences by:

(98) $$\text{Div}\, \varphi = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}(\sqrt{g}\varphi^k)\,,$$

(99) $$\text{Div}^i\, F = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^k}(\sqrt{g}F^{ik})\,,$$

(100) $$\text{Div}^{ik}\, S = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^l}(\sqrt{g}S^{ikl})\,,$$

(101) $$\text{Div}^{ikl}\, L = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x^m}(\sqrt{g}L^{iklm})\,, \text{ etc.}$$

Ordinarily, in place of relation (94), the Grad that was defined in (90) is used. The tensors that result from these operations are once again linear. The justification for the use of the terms "rotations" and "divergences" resides in the fact that the generalized theorems of Stokes and Gauss are valid for them [2]). These representations of the

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[1] M. v. Laue, Relativitätstheorie I, formula (115).

[2] W. Pauli, Jr., loc. cit., pp. 606. For the one-dimensional "relation," Stokes theorem takes the form $\int_{p_1}^{p_2} \text{Rot}_i\, p\, \delta x^i = p_2 - p_1$.

rotations are valid only for covariant components, just as the divergences are only valid for contravariant components. However, if we replace the ordinary derivatives with the covariant derivatives in (94) to (97) then we obtain the formulas:

(94′) $\mathrm{Rot}_i\, p = p_i$,

(95′) $\mathrm{Rot}_{ik}\, \varphi = \varphi_{ki} - \varphi_{ik}$,

(96′) $\mathrm{Rot}_{ikl}\, F = F_{ikl} + F_{kli} + F_{lik}$,

(97′) $\mathrm{Rot}_{iklm}\, S = S_{klmi} - S_{lmik} + S_{mikl} - S_{iklm}$, etc.,

in which, from Rules 1 and 2, we may henceforth use the same indices on both sides as we did above. If we proceed by analogy with (98) and (101), in which we observe that the covariant derivative of $\sqrt{g}$ vanishes, then we obtain the formulas:

(98′) $\mathrm{Div}\, \varphi = \varphi^k{}_k$ (cf., 91),

(99′) $\mathrm{Div}^i\, F = F^{ik}{}_k$,

(100′) $\mathrm{Div}^{ik}\, S = S^{ikl}{}_l$,

(101′) $\mathrm{Div}^{ikl}\, L = L^{iklm}{}_m$, etc.

We obtain the covariant components by lowering the indices. The tensors (94) to (101) are identical with the tensors (94′) to (101′), resp., because they agree with each other in a geodetic system.

From the representations (94′) to (101′), it follows that not only the geometric, but also the *formal* definition of rotation and divergence can be carried over from ordinary vector analysis into a more general form. It is well known that when ∇ is the operator $\left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right)$ and $\mathfrak{a}$ is a vector then:

(102) $\mathrm{rot}\, \mathfrak{a} = [\nabla\, \mathfrak{a}]$ (exterior product)

(103) $\mathrm{div}\, \mathfrak{a} = \nabla\, \mathfrak{a}$ (inner product).

The operator D appears in place of ∇. The exterior products of a vector (first rank linear tensor) A with the linear tensors φ_i, F_{ik}, S_{ikl}, resp., of rank 1, 2, 3, resp., are:

(104) $[A\, \varphi]_{ik} = A_i\, \varphi_k - A_k\, \varphi_i$,

(105) $[A\, F]_{ikl} = A_i\, F_{kl} + A_k\, F_{li} + A_l\, F_{ik}$,

(106) $[A\, S]_{iklm} = A_i\, S_{klm} - A_k\, S_{lmi} + A_l\, S_{mlk} - A_m\, S_{lki}$, etc.

If the vector A is a displacement $\xi_{(1)}$, in particular, and the linear tensor of rank s that one exterior multiplies it with is an s-dimensional space tensor that one constructs out of the s displacements $\xi_{(2)}$, $\xi_{(3)}$, ..., $\xi_{(s+1)}$, then the exterior product is an s+1-dimensional space tensor that is constructed out of the s+1 displacements $\xi_{(1)}$, $\xi_{(2)}$, ..., $\xi_{(s+1)}$. Correspondingly, one defines the exterior product of two arbitrary tensors [1]) and

[1] The exterior product of A_{ik} and B_{lm} will thus be given by developing the determinant

generally the exterior product of two tensors whose indices that are to be multiplied are completely anti-symmetric (which does not have to be the case for the remaining indices). For example, if $A_{\rho\sigma ik}$ is anti-symmetric in i and k, and $B_{\lambda lm}$ is anti-symmetric in l and m then we would like to indicate the exterior product relative to these indices by:

$$A_{\rho\sigma[i][j]}\,B_{\lambda[l][m]}\,. \tag{107}$$

With these clarifications, the general formal definition for the rotation of a linear tensor M reads like:

$$\text{Rot}\,M = [D\,M]\,. \tag{108}$$

As is well known, the interior multiplication (of arbitrary, not necessarily linear, tensors) occurs in the process of lowering the multiplied indices. One then has:

$$\text{Div}\,M = D\,M\,, \tag{109}$$

in which the multiplication is to be performed on the last index of M.

Corresponding to formulas (92) and (93), combined with (94), we generally define the *wave expression W for an arbitrary linear tensor M* by means of:

$$\text{Div Rot}\,M = \text{Rot Div}\,M \pm W\,M\,, \tag{110}$$

in which the upper or lower sign applies when M is of even or odd rank, resp.

