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A mapping principle that links plane geometry and kinematics with spatial geometry

By

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§ 1. The ordered point-pair as element of plane geometry. Two types of continua of such elements, the linear and quadratic ones.

One can introduce the point-pair into the geometry of the plane as an element. If the two points $\mathfrak{M}$ and M, which together define the pair $\mathfrak{M}M$ and are then assumed to lie at finite points, were distinguished as a "left" point $\mathfrak{M}$ and a "right" point M then such a point-pair could be called an "ordered" one. We shall speak of only such point-pairs, here; accordingly, the point-pairs $\mathfrak{M}M$ and $M\mathfrak{M}$ (as long as $\mathfrak{M}$ and M do not coincide, perhaps) shall be strictly distinguished from each other. The point-pairs that are defined by finite points in this way will be referred to as "real" point-pairs.

The totality of all ∞^4 real point-pairs can be mapped to the linear manifold of points in a four-dimensional space when one regards the rectangular coordinates $\mathfrak{x}$, $\mathfrak{y}$ of $\mathfrak{M}$ and the rectangular coordinates x, y of M as the rectangular coordinates of a point in four-dimensional space.

At least as important as this obvious map is another one, by which the point-pair $\mathfrak{M}M$ in the plane is mapped to a *quadratic* manifold of points in a five-dimensional linear space. One adds $\frac{1}{2}(\mathfrak{x}^2 + \mathfrak{y}^2 - x^2 - y^2)$ to the four quantities $\mathfrak{x}$, $\mathfrak{y}$, x, y as the fifth coordinate, and considers these five quantities, which are coupled by a quadratic equation, to be the rectangular coordinates of a point in a five-dimensional space; this point is to be regarded as the image point of the point-pair $\mathfrak{M}M$.

For both maps, the image points of real point-pairs on the point manifolds in question do not, by any means, lie at finite points. The infinitely-distant points – in the projective sense of the term – of the aforementioned manifolds are not images of real point-pairs. In order to extend the totality of real point-pairs in the plane to a *closed* continuum, one can choose one or the other of the maps above as one's basis.

If one introduces ∞^2 different "imaginary" point-pairs in the plane by choosing the first map, which corresponds to the ∞^2 different infinitely-distant points in the four-dimensional image space, then the continuum of point-pairs will be extended to a closed continuum that is, without exception, in one-to-one correspondence with the points of the four-dimensional linear space and thus has an essentially *linear* character.

If one chooses the second map as one's basis then one will introduce ∞^3 different "imaginary" point-pairs in the plane (which are entirely distinct from the previous ones), corresponding to the ∞^3 different infinitely-distant points of the quadratic point-manifold in five-dimensional space; it will then seem that the continuum of point-pairs has been extended to a closed continuum (that is different from the previous one) that is in one-to-one correspondence, without exception, with the points of the quadratic manifold in five-dimensions, and thus has an essentially *quadratic* character.

These two closed continua of point-pairs shall be distinguished from each other as the "linear" and "quadratic" continuum, respectively.

The analytical representation of point-pairs in the two continua demands the use of homogeneous coordinates.

In order to represent a point-pair of the linear continuum, one introduces the system of five homogeneous ([1]) quantities $\mathfrak{q}$, corresponding to the proportion:

$$\mathfrak{q}_0 : \mathfrak{q}_1 : \mathfrak{q}_2 : \mathfrak{q}_3 : \mathfrak{q}_4 = 1 : \mathfrak{x} : \mathfrak{y} : x : y.$$

For $\mathfrak{q}_0 \neq 0$, any system of homogeneous quantities $\mathfrak{q}$ will belong to a real point-pair of the linear continuum, and for $\mathfrak{q} = 0$, it will belong to an imaginary point-pair. The $\mathfrak{q}$ are to be regarded as homogeneous coordinates of the point-pair in question.

In order to represent the point-pairs of the quadratic continuum analytically, one must analogously regard the six homogeneous quantities q that are defined by the proportion:

$$q_0 : q_1 : q_2 : q_3 : q_4 : q_5 = 1 : \mathfrak{x} : \mathfrak{y} : x : y : \tfrac{1}{2}(\mathfrak{x}^2 + \mathfrak{y}^2 - x^2 - y^2)$$

as their homogeneous coordinates. For $q_0 \neq 0$, each system of homogeneous quantities q will belong to a real point-pair in the quadratic continuum, and for $q_0 = 0$, it will belong to an imaginary point-pair. The six homogeneous coordinates q are coupled to each other by the homogeneous, quadratic relation:

$$q_1^2 + q_2^2 - q_3^2 - q_4^2 - 2q_0 q_5 = 0.$$

If one restricts oneself to the real point-pairs then each of them, when regarded as an element of the "linear" continuum, will belong to a system of homogeneous coordinates $\mathfrak{q}$, while, when one regards it as an element of the "quadratic" continuum, it will belong to a system of homogeneous coordinates q. Thus:

$$q_0 = \lambda\, \mathfrak{q}_0\,, \qquad q_1 = \lambda\, \mathfrak{q}_1\,, \qquad q_2 = \lambda\, \mathfrak{q}_2\,, \qquad q_3 = \lambda\, \mathfrak{q}_3\,, \qquad q_4 = \lambda\, \mathfrak{q}_4\,,$$

([1]) "Homogeneous" quantities shall always refer to those quantities that are finite, not all zero, and determined only up to a common proportionality factor.

where λ is a proportionality factor: The system of homogeneous quantities q is then determined from the system of homogeneous quantities $\mathfrak{q}$, and conversely.

This changes immediately when one allows imaginary point-pairs in the two continua. The relationship between the homogeneous $\mathfrak{q}$ and the homogeneous q will no longer prove to be properly one-to-one without exception.

