"Les transformations isogonales en Mécanique," C. R. Acad. Sci. Paris **108** (1889), 446-448.

Isogonal transformations in mechanics

Note by **E. GOURSAT**, presented by Darboux.

Translated by D. H. Delphenich

Consider the motion of a material point in a plane in the case where there exists a force function $U(x, y)$. As one knows, the determination of the trajectories that correspond to the same value h of the *vis viva* constant comes down to the search for a complete integral of the partial differential equation:

$$\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2 = 2\,(U + h)\,. \tag{1}$$

Set $z = x + i\,y$, $Z = X + i\,Y$, and let $z = F(Z)$ be an analytic function of the complex variable Z. One infers from that relation that:

$$x = \varphi(X, Y)\,, \quad y = \psi(X, Y)\,, \tag{2}$$

in which $\varphi(X, Y)$, $\psi(X, Y)$ are two functions of the real variables X and Y whose derivatives verify the conditions:

$$\frac{\partial\varphi}{\partial X} = \frac{\partial\psi}{\partial Y}\,, \quad \frac{\partial\varphi}{\partial Y} = -\frac{\partial\psi}{\partial X}\,. \tag{3}$$

If one makes the change of variables that is defined by formulas (2) in equation (1) then one will see immediately that it will become:

$$\left(\frac{\partial\theta}{\partial X}\right)^2 + \left(\frac{\partial\theta}{\partial Y}\right)^2 = 2\,(U + h)\left[\left(\frac{\partial\varphi}{\partial X}\right)^2 + \left(\frac{\partial\varphi}{\partial Y}\right)^2\right]. \tag{4}$$

The new equation has the same form as the first one, so one will be led to the following general proposition:

If one considers all of the trajectories that correspond to the force function $U(x, y)$ and the value h of the vis viva constant, and one subjects those curves to the isogonal transformation (2) *then the new curves will be the trajectories that correspond to a new force function:*

$$\{U\,[\varphi(X, Y),\ \psi(X, Y)] + h\}\left[\left(\frac{\partial \varphi}{\partial X}\right)^2 + \left(\frac{\partial \varphi}{\partial Y}\right)^2\right]$$

and to the value zero for the vis viva constant.

Suppose, for example, that one has:

$$U = \frac{\alpha}{\sqrt{x^2+y^2}} + \beta, \quad h = 0,$$

in which α and β are two arbitrary constants, and that one has performed the isogonal transformation: $x = X^2 - Y^2$, $y = 2\,X\,Y$. The new value of U will be:

$$U = 4\alpha + 4\beta\,(X^2 + Y^2) .$$

One then passes from the law of Newtonian attraction to the law of attraction that is proportional to distance. One immediately verifies, moreover, that a conic that has one focus at the origin will give a conic that is concentric to the origin under the preceding transformation.

The force function U will remain the same, as well as the isogonal transformation (2), if one varies the *vis viva* constant h, so the transformed curves will not be the trajectories of a moving body for the same force function, in general. Meanwhile, there is a very extensive case in which that it true. Suppose that U verifies the relation:

$$\frac{\partial^2 \log U}{\partial x^2} + \frac{\partial^2 \log U}{\partial y^2} = 0 ,$$

or what amounts to the same thing, that it has the form $U = f(x + i\,y)\,f_0\,(x - i\,y)$, in which f_0 denotes the conjugate function to f. Further set:

$$z_0 = x - i\,y , \qquad Z_0 = X - i\,Y , \qquad \int f(z)\,dz = Z , \qquad \int f_0(z_0)\,dz_0 = Z_0 ,$$

and suppose that one has inferred:

$$z = \varphi(Z) , \qquad z_0 = \varphi_0\,(Z_0) \tag{5}$$

from the last two relations.