For a tensor φ_i of first rank, we have, from (88), (95´), (99´):

$$\text{Div}_i\text{ Rot }\varphi = \varphi^k{}_{ik} - \varphi_i{}^k{}_k = \varphi^k{}_{ik} - \Box_i\varphi\,.$$

From the fourth rule, formula (79) is:

$$\varphi^k{}_{ik} = \varphi^k{}_{ki} + R^{hk}{}_{ik} = \varphi^k{}_{ki} + R_i{}^h\,\varphi_h\,,$$

in which $-R_h{}^k{}_{ik} = R^k{}_{ihk}$ is the contracted curvature tensor. By noting (94´) and (98´) one will thus have:

$$\text{Div}_i\text{ Rot }\varphi = \text{Rot Div}_i\,\varphi - \Box_i\varphi - R_i{}^h\,\varphi_h$$

$$[A\,B]_{iklm} = \begin{vmatrix} \alpha_i & \alpha_k & \alpha_l & \alpha_m \\ \beta_i & \beta_k & \beta_l & \beta_m \\ \gamma_i & \gamma_k & \gamma_l & \gamma_m \\ \delta_i & \delta_k & \delta_l & \delta_m \end{vmatrix}$$

in the sub-determinants of the first two columns in which one puts $\begin{vmatrix} \alpha_i & \alpha_k \\ \beta_i & \beta_k \end{vmatrix} = A_{ik}$, $\begin{vmatrix} \gamma_l & \gamma_m \\ \delta_l & \delta_m \end{vmatrix} = B_{lm}$ in this development. One immediately sees that the exterior product changes its sign under the exchange of factors only when both factors are of odd rank.

and equating this with (110) teaches us that:

(111) $$W_i\,\varphi = \Box_i\,\varphi + R_i^{\ h}\,\varphi_h\,.$$

It is remarkable that the additional term here only involves the contracted curvature tensor.

For a tensor of second rank F_{ik} , one has, from (96´) and (100´):

(112) $$\mathrm{Div}_{ik}\,\mathrm{Rot}\,F = F_{ikl}^{\ \ \ l} + F_{kli}^{\ \ \ l} + F_{lik}^{\ \ \ l}\,.$$

From (89), the first term is $\Box_{ik} F$, and the other two collectively of the form $K_{ki} - K_{ik}$, if we mean that $K_{ik} = F_{ilk}^{\ \ \ l}$. From rule 4, formula (80) is:

$$K_{ik} = F_{il\ k}^{\ \ l} + R^{h\ \ l}_{\ lk}\,F_{ih} + R^{h\ \ l}_{\ lk}\,F_{hl}\,,$$

such that, from definition (95´), the rotation is:

$$K_{ki} - K_{ik} = \mathrm{Rot}_{ik}\,\mathrm{Div}\,F + R_i^{\ h}\,F_{hk} - R_k^{\ h}\,F_{hi} + (R^{h\ \ l}_{\ ki} - R^{h\ \ l}_{\ ik})\,F_{hl}\,.$$

As a result of the symmetry properties of R_{iklm} , we finally have that $R_{hkil} - R_{hikl} = R_{ikhl}$, and therefore (112) goes over into (110) for $M = F$:

(113) $$W_{ik}\,F = \Box_{ik}\,F + R_i^{\ h}\,F_{hk} - R_k^{\ h}\,F_{hi} + R_{ik}^{\ \ hl}\,F_{hl}\,.$$

From (104), one can, with these conventions, also write the notation (107) as:

(113´) $$W_{ik}\,F = \Box_{ik}\,F + R_{[i]}^{\ \ h}\,F_{h[k]} + R_{ik}^{\ \ hl}\,F_{hl}\,.$$

In exactly the same way, one deduces for a tensor S_{ikl} of third rank:

(114) $$W_{ikl}S = \Box_{ikl}S + R_i^{\ h}S_{hkl} + R_k^{\ h}S_{hli} + R_i^{\ h}S_{hik} + R_{ik}^{\ \ hm}S_{hml} + R_{kl}^{\ \ hm}S_{hmi} + R_{li}^{\ \ hm}S_{hmk}\,,$$

which, from (105), can be further written:

(114´) $$W_{ikl}\,S = \Box_{ikl}\,S + R_{[i]}^{\ \ h}\,S_{h[k][l]} + \mathrm{R}_{[i][k]}^{\ \ \ \ hl}\,S_{hl[l]}\,.$$

The *general* formula for the wave expression W of a linear tensor $M_{i_1 i_2\cdots i_s}$ of rank s reads like:

(115) $$W_{i_1\cdots i_s}M = \Box_{i_1\cdots i_s}M + R_{[i_1]}^{\ \ \ h}M_{h[i_2]\cdots[i_s]} + R_{[i_1[i_2]}^{\ \ \ \ \ hm}M_{hm[i_3]\cdots[i_s]}.$$

From formulas (94) to (101), the rotations and divergences follow immediately:

(116) $$\mathrm{Div}\,\mathrm{Div}\,M = \mathrm{Rot}\,\mathrm{Rot}\,M = 0\,.$$

Since the rank of a tensor is changed by one upon taking a divergence or a rotation, (110) gives, when applied to Div M and Rot M:

$$\mathrm{Div\,Rot\,Rot}\, M = \mp W\, \mathrm{Div}\, M\,, \qquad \mathrm{Rot\,Div\,Rot}\, W = \pm W\, \mathrm{Rot}\, M\,.$$

One thus has:

$$\mathrm{Div}\, W M = W\, \mathrm{Div}\, M\,, \qquad \mathrm{Rot}\, W M = W\, \mathrm{Rot}\, M\,. \tag{117}$$

The wave expression commutes with Rot *and* Div.

For two linear tensors M and N of equal rank (say, second), one has, from rule 3 and consideration of (98) and (98´), for the interior product $M \,\square\, N$:

$$\begin{aligned} M^{ik} \square_{ik} N &= M^{ik} N_{ik\,l}{}^{l} = (M^{ik} N_{ik}{}^{l})_l - M^{ik}{}_l N_{ik}{}^{l}\,, \\ &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^l} (\sqrt{g} M^{ik} N_{ik}{}^{l}) - M^{ik} N_{ik}{}^{l}\,, \end{aligned}$$

and therefore:

$$M \,\square\, N - N \,\square\, M = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^l} \left\{ \sqrt{g} (M^{ik} N_{ik}{}^{l} - M^{ik} N_{ik}{}^{l}) \right\}. \tag{118}$$

One can replace $\square$ with W in this; the additional term then arises, as one recognizes from (111), (113), (114), and the general formula (115), due to the symmetry of R_{ik} (R_{iklm}, resp.) in the indices i, k (in the index pair (ik), (lm), resp.). If we integrate (118) over a closed four-dimensional space (volume element $d\Sigma = \sqrt{g}\, dx^1\, dx^2\, dx^3\, dx^4$) then we obtain, from Gauss's theorem [1]):

$$\int (M\, W\, N - N\, W\, M)\, d\Sigma = \int (M\, N_n - N\, M_n)\, dS\,. \tag{119}$$

(dS is the hypersurface element for the boundary surface and n is the *covariant* derivative with respect to the external normal.) *Green's theorem is valid for the wave expressions (and Laplacian operators)* [2]).