If one seeks the q that correspond to the homogeneous coordinates $\mathfrak{q}$ of an imaginary point-pair ($\mathfrak{q}_0 = 0$) of the first (linear) continuum on the basis of the allowed formulas that were written down then one must distinguish between two cases, according to whether:

a) $$\mathfrak{q}_1^2 + \mathfrak{q}_2^2 - \mathfrak{q}_3^2 - \mathfrak{q}_4^2 = 0,$$
or
b) $$\neq 0.$$
In both cases, one must have:

$$\lambda^2 (\mathfrak{q}_1^2 + \mathfrak{q}_2^2 - \mathfrak{q}_3^2 - \mathfrak{q}_4^2) - 2\,\lambda\,\mathfrak{q}_0\,\mathfrak{q}_5 = 0,$$

from which, one can recognize that in case:

a) λ and $\mathfrak{q}_5$ can be chosen to be completely arbitrary (but finite), while in case

b) One must have $\lambda = 0$, while $\mathfrak{q}_5$ is arbitrary.

Therefore, in case *a*), the $\mathfrak{q}_0 = 0$, $\mathfrak{q}_1$, $\mathfrak{q}_2$, $\mathfrak{q}_3$, $\mathfrak{q}_4$ are determined up to a proportionality factor (that can also be zero), and $\mathfrak{q}_5$ is arbitrary, so the chosen system of $\mathfrak{q}$ will belong, not to one system of values of the homogeneous quantities q, but to ∞^1 different ones.

In case *b*), $\mathfrak{q}_0$, $\mathfrak{q}_1$, $\mathfrak{q}_2$, $\mathfrak{q}_3$, $\mathfrak{q}_4$ are all zero, and only $\mathfrak{q}_5$ differs from zero. Corresponding systems of quantities $\mathfrak{q}$ then belong to a single system of values for the homogeneous quantities in all of these *b*) cases.

Conversely, when a real point-pair ($q_0 = 0$, q_1, q_2, q_3, q_4, q_5) is given in the second (i.e., quadratic) continuum, and one seeks the corresponding $\mathfrak{q}$ then one will first find that $\mathfrak{q}_0 = 0$ and then that in the case: $q_1 = q_2 = q_3 = q_4 = 0$, the homogeneous quantities $\mathfrak{q}_1$, $\mathfrak{q}_2$, $\mathfrak{q}_3$, $\mathfrak{q}_4$ will be completely indeterminate, while in every other case, the latter system of values will be completely determinate (naturally, up to an arbitrary proportionality factor).

It emerges from these statements that the imaginary elements of the linear continuum and those of the quadratic continuum are not in one-to-on correspondence with each other. This situation is entirely analogous to the behavior of corresponding points under the stereographic projection of a plane onto a sphere, so, in general, the transition from the five homogeneous $\mathfrak{q}$ to the six homogeneous q can be interpreted as a kind of stereographic projection of a four-dimensional linear manifold onto a four-dimensional quadratic one (in a five-dimensional linear space).

One defines the "continuous" change of a point-pair of the linear (quadratic, respectively) continuum to be one for which the associated homogeneous coordinates $\mathfrak{q}$ (q, resp.) change continuously while they still remain similar, and at least one of the quantities in question remains finite and non-zero. A change in one of the two continua can very well prove to be discontinuous without that needing to be the case for the corresponding change in the other continuum. Two continuous passages to the limit in one continuum that both have the same imaginary element of the continuum for their limit element can lead to different imaginary elements in the other continuum as limit elements when they are transferred to it.

§ 2. Map of both continua of point-pairs of the plane to analogous continua of oriented spheres in space.

The ∞^4 oriented spheres of the three-dimensional space define an analogue to the ∞^4 point-pairs of the plane. An "oriented" sphere comes about when, for a sphere of finite radius and a center that lies at a finite point, one of the two equal and opposite values that the radius can assume is distinguished and is established as the value of the radius. A sphere that is defined in this way is a called a "real," oriented sphere. It is determined completely by the givens of the rectangular coordinates ξ, η, ζ of the center and the given of the radius ρ (including the choice of sign).

On the one hand, the ∞^4 oriented spheres can be mapped to the linear manifold of points in a four-dimensional space when one regards the quantities ξ, η, ζ, ρ as the rectangular coordinates of a point in four-dimensional space. On the other hand, they can be mapped to a quadratic manifold of points in a five-dimensional linear space when one adds one-half power of the sphere in question relative to the coordinate origin to the aforementioned four quantities as a fifth quantity and regards these five quantities, which are coupled to each other by a quadratic equation, as the rectangular coordinates of a point in a five-dimensional space.

For both maps, the image points of the real, oriented spheres do not at all lie on finite points of the point-manifolds in question. The infinitely-distant points – in the projective sense – are not images of real, oriented spheres.

In order to extend the totality of oriented spheres in three-dimensional space to a closed continuum, one can either base that upon one or the other of the two maps above.

If one bases it upon the first map and introduces ∞^3 different "imaginary," oriented spheres, corresponding to the ∞^3 different infinitely-distant points of the four-dimensional image space then the continuum of oriented spheres will be extended to a closed *linear* continuum.

If one bases it upon the second map and introduces ∞^3 different "imaginary" spheres (which are entirely distinct from the previous ones), corresponding to the ∞^3 different infinitely-different points of the quadratic point-manifold in five-dimensional space then that continuum of oriented spheres will seem to have been extended to a closed *quadratic* continuum (that is completely different from the previous one), which is, without exception, in one-to-one correspondence with the points of the quadratic point-manifold in five-dimensional space.

