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On dual numbers and their application to geometry

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The present paper treats a topic whose essential viewpoint was presented by the author to the Naturforschertage in Meran ([1]) in a talk: "Über gewisse geometrische Anwendungen der dualen Zahlen." The fundamental idea upon which the following development rests consists of a *map of what Study called the dual numbers to the manifold of points of a second-order cone* in a three-dimensional projective space. From the principle of reciprocity, one goes from this map to an analogous map of the dual numbers to the manifold of tangent planes to a conic section in three-dimensional projective space. In particular, one can then substitute the absolute circle at infinity of Euclidian geometry for the aforementioned conic section, and in this way one will obtain a *map of the dual numbers to the manifold of so-called "minimal planes" in Euclidian space.*

Insofar as the aforementioned minimal planes can be represented in a one-to-one way by their real carrier lines, when regarded with a well-defined sense of traversal, that will yield a *map of the dual numbers to* the manifold of real, "oriented" lines, or – following a terminology that was introduced by **E. Study** – *the manifold of* ***spears*** *in Euclidian space.*

The mapping principle ([2]) that will be described shortly in the following now leads, as will be shown, in a direct and informal way to the various geometric applications of the dual numbers that have been made up to now: Namely, on the one hand, there are the applications that **E. Study** developed in his ground-breaking book *Geometrie der Dynamen*, which was rich in new and fruitful ideas, and on the other hand, also the applications that **G. Scheffers** gave to the International Congress of Nathematicians at Heidelberg in 1904. The dual numbers prove themselves to be an entirely necessary instrument for the geometry of spears, since they permit a simple representation of certain finite and infinite (continuous) transformation groups in the manifold of spears in Euclidian space. The connection between the investigations that relate to this and **Ribacour**'s theory of isotropic congruences, as well as the theory of minimal surfaces, seems to be of especial interest.

The concluding section of the present paper presents considerations that relate to the geometry of real and imaginary oriented lines in Euclidian space.

([1]) On 27 Sept. 1905.

([2]) Which was treated at the Naturforschertage; cf., the previous remark. In the discussion that followed the presentation there, the remark was made by a valued colleague that this mapping principle had already been proposed and evaluated by **E. v. Weber** in the Berichten der bayr. Akademie (sächs. Gesellschaft, resp.), as well as in these Monatsheften. That was erroneous, since nothing at all was said about dual numbers in the cited papers [Münchener Sitzungsber. (1904), 447-483, Leipziger Ber. (1903), 384-408. Wiener Monatshefte (1905), 217-229].

I. Map of the dual numbers to the number cone.

§ 1. The concept of dual numbers of the form $u + v\varepsilon$, where $\varepsilon^2 = 0$, will be assumed in what follows. As far as that is concerned, one can refer to **E. Study**'s *Geometrie der Dynamen*, § 23. For such a number, the quantities u and v mean ordinary complex numbers; u will be referred to as the *scalar* part and $v\varepsilon$ as the *vectorial* part of the dual number $u + v\varepsilon$. The addition, subtraction, and multiplication of dual numbers results from the known rules of calculation for ordinary arithmetic; in particular, the commutative and associative laws are true for multiplication. By contrast, the law of ordinary arithmetic that a product can equal zero only when one of its factors vanishes is no longer true for dual numbers. Moreover, the product of dual numbers is equal to zero when the scalar part of at least two of the factors vanishes.

(See pp. 82 of the original article)

Figure 1.

For the clarification of the dual numbers and the calculation operations that one carries out with them, it is now particularly useful for one to map the dual numbers in the following way to the points of a second-order cone in three-dimensional projective space: One intersects such a cone, which is generally arbitrary, with a plane E that does not go through the vertex and arbitrarily chooses three distinct points on the plane that thus arises that will be denoted by p_0, p_1, and p_∞ ; the vertex of the cone will be denoted by p_ω . On the generator of the cone that goes through p_1, one chooses a point $p_{(1+\varepsilon)}$ that is arbitrary, but different from p_1 and p_ω. *The four points p_0, p_1, p_∞, and $p_{(1+\varepsilon)}$ shall be called "fundamental points."* One now chooses a projective coordinate system in the following way: Let the plane E be identical with the base plane $x_0 = 0$ of the coordinate system. Let the plane that is determined by p_0, p_∞, p_ω be identical with the base plane $x_3 = 0$ of the coordinate system. Finally, let the base planes $x_1 = 0$ and $x_2 = 0$ be defined by the tangential planes to the cone along the generators $[p_\infty\, p_\omega]$ and $[p_0\, p_\omega]$. The point $p_{(1+\varepsilon)}$ shall have the coordinates: $x_0 = x_1 = x_2 = x_3 = 1$. The coordinate system is obviously determined completely by the conventions above. The coordinates of the fundamental point and the vertex point p_ω can be represented by the following table:

(1)

	x_0	x_1	x_2	x_3
p_0	0	1	0	0
p_1	0	1	1	1
p_∞	0	0	1	0
$p_{(1+\varepsilon)}$	1	1	1	1
p_ω	1	0	0	0

The equation of the cone reads:

(2) $$x_1\, x_2 - x_3^2 = 0.$$

The coordinate system that was defined here, which is determined uniquely by the fundamental points p_0, p_1, p_∞, $p_{(1+\varepsilon)}$ in the manner that was set down above, may be referred to as the coordinate system that belongs to the aforementioned system of fundamental points.

In order to now map the manifold of dual numbers to the manifold of (real and imaginary) points of the "number cone" $x_1\,x_2 - x_3^2 = 0$, *one associates every dual number* $w = u + v\varepsilon$ *with the point whose coordinates are determined by the proportion:*

(3) $$x_0 : x_1 : x_2 : x_3 = v : 1 : u^2 : u.$$

A one-to-one relationship between the dual numbers w and the corresponding image points (w) on the number cone is described by this proportion. Any two numbers whose scalar parts agree map to the same generator of the cone as points. The numbers 0, 1, and $1 + \varepsilon$ (the first two of which are regarded as dual numbers with vanishing vectorial parts) correspond, as a glimpse at Table (1) will teach us, with precisely the fundamental points that are denoted by p_0, p_1, $p_{(1+\varepsilon)}$.

§ 2. Introduction of dual numbers at infinity.

The images of the finite dual numbers $w = u + v\varepsilon$ – viz., the dual numbers for which the coefficients u and v are finite quantities – do not cover the number cone $x_1\,x_2 - x_3^2 = 0$ completely. As long as one only looks at points of the number cone for which $x_1 \neq 0$, the proportion (3) generally determines finite values for the quantities u and v, and therefore also for the dual number $u + v\varepsilon$ that is associated with them, in any case. Things behave differently for any point of the cone for which $x_1 = 0$. Since one also has $x_3 = 0$ for these points according to equation (2): $x_1\,x_2 - x_3^2 = 0$, these points will fill up a generator Q of the cone; indeed, with hindsight of Table (1), this generator is obviously identical with the line $[p_\infty\, p_\omega]$. Now, for the points of Q (since $x_1 = x_3 = 0$) certainly either x_2 or x_3 is different from zero. In the former case, one learns from the formula $x_2 / x_1 = u^2$ that one can immediately infer from (3) that, at the very least, the quantity u cannot be finite; in the latter case, the formula $x_0 / x_1 = v$ teaches us analogously that the quantity v cannot be finite.

If a point on the cone approaches the generator Q then at least one of the coefficients u, v of the associated number $u + v\varepsilon$ will become infinitely large.

The fact that the image of the finite dual numbers does not cover the number cone completely, but leaves out the generator Q, then depends upon the fact that the manifold of finite dual numbers does not define a closed continuum. However, one can now extend the aforementioned manifold to a closed continuum, such that the various points of the generator Q of the cone correspond to various "infinitely large" dual numbers that one introduces. This extension of the number domain also seems preferable from another standpoint, namely, when one starts with the calculation operation of the division of dual numbers: If $w = u + v\varepsilon$ is a dual number whose scalar part u does not vanish then there will be one and only one dual number w' that satisfies the equation $ww' = 1$ ([1]); one can regard w' as the reciprocal value to w and denote it by $1/w$ or w^{-1}. One has:

$$w' = \frac{1}{w} = \frac{u - v\varepsilon}{u^2}. \tag{4}$$

By contrast, if the scalar part u of the number w equals zero then the equation $ww' = 1$ cannot be solved for w' as long as one restricts oneself to finite dual numbers; however, it is also in this case that one can make this equation soluble when one introduces various infinitely-large dual numbers in the manner that was described above.

Let a point q_h on the generator Q be established by the equation $x_2 / x_0 = - h$, where h is initially a finite, non-zero, ordinary, complex number. We now associate this point q_h with a well-defined, "infinitely-large" dual number, which, as we will likewise justify, we would like to denote by:

$$hJ.$$

If a variable dual number $w' = u' + v'\varepsilon$ changes in such a way that its image point (w') approaches the point q_h without bound then the number hJ that belongs to q_h shall be referred to as the limiting value of w', so we will set:

$$\lim w' = h\,J.$$

In this case, the ratio x_2 / x_0 , and thus, from the proportion (3), also the ratio u'^2 / v', approaches the limiting value $- h$, where the quantities u' and v' both become infinite. This yields the following limiting values for the reciprocal value $1 / w' = w$:

$$\lim \frac{1}{w'} = \lim \frac{1}{u' + v'\varepsilon} = \lim \frac{u' - v'\varepsilon}{u'^2} = - \varepsilon \lim \frac{v'}{u'^2} = \frac{1}{h} \cdot \varepsilon.$$

One simultaneously has the formulas:

$$\begin{cases} \lim w' = hJ \quad \text{and} \\ \lim w = \dfrac{1}{h} \cdot \varepsilon, \end{cases}$$

when w and w' are linked with each other by the relation $ww' = 1$, so one can establish that the dual numbers:

$$h\,J \qquad \text{and} \qquad \frac{1}{h} \cdot \varepsilon$$

have to be reciprocal values to each other, so, in particular, J is regarded as the reciprocal value to ε in this sense ([1]):

([1]) Cf., **E. Study**, *Geometrie der Dynamen*, pp. 197.

$$J = \varepsilon^{-1}.$$

By the introduction of infinitely-large dual numbers of the form $h\,J$ the equation:

$$ww' = 1$$

will also become soluble for the case in which:

$$w = \frac{1}{h}\cdot\varepsilon,$$

and indeed will give only the solution:

$$w' = h\,J$$

for w'.

We shall now examine what sort of modifications must enter in when the quantity h assumes the value infinity or zero.

1. $h = \infty$. In this case, the ratio x_2 / x_0 becomes infinitely large for a point (w') that approaches the point q_h without limit. Now, from (3), this ratio = u'^2 / v', so it follows that $\lim u'^2 / v' = \infty$, so $\lim v' / u'^2 = 0$, where at least the first of the quantities u' and v' is infinite. One then has $\lim 1 / w' = \lim \dfrac{u' - v'\varepsilon}{u'^2} = 0$. For $h = \infty$, q_h becomes identical with the fundamental point p_∞ on our cone. We associate this point of the number cone with an infinitely large dual number that we shall denote by the usual sign:

$$\infty.$$

The unbounded approach of the point (w') to the point p_∞ might then be represented by the formula:

$$\lim w' = \infty.$$

One then simultaneously has the formulas:

$$\begin{cases} \lim w' = \infty, \\ \lim w = 0, \end{cases}$$

if w and w' are coupled by the relation $ww' = 1$. In this sense, the numbers 0 and ∞ can be regarded as reciprocal values.

2. $h = 0$. In this case, the point q_h becomes identical with the vertex p_ω of the cone, because here the ratio x_2 / x_0, and thus also the coordinate x_2, assumes the value 0 for the

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([1]) The misgivings that **E. Study** raised against the insufficiently motivated use of the symbol $J = \varepsilon^{-1}$ by **R. de Saussure** (cf., **E. Study**, *Geom. d. Dyn.*, pp. 208), in no way affect the present use of the symbol here. Here, this symbol merely serves as a notation for the various infinitely large dual numbers that were introduced in a legitimate way.

point q_h , whereas without this – due to q_h being located on the generator Q – the coordinates x_1 and x_3 would vanish. [Cf., Table (1).] We now associate the vertex p_ω with a new infinitely-large number with the symbol ω.

If the image point (w') of a variable dual number w approaches the point p_ω without limit then that situation shall be denoted by $\lim w' = \omega$. The ratio x_2 / x_0 for the point (w'), and thus, the ratio u'^2 / v', approaches the limit of zero here, so one convinces oneself that one simultaneously has the formulas:

$$\begin{cases} \lim w' = \omega, \\ \lim w = \omega, \end{cases}$$

if w and w' are coupled by the relation:

$$ww' = 1.$$

The image points (w) and (w') on the number cone likewise return to the vertex p_ω . In this sense, one can regard the number ω as reciprocal to itself.

We neatly summarize the definition above of infinitely-large dual numbers:

These numbers correspond to the various points of the generator Q [p_∞ p_ω] of the cone in a one-to-one manner, and indeed:

	the point p_∞ corresponds to the number ∞,				
	the point p_ω	“	“	“	ω,
and	any point q_h of Q		“	“	$h\,J$,

when it is established by the equation $x_2 / x_0 = - h$. These numbers may be characterized arithmetically in the following way:

The number ∞ is to be regarded as the limiting value of a dual number $w' = u' + v'\varepsilon$ that grows infinitely large, and for which $\lim u'^2 / v' = \infty$,

The number ω, as the limiting value of a dual number that grows infinitely large, and for which $\lim u'^2 / v' = 0$,

The number $h\,J$, as the limiting value of a dual number that grows infinitely large, and for which $\lim u'^2 / v' = - h$.

The manifold of dual numbers is extended to a closed continuum by the introduction of these infinitely large dual numbers ([1]), *and indeed the latter seems to be related to the points of the number cone in a uniquely-invertible way.*

One defines the continuous change of a dual variable w as a change under which the corresponding image point (w) *changes continuously on the number cone.* In this sense, as one easily recognizes, one has the following formulas:

$$\begin{cases} \lim\limits_{h=0} h\,J = \omega, \\ \lim\limits_{h=\infty} h\,J = \infty. \end{cases}$$

([1]) For this notion, cf., the discussion in **E. Study**’s *Geometrie der Dynamen*, § 27 (pp. 247, *et seq.*).

§ 3. The fractional- linear transformations of a dual variable and the associated collineations of the number cone into itself.

If:

$$w_1 = u_1 + v_1\,\varepsilon$$

and

$$w_2 = u_2 + v_2\,\varepsilon$$

are arbitrary dual numbers, where the scalar part u_2 is non-zero, then the equation:

$$w \cdot w_2 = w_1$$

will always be soluble, and indeed in just one way, by a finite dual number $w = u + v\,\varepsilon$.

It is ([1]):

$$w = w_1 \cdot \frac{1}{w_2} = (u_1 + v_1\,\varepsilon)\,\frac{u_2 - v_2\varepsilon}{u_2^2} = \frac{u_1u_2 + (v_1u_2 - v_2u_1)\varepsilon}{u_2^2}$$

and the "quotient":

$$w = \frac{w_1}{w_2} = \frac{u_1 + v_1\varepsilon}{u_2 + v_2\varepsilon}$$

is constructed most conveniently by multiplying the numerator and denominator by $(u_2 + v_2\,\varepsilon)$:

$$w = \frac{(u_1 + v_1\varepsilon)(u_2 - v_2\varepsilon)}{(u_2 + v_2\varepsilon)(u_2 - v_2\varepsilon)},$$

from which the result above will be inferred.