3.

Field, four-potential, Hertz tensor. The differential equations of the field F may be written (again, when one omits the brackets on the indices) when one introduces the definitions (96) and (99) as:

$$\mathrm{Rot}\, F = 0, \qquad \mathrm{Div}\, F = 0\,. \tag{120}$$

[1] W. Pauli, Jr., loc. cit., formula (139a).

[2] The densities $\sqrt{g}\,\square$, $\sqrt{g}\, W$ thus represent self-adjoint differential expressions.

From (110), it then follows that:

(121) $$W\,F = 0.$$

$W\,F$ is the expression (113) (in which rule 5 is to be observed). *The field then propagates in a wavelike manner.* One proves the converse with the help of the commutation theorem (117) in a manner that is completely analogous to the classical theory [1]): A solution of the wave equation (121) that satisfies (120) at one point in time, always does so.

One can weaken or completely omit the uncomfortable reduction to the initial values, in the former case, by the introduction of the *four-potential* φ, and in the latter, by the introduction of the *six-potential* Z, which we would also like to call the *Hertz tensor*. (We will likewise justify these names.) As in (39), we set:

(122) $$F = \text{Rot}\ \varphi,$$

in which, from (116), the first equation of (120) will be satisfied. From (93), it then follows that:

(123) $$\text{Div}\ F = \text{Div Rot}\ \varphi = \text{Grad Div}\ \varphi - W\,\varphi\,,$$

such that the second equation of (120) will be satisfied, when φ is subject to the wave equation:

(124) $$W_i\ \varphi = \Box_i\,\varphi + R_i^h\ \varphi_h = 0,$$

with the condition:

(125) $$\text{Div}\ \varphi = 0\,.$$

From (116), one can finally free oneself from this condition by the Ansatz:

(126) $$\varphi = \text{Div}\ Z\,.$$

From the commutation rule (117), one has:

(127) $$W\,\varphi = W\ \text{Div}\ Z = \text{Div}\ W\,Z\,,$$

such that (124) will be satisfied through the equation:

(128) $$W\,Z = 0\,.$$

From (122) and (126), one obtains the representation for the field in terms of the six-potential:

(129) $$F = \text{Rot Div}\ Z\,,$$

[1] Cf. E. Cohn, Das elektromagnetishe Feld, pp. 410-412.

or, on account of (110) and (128):

(129´) $$F = \text{Div Rot } Z\,.$$

The four-potential and the six-potential both satisfy the wave equation.

From special relativity (the rest of this section is founded on the electron theory of the vacuum), a system of charged particles (called, more briefly, a "molecule" in the sequel) with total charge e induces, in the first approximation, a four-potential:

(130) [1]) $$\varphi_i = \frac{e u_i}{R}, \qquad R = -(x_r - \xi_r)\, u^r$$

at a world-point P (x^1, x^2, x^3, x^4). (ξ_r, u_r are the coordinates and four-velocity of the molecule at the intersection of a world-line with the forward cone $(x_r - \xi_r)(x^r - \xi^r) = 0$ at P.) Instead of the proper time s we can, however, as in (14), choose an arbitrary parameter τ, through which the position of the molecule on its world-line can be determined. The root that appears in (15) disappears in (130) in the numerator and denominator, and we can then write:

(130´) $$\varphi_i = \frac{e}{R}\frac{d\xi_i}{d\tau}, \qquad R = -(x_r - \xi_r)\frac{d\xi_i}{d\tau}\,.$$

If the total charge e vanishes then the molecule is *electrodynamically polarized* (in which the separation into electric and magnetic polarizations depends upon the polarization into space and time). In this case, (130´) is not attained, so one must go a step further into the approximation. From (126), Z is a differentiation step away from φ. We will thus suppose that, in analogy with (130´), *the six-potential of an electromagnetic dipole* will be represented by the formula:

(131) $$Z_{ik} = \frac{m_{ik}}{R}$$

(in the first approximation and in the case of a non-vanishing electromagnetic moment m). If we split space and time according to the schema:

$$\begin{array}{lll}
F_{14}\, F_{24}\, F_{34}\,, & F_{23}\, F_{31}\, F_{12} & = \mathfrak{E}, \mathfrak{H}\,, \\
\varphi_1\, \varphi_2\, \varphi_3 & ,\ \varphi_4 & = \mathfrak{A}, \varphi\,, \\
Z_{14}\, Z_{24}\, Z_{34}, & Z_{23}\, Z_{31}\, Z_{12} & = -\,\mathfrak{Z}, \mathfrak{Z}'\,, \\
m_{14}\, m_{24}\, m_{34}, & m_{23}\, m_{31}\, m_{12} & = -\,\mathfrak{p}, \mathfrak{m},
\end{array}$$

in which $\mathfrak{E}$, $\mathfrak{H}$ are the electric and magnetic field strengths, $\mathfrak{A}$, φ are the vector and scalar potential, $\mathfrak{Z}$, $\mathfrak{p}$ are polar spatial vectors, and $\mathfrak{Z}'$, $\mathfrak{m}$ are axial space vectors, then (126) and (129), (129´) decompose into the equations:

[1] M. V. Laue. Relativitätstheorie I, formula (218); W. Pauli, Jr., loc. cit., formula (238a).