The elements of the linear continuum of oriented spheres can be represented by the five homogeneous coordinates $\overline{\mathrm{q}}$ according to the proportion:

$$\overline{q}_0 : \overline{q}_1 : \overline{q}_2 : \overline{q}_3 : \overline{q}_4 = 1 : \xi : \eta : \zeta : \rho,$$

analogous to the elements of the quadratic continuum, which can be represented by six homogeneous coordinates $\overline{q}$ according to the proportion:

$$\overline{q}_0 : \overline{q}_1 : \overline{q}_2 : \overline{q}_3 : \overline{q}_4 : \overline{q}_5 = 1 : \xi : \eta : \zeta : \rho : \tfrac{1}{2}(\xi^2 + \eta^2 + \zeta^2 - \rho^2).$$

The same thing is true of the relationship between the two continua that was stated above for the relationship between the analogous continua of point-pairs in the plane.

The tangential planes to a real, oriented sphere are likewise oriented. If one establishes that the distance from the center of the sphere to one of its tangential planes, including the sign, shall have the value ρ then distance from each point in space to the tangent plane in question, including the sign, will be determined from it.

Analytically, the process of orienting a plane whose equation, when written in the running coordinates X, Y, Z reads:

$$U_0 + U_1 X + U_2 Y + U_3 Z = 0,$$

represents the idea that of the two equal and opposite values of the square root $\sqrt{U_1^2 + U_2^2 + U_3^2}$, one of them – say, U^* – will be distinguished; the five homogeneous quantities U_0, U_1, U_2, U_3, U^*, between which the homogeneous, quadratic relation exists that:

$$U_1^2 + U_2^2 + U_3^2 = U^{*2},$$

are regarded as homogeneous coordinates of the plane thus "oriented." Two oppositely oriented, coincident planes will differ only by the sign of their U^*–coordinate.

Minimal planes (viz., tangential planes to the absolute sphere-circle), for which $U^* = 0$, cannot be distinguished by opposite orientations at all, since they will have only one orientation, or – if one prefers – both orientations will be the same one.

An imaginary sphere ($\overline{q}_0 = 0$) of the linear continuum determines a system of ∞^2 oriented planes:

$$\overline{q}_1 U_1^2 + \overline{q}_2 U_2^2 + \overline{q}_3 U_3^2 = \overline{q}_4 U^*,$$

which are to be regarded as its tangent planes and which contact the absolute sphere-circle doubly. Each such system of ∞^2 oriented planes is to be regarded as a representation of a certain imaginary, oriented sphere in the linear continuum.

The imaginary spheres of the quadratic continuum are represented in a completely different way: Such an imaginary sphere, with the homogeneous coordinates $\overline{q}$ ($\overline{q}_0 = 0$), can be represented by the oriented plane with the coordinates $U_0 = \overline{q}_5$, $U_1 = \overline{q}_1$, $U_2 = \overline{q}_2$, $U_3 = \overline{q}_3$, $U^* = \overline{q}_4$. Any imaginary, oriented sphere of the quadratic continuum is to be regarded as an oriented plane in this way, and conversely.

The linear continuum of oriented spheres can be related to the linear continuum of points pairs in the plane that was considered in § 1 in a one-to-one way by the proportion:

$$\bar{\mathfrak{q}}_0 : \bar{\mathfrak{q}}_1 : \bar{\mathfrak{q}}_2 : \bar{\mathfrak{q}}_3 : \bar{\mathfrak{q}}_4 = \mathfrak{q}_0 : \mathfrak{q}_1 : \mathfrak{q}_2 : i\, \mathfrak{q}_3 : \mathfrak{q}_4 \quad (\text{where } i = \sqrt{-1}),$$

which we will call map $\mathfrak{A}$.

The quadratic continuum of oriented spheres can be put into one-to-one correspondence with the quadratic continuum of point-pairs in the plane that was considered in § 1 by the proportion:

$$\bar{q}_0 : \bar{q}_1 : \bar{q}_2 : \bar{q}_3 : \bar{q}_4 : \bar{q}_5 = q_0 : q_1 : q_2 : i\, q_3 : q_4 : q_5 \quad (\text{where } i = \sqrt{-1}),$$

which we will call map A.

The two maps $\mathfrak{A}$ and A are imaginary; one can replace them with real maps when one suppresses the factor i in the proportions above. The real maps would not have the same uses then in the following development as the imaginary maps $\mathfrak{A}$ and A that were introduced.

§ 3. Map of the quadratic continuum of point-pairs in the plane to the (Plückerian – i.e., understood in the projective sense) lines in three-dimensional space.

If one extends the rectangular coordinate system OXY in the plane to a right-handed spatial system by adding the Z-axis perpendicular to the XY-plane then one can assign any real point-pair $\mathfrak{M}(\mathfrak{x}, \mathfrak{y})$ $M(x, y)$ to the line:

$$\left\{ \begin{aligned} X &= \frac{x+\mathfrak{x}}{2} - \frac{y-\mathfrak{y}}{2} Z, \\ Y &= \frac{y+\mathfrak{y}}{2} + \frac{x-\mathfrak{x}}{2} Z, \end{aligned} \right\}$$

as an image line, where X, Y, Z are the running coordinates of a point of the line. Geometrically, this map can be exhibited as follows:

Rotate the point M through a right angle in the positive sense around the bisector H of the point-pair $\mathfrak{M}M$, and thus raise its new position that is so obtained by + 1 *in the sense of the positive Z-axis. The connecting line between the point thus-obtained and the bisector H will give the desired image line.*

The image lines of the real point-pairs fill up all of the line space, with the exception of the special linear complexes, whose guiding line $\mathfrak{Q}$ is the infinitely-distant line in the XY-plane. This complex, which shall be referred to briefly as the complex $\mathfrak{Q}$, remains free of lines that are images of real point-pairs (in the XY-plane).

Two such intersecting lines that do not belong to the complex $\mathfrak{Q}$ are images of real point-pairs $\mathfrak{M}_1 M_1$ and $\mathfrak{M}_2 M_2$ whose:

$$\text{“left” distance } \overline{\mathfrak{M}_1\mathfrak{M}_2} \text{ is equal to its “right” distance } \overline{M_1 M_2}\,,$$

and conversely, such real point-pairs correspond to two intersecting lines that do not belong to the complex $\mathfrak{Q}$.