Let $w = u + v\,\varepsilon$ be a dual variable. One subjects it to a fractional-linear transformation:

(5) $$w' = \frac{aw + b}{cw + d},$$

where the coefficients:

$$\left\{\begin{array}{ll} a = \alpha + \alpha'\varepsilon, & b = \beta + \beta'\varepsilon, \\ c = \gamma + \gamma'\varepsilon, & d = \delta + \delta'\varepsilon \end{array}\right\}$$

[1]) Cf. (4). The validity of the solution written down can be recognized immediately by a test. It is the only one, because the existence of two equations:

$$\left\{\begin{array}{l} w \cdot w_2 = w_1 \text{ and} \\ w' \cdot w_2 = w_1 \end{array}\right\}$$

would have the equation $(w - w')\,w_2 = 0$ as a consequence. Now, since the scalar part u_2 of w_2 is non-zero, by assumption, the latter product can vanish only when $w - w' = 0$.

are finite dual numbers for which the determinant of the scalar parts:

(6) $$D = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix},$$

or, what amounts to the same thing, the scalar part of the substitution determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, shall be assumed to be non-zero:

(7) $$D \neq 0.$$

We first examine the effect of the aforementioned substitution on those finite values of the variables $w = u + v\,\varepsilon$ for which $\gamma u + d$, namely, the scalar part of the denominator $cw + d$ of the fraction in (5) is non-zero. On the basis of the statements above regarding the quotients of dual numbers, that would imply that, from (5), $w' = u' + v'\varepsilon$ would be determined uniquely by w, and indeed one would find that:

$$w' = u' + v'\varepsilon = \frac{(\alpha+\alpha'\varepsilon)(u+v\varepsilon)+(\beta+\beta'\varepsilon)}{(\gamma+\gamma'\varepsilon)(u+v\varepsilon)+(\delta+\delta'\varepsilon)} =$$
$$= \frac{(\alpha u+\beta)+(\alpha v+\alpha' u+\beta')\varepsilon}{(\gamma u+\delta)+(\gamma v+\gamma' u+\delta')\varepsilon},$$

and when one multiplies the numerator and denominator by:

$$(\gamma u + \delta) - (\gamma v + \gamma' u + \delta)\,\varepsilon$$

according to the rule above, that would further imply that:

$$w' = u' + v'\varepsilon = \frac{(\alpha u+\beta)(\gamma u+\delta)+\begin{vmatrix} \alpha v+\alpha' u+\beta' & \alpha u+\beta \\ \gamma v+\gamma' u+\delta' & \gamma u+\delta \end{vmatrix}\varepsilon}{(\gamma u+\delta)^2},$$

and from this, by separating the scalar and vectorial parts:

(8) $$\begin{cases} u' = \dfrac{\alpha u+\beta}{\gamma u+\delta}, \\ v' = \dfrac{(\alpha\gamma)_1 u^2 + [(\alpha\delta)_1 + (\beta\delta)_1]u + (\beta\delta)_1 + Dv}{(\gamma u+\delta)^2}, \end{cases}$$

if $(\alpha\gamma)_1, \ldots$, is a self-explanatory abbreviation for $(\alpha'\gamma - \alpha\gamma')$.

According to (3), the coordinates x_0', x_1', x_2', x_3' of the image point (w') of the number w' on the number cone are proportional to the quantities v', 1, u'^2, u', just as the coordinates of the image point (w) of the number w are proportional to the quantities v, 1, u^2, u.

From (8), one immediately infers the connection between coordinates of the image points (w) and (w'):

$$(9)\qquad \begin{cases} \rho x_0' = Dx_0 + (\beta\delta)_1 x_1 + (\alpha\gamma)_1 x_2 + [(\alpha\delta)_1 + (\beta\gamma)_1]\, x_3, \\ \rho x_1' = \delta^2\; x_1 + \gamma^2\; x_2 + 2\gamma\delta\; x_3, \\ \rho x_2' = \beta^2\; x_1 + \alpha^2\; x_2 + 2\alpha\beta\; x_3, \\ \rho x_3' = \beta\delta\; x_1 + \alpha\gamma\; x_2 + (\alpha\delta + \beta\gamma)\; x_3. \end{cases}$$

As a simple calculation shows, the determinant of this system of equations is equal to D^3, and therefore, on the basis of the assumption that was made in (7), non-zero. Thus, if one next ignores the infinite values of $w = u + v\varepsilon$, as well as ones for which $\gamma u + \delta = 0$, then one can state the theorem:

Under the assumption (7) *that* $D \neq 0$, *the fractional-linear substitution* (5) *will transforms the number cone into itself* **collinearly**, *and indeed, that collineation is certainly not a degenerate collineation, due to the fact that* $D^3 \neq 0$.

As far as the excluded values of w are concerned, we make the following remark: The infinitely-large values of w are mapped to the generator Q, and the ones for which $\gamma u + \delta = 0$ are mapped to another generator $\bar{Q}$. Now, let $w = w_0$ be a dual number whose image point (w_0) lies on one of the two generators Q, $\bar{Q}$; the value of the function $w' = \dfrac{aw+b}{cw+d}$ is undefined, for the moment, but it can be determined by the following assignment:

$$(10)\qquad \left[\frac{aw+b}{cw+d}\right]_{w=w_0} = \lim_{w=w_0} \frac{aw+b}{cw+d}.$$

The justification for this definition comes from the fact that a continuous transformation is produced on the number cone by the substitution (5), from which the existence of the limiting values that were employed above for the definition is established beyond question. By this *a posteriori* definition, one will arrive at the fact that the transformation ($D \neq 0$) that belongs to (5) is **well-defined** in the entire number continuum, **uniquely invertible**, and **continuous**.

The collineations (8) produced on the number cone by the substitutions of the form (5), under the assumption that $D \neq 0$, define a **group** G_6 **of** ∞^6 **collineations**, of which only three of the four dual constants that appear in (5) are essential, since one of these four can be made equal to 1 by dividing the numerator and denominator in the fraction (5) with precisely those constants.

§ 4. The group G_7 of all collineations of the number cone into itself and its representation by dual numbers.

Let C be an arbitrary, non-degenerate collineation of the number cone into itself. The generators of the cone will be permuted projectively amongst themselves by it. In the group G_6, one can certainly find a collineation R that permutes the generators of the cone amongst themselves in entirely the same way (and in a still more precisely-characterized manner).

From (3), the dual numbers $w = u + v\,e$ for which u has a well-defined value, u = const., belong to the points of a certain generator of the cone. One can regard u as a projective parameter of this generator of the number cone. Any projective permutation of these generators, which therefore also belongs to C, may be represented by a fractional-linear transformation of this parameter, perhaps by:

(11) $$u' = \frac{\alpha_1 u + \beta_1}{\gamma_1 u + \delta_1},$$

where $\begin{vmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{vmatrix} \neq 0$, and $\alpha_1, \beta_1, \gamma_1, \delta_1$ are ordinary real or complex numbers.

In order to now arrive at a non-degenerate collineation R in G_6 that produces the same projective permutation of the generators of the cone as (11), one needs only to define the collineation R by the fractional-linear substitution:

(12) $$R, \ldots, w' = \frac{\alpha_1 w + \beta_1}{\gamma_1 w + \delta_1},$$

and then, from (8), the variable $w = u + v\,\varepsilon$ will go to another one $w' = u' + v'\,\varepsilon$ under this substitution, for which one will have:

$$u' = \frac{\alpha_1 u + \beta_1}{\gamma_1 u + \delta_1},$$

in precise agreement with (11). The collineation R ([1]) that belongs to (12) thus produces the same permutation of the generators of the cone as the collineation C, since the permutation of the generators that belongs to C will be represented by (11).

The collineation R^{-1} that is inverse to R, which will be represented by the substitution that is inverse to (12):

(13) $$R^{-1}, \ldots, w' = -\frac{\delta_1 w - \beta_1}{\gamma_1 w - \alpha_1},$$

and, like R, belongs to the group G_6, produces precisely the opposite permutation of the generators as the collineation C. With that, the collineation $(R^{-1}\,C)$ that is composed of

([1]) The collineation R has the characteristic property (which is easily verified by calculation) that the planes of the pencil $\dfrac{\beta_1 x_1 - \gamma_1 x_2 + (\alpha_1 - \delta_1) x_3}{x_0}$ = const. remain fixed under it.

R^{-1} and C will represent a transformation for which every generator remains at rest. (R^{-1} C) will then be central collineation with center p_ω, and it may be represented analytically by a system of equations of the form:

$$\text{(14)} \qquad (R^{-1}\,C)\ \dots \quad \left\{\begin{array}{lllll} \rho x_0' = \lambda_0 x_0 & +\lambda_1 x_1 & +\lambda_2 x_2 & +\lambda_3 x_3, \\ \rho x_1' = & x_1 & & \\ \rho x_2' = & & x_2 & \\ \rho x_3' = & & & x_3. \end{array}\right\}$$

The determinant λ_0 of this system of equations is non-zero, since neither R^{-1} nor C degenerates, and as a result, (R^{-1} C) cannot be a degenerate collineation, either.

We now compare the central collineation (R^{-1} C) with the *central collineations that are contained in the group* G_6 . *The latter define a subgroup* T *of* G_6 . Should the collineation that belongs to (5) be central (with center p_ω), so the generators of the cone are individually at rest, then, with hindsight of the first equation in (8), it will be necessary and sufficient that:

$$\beta = 0, \quad \gamma = 0, \quad \text{and} \qquad \alpha = \delta.$$

With no loss of generality, one can set $\alpha = \delta = 1$. The system of equations (9) is specialized by these assumptions, and goes to:

$$\text{(15)} \qquad T\ \dots\ (\text{subgroup of } G_6)\ \dots \left\{\begin{array}{lllll} \rho x_0' = x_0 & +\beta' x_1 & -\gamma' x_2 & +(\alpha'-\delta')x_3, \\ \rho x_1' = & x_1 & & \\ \rho x_2' = & & x_2 & \\ \rho x_3' = & & & x_3. \end{array}\right\}$$

This system (15) is the general expression for the collineations of the subgroups in G_6 ([1]). One now determines a collineation T_0 in this group by the assignment:

$$\beta' = \frac{\lambda_1}{\lambda_0}, \qquad -\gamma' = \frac{\lambda_2}{\lambda_0}, \qquad \alpha' - \delta' = \frac{\lambda_3}{\lambda_0}.$$

T_0 will then be represented by:

([1]) Geometrically, the collineations of T can be characterized as central collineations for which the "central planes" (viz., any planes that remain point-wise fixed) go through the center p_ω. The equation of such a central plane reads: $\beta' x_1 - \gamma' x_2 + (\alpha' - \delta')\, x_3 = 0$.

$$(16) \qquad T_0 \ldots \left\{ \begin{array}{l} \rho x_0' = x_0 + \frac{\lambda_1}{\lambda_0} x_1 + \frac{\lambda_2}{\lambda_0} x_2 + \frac{\lambda_3}{\lambda_0} x_3, \\ \rho x_1' = x_1 \\ \rho x_2' = x_2 \\ \rho x_3' = x_3. \end{array} \right\}$$

If one now compares T_0 with $(R^{-1} C)$, and thus, (16) with (14), then one will find that $(R^{-1} C)$ can be thought of as composed of T_0 and some collineation A_{λ_0} that is defined by the system:

$$(17) \qquad A_{\lambda_0} \ldots \left\{ \begin{array}{l} \bar{\rho} x_0' = \lambda_0 x_0 \\ \bar{\rho} x_1' = x_1 \\ \bar{\rho} x_2' = x_2 \\ \bar{\rho} x_3' = x_3. \end{array} \right\}$$

One will then have:

$$(18) \qquad (R^{-1} C) = (T_0 \, A_{\lambda_0}).$$

The collineation A_{λ_0} transforms the number cone (2), $x_1 \, x_2 - x_3^2 = 0$, into itself; on the other hand, as long as $\lambda_0 \neq 1$, A_{λ_0} cannot be contained in G_6, since in this case, as a central collineation, it must belong to the subgroup T, and the latter is impossible when one compares (15) with (17).

It follows from (18) that:

$$(19) \qquad C = (R \, T_0 \, A_{\lambda_0}).$$

The most general ([1]) *collineation C that takes the number cone to itself may be composed from a collineation* (RT_0) *of the group* G_6 *and a collineation* A_{λ_0} *of the one-parameter group A:*

$$(20) \qquad A \ldots \left\{ \begin{array}{l} \bar{\rho} x_0' = \lambda_0 x_0 \\ \bar{\rho} x_1' = x_1 \\ \bar{\rho} x_2' = x_2 \\ \bar{\rho} x_3' = x_3. \end{array} \right\}$$

The totality of all C thus defines a group G_7 *of* ∞^7 *collineations.*

The question now arises of how the collineations of the group G_7 may be represented in dual numbers. Since the collineations of the group G_6 can be represented by an equation of the form (5), it remains for us to represent the one-parameter group A in the dual numbers. If one envisions the association of points of the number cone and the associated dual numbers that is described by the proportion (3) then one recognizes that

([1]) I.e., non-degenerate.

the collineation A_λ of the number cone into itself that acts by way of (20) takes the image point of a number $w = u + v\,\varepsilon$ to the image point of the number $w' = u + \lambda\, v\,\varepsilon$. *If one defines a symbolic operation factor* $\mathfrak{A}_\lambda$ by:

$$\mathfrak{A}_\lambda\,(u + v\,\varepsilon) = u + \lambda\, v\,\varepsilon \tag{21}$$

then the collineation A_λ of the number cone can be expressed in terms of dual numbers as follows:

$$A_\lambda, \ldots, w' = \mathfrak{A}_\lambda(w). \tag{22}$$

The **most general** ([1]) *collineation C of the number cone into itself will then be represented by:*

$$C, \ldots, w' = \mathfrak{A}_\lambda\left(\frac{aw+b}{cw+d}\right). \tag{23}$$

In particular, if $\lambda = 1$ then (23) will go to (5), and the collineation C will then belong to the subgroup G_6 .

As is easy to see, G_6 is an invariant subgroup. Since G_7 arises from G_6 by the addition of the group A (eq. 20), the proof of this can be carried out simply: Let S be any collineation in G_6, and let A_λ be any collineation of the group A; one then shows that the transformation ($A_\lambda^{-1}\ S\,A_\lambda$) is again contained in G_6 . One can think of S as represented by a substitution of the form (5): $w' = \dfrac{aw+b}{cw+d}$. Now, with hindsight of (8) [and (6)], the latter equation when one multiplies the vector parts of all dual numbers, namely, the numbers $w' = u' + v'\varepsilon$, $w = u + v\,\varepsilon$, $a = \alpha_1 + \alpha_2\,\varepsilon$, $b = \beta_1 + \beta_2\,\varepsilon$, $c = \gamma_1 + \gamma_2\,\varepsilon$, $d = \delta_1 + \delta_2\,\varepsilon$, with the same factor λ. Equation (5) is thus replaceable with:

$$\mathfrak{A}_\lambda(w') = \frac{\mathfrak{A}_\lambda(a)\mathfrak{A}_\lambda(w)+\mathfrak{A}_\lambda(b)}{\mathfrak{A}_\lambda(c)\mathfrak{A}_\lambda(w)+\mathfrak{A}_\lambda(d)}. \tag{24}$$

If one now sets:

$$\begin{cases} \mathfrak{A}_\lambda(w) = z \quad \text{and} \\ \mathfrak{A}_\lambda(w') = z', \end{cases} \tag{25}$$

and further introduces the abbreviations:

$$\mathfrak{A}_\lambda(a) = \bar{a}\,, \qquad \mathfrak{A}_\lambda(b) = \bar{b}\,, \ldots$$

then one will have the following relation between z and z':

([1]) (non-degenerate)

(26) $$z' = \frac{\bar{a}z + \bar{b}}{\bar{c}z + \bar{d}}.$$

From (25), the transition from the image point (z) to the image point (w) corresponds to the collineation A_λ^{-1} of the group A.

From (5), the transition from the image point (w) to the image point (w') corresponds to a collineation S of the group G_6 .

From (25), the transition from the image point (w') to the image point (z') corresponds to the collineation A_λ of the group A.

One therefore goes from (z) to the point (z') by the collineation:

$$(A_\lambda^{-1} \; S \, A_\lambda);$$

on the other hand, from (26), the transition from (z) to (z') corresponds to a certain collineation of the group G_6, so (A_λ^{-1} $S\,A_\lambda$) will belong to G_6 ; Q. E. D.

The group G_4 of all central collineations in the group G_7 that are represented by (14) is an invariant subgroup in G_7, as immediately comes to light.

The group T that is defined by (15), which takes the form of the intersection of the groups G_6 and G_4, thus defines a likewise invariant three-parameter subgroup of G_7 .

It remains for us to show the manner by which any collineation of the group G_6 (G_7, resp.) can be established geometrically by the association of a number of corresponding points.

Let t_1, t_2, t_3 and t_1', t_2', t_3' be two point triples on the number cone, such that the planes [t_1, t_2, t_3] and [t_1', t_2', t_3'] do not go through the vertex. *There will then certainly exist one and only one collineation S in G_6 that takes the points of the first triple to the corresponding points of the second triple.*

Proof: The projective permutation of the generators of the cone by the desired collineation is determined completely by the association of both point triples. Now, let R be a collineation of the type represented by (12), through which the same permutation of the generators of the cone is effected. The points t_1, t_2, t_3 will correspond to the collineation R of three points $\bar{t}_1$, $\bar{t}_2$, $\bar{t}_3$ that lie on the same generators as the points t_1', t_2', t_3'.

If one now takes the desired collineation S in the form ($R\,P$); i.e.:

$$S = (R\,P),$$

then $P = (R^{-1}\,S)$ must be contained in G_6, since R^{-1} and S belong to this group G_6 . P is then determined by the fact that the points $\bar{t}_1$, $\bar{t}_2$, $\bar{t}_3$ correspond to the points t_1', t_2', t_3', which lie on the same generators, resp., so it is necessarily a central collineation; since it must now also belong to G_6, it must be contained in the group T (eq. 15).