(126´) $$\mathfrak{A} = \frac{\dot{\mathfrak{Z}}}{e} + \text{rot}\,\mathfrak{Z}', \qquad \varphi = -\,\text{div}\,\mathfrak{Z},$$

(129") $$\mathfrak{E} = \text{rot rot}\,\mathfrak{Z} - \frac{1}{e}\text{rot}\,\dot{\mathfrak{Z}}', \qquad \mathfrak{H} = \frac{1}{e}\text{rot}\,\dot{\mathfrak{Z}} + \text{rot rot}\,\mathfrak{Z}'$$

$$\left(\text{dot} = \frac{\partial}{\partial t}\right).$$

The usual wave equation for $\mathfrak{Z}$ and $\mathfrak{Z}'$ emerges from (128). However, when the molecule is at rest $\left(\frac{d\xi^1}{d\tau} = \frac{d\xi^2}{d\tau} = \frac{d\xi^3}{d\tau} = 0\right)$ (131) decomposes into:

(131´) $$\mathfrak{Z} = \frac{\mathfrak{p}\left(t - \frac{r}{c}\right)}{r}, \qquad \mathfrak{Z}' = \frac{\mathfrak{m}\left(t - \frac{r}{c}\right)}{r} \qquad (r = \text{distance to molecule at } P).$$

If $\mathfrak{Z}$ and therefore $\mathfrak{Z}'$ are equal to zero then, as one recognizes from (126´), (129"), and (131´), $\mathfrak{Z}$ is equal to the *Hertz vector* for an *electrically* polarized molecule of moment $\mathfrak{p}$.[1] Conversely, if $\mathfrak{p} = \mathfrak{Z} = 0$ then, from (126´), (129"), in conjunction with (131´), one will be given the potential and field of a *magnetically* polarized molecule of moment $\mathfrak{m}$.[2] $\mathfrak{Z}'$ is the magnetic counterpart of $\mathfrak{Z}$. *The six-potential is the four-dimensional summarization of the electric and magnetic Hertz vectors.*

We would like to give the Hertz tensor for an arbitrary uncharged molecule, and thus confirm formula (131), as well. Let:

(132) $$\eta(\tau, \varepsilon) = \xi^i(\tau) + \varepsilon\,\delta\xi^i, \qquad \varepsilon = 1$$

be the world-line of a charged particle of charge e, let $\xi^i(\tau)$ be the world-line of the midpoint of the molecule, and let the $\delta\xi^i$ be functions of τ, hence, of the relative coordinates of the particle. The formal introduction of the factor ε serves, in a well-known way, to facilitate the development in the $\delta\xi^i$ and their derivatives with respect to τ into a series in ε. (From now on, ε is therefore to be set equal to 1.) From (130´), the four-potential of the particle is:

(133) $$\varphi_i = \frac{e}{R}\frac{\partial\eta_i}{\partial\tau}, \qquad R = -(x_r - \eta_r)\frac{\partial\eta^r}{\partial\tau},$$

in which τ, from the equation:

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[1] Cf., e.g., M. Planck, Einführung in die Theorie der Elektrizität und des Magnetismus, § 87, 88.

[2] H. A. Lorentz, Enz. d. math. Wiss., v. 14, no. 15.

(134) $$(x_r - \eta_r)(x^r - \eta^r) = 0$$

of the null cone at P, is a given function of ε and the coordinates x^1, x^2, x^3, x^4 of P. The right-hand sides of (133) (in which the η_i are functions (132) of ε and τ), which originally were functions $\psi_i(\tau, \varepsilon, x^1, x^2, x^3, x^4)$, come about by the substitution of τ in the functions $\varphi_i(\varepsilon, x^1, x^2, x^3, x^4)$:

(135) $$\psi_i(\tau, \varepsilon, x^1, x^2, x^3, x^4) = \varphi_i(\varepsilon, x^1, x^2, x^3, x^4).$$

In order to develop this in ε, we must construct the derivatives $\frac{\partial \varphi}{\partial \varepsilon}, \frac{\partial^2 \varphi}{\partial \varepsilon^2}$, etc. Differentiation of (134) and (135) immediately gives:

(136) $$\frac{\partial \tau}{\partial x^k} = -\frac{x_k - \eta_k}{R}, \qquad \frac{\partial \tau}{\partial \varepsilon} = \frac{x_k - \eta_k}{R}\frac{\partial \eta^r}{\partial \varepsilon} = -\frac{\partial \tau}{\partial x^r}\frac{\partial \eta^r}{\partial \varepsilon},$$

(136´) $$\frac{\partial \varphi_i}{\partial \varepsilon} = \frac{\partial \psi_i}{\partial \varepsilon} + \frac{\partial \psi_i}{\partial \tau}\frac{\partial \tau}{\partial \varepsilon} = \frac{d\psi_i}{d\varepsilon}, \qquad \frac{\partial \varphi_i}{\partial x^k} = \frac{\partial \psi_i}{\partial x^k} + \frac{\partial \psi_i}{\partial \tau}\frac{\partial \tau}{\partial x^k},$$

from which, it follows that:

(137) $$\frac{\partial \varphi_i}{\partial \varepsilon} = \frac{\partial \psi_i}{\partial \varepsilon} - \frac{\partial \psi_i}{\partial \tau}\frac{\partial \tau}{\partial x^r}\frac{\partial \eta^r}{\partial \varepsilon} = \frac{\partial \psi_i}{\partial \varepsilon} + \left(\frac{\partial \psi_i}{\partial x^r} - \frac{\partial \varphi_i}{\partial x^r}\right)\frac{\partial \eta^r}{\partial \varepsilon}.$$

For the expressions that appear on the right here, one obtains, from (133):

$$\frac{\partial \psi_i}{\partial \varepsilon} = \frac{e}{R}\frac{\partial^2 \eta_i}{\partial \tau\, \partial \varepsilon} - \frac{e}{R^2}\frac{\partial \eta_i}{\partial \tau}\left\{\frac{\partial \eta_r}{\partial \varepsilon}\frac{\partial \eta^r}{\partial \tau} - (x_r - \eta_r)\frac{\partial^2 \eta^r}{\partial \tau\, \partial \varepsilon}\right\}$$

$$\frac{\partial \psi_i}{\partial x^r}\frac{\partial \eta^r}{\partial \varepsilon} = \frac{e}{R^2}\frac{\partial \eta_i}{\partial \tau}\frac{\partial \eta_r}{\partial \tau}\frac{\partial \eta^r}{\partial \varepsilon}$$

$$-\frac{\partial \varphi_i}{\partial x^r}\frac{\partial \eta^r}{\partial \varepsilon} = -\frac{\partial}{\partial x^r}\left(\varphi_i \frac{\partial \eta^r}{\partial \varepsilon}\right) + \varphi_i \frac{\partial}{\partial x^r}\left(\frac{\partial \eta^r}{\partial \varepsilon}\right).$$