This important property of the map in question can be proved in a purely geometric way with the help of the construction above that brought about the map itself, when one takes the connection, which is exhibited similarly, between the coordinates of any point-pair $\mathfrak{M}M$ and the Plückerian coordinates p_{ik} of its image lines to be the starting point.

If one introduces the latter coordinates p_{ik} in such a way that the p_{ik} that belong to the connecting line between two points (X_1, Y_1, Z_1) and (X_2, Y_2, Z_2) is defined by the proportion:

$$p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{11}$$

$$= (X_2 - X_1) : (X_2 - X_1) : (X_3 - X_1) : (Y_1\, Z_2 - Y_2\, Z_1) : (Z_1\, X_2 - Z_2\, X_1) : (X_1\, Y_2 - X_2\, Y_1)$$

then the Plückerian coordinates p_{ik} of the image line of a real point-pair MM (in the XY-plane) will be expressed in terms of its homogeneous coordinates q in the following way:

$$p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{11}$$

$$= -\,(q_4 - q_2) : (q_3 - q_1) : 2q_0 : (q_4 + q_2) : -\,(q_3 + q_1) : -\,q_1\,.$$

This proportion also associates the imaginary point-pairs of the quadratic continuum ($q_0 = 0$, q_1, q_2, q_3, q_4, q_5) with well-defined image lines that belong to the special complex $\mathfrak{Q}$ in a one-to-one way. The entire quadratic continuum of point-pairs then seems to be related to the (quadratic) continuum of lines in the five-dimensional image space in a one-to-one way. This, exceptionless one-to-one map shall be referred to as *map B*.

In place of the six homogeneous p_{ik} it is useful for many purposes to introduce the six homogeneous q, which are indeed homogeneous-linearly connected with them according to the proportions above, as the homogeneous line coordinates in space; to distinguish them from the p_{ik}, they shall be referred to as the *Q-coordinates* in line space. The quadratic relation between them reads:

$$q_1^2 + q_2^2 - q_3^2 - q_4^2 - 2\, q_0\, q_5 = 0$$

in agreement with § 1.

By employing the Q-coordinates in line space, any point-pair of the quadratic continuum of point-pairs of the plane will have the same homogeneous coordinates q as the associated image line.

Intersecting lines belong to systems of values of the Q-coordinates that are conjugate relative to the quadratic form:

$$q_1^2 + q_2^2 - q_3^2 - q_4^2 - 2\, q_0\, q_5\, .$$

If one expresses this by the equation:

$$q_1 q_1' + q_2 q_2' - q_3 q_3' - q_4 q_4' - q_0 q_5' - q_5 q_0' = 0$$

between the Q-coordinates q and q' of both lines then this equation will likewise give the relation that exists between the associated point-pairs in the XY-plane. If one introduces the inhomogeneous coordinates $\mathfrak{x}$, $\mathfrak{y}$, x, y and $\mathfrak{x}'$, $\mathfrak{y}'$, x', y' of the two point-pairs, instead of the q and q', then equation that emerges from the equation above will be written:

$$(\mathfrak{x}' - \mathfrak{x})^2 + (\mathfrak{y}' - \mathfrak{y})^2 = (x' - x)^2 + (y' - y)^2,$$

which will also provide the analytical proof that – as is often remarked – the "left" and "right" distances between real point-pairs are equal if and only if their image lines (under map B) intersect.

If one maps the point-pairs of the quadratic continuum of the plane, once by the map A (of § 2) onto the quadratic continuum of the oriented spheres in a three-dimensional space Σ, and once by the map B (this § 3) onto the (quadratic) continuum of the lines in a three-dimensional space S then the stated continua will be put into one-to-one correspondence with each other.

Intersecting lines in the space S correspond to contacting, oriented spheres in Σ, and conversely, whereby two real ([1]), oriented spheres are regarded as contacting each other if and only if the tangential planes at the contact point that belong to the two spheres coincide, not only in position, but also in orientation. The relationship between the spaces Σ and S thus-obtained will then be identical with the celebrated *Lie affinity* between the sphere and line spaces.

§ 4. Representation of the point transformations of the plane by image congruences in line space on the basis of the map *B* of § 3.

If one has any point-transformation in the XY-plane then one can exhibit a point-pair $\mathfrak{M}M$ (several point pairs, resp., when the transformation is many-valued) by means of any point $\mathfrak{M}$ and the point M that corresponds to it under the transformation (the corresponding points, resp.). The image lines of the ∞^2 real point-pairs thus-obtained – together with a likewise given restriction – will fulfill a certain line congruence in image space, which can be regarded as the image of the point transformation.

The restriction that was just mentioned relates to the fact that, at first, no rays are present in the image congruence that also belong to the special complex $\mathfrak{Q}$. Thus,

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([1]) The concept of "contact" must be defined specially for the imaginary, oriented spheres if anything but the assertion above is to remain true, in general. This definition is provided by the formulas.

E.g., two real, oriented planes, when regarded as imaginary, oriented spheres are regarded as "contacting" each other if and only if they are parallel and oriented consistently.

nothing stands in the way of subsequently adding these still-lacking rays by analytic continuation and extending the image congruence to a closed continuum, in this sense.

Now, it is typical of the importance of the map B that was introduced here that precisely the simplest point transformations of the plane will again correspond to very simple image congruences in line space.

1. A rigid motion in the plane, when regarded as a point transformation, corresponds to a bundle of rays in the image space whose carrier point is considered to be the image point of the rigid motion.

If the rigid motion is a rotation around the point C through the angle γ then the image point of this rotation will on a line that is perpendicular to the center of rotation C at a height of cot γ/ 2.