However, there is one and only one collineation in T that takes the points $\bar{t}_1$, $\bar{t}_2$, $\bar{t}_3$ to the points t_1', t_2', t_3'. P, and therefore $S = (R\ P)$ is thus determined uniquely.

A collineation of the group G_7 can be established geometrically in the following way: Let t_1, t_2, t_3 and t_1', t_2', t_3' be two point triples on the number cone, as above, let s_3 be a point of $[p_\omega, t_3]$ that is different from the vertices p_ω and t_3, and likewise let s_3' be a point of $[p_\omega, t_3']$. *There will then exist one and only collineation* Σ *of the group* G_7 *that takes the quadruple* t_1, t_2, t_3, s_3 *into the quadruple* t_1', t_2', t_3', s_3'.

The proof is connected with the preceding one: One again puts the desired collineation Σ into the form $(R\ P)$. R shall again be a collineation of the group G_6 that takes the quadruple t_1, t_2, t_3, s_3 to the quadruple $\bar{t}_1$, $\bar{t}_2$, $\bar{t}_3$, $\bar{s}_3$, and indeed, in such a way that the points of the latter quadruple lie on the same generators as t_1', t_2', t_3', s_3'.

$P = (R^{-1}\ \Sigma)$ belongs to the group G_7 and is a central collineation, so it belongs to the group G_4 (eq. 14). Since the quadruple $\bar{t}_1$, $\bar{t}_2$, $\bar{t}_3$, $\bar{s}_3$ and the quadruple t_1', t_2', t_3', s_3' must correspond at P, from (14), the collineation P, and therefore also Σ, is determined uniquely.

§ 5. Coordinate transformation. Double ratio.

The opposite assignment of dual numbers and the associated points of the number cone was defined in § 1 by the proportion (3) after establishing a projective coordinate system by means of the so-called fundamental points p_0, p_∞, p_1, $p_{(1+\varepsilon)}$, the last two of which lie on a generator of the cone, while the plane E that is laid through p_0, p_∞, p_1, does not go through the vertex p_∞.

If one chooses the fundamental points in another way – e.g., $\mathfrak{p}_0$, $\mathfrak{p}_\infty$, $\mathfrak{p}_1$, $\mathfrak{p}_{(1+\varepsilon)}$ – while observing analogous conditions, and one again defines the projective coordinates (x_0', x_1', x_2', x_3') that belong to that system of fundamental points in the same way as in § 1, and one then assigns the dual number $w' = u' + v'\,\varepsilon$ to those points whose coordinates in the new system are (x_0', x_1', x_2', x_3') by means of the proportion:

$$x_0' : x_1' : x_2' : x_3' = v' : 1 : u'^2 : u', \tag{3'}$$

which is analogous to (3), then one will obtain an association of points on the number cone and dual numbers that is different from the one in § 1; the former association will be denoted by II, in contrast to the latter, which will be denoted by I. The image point of dual number under the assignment II can be denoted by enclosing the dual number in question in square brackets.

The question now arises of *the manner in which the dual numbers:*

$$w = u + v\,\varepsilon \qquad \text{and} \qquad w' = u' + v'\,\varepsilon$$

that belong to the same point p of the cone according to the associations I *and* II *are connected with each other.* One finds that *a relationship* of precisely the same form as (23):

$$w = \mathfrak{A}_\lambda \left(\frac{aw+b}{cw+d}\right) \tag{23′}$$

exists between these numbers, in which the operator $\mathfrak{A}_\lambda$ has the meaning that is represented by (21):

$$\mathfrak{A}_\lambda (u + v\,\varepsilon) = u + \lambda\, v\, \varepsilon\,. \tag{21}$$

Proof: A system of homogeneous, linear equations exists between the coordinates x_i (i = 0, 1, 2, 3) of a point *p* in the original coordinate system and the coordinates x_i' of the same points in the new system. The equation of the number cone in the new system must again have the form:

$$x_1' x_2' - x_3'^2 = 0. \tag{2′}$$

One can now also regard the x_i' as the coordinates of a point *p*′, when referred to the original coordinate system. If one chooses *p* to be on the number cone then, from (2′), the point *p*′ must also lie on the number cone, and the relationship between *p* and *p*′ will be a collineation *C* of the number cone *C* to itself. It then follows from (3) and (3′) that *the dual number w*′ *that belongs to p under the association* II *is identical to the number that belongs to the point p*′ *under the association* I. However, from §4, the latter is linked with the number *w* that belongs to the point *p* under the association I through an equation of the form (23), which proves the assertion above.

One can infer a consequence of this proof: The point *p*′ has the same coordinates in the original coordinate system that the point *p* did in the new coordinate system. Now, the fundamental points p_0, p_∞, p_1, $p_{(1+\varepsilon)}$ have the same coordinates in the original coordinate system that the fundamental points $\mathfrak{p}_0$, $\mathfrak{p}_\infty$, $\mathfrak{p}_1$, $\mathfrak{p}_{(1+\varepsilon)}$ have in the new one; cf., Table (1). If one then shifts the point *p* to the points $\mathfrak{p}_0$, $\mathfrak{p}_\infty$, $\mathfrak{p}_1$, $\mathfrak{p}_{(1+\varepsilon)}$, in sequence, then *p*′ will shift to the points p_0, p_∞, p_1, $p_{(1+\varepsilon)}$, in sequence. Now, since *p*′ corresponds to the point *p* under the collineation *C*, it will follow that the new fundamental points $\mathfrak{p}_0$, … will go to the old ones p_0, … under the collineation *C*. From § 4, the collineation *C* will already be determined completely with that.

If one represents *C* in terms of dual numbers in the sense of the association I then that must yield equation (23′) precisely, so one can then *obtain the connection between the dual numbers w and w*′ *that belong to the same point p of the number-cone under the associations* I *and* II *quite simply by determining that collineation C on the number-cone that takes the new fundamental points to the old ones, and that collineation will then be represented by dual numbers in the sense of the association* I, and *the quantities w and w*′ in the equation that then arises *are interpreted as the dual numbers that belong to the point p* (*under* I *and* II).

In particular, if the collineation C that takes the new fundamental points to the old ones belongs to the group G_6 then one can refer to the point-quadruples that are defined by the old and new fundamental points as *equivalent relative to the group* G_6. In that case, in place of equation (23), which represents C in the sense of the association I, one will find the simpler equation:

$$w' = \frac{aw+b}{cw+d}, \tag{5′}$$

which represent the connection above between the numbers that are associated with a point p under the associations I and II in the present special case.

Let w_1, w_2, w_3, w_4 be four dual numbers, about which, one might initially assume:

Condition α): viz., that they are finite and that the scalar parts of the two differences $(w_3 - w_2)$ and $(w_4 - w_1)$ are non-zero.

We understand the *double ratio* $\{w_1, w_2, w_3, w_4\}$ to mean the expression:

$$\{w_1, w_2, w_3, w_4\} = \frac{(w_3 - w_1)(w_4 - w_2)}{(w_3 - w_2)(w_4 - w_2)}, \tag{27}$$

which is well-defined and uniquely-determined, on the basis of the assumption (α). If condition (α) is not fulfilled then one can initially think of the numbers w_1, … as variable, and then let their given values become unbounded, in such a way that condition (α) will indeed remain true during the limiting process. *If the double ratio that is defined by* (27) *tends to a definite limiting value under all such limiting processes then it will be equal to the double ratio of the four numbers.* One will then have the definition:

$$\{w_1, w_2, w_3, w_4\} = \lim_{\bar{w}_i = w_i} \frac{(\bar{w}_3 - \bar{w}_1)(\bar{w}_4 - \bar{w}_2)}{(\bar{w}_3 - \bar{w}_2)(\bar{w}_4 - \bar{w}_2)}. \tag{28}$$

If the limiting processes above yield different limiting values according to the way in which they are present then the double ratio shall be regarded as indeterminate, and the scope of its indeterminacy can be characterized by the different values that can be reached. – If the image points of the four numbers w_i all lie on a plane that goes through the vertex p_∞ of the number cone then the quadruple of those image points, and likewise the quadruple of the four numbers w_i, shall be called *special*. *For non-special number-quadruples, the double ratio is determined completely and uniquely by the definition* (27), (28) *above.*

If one transforms the four numbers w_i, which we will assume do not define a special triple, into the numbers w_i' by a fractional-linear substitution (5) $w' = \frac{aw+b}{cw+d}$ then the double ratio of those numbers will be preserved, which can be proved in the usual way that one uses for ordinary numbers.

If one maps the w_i in a well-defined way onto the number cone (perhaps by the association I) then one can also define the *double ratio of the four points* (w_i) *on the number cone, in which one must, however, make reference to the association* I *explicitly.* If one goes from the association I to another association II whose fundamental points are equivalent to those of I relative to the group G_6 (cf., pp. 17) then the dual numbers that belong to the four points of the number cone above will transform by a fractional-linear substitution of the form (5) = (5′), so the double ratio of the four points above will also be the same for the new association II as its was for the association I. However, if the quadruple of the new fundamental points under the new association II is *no longer equivalent* (relative to the group G_6) to the original quadruple of fundamental points that belongs to I then the w_i will transform according to formula (23′) of this paragraph $w' = \mathfrak{A}_\lambda\left(\frac{aw+b}{cw+d}\right)$, and one will determine the double ratio of the transformed numbers from:

(29) $$\{w_1', w_2', w_3', w_4'\} = \mathfrak{A}_\lambda\{w_1, w_2, w_3, w_4\}.$$

That is, the double ratio of the four points of the cone that belongs to the association II *emerges from the double ratio that belongs to* I *by leaving its scalar part unchanged, while multiplying the vectorial part with a factor l, in which the factor l is independent of the position of the four points* (w_i) *on the number cone*, and depends upon only the relationship (23′) that exists between the associations I and II.

If one chooses four points on the number cone in such a way that they do not define a special quadruple and forms their double ratios relative to all possible associations I, II, … then the scalar part of the those double ratios will be *the same*, but the vectorial component will change by a factor λ under the transition from one association to another. If the number w belongs to a point of the number cone under one association I, while the number w' belongs to it under another II, such that w and w' are connected by (23′) then the factor λ that appears in that formula will specify precisely the *factor* that the vector part of *any* point-quadruple will take on under the transition from I to II. Since the factor λ depends upon *only* the associations I and II, we would like to refer to it as the *modulus* of the association II relative to the association I. We call the totality of all associations that have the same modulus relative to I a *class of associations* K_λ . The double ratio of a point-quadruple of the number cone has the same value relative to all associations of the same class under the transition from an association of the class K_λ to an association of the class K_μ, while the vectorial part of the double ratio changes merely by the factor μ / λ for any point-quadruple.

The *scalar part* of the double ratio of a point-quadruple is the same for the associations of all classes, so one finds the value:

$$\frac{(u_3-u_1)(u_4-u_2)}{(u_3-u_2)(u_4-u_1)}$$

for it, in which the u_i are the scalar parts of the dual numbers that belong to the points of the quadruple. Now, since, from § 4, the u_i are the projective parameters of the cone generators that are determined by the individual points of the quadruple, it follows that the scalar part of the double ratio of four points of the number cone is *identical* (for all associations) with the ordinary projective double ratio of the associated cone generators.

As far as the *vectorial part* of the double ratio is concerned, one can choose it to be arbitrary ([1]), but *non-zero*, for *one* quadruple of points that do not lie in a plane. In that way, one determines a class of associations, and from it one will then establish the double ratio of all quadruples on the number cone. *The vectorial part of the double ratio* of a non-special point-quadruple *has the value zero* if and only if its four points lie *in a plane.* (The proof is achieved by transforming the quadruple by a collineation of G_6 in such a way that three mutually-distinct points of it will lie in the plane $E = [p_\infty, p_1, p_0]$. The double ratio of the transformed quadruple is a pure scalar quantity if and only if the fourth point also lies in the plane E after the transformation.)

§ 6. Synectic functions of dual variables.

In agreement with the terminology that was introduced by **E. Study** (*Geom. d. Dyn.*, pp. 199) a dual variable $w' = u' + v'\,\varepsilon$ can be referred to as a synectic function of another dual variable $w = u + v\,\varepsilon$ if and only if u' and v' are *analytic functions* of u and v, in such a way that the *differential quotient* ([2]):

$$\frac{dw'}{dw} = \frac{du' + \varepsilon\,dv'}{du + \varepsilon\,dv}$$

is independent of the differentials that are found in the denominator (except for possible exceptional points), and thus depends upon only u and v:

$$\frac{dw'}{dw} = \frac{du' + \varepsilon\,dv'}{du + \varepsilon\,dv} = F(u, v). \tag{30}$$

One finds that:

$$\frac{dw'}{dw} = \frac{\dfrac{\partial u'}{\partial u}du + \dfrac{\partial u'}{\partial v}dv + \varepsilon\left(\dfrac{\partial v'}{\partial u}du + \dfrac{\partial v'}{\partial v}dv\right)}{du + \varepsilon\,dv}$$

$$= \frac{\left(\dfrac{\partial u'}{\partial u} + \varepsilon\dfrac{\partial v'}{\partial u}\right)du + \left(\dfrac{\partial u'}{\partial v} + \varepsilon\dfrac{\partial v'}{\partial v}\right)dv}{du + \varepsilon\,dv},$$

and sees from this that the differential quotient dw' / dw is independent of the differentials du and dv that appear in the denominator if and only if:

([1]) It is arbitrary because the vectorial part of the double ratio proves to be different for the various classes of associations.

([2]) This differential quotient is initially defined only when the differential du that appears in the denominator is non-zero, so the scalar part of dw must be non-zero.

$$\frac{\partial u'}{\partial v} + \varepsilon \frac{\partial v'}{\partial v} = \varepsilon \left(\frac{\partial u'}{\partial u} + \varepsilon \frac{\partial u'}{\partial v} \right),$$

so

(31) $$\left\{ \frac{\partial u'}{\partial u} = 0 \quad \text{and} \quad \frac{\partial v'}{\partial v} = \frac{\partial u'}{\partial u} \right\}.$$

One finds from this, by integration:

(32) $$\left\{ u' = \varphi(u) \quad \text{and} \quad v' = \psi(u) + v \cdot \frac{d}{du} \varphi(u) \right\},$$

that:

(33) $$w' = u' + v' \varepsilon = f(w) = f(u + v \varepsilon)$$

are equations that characterize synectic functions of $w = u + v\,\varepsilon$.

One can also define such a synectic function by a *power series* in the dual variables w with dual coefficients $a_k = \alpha_k + \alpha'_k \varepsilon$:

$$\begin{aligned} w' = \sum a_k w^k &= \sum (\alpha_k + \alpha'_k \varepsilon)(u + v\,\varepsilon) \\ &= \sum \alpha_k u^k + \varepsilon \left\{ \sum \alpha'_k u^k + v \cdot \sum k\, \alpha_k u^{-1} \right\} \\ &= \varphi(u) \quad + \varepsilon \left\{ \psi(u) \quad + v \cdot \frac{d}{du} \varphi(u) \quad \right\}, \end{aligned}$$

in which we have set $\varphi(u) = \sum \alpha_k u^k$ and $\psi(u) = \sum \alpha'_k u^k$, and in which $w' = f(w)$ is shown to be a synectic function of w in the previous sense. (Naturally, the convergence of the powers series for $\varphi(u)$ and $\psi(u)$ is assumed.)

A synectic function will induce a transformation of the number cone into itself that takes the points of a cone generator to the points of a generator ([1]), so, from (32), it will follow from u = const. that u' = const.

One can establish such a transformation by giving the transformed points on the cone that correspond to the points of an analytic curve on the cone, for which the type of association of points must be mediated by analytic functions. Namely, from (32), the functions $\varphi(u)$ and $\psi(u)$ will then be determined from the given association.

In what follows, special use will be made of synectic functions that correspond to the ordinary function of analysis – e.g., sin w, cos w, etc., e^w, $l(w)$ – by allowing dual values for the arguments w. One must think of those functions as being defined by the associated power series, such that one will have, e.g., $\sin w = \sin(u + v\,\varepsilon) = \sin + \varepsilon\, v \cos u$, etc. (Cf., **E. Study**, *Geom. d. Dyn.*, § 23.)

([1]) The two generators will be related to each other perspectively by the transformation.

§ 7. Stereographic projection.