By summing these three expressions, the second term of the first expression cancels the right-hand side of the second expression. The last term of the first expression is, from (133) and the first equation in (136), $-\varphi_i \frac{\partial \tau}{\partial x^r}\frac{\partial^2 \eta^r}{\partial \tau \partial \varepsilon} = -\varphi_i \frac{\partial}{\partial x^r}\left(\frac{\partial \eta^r}{\partial \varepsilon}\right)$, and will thus cancel the last term of the third expression. From the first equation of (136) and the definition of R in (133), one has $\frac{\partial}{\partial x^r}\left(\frac{\partial \eta^i}{\partial \varepsilon}\right) \bullet \frac{\partial \eta^r}{\partial \tau} = \frac{\partial^2 \eta^i}{\partial \tau \partial \varepsilon}\frac{\partial \tau}{\partial x^r}\frac{\partial \eta^r}{\partial \tau} = \frac{\partial^2 \eta^i}{\partial \tau \partial \varepsilon}$, such that one can write $\frac{\partial}{\partial x^r}\left(\frac{\partial \eta^i}{\partial \varepsilon}\right)\bullet \varphi^r$ for the first term of the first expression, or $\frac{\partial}{\partial x^r}\left(\frac{\partial \eta^i}{\partial \varepsilon}\varphi^r\right)$, due to the vanishing divergence (125) of the four-potential. We therefore ultimately obtain:

(138) $$\frac{\partial \varphi^i}{\partial \varepsilon} = \frac{\partial}{\partial x^r}\left(\frac{\partial \eta^i}{\partial \varepsilon}\varphi^r - \frac{\partial \eta^r}{\partial \varepsilon}\varphi^i\right),$$

or, when we introduce the definition of the exterior product of two vectors:

(138´) $$\frac{\partial \varphi}{\partial \varepsilon} = \mathrm{Div}\left[\frac{\partial \eta^i}{\partial \varepsilon}\varphi^r\right].$$

From the first equation in (136´), applied to $\left[\frac{\partial \eta^i}{\partial \varepsilon}\varphi^r\right]$, one further has:

(139) $$\frac{\partial^2 \varphi^i}{\partial \varepsilon^2} = \frac{\partial}{\partial \varepsilon}\mathrm{Div}\left[\frac{\partial \eta}{\partial \varepsilon}\varphi\right] = \mathrm{Div}\frac{\partial}{\partial \varepsilon}\left[\frac{\partial \eta}{\partial \varepsilon}\varphi\right] = \mathrm{Div}\frac{d}{d\varepsilon}\left[\frac{\partial \eta}{\partial \varepsilon}\psi\right],$$

in which $\partial\eta/\partial\varepsilon$ is first thought of as a function of ε and the x, and finally, as a function of ε and τ. From (138´) and (139), one sees that the *six-potential of a polarized molecule* will be represented by:

(140) $$Z = z + \frac{1}{2!}\frac{dx}{d\varepsilon} + \frac{1}{3!}\frac{d^2 x}{d\varepsilon^2} + \ldots,$$

in which one replaces z, from (132), (133), and (138´), with:

(140´) $$\begin{cases} z = \left[\frac{\partial \eta}{\partial \varepsilon}\varphi\right] = \frac{e}{R}\left(\left[\delta\xi\frac{d\xi}{d\tau}\right] + \varepsilon\left[\delta\xi\frac{d\delta\xi}{d\tau}\right]\right) \\ R = -(x_r - \xi_r)\frac{d\xi^r}{d\tau} - \varepsilon\frac{d}{d\tau}[(x_r - \xi_r)\delta\xi^r] + \frac{e^2}{2}\frac{d}{d\tau}(\delta\xi_r\delta\xi^r). \end{cases}$$

(τ is a function of ε and x that is given implicitly by (134)); z and its differential quotients are to be taken at $\varepsilon = 0$. One must sum over all particles in the molecule.

In order to construct $dz/d\varepsilon$, one observes that, from (132), (136), and (140´):

$$\frac{\partial \tau}{\partial \varepsilon} = \frac{1}{R}[(x_r - \xi_r)\,\delta\xi^r - \varepsilon\,\delta\xi_r\,\delta\xi^r];$$
$$\frac{\partial R}{\partial \varepsilon} = -\frac{d}{d\tau}\{(x_r - \xi_r)\,\delta\xi^r\} + \varepsilon\frac{d}{d\tau}(\delta\xi_r\,\delta\xi^r),$$

such that:

$$\frac{\partial R}{\partial \varepsilon} = -\frac{\partial}{\partial \tau}\left(R\frac{\partial \tau}{\partial \varepsilon}\right), \qquad \frac{dR}{d\varepsilon} = \frac{\partial R}{\partial \varepsilon} + \frac{\partial R}{\partial \tau}\frac{\partial \tau}{\partial \varepsilon} = -R\frac{\partial}{\partial \tau}\left(\frac{\partial \tau}{\partial \varepsilon}\right).$$

One thus obtains, from (140´), with no further computation:

$$\frac{dz}{d\varepsilon} = \frac{e}{R}\left\{\left[\delta\xi\frac{d\delta\xi}{d\tau}\right] + \frac{\partial}{\partial \tau}\left(\left(\left[\delta\xi\frac{d\xi}{d\tau}\right] + \varepsilon\left[\delta\xi\frac{d\delta\xi}{d\tau}\right]\right)\frac{\partial \tau}{\partial \varepsilon}\right)\right\}.$$

For $\varepsilon = 0$, one has:

$$(141) \qquad \begin{cases} z = \dfrac{e}{R}\left[\delta\xi\dfrac{d\xi}{d\tau}\right], \\ \dfrac{dx}{d\varepsilon} = \varepsilon\left[\delta\xi\dfrac{d\delta\xi}{d\tau}\right] + \dfrac{e^2}{R}\left\{\dfrac{d}{d\tau}\left((x_r - \xi_r)\delta\xi^r\left[\delta\xi\dfrac{d\xi}{d\tau}\right]\right)\right. \\ \qquad \left. + \dfrac{\zeta}{R}(x_r - \xi_r)\delta\xi^r\left[\delta\xi\dfrac{d\xi}{d\tau}\right]\right\} \\ R = -(x_r - \xi_r)\dfrac{d\xi^r}{d\tau}, \end{cases}$$

with the abbreviation:

$$(141´) \qquad \zeta = -\left(\frac{\partial R}{\partial \tau}\right)_{\varepsilon=0} = -\frac{d\xi_r}{d\tau}\frac{d\xi^r}{d\tau} + (x_r - \xi_r)\frac{d^2\xi^r}{d\tau^2}.$$