If the rigid motion is a translation then the image point will be an infinitely-distant point (in the sense of projective geometry) and simple to construct when one defines a point-pair $\mathfrak{M}M$ from an arbitrary point $\mathfrak{M}$ and any point M to which $\mathfrak{M}$ goes under the translation, determines the image line that belongs to the point-pair $\mathfrak{M}M$, from § 3, and looks for its infinitely-distant point. That will be the desired image point.

The image points of the rigid motions fill up the image space, except for the two conjugate-imaginary planes:

$$I_1 \quad (Z = +i) \quad \text{and} \quad I_2 \quad (Z = -i),$$

which remain free.

2. A *transfer* in the plane (i.e., a point transformation that consists of the composition of a rigid motion with a reflection in a line in the plane), which can always be replaced with a reflection in a certain line in the plane – viz., the “transfer axis,” c – in conjunction with a translation in the direction of the transfer axis, corresponds to a ray field in image space whose carrier plane is regarded as the image plane of the transfer.

The image plane goes through the transfer axis c and is simple to construct when one chooses a point $\mathfrak{M}$ in the transfer axis arbitrarily and defines a point-pair from any point M (which also lies on the transfer axis) that $\mathfrak{M}$ goes to under the transfer, determines the image line of this point-pair, as in § 3, and connects it to the transfer axis with a plane. This plane will be the desired image plane. Its angle γ with the XY-plane is thus determined by:

$$\cot \frac{\gamma}{2} = \text{the magnitude of the translation parallel to } c.$$

The image planes of the transfers in the plane fill up the space of planes, except for the two bundles of planes that go through the absolute circle points:

$$k_1 \quad (y = ix) \quad \text{and} \quad k_2 \quad (y = -ix)$$

in the XY-plane. The latter plane bundle remains free of image planes.

3. A direct similarity transformation in the plane corresponds to a linear congruence in image space whose skew guiding lines go through k_1 and k_2, and conversely, any such congruence corresponds to a direct similarity transformation.

4. An indirect similarity transformation in the plane corresponds to a linear congruence in image space whose skew guiding lines lie in the conjugate-imaginary planes I_1 and I_2, and conversely.

5. An affine transformation in the plane corresponds to a linear congruence in image space whose guiding lines lie in the special complex $\mathfrak{Q}$, and are therefore parallel to the XY-plane.

6. A bifocal conversion of the kind that was considered by L. Burmester in his textbook on kinematics that has the focal points $F_1\,L_1$ and $F_2\,L_2$ – i.e., any conversion under which any point $\mathfrak{M}$ will be assigned to two corresponding point M by the demand that:

$$\overline{F_1\mathfrak{M}} = \overline{F_2\mathfrak{M}} \qquad \text{and} \qquad \overline{L_1\mathfrak{M}} = \overline{L_2\mathfrak{M}},$$

corresponds to a linear congruence in image space whose guiding lines do not belong to the special complex $\mathfrak{Q}$ (and thus, are not parallel to the XY-plane), and will be represented by the real point-pairs $F_1\,F_2$ and $L_1\,L_2$, in the sense of the map B.

7. A direct Moebius circle conversion corresponds to a congruence in image space that consists of the common tangents to two (in the case of a real conjugate-imaginary conversion) cones of second order whose vertices lie at the points k_1, k_2, resp., and which contact the planes I_1 and I_2, resp.

8. An indirect Moebius circle conversion corresponds to a congruence in image space that consists of the common secants to two conic sections that lie in the planes I_1, I_2, resp., and go through the two points k_1, k_2 .

Along with those congruences that can be regarded as the image congruences of (non-degenerate) point transformations, we still have to distinguish the ones for which that is not the case. The ∞^2 point-pairs $\mathfrak{M}M$ that belong to the rays of a congruence can have the peculiarity that one of the two points of the pair is bound to a curve while the other one correspondingly describes the ∞^1 possible positions of the former ∞^1 curves, or also the peculiarity that one of the two points of the pair is fixed while the other one can range through the entire XY-plane (cf., § 7). Furthermore, we point out here the congruences that are contained in the special complex $\mathfrak{Q}$ whose rays do not belong to any real point-pairs, at all.

§ 5. Representation of the ∞^3 lines of a special linear complex in image space by the corresponding ∞^3 point-pairs in the *XY*-plane. Types of imaginary point-pairs in the quadratic continuum.

If $\mathfrak{G}$ is a line that is skew to the infinitely-distant line $\mathfrak{Q}$ in the *XY*-plane then one can represent the ∞^3 lines of the special linear complex that is associated with $\mathfrak{G}$ as its guiding line by the corresponding ∞^3 point-pairs in the plane. The system that is defined by them might likewise be referred as a "special linear complex of point-pairs"; the (real) point-pair $\mathfrak{M}M$ that corresponds to the line $\mathfrak{G}$ itself will be called the "guiding pair" of the complex.

The system of ∞^3 point-pairs $\mathfrak{M}M'$ of the complex is determined from the guiding pair $\mathfrak{M}M$ in the simplest way: Define the family of concentric circles around $\mathfrak{M}$ to be the "left" family of circles and the analogous family of circles around M to be the "right" family of circles, and if one calls these two families congruent to each other when one associates each circle of the left family with the circle in the left family that is conjugate to it then the two points of each point-pair $\mathfrak{M}'M'$ that belongs to the complex will lie upon corresponding circles of the two congruently-related families of circles, and indeed each of the points will lie upon the circle of the family of circles that has the same name.

The perpendicular projection $\mathfrak{g}$ of the line $\mathfrak{G}$ on the *XY*-plane is likewise symmetric for the points $\mathfrak{M}$ and M; it includes those ∞^1 point-pairs of the complex whose left and right points cover it, namely, the "covering pairs" of the complex.