One obtains an especially simple kind of map of dual numbers to the points of a second-order cone by the following process:

One interprets the coefficients *u* and *v* of a dual number $w = u + v\,\varepsilon$ as the ordinary rectangular coordinates of a point in a plane, describes a cylinder of rotation of arbitrary radius (say, 1) around the *v*-axis ($u = 0$) of the coordinate system, determines one of the two points on the surface of the cylinder that projects rectangularly onto the chosen *uv*-plane at the coordinate origin *O* ($u = 0$, $v = 0$), and denotes it by p_∞. The point with the coordinates *u*, *v* in the chosen plane will then be regarded as projected centrally from the point p_∞ on the surface of the cylinder to a point *p*, and it can then be regarded as the image point (*w*) of the dual number $w = u + v\,\varepsilon$.

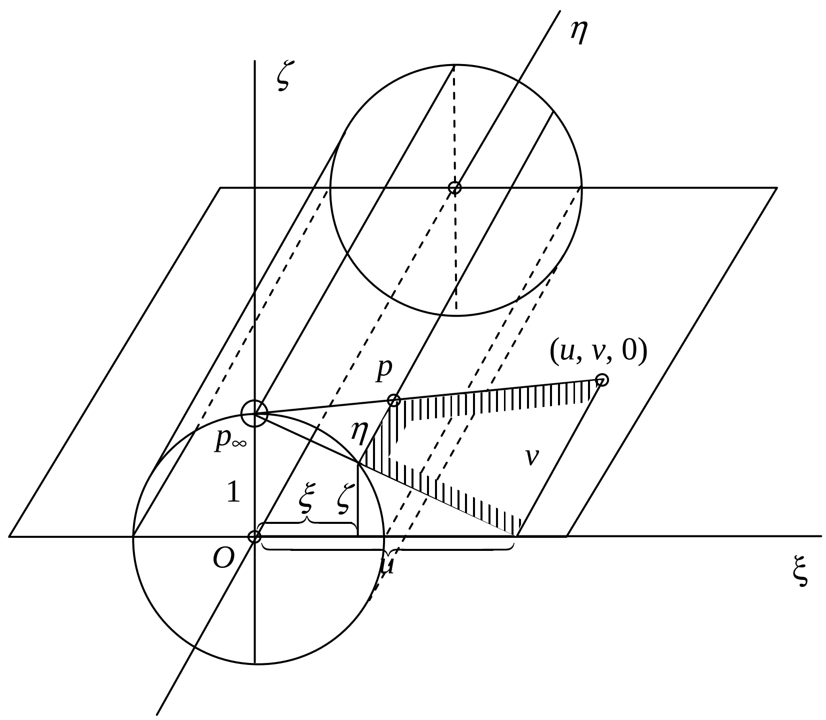

Figure 2.

We choose Op_∞ to be the third axis ζ of a spatial orthogonal coordinate system (ξ, η, ζ) that extends the previous one, and whose ξ and η axes coincide with the *u*, *v* axes, resp., such that the equation of the cylinder reads:

$$\xi^2 + \zeta^2 = 1.$$

One finds by calculation that the coordinates ξ, η, ζ of the point $p = (w)$ have the proportions:

$$\left\{\begin{array}{lcccc} \eta : & (1-\zeta): & (1+\zeta): & \xi \\ = v : & 1 & : & u^2 & : & u \end{array}\right\}.$$

One can now regard four quantities that are proportional to η, $1 - \zeta$, $1 + \zeta$, ξ as the homogeneous projective coordinates:

$$x_0\,,\quad x_1,\quad x_2,\quad x_3\,,$$

of the point $p = (w)$. The association between the dual number $w = u + v\,\varepsilon$ and the associated image point $p = (w)$ of the cylinder will then be mediated by the proportion:

$$x_0 : x_1 : x_2 : x_3 = v : 1 : u^2 : u, \tag{3}$$

with which, it is established that the *present map is a (metrically-specialized) special case of the general map that was treated in* § 1.

The present map is completely *analogous to the Riemann map of the ordinary complex numbers to the number cone by means of stereographic projection.*

II. Map of the dual numbers to the minimal planes in Euclidian space, and then to the "spears" in them.

§ 8. Principle of the map.

From the principle of reciprocity, one can go from the map of dual numbers to the points of a second-order cone in Part I to an entirely analogous map of those numbers to the tangent planes to a second class curve, namely, a conic section. One can choose the latter to be, in particular, the sphere circle at infinity – viz., the absolute circle of Euclidian geometry – and thus arrive at a map of dual numbers to the manifold of "minimal planes" in Euclidian space, each of which can be represented in a well-known way by the real line that carries it when it is provided with a sense of traversal, and thus by a "spear" ([1]).

We now establish that map in a well-defined way. We choose a rectangular coordinate system with origin O, and x, y, and z axes, and denote the coordinates of a plane P in it by T, U, V, W ([2]). By analogy with (3), one can then assign the dual number $w = u + v\,\varepsilon$ as the image of that plane, whose coordinates are determined by the proportion:

$$\left\{\begin{array}{r} (-T):(V+iU):(V-iU):(-iW) = v:1:u^2:u \\ i=\sqrt{-1}, \end{array}\right\} \tag{34}$$

then the quantities on the left of the equal sign themselves – or even better, any quantities that are proportional to them – can be regarded as homogeneous, projective coordinates of the plane P, by which, the analogy with (3) comes to light. The proportion (34) will be replaced with the following one:

([1]) Cf., **E. Study**, "Über Nicht-Euklidische und Liniengeometrie," in the Jahresbericht der deutschen Mathematikervereinigung (1902), pp. 319.

([2]) $-T/U$, $-T/V$, $-T/W$ are the sections of that plane along the coordinate axes.

(35) $$T : U : V : W = -v : i\,\frac{u^2-1}{2} : \frac{u^2+1}{2} : iu.$$

The planes $P(T, U, V, W)$ that belong to the various dual numbers according to (35) satisfy the equation:

(36) $$U^2 + V^2 + W^2 = 0,$$

and are thus minimal planes. *The manifold of all minimal planes appears to be related to the continuum of dual numbers (which, from* § 2, *is closed) in a single-valued and invertible way.*

We now address the problem of constructing the associated image plane P to any given dual number $w = u + v\,\varepsilon$; i.e., of finding the associated spear. The direction of the latter spear depends upon only u and can be found by the following construction:

$$u = u' + i\,u'' \quad (i = \sqrt{-1}),$$

in which u' and u'' mean ordinary real numbers. One determines the rectangular system of coordinates of the point $(u', u'', 0)$ in the xy-plane, and projects it stereographically from $M(\xi, \eta, \zeta)$ onto the points S_0 (0, 0, –1) on the sphere $x^2 + y^2 + z^2 = 1$. The radius OM that belongs to the latter point of the sphere will then represent the *direction* of the desired spear.

Proof: One finds the following expression for the coordinates of the point M:

(37) $$M\left(\xi = \frac{2u'}{1+u'^2+u''^2}, \quad \eta = \frac{2u''}{1+u'^2+u''^2}, \quad \zeta = \frac{1-u'^2-u''^2}{1+u'^2+u''^2}\right),$$

and convinces oneself by calculation of the validity of the equation:

(38) $$\xi\left(i\,\frac{u^2-1}{2}\right) + \eta\left(\frac{u^2+1}{2}\right) + \zeta\,(iu) = 0,$$

from which, it will follow, when one recalls (35), that:

$$\xi\,U + \eta\,V + \zeta\,W = 0\,;$$

i.e., the real direction OM belongs to the minimal plane $P(T, U, V, W)$ and thus gives the direction of the spear that belongs to P ([1]).

If the vectorial part v of a dual number $w = u + v\,\varepsilon$ is equal to zero, so w reduces to u, then, from (35), the associated minimal plane P, and therefore, the associated spear, as well, will go through the origin O. The aforementioned spear is then determined

([1]) The radius OM' that is opposite to OM gives the direction of the spear that belongs to the minimal plane $\bar{P}$, which is conjugate-imaginary to P.

completely, since its direction would be given by just that, and will then be represented by *OM* itself.

If the scalar part u of w is equal zero then, from (35), the associated minimal plane is parallel to the z-axis and cuts out pieces $2iv$ and $2v$ from the x and y axes. Let:

$$v = v' + i\,v'',$$

so the spear of that plane will pierce the xy-plane at the point $(-\,2v', 2v'', 0)$ and have the direction of the positive z-axis, as one learns from the construction of *OM* in the present case.

We shall now determine the minimal plane that is associated with a general, finite, dual number $w = u + v\,\varepsilon$ $(u \neq 0, v \neq 0)$ of finite magnitude, and therefore its *spear*. One can then put w into the form:

$$w = u\,[1 + \varepsilon\, i\,(f' + i\,f'')],$$

such that one will have:

$$v = u\,i \cdot (f' + i\,f'')\,.$$

From (35), the minimal plane that belongs to w cuts out the piece $(f' + i\,f'') = -\,v\,/\,u\,i$ from the z-axis. One finds the associated spear in the following way: One first constructs the spherical radius *OM*, from the above, which will give the direction of the desired spear, then displaces the piece f'' parallel to the z-axis to O_1M_1, and then rotates O_1M_1 around the axis *OM* through a right angle in the positive sense to O_2M_2 , in which the positive sense is to be regarded as the sense that makes the rotation of *Ox* coincide with *Oy* relative to the *Oz* axis. Finally, one displaces O_2M_2 along the piece f' that is parallel to the z-axis to $O'M'$. *$O'M'$ will then represent the desired spear.*

From (35), the dual number ∞ corresponds to a minimal plane whose spear will be given by the *negative* z-axis.

The dual infinitely-large numbers of the form hJ correspond to minimal planes whose spears are parallel to the negative z-axis.

The dual, infinitely-large number ω corresponds to the plane at infinity.

§ 9. The transformation of the minimal planes that is represented by a fractional-linear substitution of *w*.

One can think of a fractional-linear substitution of the dual variables w [cf. (5)] as being represented by an equation of the form:

$$\mathfrak{a}_0\,(w' - w) + \mathfrak{a}_1 \cdot i\,(ww' - 1) + \mathfrak{a}_2 \cdot (ww' + 1) + \mathfrak{a}_3 \cdot i\,(w + w') = 0 \tag{39}$$

in which the:

$$\mathfrak{a}_k = \alpha_k + \beta_k\,\varepsilon \qquad (k = 0, 1, 2, 3) \tag{40}$$

are finite dual real or imaginary numbers. Namely, equation (39) is equivalent to:

(41) $$w' = \frac{(\mathfrak{a}_0 - \mathfrak{a}_3\, i)\, w + (\mathfrak{a}_1\, i - \mathfrak{a}_2)}{(\mathfrak{a}_1\, i + \mathfrak{a}_2)\, w + (\mathfrak{a}_3\, i + \mathfrak{a}_0)},$$

which is an equation that has the form:

(5) $$w' = \frac{aw + b}{cw + d},$$

in which the coefficients *a*, *b*, *c*, *d* are given by:

(42) $$\left\{ \begin{array}{ll} a = \mathfrak{a}_0 - \mathfrak{a}_3\, i, & b = \mathfrak{a}_1\, i - \mathfrak{a}_2, \\ c = \mathfrak{a}_1\, i + \mathfrak{a}_3, & d = \mathfrak{a}_3\, i + \mathfrak{a}_0, \end{array} \right\}$$

or the $\mathfrak{a}_k$ are given by:

(42′) $$\left\{ \begin{array}{ll} \mathfrak{a}_0 = \frac{1}{2}(a + d), & \mathfrak{a}_1 = -\frac{i}{2}(b + c), \\ \mathfrak{a}_2 = \frac{1}{2}(c - b), & \mathfrak{a}_3 = \frac{i}{2}(a - d). \end{array} \right\}$$

The scalar part of the determinant of the substitution (5), which was denoted by *D* in (6), has the value:

(43) $$D = \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2$$

here.

The fractional-linear substitutions for which $D \neq 0$ correspond to nondegenerate collinear transformations in the manifold of minimal planes. A calculation that is analogous to the one that was presented in Part I (§ 3) will yield the following representation of the collineation that belongs to (39) in plane coordinates *T*, *U*, *V*, *W*:

(44) $$\left\{ \begin{array}{rllll} \rho T' = & a_{00} T + & a_{01} U + & a_{02} V + & a_{03} W, \\ \rho U' = & & a_{11} U + & a_{12} V + & a_{13} W, \\ \rho V' = & & a_{21} U + & a_{22} V + & a_{23} W, \\ \rho W' = & & a_{31} U + & a_{32} V + & a_{33} W, \end{array} \right\}$$

in which the coefficients a_{jk} are determined from:

$$(45)\quad \left\{\begin{array}{l|l|l} a_{00} = \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2, & & \\ a_{11} = \alpha_0^2 + \alpha_1^2 - \alpha_2^2 - \alpha_3^2, & a_{11} = 2(\alpha_2\alpha_3 + \alpha_0\alpha_1), & a_{32} = 2(\alpha_2\alpha_3 - \alpha_0\alpha_1), \\ a_{22} = \alpha_0^2 + \alpha_1^2 + \alpha_2^2 - \alpha_3^2, & a_{31} = 2(\alpha_3\alpha_1 + \alpha_0\alpha_2), & a_{13} = 2(\alpha_3\alpha_1 - \alpha_0\alpha_2), \\ a_{33} = \alpha_0^2 - \alpha_1^2 - \alpha_2^2 + \alpha_3^2, & a_{12} = 2(\alpha_1\alpha_2 + \alpha_0\alpha_3), & a_{21} = 2(\alpha_1\alpha_2 - \alpha_0\alpha_3), \\ \multicolumn{3}{c}{a_{01} = 2(\alpha_2\beta_3 - \alpha_3\beta_2 + \alpha_0\beta_1 - \alpha_1\beta_0),} \\ \multicolumn{3}{c}{a_{02} = 2(\alpha_3\beta_1 - \alpha_1\beta_3 + \alpha_0\beta_2 - \alpha_2\beta_0),} \\ \multicolumn{3}{c}{a_{03} = 2(\alpha_1\beta_2 - \alpha_2\beta_1 + \alpha_0\beta_3 - \alpha_3\beta_0).} \end{array}\right\}$$

Equations (44), (45) are completely identical to the parametric representation of the motions in Euclidian space that was presented by **E. Study** in *Geom. d. Dyn.*, § 21, pp. 174, etc., by equations (3), pp. 175, and (10), (11), pp. 176 there. In fact, the cited equations seem to be coupled with the condition $\sum_{k=0}^{3} \alpha_k \beta_k = 0$ there; however, that condition proved to be inessential in the cited work in § 25, pp. 120.

The transformations that are represented by (39) then prove to be *identical with the motions in Euclidian space*, and in fact, *the real values of the parameters* α_k , β_k *correspond to real motions.* The six-parameter group that is defined by the motions corresponds to precisely the group G_6 of transformations of the number cone in Part I. *The* $\mathfrak{a}_k$ (40) *shall be referred to as the homogeneous dual coordinates of the motion* (39) [(39), resp.].

The transformation of the group *A* of Part I (20), which is expressed in terms of dual numbers by the operation (21) $\mathfrak{A}_\lambda (w) = \mathfrak{A}_\lambda (u + v\,\varepsilon) = u + \lambda\, v\, \varepsilon$, will correspond to the similarity transformation about the origin here.

$$(46)\quad \left\{\begin{array}{llllll} \rho T' = & \lambda T & & & \\ \rho U' = & & U & & \\ \rho V' = & & & V & \\ \rho W' = & & & & W \end{array}\right\}.$$

The general transformation $w' = \mathfrak{A}_\lambda \left(\frac{aw+b}{cw+d}\right)$ [eq. (23)] represents the *group of all similarity transformations of Euclidian space*, which is precisely analogous to G_7 .

In what follows, as we did already, we will now refer to the results in the aforementioned ground-breaking book of **E. Study**. In fact, the foregoing explanations will give us the means to derive those results in a simple, self-explanatory way, so in the interests of brevity, we shall omit presenting them.

§ 10. The connection between the Plücker coordinates of a real line and the dual numbers that belong to the spears of that line.

Let l be a real line at infinity with Plücker coordinates:

$$X_{01}, X_{02}, X_{03}, X_{23}, X_{31}, X_{12}$$

(the first three of which are not all zero), which are assumed to be real.

From **E. Study**, *Geom. d. Dyn.*, § 23, pp. 200, etc., the "ray coordinates" of l, X_1, X_2, X_3 are defined by:

$$\{\rho X_1 = X_{01} + X_{23}\,\varepsilon,\ \rho X_2 = X_{02} + X_{31}\,\varepsilon,\ \rho X_3 = X_{03} + X_{12}\,\varepsilon\}, \tag{47}$$

in which $\rho = \sigma + \tau\varepsilon$ ($\sigma \neq 0$) is a dual proportionality factor. Here, σ and τ shall be assumed to be ordinary *real* numbers.