If one makes the transformation that is defined by formulas (5) on the equation:

$$\left(\frac{\partial \theta}{\partial x}\right)^2 + \left(\frac{\partial \theta}{\partial y}\right)^2 = 2\,[\alpha\,f(x + i\,y)\,f_0\,(x - i\,y) + \beta] ,$$

in which α and β are two arbitrary constants, then it will become:

$$\left(\frac{\partial\theta}{\partial Y}\right)^2 = \left(\frac{\partial\theta}{\partial Y}\right)^2 = 2[\alpha + \beta\,\varphi'(Z)\,\varphi_0'(Z_0)] \; .$$

As a result, the trajectories that correspond to the force function:

$$f(x + i\,y)\,f_0\,(x - i\,y)$$

will give the trajectories that correspond to the force function $\varphi'(X + iY)\,\varphi_0'(X - iY)$, and that will be true no matter what the *vis viva* constant is.

Upon taking $f(z) = z^m$, one will be led to a result that includes the result that was pointed out above as a special case. If one considers a material point that is subject to the action of a central force that is proportional to the n^{th} power of distance then the two systems of trajectories will correspond to the two values μ, ν of n that are deduced from each other by an isogonal transformation when μ, ν verify the relation:

$$\mu\,\nu + 3\,(\mu + \nu) + 5 = 0 \; .$$

(One must exclude the cases of $\mu = -1$, $\mu = -3$.)

The remarks above also apply to the motion of a material point in space, and in general to all problems in dynamics for which there exist a force function and in which the constraints are independent of time. If $q_1, q_2, \ldots, q_n$ are the variables that fix the position of the system of the system and $2T$ is the homogeneous quadratic form in $q_1', q_2', \ldots, q_n'$, which is equal to the total *vis viva*, then it will suffice to know a transformation such that T is reproduced up to a factor that depends upon only $q_1, q_2, \ldots, q_n$ in order to get a theorem that is analogous to the one that was stated above.

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“Remarque sur la Communication précédente,” C. R. Acad. Sci. Paris 108 (1889), 449-450.

Remark on the preceding communication

By **G. DARBOUX**

Translated by D. H. Delphenich

Consider a material point that is subject to move on a surface whose linear element is defined by the formula:

(1) $$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 .$$

If it is subject to forces that admit a potential:

(2) $$U = F(u, v)$$

then one will have the *vis viva* integral:

$$v^2 = 2(U + h),$$

and from the principle of least action, the search for the trajectories of a moving body will reduce to the search for geodesics on a new surface whose linear element is determined by the formula:

$$ds'^2 = (U + h)(E\,du^2 + 2F\,du\,dv + G\,dv^2) .$$

If one replaces h with β / α, in which β and α denote two constants, then one will replace the preceding expression for ds^2 with the following one:

$$ds^2 = (\alpha U + \beta)(E\,du^2 + 2F\,du\,dv + G\,dv^2),$$

which is linear in α and β. That implies the following consequence:

If one knows how to solve the proposed problem in mechanics for the surface that is defined by formula (1) *and the force function* (2) *then one will also know how to solve it for the surface whose linear element is given by the formula:*

(3) $$ds^2 = U(E\,du^2 + 2F\,du\,dv + G\,dv^2),$$

when the force function is now:

$$U' = \frac{1}{U} = \frac{1}{F(u,v)} . \tag{4}$$

For example, suppose that the original surface is planar. One can take:

$$ds^2 = dx^2 + dy^2 ,$$

in which x, y are the rectangular coordinates of the moving body.

If one knows how to determine the motion of that moving body when the force function is:

$$U = F(x, y)$$

then one will also know how to determine the motion of the moving body when it is subject to remain on the surface whose linear element has the expression:

$$ds^2 = F(x, y)(dx^2 + dy^2) ,$$

in which the force is now $1 / F$.

If one desires that the new surface should be planar then one must take:

$$F = \varphi(x + i\,y)\ \psi(x - i\,y) ,$$

and one will then recover Goursat's theorem ([1]).

The reader can easily attach the following remark to the preceding developments, which we shall be content to state:

Whenever one has the complete solution to a problem in mechanics on a surface that corresponds to a force function that is given, but arbitrary, one will know how to find the geodesic lines on that surface from that fact itself.

([1]) One can also obtain some interesting results by studying the case in which one of the two surfaces is planar and the other is spherical.