The complex is associated with ∞^1 motions, and likewise ∞^1 transfers, that take the point $\mathfrak{M}$ to the point M, and thus each circle of the left family of circles to the congruent circle of the right family of circles; the image points of these ∞^1 motions lie on $\mathfrak{G}$, while the image planes of the ∞^1 transfers go through $\mathfrak{G}$.

If the guiding line $\mathfrak{G}$ of a special linear complex in space cuts the infinitely-distant line $\mathfrak{Q}$ in the *XY*-plane then the corresponding congruence of point-pairs will no longer have any real guiding pair; the guiding pair that corresponds to the guiding line $\mathfrak{G}$ is a certain *imaginary point-pair* of the quadratic continuum. Each such point-pair can be represented intuitively by the special linear complex that has it for a guiding pair.

In order to assess the various possible special cases, for the time being, we shall start with the equations:

$$q_0 = 0, \qquad q_1^2 + q_2^2 - q_3^2 - q_4^2 = 0$$

between the coordinates q of the line $\mathfrak{G}$ (and the corresponding imaginary point-pair).

The left point $\mathfrak{M}(\mathfrak{x}, \mathfrak{y})$ and the right point $M(x, y)$ of the imaginary point-pair are defined, as for real point-pairs, by the proportion:

$$1 : \mathfrak{x} : \mathfrak{y} = q_0 : q_1 : q_2 \qquad\qquad (1 : x : y = q_0 : q_3 : q_4\text{ , resp.}).$$

One must infer from this that the left point is completely indeterminate in the case of the simultaneous vanishing of q_1, q_2 (q_0 is zero, anyway), while the right point is completely indeterminate in the case of the simultaneous vanishing of q_3, q_4 (q_0 is zero, anyway).

If the two aforementioned special cases are excluded, for the moment, then the left (and likewise the right) point of such an imaginary point-pair will be a well-defined infinitely-distant point $\mathfrak{M}_\infty$ (M_∞ , resp.).

However, the imaginary point-pair in question is in no way determined by the aforementioned infinitely-distant points. Moreover, there is an entire class of infinitely many different imaginary points of the quadratic continuum that all have the stated points for their left and right points; let this class be denoted by ($\mathfrak{M}_\infty$, M_∞).

The relation $q_1^2 + q_2^2 - q_3^2 - q_4^2 = 0$ teaches us that the points $\mathfrak{M}_\infty$ and M_∞ are either:

a) both different from the absolute circle points k_1, k_2 or

b) they both agree with the aforementioned circle points, and indeed, either both of them are at the same circle point or one at the one circle point and the other at the other.

When case *b*) is present, the imaginary point-pair in question will belong to one of four classes:

$$(k_1, k_1), (k_2, k_2), (k_1, k_2), (k_2, k_1).$$

As far as the previously-excluded cases are concerned, for which the left or right point is completely indeterminate, it follows from the relation $q_1^2 + q_2^2 - q_3^2 - q_4^2 = 0$ that the respective other point of the imaginary point-pair can only be one of the two absolute circle points k_1 and k_2 , if it is not perhaps also itself indeterminate. It thus yields:

$$(k_1, 0), (k_2, 0)$$

for the completely-indeterminate classes of the right point, and

$$(0, k_1), (0, k_2)$$

for the completely-indeterminate classes of the left point.

The left and right point of an imaginary point-pair of the quadratic continuum can likewise be indeterminate only when all of its coordinates up to q_5 vanish. This imaginary point-pair, whose image line is the infinitely-distant line $\mathfrak{Q}$ in the *XY*-plane, shall be referred to as the point-pair (0).

The special linear complexes that are associated with the types of imaginary point-pairs that were just enumerated and which have the point-pairs in question for their guiding pairs will now be considered, which will yield an intuitive representation for the stated imaginary point-pairs:

For the imaginary point-pairs $\mathfrak{A}$ whose left and right points, $\mathfrak{M}_\infty$ and M_∞, resp., are infinitely-distant and different from both of the absolute circle points k_1 and k_2, the associated special linear complex of point-pairs $\mathfrak{M}'M'$ can be exhibited most simply by means of certain congruence relation between the "left" bundle of parallel rays in the XY-plane that are perpendicular to the direction $\mathfrak{M}_\infty$ and the "right" bundle of parallel rays in that plane that are perpendicular to the M_∞ : The left point $\mathfrak{M}'$ and the right point M' of an arbitrary point-pair that belongs to the complex will always lie on corresponding rays (in the congruence relation) of the left and right bundles of parallel rays.

Accordingly, the figure of two congruently-related bundles of parallel rays that are perpendicular to the infinitely-distant points $\mathfrak{M}_\infty$ and M_∞ can serve as an intuitive representation of such an imaginary point-pair $\mathfrak{A}$.

If the imaginary point-pair $\mathfrak{A}$ belongs to one of the classes:

$$(k_1, k_1), (k_2, k_2), (k_1, k_2), (k_2, k_1)$$

then, instead of the two congruently-related bundles of parallel rays above, one will get similarly-related minimal bundles of rays, such that the carrier points of the left and right minimal bundles of rays can be regarded as the left (right, resp.) point of the class symbol in question. One can regard the figure of the two similarly-related minimal bundles of rays as an intuitive representation of such a point-pair $\mathfrak{A}$.

For imaginary point-pairs that belong to one of the classes:

$$(\mathfrak{M}_\infty, M_\infty), (k_1, k_1), (k_2, k_2),$$

where $\mathfrak{M}_\infty$ M_∞ mean any two infinitely-distant points that are different from k_1 and k_2, the congruence or (in the last two cases) similarity relation between the left and right associated bundles of parallel rays will be exhibited by an arbitrary motion of a certain bundle of ∞^1 motions. The image points of these ∞^1 motions trace out the image line of the imaginary point-pair, which (corresponding to the three enumerated classes) meets the infinitely-distant line $\mathfrak{Q}$ in the XY-plane at k_1, k_2, or a point that is different from k_1 and k_2, and lies either in the plane I_1 or in the plane I_2 .