If we restrict ourselves in (140) to the first two terms then it results that:

$$(140'') \qquad Z = \frac{m}{R} + \frac{e}{2R^2}\left\{\frac{d}{d\tau}\left((x_r - \xi_r)\delta\xi^r\left[\delta\xi\frac{d\xi}{d\tau}\right]\right) + \frac{\zeta}{R}(x_r - \xi_r)\delta\xi^r\left[\delta\xi\frac{d\xi}{d\tau}\right]\right\}$$

with:

$$(140´´´) \qquad m = e\left[\delta\xi\frac{d\xi}{d\tau}\right] + \frac{e}{2}\left[\delta\xi\frac{d\delta\xi}{d\tau}\right].$$

By neglecting the moments $\delta\xi^r\ \delta\xi^s$ of second degree, one obtains, in fact, formula (131). If we split into space and time, in which we choose the variable $\xi^4 = c\ t$ for the parameter τ (which then makes $\frac{d\xi^4}{d\tau} = -\frac{d\xi_4}{d\tau} = 1$, $\delta\xi^4 = -\delta\xi_4 = 0$), then we arrive at the representations:

(142) $$\mathfrak{p} = \sum e\,\mathfrak{s}, \qquad \mathfrak{m} = \left[\mathfrak{p}\frac{\mathfrak{v}}{c}\right] + \tfrac{1}{2}\sum e\left[\mathfrak{s}\frac{\mathfrak{u}}{c}\right]$$

for the electric and magnetic moments. Here, $\mathfrak{s}$ means the position vector of the particle from the midpoint of the molecule, $\mathfrak{v}$, its velocity, and $\mathfrak{u}$, the relative velocity of the particle. The sums are to be taken over the figure of the molecule at the time $t - \frac{r}{c}$, in which r is the observer-molecule distance at this time. For the two vectors $\mathfrak{Z}$ and $\mathfrak{Z}'$, we find, from (140"), (141_3), and (141´):

(143) $$\mathfrak{Z} = \frac{\mathfrak{p}}{r\left(1-\frac{v_r}{c}\right)} + \frac{1}{2r^2\left(1-\frac{v_r}{c}\right)^2}\left\{\frac{1}{c}\frac{d}{dt}\sum e\,\mathfrak{s}(\mathfrak{r}\,\mathfrak{s}) + \frac{\zeta}{r\left(1-\frac{v_r}{c}\right)}\sum e\,\mathfrak{s}(\mathfrak{r}\,\mathfrak{s})\right\},$$

(143´) $$\begin{cases} \mathfrak{Z}' = \dfrac{\mathfrak{m}}{r\left(1-\frac{v_r}{c}\right)} + \dfrac{1}{2r^2\left(1-\frac{v_r}{c}\right)^2} \\ \qquad \left\{\dfrac{1}{c}\dfrac{d}{dt}\sum e\left[\mathfrak{s}\dfrac{\mathfrak{v}}{c}\right](\mathfrak{r}\,\mathfrak{s}) + \dfrac{\zeta}{r\left(1-\frac{v_r}{c}\right)}\sum e\left[\mathfrak{s}\dfrac{\mathfrak{v}}{c}\right](\mathfrak{r}\,\mathfrak{s})\right\}, \end{cases}$$

(143") [1]) $$\zeta = 1 - \frac{v^2}{c^2} + \frac{(\dot{\mathfrak{v}}\,\mathfrak{r})}{c^2},$$

in which $\mathfrak{r}$ is the radius vector of the molecule-observer, v_r are the components of $\mathfrak{v}$ in this direction, and $\dot{\mathfrak{v}} = \frac{d\mathfrak{v}}{dt}$. (142) shows that a *moving* Hertzian oscillator possesses a magnetic moment $\left[\mathfrak{p}\frac{\mathfrak{v}}{c}\right]$, thus *both* the vectors $\mathfrak{Z}$ and $\mathfrak{Z}'$ are necessary for the representation of its field.

If we set $H = F - M$ then the field equations (2) take the form Div F = s + Div M. From (123), one then has, due to (125), $W\,\varphi = -\,s$ – Div M. If we separate φ into two pieces $\varphi_1 + \varphi_2$ in such a way that $W\,\varphi_1 = -\,s$, $W\,\varphi_2 =$ – Div M, and we set φ_2 = Div Z, then we get Div $W\,Z$ = – Div M, which will be satisfied because $W\,Z = -\,M$. From this, we conclude: *The field F may be represented by a four-potential and a six-potential:*

[1] Cf., M. Abraham, Theorie der Elektrizität 2, 4th ed., formula (72c).

(144) $F = \text{Rot}\ \varphi + \text{Rot Div}\ Z, \qquad \varphi = \int \frac{[s]}{r} dV, \qquad Z = \int \frac{[M]}{r} dV,$

(dV = spatial volume element)

([ ] means the value at the time $t - \frac{r}{c}$), *the first of which relates to the four-current and the second of which to the six-polarization*. From (143) and (143′), one obtains (by neglecting the moments of second degree):

(144′) $\mathfrak{P} = N \sum e\,\mathfrak{s}, \qquad \mathfrak{M} = \left[\mathfrak{P}\frac{\mathfrak{v}}{c}\right] + \frac{1}{2} N \sum e \left[\mathfrak{s}\frac{\mathfrak{u}}{c}\right].$

(N is the number of molecules in the volume element.) This is the well-known representation of the field from electron theory. [1]) The magnetic effect of the Röntgen current originates in the magnetic moment of the moving polarized molecule above.

4.

Rays. The case that shall be determined in full detail for given initial conditions of the field does not correspond to the one in optics of general pre-existing circumstances. There, on the contrary, one must know the velocity of the "rays" in order to assess the motion and the interference effects that are evoked by the gravitational field.