One can think of a real motion with the screw axis l as being represented by four homogeneous $\mathfrak{a}_k$ (40), the last three of which $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{a}_3$ are proportional to the quantities X_1, X_2, X_3 that appear in (47). (Cf., **Study**, *Geom. d. Dyn.*, § 25) The minimal planes that go through l remain fixed under such a motion. The dual numbers that belong to them can then be found from (39), when one sets $w' = w$ in it, instead of $\mathfrak{a}_1$, $\mathfrak{a}_2$, $\mathfrak{a}_3$ (X_1, X_2, X_3, resp.), and solves the quadratic equation in w that it yields. One finds that:

$$X_1 \cdot i\,(w^2 - 1) + X_2 \cdot (w^2 + 1) + X_3 \cdot (2wi) = 0. \tag{48}$$

The two roots ([1]) *of that equation w and w^* are those dual numbers that are assigned to the minimal plane that goes through l.* Solution gives:

$$w\ (w^*, \text{resp.}) = \frac{X_1 + iX_2}{X_3 \pm \sqrt{X_1^2 + X_2^2 + X_3^2}} = -\frac{X_3 \mp \sqrt{X_1^2 + X_2^2 + X_3^2}}{X_1 - iX_2}, \tag{49}$$

in which taking the square root will yield the following formula:

$$\sqrt{m + n\varepsilon} = \sqrt{m}\sqrt{1 + \frac{n}{m}\varepsilon} = \sqrt{m}\left(1 + \frac{n}{2m}\varepsilon\right).$$

([1]) The existence of those roots is ensured by the fact that two minimal planes go through l. In general, the case in which the line l is parallel to the z-axis demands special treatment. In that case, the scalar parts of X_1 and X_2 will vanish, and formula (49) will provide only *one finite* root $w_1 = \frac{X_1 + iX_2}{2X_3} = v_1\,\varepsilon$. As a passage to the limit will show, when one recalls (51), the minimal planes that go through l will then correspond to the dual numbers w_1 and $w_2 = -1 / \bar{w}_1$, the first of which has a vanishing scalar part, from the above, while the latter is infinitely large. The latter number is to be regarded as the second root of (48) in the present case.

In (49), the quantities X_1, X_2, X_3 are, by definition, free of *i*; an important relationship between the roots *w* and w^* can be inferred from that fact.

Let:

(50) $$w = \mathfrak{u}' + i\,\mathfrak{u}'', \quad \text{in which} \quad \mathfrak{u}' = u' + v'\,\varepsilon \quad \text{and} \quad \mathfrak{u}'' = u'' + v''\,\varepsilon,$$

be such that $\mathfrak{u}'$ *and* $\mathfrak{u}''$ *are free of i, and thus "real-dual"* ([1]), and let:

(50′) $$\bar{w} = \mathfrak{u}' - i\,\mathfrak{u}''.$$

The other root w^* is then given by:

(51) $$w^* = -\frac{1}{\mathfrak{u}' - i\mathfrak{u}''} = -\frac{1}{\bar{w}}.$$

In this, one finds the solution to the problem of finding the two dual numbers that belong to the minimal planes through any real line *l* that is given by its **Plückerian** line (**Study** ray, resp.) coordinates.

We now turn to the inverse problem of *analytically determining* (i.e., representing by its coordinates) *the real carrier line of the minimal plane that is associated with a given dual number* $w = \mathfrak{u}' + i\,\mathfrak{u}''$ (50), or what amounts to the same thing, *the carrier line of the spear that belongs to w.*

From (51), one can determine from the number *w* the w^* whose spear belongs to the same line as *w*, but has the opposite sense. One can pose the quadratic equation whose roots are *w* and w^* and bring that equation into the form (48); the corresponding coefficients of the latter will then yield the ray coordinates X_1, X_2, X_3 of the desired carrier line. The following path leads to that goal even faster: One substitutes $w = \mathfrak{u}' + i\,\mathfrak{u}''$ in (49); by separating the real imaginary parts, one will then arrive at the equations:

(49′) $$\mathfrak{u}' = \frac{X_1}{X_3 \pm \sqrt{X_1^2 + X_2^2 + X_3^2}}, \quad \mathfrak{u}'' = \frac{X_2}{X_3 \pm \sqrt{X_1^2 + X_2^2 + X_3^2}},$$

from which, the ratios $X_1 : X_2 : X_3$ can be calculated. That will yield the fact that when ρ' is understood to mean a proportionality factor that should be taken to be real-dual (i.e., free of *i*):

(52) $$\rho' X_1 = 2\mathfrak{u}', \quad \rho' X_2 = 2\mathfrak{u}'', \quad \rho' X_3 = 1 - \mathfrak{u}'^2 - \mathfrak{u}''^2$$

will be the representation of the desired line in ray coordinates. [In order to obtain its **Plückerian** line coordinates, one must (cf., **Study**, *Geom. d. Dyn.*, § 23, pp. 200) substitute the values X_1, X_2, X_3 that were found in (47) and then determine the dual proportionality factor ρ that appears in it in such a ways that the X_{ik} that (47) implies will

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([1]) One should confer formulas (49′) (cf., *infra*) for the dependency of the quantities $\mathfrak{u}'$ and $\mathfrak{u}''$ on X_1, X_2, X_3 .

satisfy the **Plücker** relation $X_{01}\,X_{23} + X_{02}\,X_{31} + X_{03}\,X_{31} = 0$. The X_{ik} thus-calculated are then the desired **Plückerian** line coordinates.]

Formulas (52) are initially derived only under the assumption that *w* is a finite dual number with non-vanishing scalar parts. The general validity of those formulas can be established easily by a limiting argument.

§ 11. Map of the spears to the "dual points" of a sphere.

It follows from (52) that:

$$\pm\rho'\sqrt{X_1^2 + X_2^2 + X_3^2} = 1 + \mathfrak{u}'^2 + \mathfrak{u}''^2,$$

in which the sign on the left is taken to agree with the sign of the root in (49′).

If one introduces the notation:

$$\pm\sqrt{X_1^2 + X_2^2 + X_3^2} = X_0 \tag{53}$$

for the square roots that one obtains in (49′) and (49) then one will get:

$$\rho' X_0 = 1 + \mathfrak{u}'^2 + \mathfrak{u}''^2. \tag{54}$$

The four quantities X_0, X_1, X_2, X_3 are then related by:

$$X_1^2 + X_2^2 + X_3^2 = X_0^2, \tag{55}$$

and are connected with $\mathfrak{u}'$ and $\mathfrak{u}''$ rationally by means of formulas (52) and (54) whose solution in terms of $\mathfrak{u}'$ and $\mathfrak{u}''$ will yield the equations:

$$\mathfrak{u}' = \frac{X_1}{X_3 + X_0}, \qquad \mathfrak{u}'' = \frac{X_2}{X_3 + X_0}, \tag{56}$$

which are, in essence, identical to (49′).

The four dual quantities X_0, X_1, X_2, X_3 can be regarded as the homogeneous coordinates of a "dual point," in such a way that the ordinary Cartesian coordinates of the latter relative to the coordinate system *Oxyz* will be given by the ratios X_1 / X_0, X_2 / X_0, X_3 / X_0. *Any dual number* $w = \mathfrak{i} + i\,\mathfrak{u}$, *and therefore any spear* (w), *as well, belongs to a "dual point" in this way* [from (52) and (54)] *as the image point. From (55), all of those points are to be regarded as lying on a sphere* (of radius 1 about *O*), which we would like to call the "image sphere." The spear (w) that belongs to the number *w* lies on a line whose **Study** ray coordinates are X_1, X_2, X_3, and the direction of the spear will be established by adjoining the root in (53); i.e., X_0. This kind of representation of a spear is then one that **Study** used as a basis in the book that we have cited repeatedly (cf., § 23, in particular). In it, it was shown that the metric relationships between spears are identical with the metric relationships between the "dual points" of a sphere. By a corresponding definition of the dual angle between two spears, one will arrive at the fact that this angle agrees with the angle (that is defined by the usual analytical formula) that determines the associated "dual points" on the sphere. Since the identities upon which the analytical treatment of spherical trigonometry are based will not lose their validity when the quantities in them are assumed to be dual numbers, spherical trigonometry can be carried over to spherical triangles whose angles are "dual points" with no further assumptions, and therefore, to triples of spears, since from the above, the same metric will be true for them that is true for spherical triangles of the latter kind.

Since the concept of a dual angle should be employed hereinafter, this is probably the place to briefly recall its definition. Let l and m be two spears, and let t be one of the two spears that lie on the shortest transversal to l and m. Let ϑ be the angle through which one must rotate l around t in the positive sense, and let ϑ_1 be the distance through which one must displace l in the positive sense along t if one is to make l overlap with m; $\Theta = \vartheta + \vartheta_1 \varepsilon$ *will then be referred to as the dual angle between the spear l and m*. That angle $\Theta = \sphericalangle lm$ is then determined modulo 2π when the direction of t is fixed on the shortest transversal to l and m. If one leaves the latter direction undetermined then the sign of Θ will remain undetermined; i.e., one can consider Θ, as well as $(-\Theta)$ to be the value of the angle $\sphericalangle (l, m)$.

From the above, any two spears of a triple of spears (l, m, n) will determine a dual angle (after one subsequently establishes the sense of direction of the shortest transversals $\bar{l}$, $\bar{m}$, $\bar{n}$ between m and n, n and l, and l and m); we would like to call these three angles ([1]) the *angles* of the triple. If $\bar{\bar{m}}$ is the spear that is opposite to $\bar{m}$ then the dual angle $\sphericalangle \bar{n}\,\bar{\bar{m}}$ shall be referred to as the *complementary angle* (Ger. Beiwinkel) $\sphericalangle l = \sphericalangle lmn$ of the triple (l, m, n), and the complementary angles $\sphericalangle m$ and $\sphericalangle n$ are defined analogously ([2]). *The angle and complementary angle of a triple of spears are completely analogous to the sides and angles of a spherical triangle and are connected with them by the same equations.* (Cf., **Study**, *Geom. d. Dyn.*, § 24, pp. 209, 210.)

One can think of exhibiting the map that was developed above from the spear (w) to the "dual points" of a sphere by way of a stereographic projection: If $w = \mathfrak{u}' + i\,\mathfrak{u}''$ ([3]) then one projects (stereographically) the "dual point" $(x = \mathfrak{u}', y = \mathfrak{u}'', z = 0)$ from the point $(x = 0, y = 0, z = -1)$ onto the sphere $x^2 + y^2 + z^2 = 1$; i.e., onto the sphere that is represented by (55). As a simple calculation will imply, the "dual point" thus-obtained will have precisely the homogeneous coordinates X_0, X_1, X_2, X_3 that are defined by (52) and (54), so it will be the "dual point" of the sphere above that belongs to (w).

A fractional-linear transformation:

$$w' = \frac{aw+b}{cw+d}, \tag{5}$$

([1]) $\sphericalangle lm$, $\sphericalangle mn$, and $\sphericalangle nl$.

([2]) The definitions in question are obtained by cyclic permutations of the symbols l, m, n; in the definition of $\sphericalangle l$, the direction of the shortest transversal to $\bar{n}$ and $\bar{\bar{m}}$ is thought of as being fixed by the spear l itself.

([3]) Cf., (50).

in which one might have $a = a_1 + i\, a_2$, etc., and a_1, a_2 , etc., represent real-dual numbers, effects a transformation of the quantities X_0, X_1, X_2, X_3 that belong to w into the corresponding quantities X_0', X_1', X_2', X_3', resp. In fact, one expresses the latter as *homogeneous, linear functions* of the latter with *real-dual coefficients;* i.e., ones that are free of i; the transformation above will then represent a *real-dual collineation* on the sphere. One illuminates the fact that this is true most directly in the special case for which $\mathfrak{u}'$, $\mathfrak{u}''$, a_1, a_2, b_1, b_2, … have no vectorial parts. In that case, the collineation above will be an ordinary real collineation that is represented by a system of equations with ordinary, real numbers for its coefficients, and those coefficients will be entire, rational, real functions of the a_1, a_2, b_1, b_2, … If one now allows real-dual values of $\mathfrak{u}'$, $\mathfrak{u}''$, a_1, a_2, b_1, b_2, … whose vectorial parts do not vanish then the coefficients will assume real-dual values, although nothing will change in regard to the form of the equations that express the X_i' in terms of the X_i ([1]). The map of the spear (w) to the dual points of the sphere in question is, in a certain sense, an extended counterpart to the map from the spear to the tangential planes of the absolute circle, which was the starting point of our considerations about spears here. Both mapping principles seem to be coupled by means of the dual number w, except that one of them prefers the decomposition $w = u + v\,\varepsilon$, while the other one prefers the decomposition $w = \mathfrak{u}' + i\,\mathfrak{u}''$.

Remark: Along with those collineations of the image sphere that correspond to a fractional-linear substitution $w' = \dfrac{aw+b}{cw+d}$, there is yet another family of real-dual collineations that is represented by $w' = \dfrac{a\bar{w}+b}{c\bar{w}+d}$, if $\bar{w} = \mathfrak{u}' - i\,\mathfrak{u}''$ is the dual number that is conjugate-imaginary to w. The latter family is to be regarded as an extension of the so-called indirect circle conversions of the image sphere in the dual domain, just as the group of collineations of the image sphere to itself that belongs to $w' = \dfrac{aw+b}{cw+d}$ is regarded as an extension of the group of direct circle conversions in the dual domain. The real-dual collineations of the image sphere can then be referred to as direct (indirect, resp.) real-dual circle conversions on the image sphere. The transformation of the minimal plane that belongs to $w' = \dfrac{a\bar{w}+b}{c\bar{w}+d}$ is an "anti-collinear" transformation, with the terminology of **Segre** ([2]).

§ 12. Defining a spear by two dual numbers.

Let s be any spear, and let w be the associated dual number. The spear s, along with the spears of the negative y-axis Oy' and the positive z-axis Oz', defines a triple of spears

(1) The argument that is employed corresponds to **E. Study**'s transition principle (*Geom. d. Dyn.*, § 25).
(2) Cf., say, Mathematische Annalen, Jahrgang 1892.

([1]). Let the complementary angle of that triple that belongs to *Oz* be $\Phi = \varphi + \zeta\,\varepsilon$, and let the angle of *s* with *Oz* be $\Theta = \vartheta + \rho\,\varepsilon$; in this, these dual angles shall be chosen in such a way that $0 \leq \varphi < 2\pi$, $0 \leq \vartheta \leq \pi$ ([2]). Let *t* be a spear on the shortest transversal from *Oz* to *s*, in such a way that the rotation of *Oz* in the direction of *s* (along the shortest path) will possess the positive sense of rotation when one considers that rotation from the side from which *t* points. The spear *Ox* will emerge from *Oz* and *Oy′* by the same construction that gave the spear *t* from *Oz* and *s*. *t* can be obtained from the definition of the dual angle $\Phi = \varphi + \zeta\,\varepsilon$ and $\Theta = \vartheta + \rho\,\varepsilon$ in such a way that one rotates the spear *Ox* around the *Oz* axis in the positive sense through an angle φ and then displaces along *Oz* in the positive sense by the quantity ζ. *s* will be obtained when one rotates the spear *Oz* around the *t* axis in the positive sense through the angle ϑ and then displaces along *t* in the positive sense by the quantity ρ. When one is given the dual angles Θ and Φ, the spear *s* = (*w*) will be determined uniquely in that way, and it will be easy to construct.

That raises the question of *how the angles* Θ *and* Φ *are connected with the dual number w that belongs to the spear s* = (*w*). In the special case where the vectorial parts of Θ and Φ vanish, so the stated angles reduce to ϑ and φ, the spear *s* will go through the origin *O*, and the construction that was given in § 8 of the spear that belongs to a pure scalar quantity will give the dual number that belongs to *s* as the pure scalar value $\tan\frac{\vartheta}{2}e^{\varphi i}$. From the oft-cited transition principle, one would expect that in the general case, the *dual number w that belongs to s will be represented by the formula:*

$$w = \tan\frac{\Theta}{2}e^{\Phi i}. \tag{57}$$

This formula can be verified easily as follows: One thinks of *v* as represented in the form:

$$w = u\,[1 + \varepsilon\, i\,(f' + i\,f'')]$$

and constructs the associated spear (*w*) = *s* from § 8. It is easy to infer from the construction that the angles Θ and Φ that belong to *s* have the values:

$$\Phi = \varphi + f'\varepsilon, \quad \Theta = \vartheta - f''\sin\vartheta \cdot \varepsilon.$$

If one develops tan Θ / 2 in a **Taylor** series then one will get:

$$\tan\frac{\Theta}{2} = \tan\frac{\vartheta - (f''\sin\varphi)\varepsilon}{2} = \tan\frac{\vartheta}{2} + \frac{1}{2\cos^2\frac{\vartheta}{2}}(-f''\sin\vartheta)\,\varepsilon,$$

since the higher powers of ε give zero, and furthermore:

([1]) The spears *l*, *m*, *n* in § 11 are *Oz*, *Oy′*, *s*, resp., in the present case. The shortest transversals $\bar{n}$ and $\bar{\bar{m}}$ in § 11 correspond to the spears *Ox* and *t*, here (see below).