For imaginary point-pairs that belong to one of the classes $(\mathfrak{M}_\infty, M_\infty)$, (k_1, k_2), (k_2, k_1), the congruence or (in the last two cases) similarity relation between the two bundles of parallel rays can be exhibited by an arbitrary transfer of a certain bundle of ∞^1 transfers. The image planes of these ∞^1 transfers go through the image line of the imaginary point-pair that (corresponding to the three enumerated classes) belongs to the plane I_1 or to the plane I_2 , or neither of the two planes I_1, I_2 , and goes through neither of the points k_1 and k_2 .

The image lines of the imaginary point-pair that was considered up to now, which possess certain left and right points, trace out the special linear complex $\mathfrak{Q}$ with the exclusion of the four ray bundles:

$(k_1\ I_1)$, $(k_2\ I_2)$: (Left pair of ray bundles in image space),
$(k_1\ I_2)$, $(k_2\ I_1)$: (Right pair of ray bundles in image space).

In particular, the imaginary point-pairs with the classes:

$$(k_1\ k_1),\ (k_2\ k_2),\ (k_1\ k_2),\ (k_2\ k_1)$$

belong to lines in image space that belong to the ray bundle k_1 or k_2 or to the ray field I_1 or the field I_2 .

It remains for us to consider those imaginary point-pairs for which the left or right point remains indeterminate, and thus, point-pairs of one of the four classes:

$$\left\{ \begin{matrix} (k_1, 0), & (k_2, 0) \\ (0, k_1), & (0, k_2). \end{matrix} \right\}$$

Any such imaginary point-pair $\mathfrak{A}$ is the guiding pair of a special linear complex of point-pairs $\mathfrak{M}'M'$.

If $\mathfrak{A}$ belongs to one of the classes $(k_1, 0)$, $(k_2, 0)$ then $\mathfrak{M}'$ will be linked to a certain minimal line through k_1 (k_2, resp.), while M' will remain completely indeterminate; the aforementioned minimal line, which is regarded as the locus of the left points, can be regarded as an intuitive representation of $\mathfrak{A}$.

If $\mathfrak{A}$ belongs to one of the classes $(0, k_1)$, $(0, k_2)$ then M' will be linked to a certain minimal line through k_1 (k_2, resp.), while $\mathfrak{M}'$ will remain completely indeterminate; the aforementioned minimal line, which is regarded as the locus of the right points, can be regarded as an intuitive representation of $\mathfrak{A}$.

The image line of an imaginary point-pair $\mathfrak{A}$, which is represented by a minimal line that is regarded as a $\begin{matrix} \text{left line, i.e., the locus of left points} \\ \text{right line, i.e., the locus of right points} \end{matrix}$, lies perpendicular to the minimal line in question of a ray bundle of the $\begin{matrix} \text{left} \\ \text{right} \end{matrix}$ ray-bundle-pair $\begin{matrix} (k_1 I_1), (k_2 I_2) \\ (k_1 I_2), (k_2 I_1) \end{matrix}$.

The line $\mathfrak{Q}$ in image space is the image line of the point-pair that was denoted by (o) above, whose associated special complex consists of the totality of all imaginary point-pairs.

Remark. The special linear complex that belongs to an imaginary point-pair A with class (k_1, k_2) as a guiding pair contains a "congruence" of ∞^2 real point-pairs of a certain direct similarity transformation that belongs to the congruence of real lines that meet the image line $\mathfrak{G}$ of $\mathfrak{Q}$ (and the line $\bar{\mathfrak{G}}$ that is conjugate-imaginary to $\mathfrak{G}$) as image congruence.

$\mathfrak{G}$ goes through k_1, projects perpendicular to the XY-plane onto that line that connects the center of the similarity to k_1, and has the altitude:

$$\cot \tfrac{1}{2}(\alpha - i \ln \kappa),$$

if α denotes the angle of rotation and κ denotes the dilatation factor of the similarity transformation.

The special linear complex that belongs to an imaginary point-pair $\mathfrak{A}$ with class (k_2, k_1) as its guiding pair contains a "congruence" of ∞^2 point-pairs of a certain indirect similarity transformation that belongs to the congruence of real lines that meet the image line $\mathfrak{G}$ of $\mathfrak{A}$ (and the line $\bar{\mathfrak{G}}$ that is conjugate-imaginary to $\mathfrak{G}$). $\mathfrak{G}$ lies in I_1 and projects perpendicularly to the XY-plane onto a line that goes through the similarity center and subtends the angle:

$$\tfrac{1}{2}(\alpha - i \ln \kappa)$$

with the X-axis, if the indirect similarity transformation can be composed of a dilatation with modulus κ from the center and a reflection in a line that goes through the center and is inclined to the X-axis by an angle of $\alpha / 2$.

§ 6. General remarks about complexes and ruled surfaces in image space regarding their representation by point-pairs in the plane.

A complex of lines in image space, or the complex point-pairs that corresponds to it, can be defined by a homogeneous equation $\Omega(q_0, q_1, q_2, q_3, q_4, q_5) = 0$, according to whether the six homogeneous q are interpreted as the coordinates of a line in space or the coordinates of the corresponding point-pair in the XY-plane.

Here, we might consider an example of those (linear) complexes of points-pairs that do not correspond to special linear complexes of rays in image space.

If the infinitely-distant line $\mathfrak{Q}$ in the XY-plane is not contained in the spatial complex then it will be conjugate to a line $\mathfrak{G}$ (which is skew to it) relative to the complex. It will correspond to a real point-pair $\mathfrak{M}M$ in the XY-plane. The real point-pair $\mathfrak{M}'M'$ of the complex in the XY-plane can then be represented by a relation between the distances $\overline{\mathfrak{M}\mathfrak{M}'}$ and $\overline{MM'}$ that takes the form:

$$\overline{\mathfrak{M}\mathfrak{M}'}^2 - \overline{MM'}^2 = \text{const.} \neq 0.$$

(The value zero of the constant that appears here would correspond to a special complex.)