Throughout the preceding development we have arrived at the formal apparatus that is employed in classical electromagnetic optics. We can therefore cease with the classical methods in our treatment of our present problem completely. [2]) As we have seen, when we choose the Hertzian tensor as the means of representation of the electromagnetic field, we need only to concern ourselves with the wave equation, and need not consider any further conditions. We make the Ansatz for *Z:*

$$Z_l = A_l \cos \chi E - \chi A\, E_l \sin \pi E + a_l$$

$$\Box Z = Z_l{}^l - A_l{}^l \cos \chi E - 2\pi A_l E^l \sin \pi E - \chi A\, E_l{}^l \sin \pi E - \chi^2 A\, E_l E^l \cos \pi E + \sigma_l{}^l$$

and therefore, from (92) and (113):

(146) $W Z = -\chi^2 A E_l E^l \cos \chi E - 2\pi (A_l E^l + \frac{1}{2} AWE) \sin \pi E + \cos \pi E \cdot WA + Wa = 0.$

One understands the term "rays" to mean lines that can bound light complexes and behave independently of each other, when one disregards diffraction phenomena. In order to ignore diffraction, the wavelength must be small compared to the dimensions of

[1] H. A. Lorentz, loc. cit.; W. Dällenbach, Ann. d. Phys. 58, pp. 523, 1919.

[2] Cf., e.g., J. Hadamard, Leçons sur la propagation des ondes, Paris 1902, pp. 331 et seq.; H.A. Lorentz, Abh. über theor. Physik, pp. 415.

the apparatus. In order to formulate this assumption mathematically, we give the parameter χ the dimension and order of magnitude of a reciprocal wavelength λ. The first derivatives of E then have the order of magnitude of direction cosines and indices of refraction, hence, the order of magnitude 1. We would like to further assume the simpler way of speaking that ds (and therefore $d\sigma$, as well) and the coordinates have the dimensions of length. (g_{ik}, g^{ik}, γ_{ik}, γ^{ik}, are then dimensionless). We call a quantity slowly varying [1]) when its relative variation and its (ordinary) derivatives are small compared to the length λ, i.e., $\frac{\lambda P'}{P} \ll 1, \frac{\lambda P''}{P'} \ll 1$, etc., when P, P', P'', etc. have the order of magnitude of the quantities concerned and their derivatives. The assumption that we must make in order to speak of rays then takes the form: A, E', γ_{ik} are slowly varying and a is small compared to A. Furthermore, the coordinates may be chosen in such a way that g_{ik} and u_l are of order (at most) 1. (The same is then true for g^{ik} and u^l.) If γ is this magnitude then we have:

$$(147) \qquad \begin{cases} \dfrac{\lambda A'}{A} \ll 1, \quad E' \sim 1, \quad \lambda E'' \ll 1, \quad \gamma \sim 1, \quad \lambda \gamma' \ll 1, \\ \qquad\qquad\qquad\qquad \dfrac{\lambda \gamma''}{\gamma'} \ll 1, \quad a \ll A. \end{cases}$$

The slowly varying character of A and the smallness of a (which will be rapidly varying) means the neglect of the edge effects, the assumption on E restricts the curvature of the wave fronts (diffraction in the neighborhood of image points), the slowly varying character of γ_{ik} means: the velocities and gravitational fields that are produced by the mechanics and weight effect (Schwerewirkung) of light (which are rapidly varying) are so small that one can neglect their reaction on the propagation of light, and by g_{ik} and u_i, one has to understand only the externally excited (?) (slowly varying) quantities.

The order of magnitude of the first covariant derivative is, from (73), $P' + \gamma' P$, that of the second is thus $(P' + \gamma' P)' + \gamma' (P' + \gamma' P) \sim P'' + \gamma'' P + \gamma' P' + \gamma'^2 P$ (for a scalar, one sets $P = 0$), that of the curvature is, from (82), $\gamma'' + \gamma'^2$, thus, from (92), (113), (146), and (147), that of $W Z$ is:

$$(146') \qquad \frac{A}{\lambda^2} + \frac{1}{\lambda}(A' + \gamma' A + A E') + (A'' + \gamma' A' + \gamma'' A + \gamma'^2 A) + (a)$$

((a) is to be constructed by analogy with the previous expression). When we thus neglect the second derivatives and the products of the first derivatives of the slowly-varying quantities, along with the diffraction term a, then we only have to consider the terms in χ and χ^2 in (146), which, as one can see from (147) and (146′), are of different order of magnitude. For that reason, (146) decomposes into two equations:

[1] H. A. Lorentz, loc. cit.

(148) $$E_l\,E^l = \gamma^{lr}\,\frac{\partial E}{\partial x^l}\,\frac{\partial E}{\partial x^r} = 0\,,$$

(149) $$A_l\,E^l = -\tfrac{1}{2} A\,W E\,.$$

(148) is the Jacobi differential equation of a "mechanical" problem with the Hamiltonian function $H = \frac{1}{2}\,\gamma^{lr}\,p_l\,p_r$, in which $p_l = \frac{\partial E}{\partial x^l}$ is the impulse. From the canonical equations, one has:

(150) $$\frac{dx^i}{d\tau} = \frac{\partial H}{\partial p^i} = \gamma^{ir}\,p_r = p^i = E^i\,, \qquad (\tau = \text{parameter}).$$

Thus, $H = \frac{1}{2}\gamma_{lr}\,\frac{dx^l}{d\tau}\,\frac{dx^r}{d\tau}$, such that the Lagrange equations that arise from the variational problem $\delta\int H\,d\tau$ are the equations:

(151) $$\frac{d^2x^i}{d\tau^2} + \begin{Bmatrix} k\,l \\ i \end{Bmatrix} \frac{dx^k}{d\tau}\,\frac{dx^l}{d\tau} = 0$$

of geodetic lines [1]) (for a manifold with a line element $d\sigma$), and indeed, from (148) and (150):

(152) $$\gamma_{lr}\,\frac{dx^l}{d\tau}\,\frac{dx^r}{d\tau} = 0$$

so one is dealing with *null geodetic lines.*

Due to the slowly-varying character of the γ^{lr}, one can solve (148) in terms of functions of E with slowly-varying derivatives. If we substitute the solution in (149) then, from (150) and recalling the definition (72) of the covariant derivative, we obtain equations of the form:

(153) $$\frac{dA}{d\tau} = \text{linear homogeneous function of } A,$$

i.e., ordinary linear homogeneous differential equations for the components of A, whose coefficients can be summarized in terms of the first and second derivatives of E and three-index symbols (constructed from the γ_{ik}), are also known slowly-varying functions. From (153), one can determine the variation in the amplitude A along null geodetic lines, as long as the initial values are known for some τ. These initial values can be chosen *arbitrarily* from null line to null line, under the restriction that they be slowly varying. This is not to say that the amplitudes are completely independent of each other. Due to the linearity and homogeneity of the equations (153), their vanishing at one point has, as

[1] Cf., e.g., W. Pauli, Jr., loc. cit., no. 15.

a consequence, their vanishing on all of the null lines that go through that point. *Thus, the world-domain of the null geodetic lines can be bounded, and outside of that domain the amplitudes vanish.* From (148) and (150), it follows, moreover, that:

$$\frac{dE}{d\tau} = 0\,, \tag{148´}$$

i.e., the phase E remains constant on each null line.