([2]) Θ corresponds to the angle ∢ *nl* in § 11, and Φ, to the complementary angle ∢ *l*.

$$\tan\frac{\Theta}{2} = \tan\frac{\vartheta}{2} - f''\tan\frac{\vartheta}{2}\,\varepsilon = \tan\frac{\vartheta}{2}\,(1 - f''\varepsilon).$$

Analogously, one finds by an application of **Taylor**’s theorem:

$$e^{\Phi i} = e^{(\varphi + f'\varepsilon)\,i} = e^{\varphi i} + e^{\varphi i} f'\varepsilon\, i = e^{\varphi i}\,(1 + f'\varepsilon\, i).$$

It follows further that:

$$\tan\frac{\Theta}{2} e^{\Phi i} = \tan\frac{\vartheta}{2}\; e^{\varphi i}\,(1 - f''\varepsilon)\,(1 + f'\varepsilon\, i) = u\,[1 + \varepsilon\, i\,(f' + i\,f'')] = w,$$

which was to be proved.

w is assumed to be a finite dual number whose scalar part is non-zero in the proof. The general validity of formula (57) is proved by passing to the limit.

If one subjects a spear $s = (w)$ to a screwing motion around the Z-axis, when one rotates it around Oz in the positive sense through the angle γ and likewise displaces along Oz in the positive sense through the quantity γ_1, then one will multiply the associated dual number w from (57) by the factor $e^{\Gamma i}$, where $\Gamma = \gamma + \gamma_1\,\varepsilon$. If one then subjects two spears $s = (w)$ and $s_1 = (w_1)$ to such a screwing motion then the ratio $w_1 / w_2 = q$ of the associated dual numbers will remain unchanged. Conversely, if one screws the spear $s = (w)$ around Oz as axis in any way then the spear that belongs to the number $qw = w_1$, when q is a constant (dual) quantity, will move in such a way that it participates in the screwing of s around Oz as if it were rigidly coupled with s.

§ 13. Infinitesimal spear triples.

One might now understand δw_1^* and δw_2^* to mean two infinitely-small dual numbers ([1]); the associated spears s_1^* and s_2^* differ infinitely little from the spear that the positive Z-axis represents, which will be denoted by s_0^*, and together with it, they define an *infinitesimal triple of spears* (s_0^*, s_1^*, s_2^*).

The dual angles Θ and Φ might have the values Θ_1, Φ_1 for s_1^*; correspondingly, one might get the values Θ_2, Φ_2 for s_2^*. The quantities Θ_2 and Φ_2 (*but not* Φ_1 and Φ_2) are then infinitely small in any event, and if one recalls (57) then:

$$\delta w_1^* = \frac{\Theta_1}{2} e^{\Phi_1 i}, \qquad \delta w_2^* = \frac{\Theta_2}{2} e^{\Phi_2 i},$$

so

([1]) The scalar parts of δw_1^*, δw_2^*, and likewise those of δw_1, δw_2, are assumed to be non-zero.

$$\frac{\delta w_2^*}{\delta w_1^*} = \frac{\Theta_2}{\Theta_1} e^{(\Phi_2 - \Phi_1)i}.$$

If $\frac{\delta w_2^*}{\delta w_1^*} = q$ then the dual number q can be put into the form $q = R\, e^{\Phi i}$, where R and Φ are real-dual quantities – i.e., they are free of i – and in fact R and Φ are determined completely when one assumes that the scalar part of R is not negative, while the scalar part of Φ is non-negative and smaller than 2π. One has: $R\, e^{\Phi i} = \frac{\Theta_2}{\Theta_1} e^{(\Phi_2 - \Phi_1)i}$, which will imply that $\Theta_2/\Theta_1 = R$, $\Phi_2 - \Phi_1 \equiv \Phi \pmod{2\pi}$. Now, $\Phi_2 - \Phi_1$ is nothing but the complementary angle $\sphericalangle s_1^*\, s_0^*\, s_2^*$ of the triple (s_0^*, s_1^*, s_2^*); Φ_1 and Φ_2 are the angles $\sphericalangle s_1^*\, s_0^*$ and $\sphericalangle s_2^*\, s_0^*$ of that triple [1]. One then has, moreover: $\frac{\sphericalangle s_2^*\, s_0^*}{\sphericalangle s_0^*\, s_1^*} = R$, $\sphericalangle s_1^*\, s_0^*\, s_2^* \equiv \Phi \pmod{2\pi}$. *The complementary angle* $\sphericalangle s_1^*\, s_0^*\, s_2^*$ *of the spear triple in question that belongs to* s_0^* *and the ratio of the angle that is subtended at* s_0^* *are determined completely by the quotient* $\frac{\delta w_2^*}{\delta w_1^*} = q$.

Now, let $s_0 = (w_0)$ be an arbitrary spear, let $s_1 = (w_0 + \delta w_1)$ and $s_2 = (w_0 + \delta w_2)$ be two spears that differ from it infinitely little [1], which then defines an infinitesimal spear-triple in general position, along with s. One can think of the latter triple (s_0, s_1, s_2) as being brought into its position in such a way that one subjects another triple (s_0^*, s_1^*, s_2^*) for which s_0^* coincides with the spear Oz (as above) to a real motion. Obviously, the angle and complementary angle will not differ under this motion, so they will correspondingly be the same for both triples. If w_0^*, $w_0^* + \delta w_1^*$, $w_0^* + \delta w_2^*$ are the dual numbers that belong to s_0^*, s_1^*, s_2^* (in which one must obviously have $w_0^* = 0$) then it must follow from § 9 that the numbers w, $w + \delta w_1$, $w + \delta w_2$ that belong to s_0, s_1, s_2, resp. will emerge from those numbers by a fractional-linear transformation of the form $w' = \frac{aw+b}{cw+d}$. The value of the differential quotient dw' / dw for $w = 0$ must now coincide with $\delta w_1 / \delta w_1^*$, on the one hand, and $\delta w_2 / \delta w_2^*$, on the other [2]. The last two quotients will then be equal. However, it follows immediately from $\frac{\delta w_1}{\delta w_1^*} = \frac{\delta w_2}{\delta w_2^*}$ that $\frac{\delta w_2}{\delta w_1} = \frac{\delta w_2^*}{\delta w_1^*}$, say, $= R\, e^{\Phi i}$. One will then have:

$$\frac{\sphericalangle s_2\, s_0}{\sphericalangle s_0\, s_1} = \frac{\sphericalangle s_2^*\, s_0^*}{\sphericalangle s_0^*\, s_1^*} = R$$

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(1) With corresponding orientations for the shortest transversals to the spears of the triple.
(2) This follows immediately from the fact that w' is a "synectic" function of w (cf., § 6).

[if one recalls the special position of the triple (s_0^*, s_1^*, s_2^*)] and likewise:

$$\sphericalangle\ s_1\, s_0\, s_2 = \sphericalangle\, s_1^*\, s_0^*\, s_2^* = \Phi.$$

One then also finds the ratio of the angle that is subtended at s_0 and the complementary angle s_0 at s_0 for *an arbitrary infinitesimal spear-triple* (s_0, s_1, s_2) *from the quotient* $\frac{\delta w_2}{\delta w_1} = R\, e^{\Phi i}$ *in the same way as before.*

§ 14. Dual-conformal transformations of the spear space. (An important infinite group of spear transformations)

If one dual variable depends upon another according to the equation:

(58) $$w' = \varphi(w),$$

in which $\varphi(w)$ is a synectic function (cf., § 6), then every spear (w) will be assigned to another corresponding spear (w'). The transformation of spear space thus-defined might be referred to as a *synectic transformation.* The synectic transformations define an infinite group.

The transformations that we speak of possess the important characteristic property of being dually-conformal; i.e., *any infinitesimal spear-triple* (s_0, s_1, s_2) (except for possible exceptional ones) *will go to another* (s_0', s_1', s_2') *whose complementary angles are equal to the corresponding angles of the former under a transformation of that group, while its angles will be equal to the corresponding angles in the former.* The proportionality factor is (in general) a dual number.

Proof: Let w_0, $w_0 = \delta w_1$, $w + \delta w_2$ be the dual numbers that belong to the spears s_0, s_1, s_2 of an infinitesimal triple, and let w_0', $w_0' + \delta w_1'$, $w_0' + \delta w_2'$ be the corresponding numbers of the transformed triple s_0', s_1', s_2' ([1]). The value of the differential quotient dw' / dw for $w = w_0$ must then coincide with $\delta w_1' / \delta w_1$, on the one hand, and with $\delta w_2' / \delta w_2$, on the other. The last two quotients are then equal, which yields the relation $\frac{\delta w_2'}{\delta w_1'} = \frac{\delta w_2}{\delta w_1}$ directly. From § 13, that will imply coincident values for the complementary angles $\sphericalangle\, s_1\, s_0\, s_2$ and $\sphericalangle\, s_1'\, s_0'\, s_2'$, as well as for the ratios of the angles $\frac{\sphericalangle\, s_2\, s_0}{\sphericalangle\, s_0\, s_1}$ and $\frac{\sphericalangle\, s_2'\, s_0'}{\sphericalangle\, s_0'\, s_1'}$. The proof is complete with that.

([1]) The dual numbers that belong to s_0, s_1, s_2 and s_0', s_1', s_2' can be assumed to be finite dual numbers, because, if necessary, that can always be achieved by changing the coordinate system. The scalar parts of δw^1 and δw^2 are assumed to be non-zero in the text. Except for possible exceptions, the scalar part of dw' / dw will be non-zero, and therefore the scalar parts of $\delta w_1'$ and $\delta w_2'$ will also be non-zero.

The invariance of the complementary angle of an infinitesimal spear-triple under a synectic transformation can be formulated in the usual terminology as follows: Let s_0 be any spear, and let s be a spear that differs from it infinitely little. Let t be a spear along the shortest transversal from s_0 to s, where the direction of t is determined in such a way that the sense of rotation of s_0 to s will appear positive for an observer that looks in the direction t. s_0, s might go to s_0', s' under the synectic transformation considered; let the corresponding (to the above) oriented shortest transversal from s_0' to s_1' be $\bar{t}$. If one changes s arbitrarily in such a way that the spear remains infinitely close to s_0 then s' will behave analogously, as opposed to s_0'. The transversal t runs through the normal net of s_0, and $\bar{t}$ runs through the normal net of s_0'. *Therefore, t and $\bar{t}$ will now be coupled to each other in such a way that the systems that are described in the associated normal nets will be* **congruent**, so any t can be made to overlap the corresponding $\bar{t}$ by one and the same real motion of spear space.

The invariance of the ratios of the angles, like the invariance of the complementary angles, can also be translated into the usual terminology. Let the infinitely-small dual angle that s defines with s_0 be $\Theta = \vartheta + \rho\,\varepsilon$, such that s emerges from s_0 by a rotation around t in the positive sense through the angle ϑ and a displacement through ρ along t in the positive sense; one can also say that s arises from s_0 by a infinitesimal screw $[s_0, s]$ about t with angle ϑ and parameter $k = \rho / \vartheta$. If the infinitely-small dual angle that s_0' makes with s is correspondingly $\Theta' = \vartheta' + \rho'\varepsilon$ then s will emerge from s_0 by an infinitesimal screw $[s_0', s']$ about the axis t, and the angle of the screw will be given by ϑ', while the parameter is given by $k' = \rho' / \vartheta'$. From the above, the ratio Θ' / Θ will remain unchanged when s changes. Now, one has $\frac{\Theta'}{\Theta} = \frac{\vartheta'(1+k'\varepsilon)}{\vartheta(1+k\varepsilon)} = \frac{\vartheta'}{\vartheta}[1 + (k' - k)\,\varepsilon]$; the quantities ϑ' / ϑ and $(k' - k)$ will remain unchanged when spear s that is infinitely close to s_0 rotates around s in any way, and the spear s', which is correspondingly infinitely close to s_0'. In other words: The infinitesimal screws $[s_0, s]$ and $[s_0', s]$ about t ($\bar{t}$, resp.) are related to each in such a way that the angles ϑ and ϑ' of those screws will differ by only a *constant factor*, while *the parameters k and k' will differ by only an additive constant.* In that statement, "constant" means the same thing as "independent of the respective position of the spears s that are infinitely close to s_0."

If one restricts one's consideration to the spears of a congruence and considers the effect of a synectic transformation on just them then the statements will be specialized in the following way: The shortest transversals t from s_0 to s will fill up a ruled surface, which is known by the name of a *cylindroid*; the associated shortest transversals $\bar{t}$ from s_0' to s' describe a cylindroid that is congruent to the one above, and the screw parameters that belong to the corresponding generators of both cylindroids have a constant difference.

§ 15. The "equi-long" transformations of G. Scheffers.

If a synectic transformation (58) permutes the spears of a plane with each other then one will obtain an *equi-long transformation* of the plane, in the sense of **G. Scheffers** ([1]); *in that way, the equi-long transformations are included in the dual-conformal transformations that were just considered*. In order to see that, it is only necessary to apply the concept of the dual complementary angle ([2]) to an infinitesimal spear-triple (s_0, s_1, s_2) in the aforementioned plane. Here, the general definition implies the expression $\sphericalangle s_1\, s_0\, s_2 = \overline{P_1P_2} \cdot \varepsilon$ for the dual complementary angle $\sphericalangle s_1\, s_0\, s_2$, if one understands $\overline{P_1P_2}$ to means the distance between those points P_1 and P_2 that are cut out of s_0 by s_1 and s_2, resp., and must be chosen to be positive or negative according to whether the sequence of points P_1 and P_2 does or does not coincide with the sense of direction of the spear s_0. Now, the dual-conformal transformations that permute the spears of the plane with each other leave the complementary angle $\sphericalangle s_1\, s_0\, s_2$ unchanged; i.e., they leave the distance between the points that are cut out of any spear s_0 by two infinitely-neighboring spears s_1 and s_2 unchanged in length and direction. However, that is precisely the characteristic property of the equi-long transformations.

It only remains to show *what the relationship is between the assignment of real-dual numbers and spears in a plane that was given by G. Scheffers and the assignment of dual numbers and spears in space that was established here.* To that end, we consider the spears in the *xy*-plane of our coordinate system. The angles Θ and Φ that were introduced in § 12 have the values:

$$\Theta = \frac{\pi}{2} + \rho\,\varepsilon, \quad \Phi = \varphi,$$

resp., for any spear s in that plane. From § 12, the quantities ρ and φ have the following meaning: If t is a spear along the altitude ON that is dropped from s to O, in such a way that the sequence of spears t and s will correspond to a rotation in the positive sense (through a right angle) when considered from the positive z-axis, then φ means the angle between t and the positive x-axis Ox, while ρ means the length of the altitude ON, which will be positive or negative according to whether ON and t have the same or opposite directions, respectively.

From formula (57) in § 12, the dual number w that belongs to s will be $w = \tan\frac{\Theta}{2} e^{\varphi i}$. If one develops $\tan\frac{\Theta}{2} = \tan\left(\frac{\pi}{4} + \frac{\rho}{2}\varepsilon\right)$ according to **Taylor**'s theorem then that will give $\tan\frac{\Theta}{2} = 1 + \rho\,\varepsilon$, so:

$$w = (1 + \rho\,\varepsilon)\, e^{\varphi i} = e^{\rho\varepsilon} \cdot e^{\varphi i} = e^{\varphi i + \rho\varepsilon},$$

and that will further imply that:

([1]) Cf., the Verhandlungen des internationalen Mathematikerkongresses zu Heidelberg (1894), 349-356, as well as volume 60 of Mathematischen Annalen, 491-531.

([2]) See above, pp. 30.

$$l(w) = \varphi\, i + \rho\, \varepsilon = i\,(\varphi - \rho\, i\, \varepsilon).$$

If one writes $\bar{\varepsilon}$ in this, instead of $-\,i\,\varepsilon$, then this number will have the property that $\bar{\varepsilon}^2 = 0$, which is analogous to that of ε; the formula above can be written:

$$(58) \qquad -\,i\,l(w) = \varphi + \rho\,\bar{\varepsilon} = \mathfrak{w}.$$

Now, *the number* $\mathfrak{w} = \varphi + \rho\,\bar{\varepsilon}$ *is completely identical with Scheffer's number for the spear*, except that $\bar{\varepsilon}$ in the **Scheffer** number differs from the previous ε by the factor $(-i)$:

$$(58a) \qquad \bar{\varepsilon} = -\,i\,\varepsilon.$$

The relationship between the number w that is assigned to the spear s under our association and Scheffer's number $\mathfrak{w}$ *is mediated by* (58), and thus, by a transcendental equation. That is connected with the fact that under our association, the spear s and number w are in one-to-one correspondence with each other, while the **Scheffer** number $\mathfrak{w}$ of the spear s is determined only modulo 2π ([1]).

The association of spears in the xy-plane and the numbers $w = u + v\,\varepsilon$ has the disadvantage that the coefficients u and v that appear in w are not real quantities. That inconvenience can be avoided by going to another association, under which the spear s that belongs to the number w under the original assignment will correspond to the number:

$$(59) \qquad \tilde{w} = \frac{1}{i} \cdot \frac{w-1}{w+1}.$$

The new assignment belongs to the same "class" as the original one, from the definition of the concept of the class of an assignment that was given in § 5. If one recalls the representation $w = e^{\varphi i + \rho \varepsilon}$ that was employed above, in which one can write $i\bar{\varepsilon}$, in place of ε, then that will yield $w = e^{i(\varphi + \rho\bar{\varepsilon})}$ and:

$$\tilde{w} = \frac{1}{i} \cdot \frac{e^{i(\varphi+\rho\bar{\varepsilon})} - 1}{e^{i(\varphi+\rho\bar{\varepsilon})} + 1} = \frac{\dfrac{1}{2i}\left\{e^{i\frac{\varphi+\rho\bar{\varepsilon}}{2}} - e^{i\frac{\varphi-\rho\bar{\varepsilon}}{2}}\right\}}{\dfrac{1}{2}\left\{e^{i\frac{\varphi+\rho\bar{\varepsilon}}{2}} + e^{i\frac{\varphi-\rho\bar{\varepsilon}}{2}}\right\}} = \frac{\sin\left(\dfrac{\varphi+\rho\bar{\varepsilon}}{2}\right)}{\cos\left(\dfrac{\varphi+\rho\bar{\varepsilon}}{2}\right)},$$

so one will have:

$$(60) \qquad \tilde{w} = \tan\frac{\varphi+\rho\bar{\varepsilon}}{2} = \tilde{u} + \tilde{v}\,\bar{\varepsilon},$$

([1]) If one regards equation (58) as the definition of the **Scheffer** number $\mathfrak{w}$ of the spear s then one will see that along with $\mathfrak{w} = \varphi + \rho\bar{\varepsilon}$, the numbers of the form $[(\varphi + 2k\pi) + \rho\bar{\varepsilon}]$ will also belong to s as **Scheffer** numbers, where k is any whole number.

moreover, when one sets $\tilde{u} = \tan \varphi / 2$, $\tilde{v} = \frac{1}{2\cos^2 \frac{\varphi}{2}} \cdot \rho$. ($\tilde{u}$ and $\tilde{v}$ are *ordinary real numbers* then.)

A fractional-linear substitution: $\tilde{w}' = \frac{a\tilde{w}+b}{c\tilde{w}+d}$, for which the coefficients a, b, c, d have the form:

$$a = \alpha + \alpha' \bar{\varepsilon}, \quad b = \beta + \beta' \bar{\varepsilon}, \quad \text{etc.},$$

in which α and α', β and β', etc., describe ordinary real numbers, will then permute the spears in the *xy*-plane. One can show that under the group of spear transformations of the *xy*-plane thus-defined, the spears that contact an "oriented" circle will again go to another such spear that contacts an oriented circle, and the group in question will prove to be identical to the group of *Laguerre circle conversions* (**Laguerre**, Annales de Mathématiques, 1882, 1883). The *dual numbers* will then play the same role for the latter group that *the ordinary complex numbers do for the Möbius circle conversions.*

A synectic transformation $\tilde{w}' = \sum_{k=1}^{\infty} a_k (\tilde{w} - c)^k$, in which $a_k = \alpha_k + \alpha'_k \bar{\varepsilon}$, $c = \gamma + \gamma' \bar{\varepsilon}$, and the α_k, α'_k, γ, γ' are ordinary real numbers, represents *the most general (real, analytic) equi-long transformation of the xy-plane.*

§ 16. The isotropic congruences of Ribaucour.

The map of dual numbers to the minimal planes, and then to the spears in space, admits an especially simple representation in terms of **Ribaucour**'s theory of isotropic congruences. [Cf., **Ribaucour**, "Étude des Élassoides ou Surfaces à Courbure Nulle," in the Mémoires couronnées par l'Académie de Belgique **44** (1881).]

One can define an isotropic congruence with ∞^2 real lines as follows: On circumscribes any developable D by the absolute circle of Euclidian geometry ([1]). The real carrier lines of the general planes of D, since the latter are obviously minimal planes, then fill up an isotropic congruence Q. If dual numbers $w = u + v\,\varepsilon$ are assigned to minimal planes according to (35) then (35) can be regarded as a parametric representation of the minimal planes by the parameters u and v. The developable D is then characterized analytically by an equation between the parameters u and v, such as:

$$v = \varphi(u). \tag{61}$$

The minimal plane that corresponds to the number $w = u + v\,\varepsilon$ then belongs to the developable D if and only if u and v are coupled by the relation above. The spear (w), as the real representative of the minimal plane, will then lie on a line of the isotropic congruence Q. Conversely, if the spear ($w = u + v\,\varepsilon$) lies on a line of the isotropic

([1]) The name of "minimal developable" has recently become useful for such a developable whose generating planes are minimal planes.

congruence Q in such a way that the associated minimal plane belongs to D then the above relation must exist between u and v. Of the two spears of a line G of the isotropic congruence, the only one that is referred to as belonging to the congruence will always be the one that represents the plane of D that goes through G. *With that assumption,* (61) *gives the necessary and sufficient condition for the spears* ($w = u + v\,\varepsilon$) *to be assigned to the isotropic congruence in question.*

From (61), one can think of any spear ($u + v\,\varepsilon$) of the congruence Q as being derived from the spear ($u + 0\,\varepsilon$) (which is parallel to it and goes through the origin O) by means of the synectic transformation $w' = w + \varepsilon\,\varphi(w)$. If one then assigns any spear of the congruence to the one through O that is parallel to it as its correspondent then, from § 15, the congruence will be mapped dual-conformally to the spear bundle through O in that way. That sheds light upon the fact that *the shortest transversals between a spear s of the congruence and spears that are infinitely-close to it will generally define a pencil of rays*, so the "limiting point" on an s of the congruence will coincide with the "center" on s.

One will obtain an especially simple special case of this when one takes the relation (61) in the form:

$$v = A + B\,u + C\,u^2.$$

In this case, if one recalls (35), or even better (34), the coordinates of the minimal plane (T, U, V, W) that is coupled with $w = u + v\,\varepsilon$ will be coupled by:

$$-T = A\,(V + i\,U) - B\,i\,W + C\,(V - i\,U)$$

or

$$T + i\,(A - C)\,U + (A + C)\,V - B\,i\,W = 0.$$

That is, the spears of the congruence in question represent minimal planes that all go through a fixed point (a, b, c) whose coordinates are:

$$a = i\,(A - C), \quad b = (A + C), \quad c = -B\,i.$$

Following **E. v. Weber** ([1]), the congruence shall be referred to as a *cycle* of spears. If $a = a' + a''i$, $b = b' + b''i$, $c = c' + c''i$ ($a', a'', \ldots$, real) then the real point (a', b', c') shall be called the *center of the cycle*; we would like to use the terminology *nucleus of the cycle* for the imaginary point (a, b, c) ([2]).

The special congruence considered is of great utility for the investigation of the general isotropic congruences, as one can see from the following fact: *In the neighborhood of any spear* $s_0 = (w_0)$ *of the congruence* $v = \varphi(u)$, *the latter can be replaced with a cycle that has a second-order contact with* s_0 .

([1]) Cf., the papers that were cited in the remark on pp. 1.

([2]) One easily verifies that a cycle of spears *consists of the generators of a confocal family of hyperboloids of rotation* that are oriented in a certain way, and the orientation of the generators is such that the orthogonal projections of the oriented generators onto the plane of the throat circle of the hyperboloid will give tangents to the throat circle, *which are oriented in a consistent way*. (Cf., **E. v. Weber**, *loc. cit.*) The center of the cycle is identical with the common center of the confocal family whose nucleus lies on the common rotational axis of the confocal family.

In fact, if one develops $v = \varphi(u)$ in powers of $(u - u_0)$ using **Taylor**'s theorem then when one sets $\left(\frac{dv}{du}\right)_{u=u_0} = v_0'$, $\left(\frac{d^2v}{du^2}\right)_{u=u_0} = v_0''$, …, that will give:

$$v = v_0 + v_0' \,(u - u_0) + \tfrac{1}{2} v_0'' (u - u_0)^2 + \dots$$

If one now truncates the right-hand side at the terms in $(u - u_0)^2$ then one will get the equation of a cycle $v = v_0 + v_0' \,(u - u_0) + \tfrac{1}{2} v_0'' (u - u_0)^2$ or:

$$v = \left(v_0 - u_0 v_0' + \tfrac{1}{2} v_0'' u_0^2\right) + (v_0' - u_0 v_0'')u + \tfrac{1}{2} v_0'' u^2,$$

which agrees with the equation of the congruence $v = \varphi(u)$, up to (exclusively) third-order infinitesimals in $(u - u_0)$.

That cycle shall be called the *osculating cycle* of the congruence for the spear s_0 . The coordinates of its nucleus (which shall be denoted by ξ_0, η_0, ζ_0) are provided by the formula above for a, b, c when one sets $A = v_0 - u_0 v_0' + \tfrac{1}{2} v_0'' u_0^2$, $B = v_0' - u_0 v_0''$, $C = \tfrac{1}{2} v_0''$ on its right-hand side:

$$\xi_0 = \left(-\tfrac{1}{2} + \tfrac{1}{2} u_0^2\right) i\, v_0'' - u_0\, i\, v_0' + i\, v_0,$$

$$\eta_0 = \left(\tfrac{1}{2} + \tfrac{1}{2} u_0^2\right) \cdot v_0'' - u_0\, v_0' + v_0,$$

$$\zeta_0 = + u_0\, i\, v_0'' - i v_0'.$$

If one sets $\varphi(u) = 2i\, F(u)$ for the function that appears in equation (61) then equation of the congruence will assume the form:

$$v = 2i\, F(u), \tag{62}$$

and the coordinates of the nucleus of the osculating cycle that belongs to an arbitrary spear $s = (w) = (u + v\,\varepsilon)$ can be expressed in terms of the function $F(u)$ and its derivatives $F'(u)$ and $F''(u)$ in the following way:

$$\left.\begin{aligned} \xi &= (1-u^2)\, F''(u) + 2u\, F'(u) - 2\, F(u), \\ \eta &= i(1+u^2) F''(u) - 2iu\, F'(u) + 2i\, F(u), \\ \zeta &= -2u\, F''(u) + 2\, F'(u). \end{aligned}\right\} \quad (\mathfrak{C}) \tag{63}$$

One will obtain the coordinates x, y, z of the center of the osculating cycle in s when one takes the real parts of the complex quantities ξ, η, ζ; thus:

$$(64)\qquad \left.\begin{aligned} x &= \Re\{(1-u^2)F''(u) + 2u\,F'(u) - 2F(u)\},\\ y &= \Re\{i(1+u^2)F''(u) - 2iu\,F'(u) + 2i\,F(u)\},\\ z &= \Re\{-2u\,F''(u) + 2\,F'(u)\}. \end{aligned}\right\}\qquad (K)$$

The geometric locus of all nuclei (ξ, η, ζ) for the spear s of the congruence in question is the *imaginary curve* $\mathfrak{C}$ *that is represented by* (63). *That curve is identical with the edge of regression of the developable D.* All generating minimal planes of the developable D that belong to the spears that are infinitely close to s will then go through any point (ξ, η, ζ) of that curve. Naturally, since $\mathfrak{C}$ is an edge of regression, it will be a minimal curve.

The geometric locus of all centers (x, y, z) of the osculating cycles that belong to the congruence Q is the *real surface K that is represented by* (64) ([1]). The analytic representation (64) of that surface is identical with the **Weierstrass** representation of an arbitrary *minimal surface* (cf., e.g., the *Enzyklopädie der mathematischen Wissenschaften*, III, D5, **R. v. Lilienthal**, "Besondere Flächen," pp. 312). The fact that z has the opposite value in the **Weierstrass** formula to the one above (64) is inessential, since a minimal plane will go to another minimal plane under a reflection in the xy-plane of our coordinate system. *The geometric locus K of the centers of all osculating cycles of an isotropic congruence Q will then be a minimal surface.*

§ 17. Isotropic congruences and minimal surfaces.

On closer examination, the theorem that was just mentioned proves, to be basically identical to the following theorem of **Ribaucour:**

If one lays planes through the various rays of a congruence that are perpendicular to them, and which are the so-called "middle planes" of the rays in question, then all of the middle planes will envelope a surface, namely, the so-called *middle envelope* of the congruence. *According to Ribaucour, the middle envelope of an isotropic congruence is a minimal surface.* (Cf., the article in the *Enzklopädie* that was cited above, pp. 330.)

It can be proved that the middle plane of any spear s of the isotropic congruence Q contacts the middle envelope of the congruence Q at precisely the center (x, y, z) of the osculating cycle q that belongs to s, and with that one likewise shows that the *middle envelope of Q is identical with the surface K of the preceding paragraph.*

We present the following argument for the purpose of proving that: A cycle of spears consists of the generators of a family of confocal hyperboloids of rotation of one sheet, which are oriented in a certain way ([2]). The middle planes of all spears of a cycle all go through the common midpoint of the aforementioned hyperboloid, and thus through the center of the cycle. If one determines the point of intersection of the middle planes to all spears of Q that are infinitely close to s, on the one hand, and then the intersection point of the middle planes of all spears of q that are infinitely close to s, on the other hand, then

(1) Obviously, K can also be defined to be the geometric locus of the bisecting points of the line segments that connect the different points of $\mathfrak{C}$ with the respective associated conjugate-image points.

(2) Cf., remark (2) on pp. 40.

it will follow from the fact that the osculating cycle q has second-order contact with Q at s that the two points thus-obtained will be identical. However, the first point is identical with the point at which the middle plane of s contacts the middle envelope of Q, while the second point is, from the above, identical with the center (x, y, z) of the osculating cycle q. With that, the theorem is proved.

The way that the parametric representation (64) of the surface K was derived seems to be of interest for the fact that it demonstrates the *connection* that exists between the complex function $F(u)$ in the **Weierstrass** *representation* (64) *of a minimal surface K and the isotropic congruence A that belongs to K according to Ribaucour.*

In fact, if any minimal surface K is given in the **Weierstrass** representation (64), and if one would like to obtain an isotropic congruence for which K is the middle envelope then one will need only to establish the relation:

$$v = 2i\,F(u)$$

between the defining numbers u and v of a dual variable $w = u + v\,\varepsilon$; the spears $s = (w)$ then fill up an isotropic congruence Q that has the desired relationship with K ([1]).

Following **Ribaucour**, there are, in general, ∞^3 isotropic congruences $\bar{Q}$ whose middle envelopes are identical with the given minimal surface K, and one can find the remaining ones from one of those congruences Q by a simple construction. If one subjects the minimal planes that belong to the spears s of Q to an imaginary translational motion $(i\alpha, i\beta, i\gamma)$ that consists of a translation by $i\alpha$ parallel to the positive x-axis, one by $i\beta$ parallel to the positive y-axis, and one by $i\gamma$ that is parallel to the positive z-axis, where α, β, γ should be real numbers then the developable D of those minimal planes will go to D, and the point (ξ, η, ζ) of the edge of regression $\mathfrak{C}$ of D will go to the point $(\bar{\xi}, \bar{\eta}, \bar{\zeta})$ of the edge of regression $\bar{\mathfrak{C}}$ of $\bar{D}$ in such a way that $\bar{\xi} = \xi + i\,\alpha$, $\bar{\eta} = \eta + i\,\beta$, $\bar{\zeta} = \zeta + i\,\gamma$. However, the real parts $\bar{x}, \bar{y}, \bar{z}$ of $(\bar{\xi}, \bar{\eta}, \bar{\zeta})$, resp., are obviously identical with the real parts x, y, z of (ξ, η, ζ), respectively.