We must now show that the last of the inequalities (147) can be satisfied. Due to (148) and (149), (146) reduces to:

$$W\,a = -\,W\,A \cdot \cos\chi E. \tag{154}$$

We thus have to solve the inhomogeneous wave equation. This solution can be arrived at by means of Green's theorem (119) in essentially the same way as in the classical theory. The solution will be representable in terms of integrals of the form [1]):

$$a = \int (G \cos\chi E\; W A)_L\, dx^1\, dx^2\, dx^3\,,$$

in which G is a function that, as is the classical theory, becomes infinite in first order at the reference point, and the index L refers to the forward cone that originates at the reference point. The theory of Fourier series teaches us that a can be made arbitrarily small by enlarging χ, i.e., diminishing the wavelength. [2]) We have thus arrived at the result that *the rays in moving bodies are represented by null geodetic lines in a manifold with a line element* $d\sigma^2 = \gamma_{ik}\, dx^i\, dx^k$. From (148´), the ray velocity is therefore equal to the phase velocity along the ray.

From $d\sigma^2 = 0$, it follows, from (26), that $ds^2 = -\left(1-\frac{1}{\varepsilon\mu}\right)(u_i\, dx^i)^2$. The world lines of the rays therefore have a timelike direction for $\varepsilon\mu > 1$. There thus exists a *ray four-velocity*, which, from (15) and (150) (when we again introduce the brackets around the indices), equals:

$$w^i = \frac{E^{(i)}}{\sqrt{-g_{\mu\nu}E^{(\mu)}E^{(\nu)}}}\,, \qquad E^{(i)} = \gamma^{ir}\frac{\partial E}{\partial x^r}. \tag{155}$$

The *validity of the theorem for the addition of velocities* follows from the existence of a four-velocity. [3])

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[1] M. v. Laue, Berl. Ber., 1922, pp. 118.

[2] Cf., e.g., M. Born, Dynamik der Kristalgitter, Appendix.

[3] W. Pauli, Jr., loc. cit., no. 25.

As an example of the propagation of light in a body in the presence of a gravitational field, we take the case of a medium in which there is a centrifugal force field (Harress (?) experiment). By the use of polar coordinates and restriction to the plane, one has:

(156) $$ds^2 = dx_1^2 + x_1^2\, dx_2^2 - \frac{2\omega}{c} x_1^2\, dx_2\, dx_4 - \left(1 - \frac{\omega^2 x_1^2}{c^2}\right) dx_4^2,$$

(x_1 is the radius vector, x_2 is the polar angle, ω is the angular velocity). Where one finds matter, one has $u^1 = u^2 = 0$, $u^4 = 1/\sqrt{-g_{44}} = 1/\sqrt{1 - \frac{\omega^2 x_1^2}{c^2}}$. The covariant components are:

$$u_1 = u_3 = 0, \qquad u_2 = g_{24}\, u^4 = \frac{-\frac{\omega}{c} x_1^2}{\sqrt{1 - \frac{\omega^2 x_1^2}{c^2}}}, \qquad u_4 = g_{44}\, dx^4 = -\sqrt{1 - \frac{\omega^2 x_1^2}{c^2}}.$$

From (18), one finds that the γ_{ik} are:

$$\gamma_{ik} = \frac{\left(1 - \frac{1}{\varepsilon\mu}\frac{\omega^2 x_1^2}{c^2}\right) x_1^2}{1 - \frac{\omega^2 x_1^2}{c^2}}, \qquad \gamma_{24} = -\frac{1}{\varepsilon\mu}\frac{\omega}{c} x_1^2, \qquad \gamma_{44} = -\frac{1}{\varepsilon\mu}\left(1 - \frac{\omega^2 x_1^2}{c^2}\right).$$

The remaining γ_{ik} are equal to the g_{ik}. One will thus have:

(157) $$d\sigma^2 = dx_1^2 + x_1^2 \cdot \frac{1 - \frac{1}{\varepsilon\mu}\frac{\omega^2 x_1^2}{c^2}}{1 - \frac{\omega^2 x_1^2}{c^2}} dx_2^2 - \frac{2\omega}{\varepsilon\mu c} x_1^2 dx_2 dx_4 - \frac{1}{\varepsilon\mu}\left(1 - \frac{\omega^2 x_1^2}{c^2}\right) dx_4^2.$$

By neglecting ω^2, (156) and (157) reduce to:

(156′) $$ds^2 = dx_1^2 + x_1^2\, dx_2^2 - \frac{2\omega}{c} x_1^2\, dx_2\, dx_4 - dx_4^2,$$

(157′) $$d\sigma^2 = dx_1^2 + x_1^2\, dx_2^2 - \frac{2\omega}{\varepsilon\mu c} x_1^2\, dx_2\, dx_4 - \frac{dx_4^2}{\varepsilon\mu}.$$

One derives the vacuum phenomena from $ds^2 = 0$ (Sagnac experiment). [1]) In this case, there is a difference $\Delta t = \frac{4\omega F}{c^2}$ between the travel times for two rays that travel in

[1] P. Langevin, Compt. Rend. 173, 831, 1921; R. Orsay, Phys. Zeitschr. 23, 176, 1922.

opposite directions (F is the surface of motion (?)). If one now exchanges the x_4 in (156´) with $x_4/\sqrt{\varepsilon\mu}$ and ω with $\omega/\sqrt{\varepsilon\mu}$ then one obtains (157´). By this exchange, however, the formulas for Δt go to each other. They are thus valid in a ponderable medium. [1])

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(Received 28 May 1923)

[1] M. V. Laue, Relativitätstheorie I, § 24d.