The spears of the minimal planes of the developable $\bar{D}$ define an isotropic congruence $\bar{Q}$. Since $\bar{x} = x$, $\bar{y} = y$, $\bar{z} = z$, the centers of all osculating cycles of a surface $\bar{K}$ that is identical with the corresponding surface K for the original congruence Q belong to that congruence as a geometric locus.

The middle envelope of the congruence $\bar{Q}$ is then identical with the middle envelope of the congruence Q. Corresponding to the different values that the real parameters α, β, γ can take on, one will obtain ∞^3 isotropic congruences $\bar{Q}$ that have the same relationship to a given minimal surface K from a congruence A has K for its middle envelope.

It still remains for us to clarify the connection between Q and $\bar{Q}$, first constructively, and then also analytically.

([1]) In particular, *if the function $F(u)$ that appears in the Weierstrass representation* (64) *of a minimal surface K is an entire, quadratic function of u*: $F(u) = A_1 + B_1u + C_1u^2$ then the associated isotropic congruence $v = 2i\,F(u)$ will be a cycle, the middle envelope of that congruence will reduce to the center of the cycle, and *the minimal surface K will then reduce to a single point.*

The minimal plane that is represented by an arbitrary spear s and the dual number w might go to another one that belongs to spear s' and the dual number w' by the imaginary translational motion $(i\,\alpha, i\,\beta, i\,\gamma)$.

As one easily verifies, the spear s' can be found from the spear s by the following construction: One displaces s by the real vector (α, β, γ), with which, that spear will come to a position s_1 that is parallel to the original one, and then rotates s_1 to s in the positive sense through a right angle; the spear thus-arrived-at will then be the desired s'. Due to its simplicity and its repeated use, the construction by which the spear s' that corresponds to any spear s by means of the vector (α, β, γ) can be found deserves a special name: We would like to refer to it as a *lateral displacement of the spear s by the vector* (α, β, γ). *One then finds from any isotropic congruence Q whose middle envelope is the given minimal surface K the other* ∞^3 *isotropic congruences* $\bar{Q}$ *that have the same relationship to K* quite simply when one *subjects the spears of Q to a lateral displacement through an arbitrarily-given vector* (α, β, γ).

Let (T, U, V, W) be the coordinates of the minimal plane of the spear s, and let $w = u + v\,\varepsilon$ be the associated number; that minimal plane will go to that of the spear s' by the imaginary translation $(i\,\alpha, i\,\beta, i\,\gamma)$. The coordinates of the latter minimal plane are:

$$T' = T - i\,\alpha\,U - i\,\beta\,V - i\,\gamma\,W, \quad U' = U, \quad V' = V, \quad W' = W;$$

from (35), the dual number $w' = u' + v'\,\varepsilon$ that is associated with it is determined from:

$$u' = u, \;\; v' = v + i\left[\alpha i\frac{u^2-1}{2} + \beta\frac{u^2+1}{2} + \gamma i u\right].$$

The dual number w' of any spear of $\bar{Q}$ depends upon the dual number w of the corresponding spear in Q in that way ([1]). If one recalls (62) then, from the above, the relation:

$$v' = 2i\,F(u') + i\left[\alpha i\frac{u'^2-1}{2} + \beta\frac{u'^2+1}{2} + \gamma i u'\right]$$

exists between the defining numbers u' and v' of w'. *The characteristic condition for a spear* $(w) = (u + v\,\varepsilon)$ (in which, the primes on w, u, v are now omitted) *to belong to an isotropic congruence* $\bar{Q}$ will then be given by:

([1]) The numbers w and w' are coupled to each other by a fractional-linear substitution. In fact, the latter can be represented by the bilinear equation:

$$(w' - w) - \frac{\varepsilon\,i}{2}\,[\alpha\,i\,(w\,w' - 1) + \beta\,(w\,w' + 1) + \gamma\cdot i\,(w + w')]$$

[which is a special case of (39)].

$$v = 2i\,F(u) + i\left[\alpha i\frac{u^2-1}{2} + \beta\frac{u^2+1}{2} + \gamma iu\right],$$

or by:

$$v = 2i\;\bar{F}(u),$$

in which:

$$\bar{F}(u) = F(u) + \frac{1}{2}\left[\alpha i\frac{u^2-1}{2} + \beta\frac{u^2+1}{2} + \gamma iu\right].$$

On the grounds of the assumption that was made, the numbers α, β, γ are ordinary *real* numbers, as we once more emphasize.

From what was said, the middle envelope of $\bar{Q}$ is identical with the middle envelope of Q. The equation of Q is $v = 2i\,F(u)$, while the equation of $\bar{Q}$ is $v = 2i\;\bar{F}(u)$.

If one then replaces the function $F(u)$ with the function $\bar{F}(u)$ that was defined above and contains three arbitrary real parameters α, β, γ in equations (64), which give the parametric representation of the middle envelope K of Q, then those equations will still represent *the same* minimal surface K.

The function $F(u)$ that appears in the **Weierstrass** representation of the surface is not determined completely by a given minimal surface (K), since one can still add arbitrary real multiples of the expressions $i\frac{u^2-1}{2}$, $i\frac{u^2+1}{2}$, $i\,u$ to such a function. (This state of affairs is not strictly valid in the report in the *Enzklopädie der Mathematischen Wissenschaften* that was cited above. There [pp. 312, line 6 from the bottom], when it is said that any analytic function belongs to a minimal surface, and conversely, that would give rise to the opinion that only one function belongs to a minimal surface, and, from the above, that opinion would be incorrect.)

§ 18. Associated minimal surfaces.

If one replaces the function $F(u)$ in the **Weierstrass** representation of a minimal surface K with $e^{i\mu} F(u)$, in which μ means a real constant, then one will obtain a minimal surface $\bar{K}$ that will be referred to as *associated with* K; corresponding to the various values of μ, one will obtain a simply-infinite family of minimal surfaces that are associated with K, and likewise with each other. It is known that these associated surfaces are developable from each other, and still other simple relationships exist between them. Only the question that was treated and solved by **Ribaucour** will be discussed here of the way by which the isotropic congruences that correspond to the associated surfaces $\bar{K}$ can be determined from an isotropic congruence Q (for which, K is the middle envelope) that belongs to K.

An isotropic congruence $\bar{\bar{Q}}$ that belongs to $\bar{\bar{K}}$ can be obtained when one establishes the relation $v' = 2i\,e^{i\mu}F(u')$ between the defining numbers u and v of a dual number $w' = u' + v'\varepsilon$. The spears s' that belong to w' then fill up an isotropic congruence whose middle envelope (from § 16) is obviously $\bar{\bar{K}}$. Now, dual numbers $w = u + v\,e$ for which the relation $v = 2i\,F(u)$ exists between u and v belong to the spears s of Q. If one then multiplies the vectorial part of the number w of a spear s of Q by $e^{i\mu}$ then one will obtain the dual number of a spear of $\bar{\bar{Q}}$. If one then denotes the latter number by w' and the associated spear by s then one will have $w' = \mathfrak{A}_\lambda(w)$, in which $\lambda = e^{i\mu}$, with the use of the symbols that were introduced in formula (21).

The transformation that belongs to $w' = \mathfrak{A}_\lambda(w)$, *which takes the spear* $s = (w)$ *of* Q *to a corresponding spear* $s' = (w')$ *of* $\bar{\bar{Q}}$, *can now be exhibited by a very simple construction.* If one recalls (35) then the minimal planes that belong to w will suffer a central similarity transformation with center O and modulus $\lambda = e^{i\mu}$. The spears s of those minimal planes will be transformed by the following construction: *One draws a spear* s_0 *through* O *and parallel to* s, *and then rotates* s *around* s_0 *in the positive sense through the angle* μ; *the spear thus-obtained is then the desired* s', *in which* s *goes to* $w' = \mathfrak{A}_\lambda(w)$ *under the transformation.*

The validity of this construction is deduced effortlessly from the solution to the problem that was posed in § 8 of ascertaining the associated spear to a given dual number. The construction above by which one found the transformed spear s' from s shall be referred to briefly as a *slewing* of the spear *s about the point O through the angle* μ. Slewing about other points is defined analogously.

One obtains the isotropic congruence whose middle envelopes are the associated surfaces of K from an isotropic congruence Q that belongs to K by slewing through a constant angle about O (or about any other point of space, since the coordinate origin can, of course, be changed arbitrarily).

§ 19. Special finite groups of spear transformations.

A fractional-linear substitution of a dual variable w: $w' = \dfrac{aw+b}{cw+d}$ represents a transformation of minimal planes that correspond to a (generally) complex motion in space. These transformations depend upon six essential complex parameters (Cf., § 9.)

Likewise, since any minimal plane can be represented by its spears, the above transformation will represent a *real transformation of the spears in space.* The group of those transformations contains $6 \times 2 = 12$ *real* parameters (since a complex parameter is equivalent to two real ones); *that group might be denoted by* Γ_{12} ([1]).

The real motions of space, and therefore, the minimal planes will again yield (real) motions of the spears. The group of motions of spear-space contains six real parameters, and is a subgroup Γ_6 of Γ_{12}.

([1]) For that group, one can confer: **E. v. Weber**, "Die komplexen Bewegungen," Berichte der sächsischen Gesellschaft der Wissenschaften (1903). That group belongs to the *dual-conformal transformations* of § 14, which emerges immediately from the above.

A certain three-parameter group of the transformations of Γ_{12} that are not contained in Γ_6 appears in § 17, namely, the one that corresponds to pure-imaginary translations of space; it is defined by the so-called *lateral displacements.*

Obviously, the double ratio of the dual numbers that belong to four spears ([1]) *will remain invariant under the transformations of* Γ_{12} , and one can regard that double ratio as the double ratio of the four spears, but one must observe that the vectorial part of that double ratio will be multiplied by a constant factor when one replaces the assignment between dual numbers, on the one hand, and the minimal planes and spears, on the other, that was established in § 8 with another assignment. (Cf., § 5.)

The totality of all spears s that determine a pure scalar double ratio with three fixed spears s_1, s_2, s_3, no two of which are parallel, will be given by the spears of the cycle that is determined by s_1, s_2, s_3 . (If there is only one such cycle that belongs to s_1, s_2, s_3 then the minimal planes of s_1, s_2, s_3 will have a common point, and it will then be the nucleus of the cycle.)

Any four spears of a cycle have a *pure scalar* double ratio.

Analogously, there are congruences of spears with the property that *any four spears of such a congruence determines a double ratio that is real-dual;* i.e., free of i. Such a congruence will be obtained when one screws a spear s around a fixed spear s_0 (that is not parallel to s) in an arbitrary way. Such a congruence might be referred to as a *vortex* ([2]). One can also define a vortex of spears as the totality of all spears that define the same dual angle with a fixed spear. If one maps the spears in space to the dual points of the number cone as in § 11 then the spears of a vortex will correspond to a planar section of the sphere with a "dual plane" and thus, to a "dual circle."

(However, the property of a vortex that any four of its spears must have a real-dual double ratio is by no means true for all classes of associations of spears and dual numbers, since under the transition in (30) from a chosen assignment to an assignment of another class, the vectorial part of the double ratio will be multiplied by a factor that can even contain the imaginary unit in the general case.)

Another remarkable manifold of spears possesses the property that *any four spears of it have a pure real scalar double ratio;* the latter is then an ordinary real number. One obtains such a manifold when one moves a spear s around a fixed spear s_0 in such a way that one rotates it around s_0 in the positive sense through the angle t and likewise displaces it parallel to s_0 in the positive sense by (b sin t), in which t is a variable parameter, and b is a constant proportionality factor. This manifold of ∞^1 spears shall be referred to as a *chain of spears.* The special kind of motion around s_0 that the spear s was subjected to above can be given the name of "reversal." All points in space describe ellipses under that motion that project onto any plane that is perpendicular to s_0 as concentric circles whose centers lie along s_0 . (These reversals represent limiting cases of

([1]) In what follows, this will always be thought of as having been chosen in such a way that the associated double ratio possesses a well-defined value.

([2]) **E. Study** used the term "cyclic congruence" for the analogous structure in non-Euclidian geometry. ["Über Nicht-Eucklidische und Liniengeometrie," Jahresbericht der deutschen Mathematikervereinigung (1902), pp. 333.] Here, we must avoid that terminology if we are to prevent confusion.

the general **Darboux-Mannheim** motion, which is the most general motion under which all points describe ellipses. ([1]))

One immediately infers from the invariance of the double ratio of four spears under the group Γ_{12}, *that cycles will go to cycles, vortices to vortices, and chains to chains under the transformations of that group.*

If one transforms a dual variable w according to the equation:

$$w' = \mathfrak{A}_\lambda \left(\frac{aw+b}{cw+d}\right)$$

then, from § 9, the minimal planes that belong to the various w will be subjected similarity transformations. The group that is defined by all such transformations depends upon seven complex parameters. The corresponding group of *real* transformation of the spear that belong to the various minimal planes correspondingly contains 14 real parameters, and let it be denoted by Γ_{14} . In particular, there are also transformations contained in Γ_{14} that were considered in § 18 and referred to as *slewings* (through a well-defined angle about a well-defined point). The double ratio of any four spears (cf., § 5) will be modified by the group Γ_{14} in such a way that its scalar part will remain unchanged, while the vectorial part will be multiplied by a constant factor. *Correspondingly, under the transformations of* Γ_{14} , *cycles will always go to cycles, and chains to chains, but generally vortices do not go to vortices.* Rather, a vortex will go to a congruence that can be generated by rotating a pencil of parallel spears about a fixed axis, and in the absence of a more suitable name, it shall be called a *rotation field of spears* ([2]).

The groups Γ_{12} and Γ_{14} can be extended to mixed group $\bar{\Gamma}_{12}$ and $\bar{\Gamma}_{14}$ by adjoining the transformation $w' = \bar{w}$, in which $\bar{w}$ is the number that is conjugate-imaginary to w [which arises from w when one switches i with $(-i)$].

Instead of the transformation that was chosen above, one can adjoin the one that is represented by $w' = -1/\bar{w}$, and from (51), any spear will be converted into the opposite one on the same line. The behavior of the cycles, chains, and vortices of spears under $\bar{\Gamma}_{12}$ and $\bar{\Gamma}_{14}$ is completely analogous to their behavior under Γ_{12} and Γ_{14} .

([1]) Cf., the Note by **Darboux** in **Koenigs** *Leçons de Cinématique*, Paris, 1897, pp. 352, as well as **A. Grünwald**'s treatise "Die Mannheim-Darbouxsche Umschwungsbewegung eines starren Körpers," Zeitschrift für Math. u. Physik (1906). One will obtain the *most general* motion of that kind by the inward rolling of a right circular cylinder of diameter a onto one that is twice as big (while allowing it to slide only in the direction of the generators of the cylinder), when a point that is rigidly bound to the first cylinder is forced to stay on a plane that is rigidly bound to the second cylinder. The "reversal" in the text will then correspond to the limiting case of $a = 0$.

([2]) The closely-related term "whorl" of spears cannot be used without reservations, if one recalls the different use of that term by **E. Study**.

§ 20. Concluding remarks.

In the foregoing, the application of dual numbers to the geometry of spears in space was treated in a rough outline, and by no means exhaustively. Certain geometric considerations lead one to ascend from the concept of a spear – i.e., a real, proper line with a sense of traversal – to another general concept – that of an *oriented real or imaginary line.* Similar to spears, they can be represented by ordinary dual numbers $w = u + v\,\varepsilon$, and thus, by higher complex numbers of the form $a + bj + c\varepsilon + d\,\varepsilon j$, in which one has $j^2 = -1$, $\varepsilon^2 = 0$, and a, b, c, d are complex numbers. (These higher complex numbers can be found in the *Enzyklopädie der mathematischen Wissenschaften*, II A 4, in the report on "Höhere komplexe Grössen," pp. 167, as the imaginary-reducible type, with the enumeration that is given there without a number.) On the other hand, the oriented lines can be mapped to the points of a four-dimensional manifold $M^{(4)}$ in a four-dimensional space $R^{(14)}$ in such a way that the spears (i.e., the real, oriented lines) would correspond to the real points of $M^{(4)}$, and groups of spear transformations that were denoted by Γ_{12}, Γ_{14}, $\bar{\Gamma}_{12}$, $\bar{\Gamma}_{14}$ in § 19 would be mapped to collineations of $M^{(4)}$ to itself.

One will succeed in extending the manifold of oriented lines to a closed continuum in a suitable way by this map. Understandably, that continuum will differ from both the **Plücker** continuum of lines and the **Study** continuum of rays.

The metric relationships in the manifold of oriented lines can be introduced by fixing a certain "vortex" of imaginary, oriented lines, just as metric relationships are introduced into projective point spaces when one distinguishes a conic section, namely, the absolute spherical circle.

A more detailed discussion of these investigations shall be reserved for a later occasion.