By contrast, if the line $\mathfrak{Q}$ is contained in the spatial complex then – if one temporarily overlooks a specialization that will be mentioned below – the associated complex of point-pairs in the XY-plane can be generated by a certain affine relationship between two bundles of parallel rays in the XY-plane, one of which is left and the other of which is right: The points $\mathfrak{M}'$ and M' of the real point-pairs of the complex will lie on corresponding rays of the two affinely-related bundles of parallel rays.

If all lines of the $\begin{matrix}\text{left}\\ \text{right}\end{matrix}$ pair of ray bundles $\begin{matrix}(k_1 I_1), (k_2 I_2)\\ (k_1 I_2), (k_2 I_1)\end{matrix}$ are contained in the complex, except for the line $\mathfrak{Q}$, then the corresponding complex of point-pairs will have an especially simple property: It will then consist of all point-pairs $\mathfrak{M}'M'$ whose $\begin{matrix}\text{right}\\ \text{left}\end{matrix}$ point is linked to a certain line in the *XY*-plane (that lies at finite points), while the other one – viz., the $\begin{matrix}\text{left}\\ \text{right}\end{matrix}$ point – will remain completely arbitrary. (The line in question can go through either k_1 or k_2, since otherwise the complex would be special.)

Under the map *B* in the *XY*-plane, the lines of a ruled surface in image space will correspond to two point-wise related curves, in such a way that any point $\mathfrak{M}$ of the first curve will define a pair $\mathfrak{M}M$ with the associated point (or the associated points) *M* on the second curve whose image line is a generator of the ruled surface. Thus, a fixed point can also appear in place of a curve that is to be combined into a point-pair with any point of the other curve.

Developable ruled surfaces are distinguished under the map by the fact that under the point-wise relationship between the two curves above corresponding arc elements of the two curves will have equal length, and conversely, when that latter condition is present, the image lines of the point-pairs of the system that consists of corresponding points of the two curves will define the generators of a developable ruled surface.

In regard to the skew second-order ruled surfaces, one must decide whether they do or do not contain the infinitely-distant line $\mathfrak{Q}$ in the *XY*-plane.

In the first case, the two aforementioned curves will be straight lines, and the point-wise relationship between them will be mediated by an affine transformation; in particular, it can happen that a single point appears in place of a line, which is then to be combined with any point o the other line. If the two affine point-sequences on the two lines are congruent then they will be associated, not with a skew ruled surface in image space, but with a planar ray bundle in space.

In the second case, we shall next make the restriction that the skew ruled surface in question belongs to a second-order surface that is symmetric with respect to the *XY*-plane. The two aforementioned curves will be confocal conic sections [$\mathfrak{M}$] and [*M*] here; the point-wise relationship between them will be one of the (four, for the center conic sections, two, for parabolas) affinities that will be exhibited between [$\mathfrak{M}$] and [*M*] by the other conic section of the confocal family. The lines of the other ruled family of the skew ruled surface will be represented by point-pairs $\mathfrak{N}N$, for which:

the conic section	[$\mathfrak{N}$]	that is described by	$\mathfrak{N}$	covers	[*M*],
“ “	[*N*]	“ “	*N*	“	[*N*],

and the affine relation between these conic relations will agree with the one above.

Since their image lines (in space) will intersect, any two point-pairs $\mathfrak{M}M$ of the first and $\mathfrak{N}N$ of the second family must have equal left and right distances:

$$\overline{\mathfrak{M}\mathfrak{N}} = \overline{MN},$$

which says nothing but the well-known theorem of Ivory on confocal conic sections. The two conic sections that appear here can degenerate into two pairs of lines in the *XY*-plane that have common symmetry lines (corresponding to the degeneracy of both ruled families of the surface into two pairs of ray bundles that lie on a hyperboloid). They can, moreover, also be concentric circles that are related to each other point-wise by their diameters; therefore, a point can also appear in place of one of the two circles, namely, the center of the other circle.

From now on, the restriction above shall be dropped, and a general second-order ruled surface that is not symmetric with respect to the *XY*-plane will be considered that does not contain the infinitely-distant line $\mathfrak{Q}$ of the *XY*-plane. The figure of the two families of ∞^1 point-pairs that correspond to the two ruled families of the surface can be described most simply when one starts with the figure of the case that was considered above (the symmetric position of the surface relative to the *XY*-plane). If one thinks of the ∞^1 point-pairs $\mathfrak{M}M$ ($\mathfrak{N}N$, resp.) that correspond to the lines of one and the other family in the case of the previously-considered symmetric surface position, then establishes the left point $\mathfrak{M}$ ($\mathfrak{N}$, resp.) and the right point M (N, resp.), and subjects them to a well-defined rigid motion (that is the same for all points) by which one arrives at M' (N', resp.) then the image lines of the point-pairs:

$$\mathfrak{M}M' \quad \text{and} \quad \mathfrak{N}N'$$

will represent the two line families of a second-order ruled surface of general case that is considered here. The points M' (N', resp.) will describe conic sections $[M']$ ($[N']$, resp.) that are congruent to the conic sections $[\mathfrak{N}]$ ($[\mathfrak{M}]$, resp.). For any two point-pairs $\mathfrak{M}M'$ and $\mathfrak{N}N'$ of one and the other family, the right and left distances will be, in turn, equal to:

$$\overline{\mathfrak{M}\mathfrak{N}} = \overline{M'N'}.$$

With that, the representation of the two ruled families of a skew ruled surface is dealt with in terms of point-pairs in the *XY*-plane.

The general figure in the *XY*-plane that one arrives at has especial interest insofar as it contains a solution of the following "singular problem":

Four points $\mathfrak{M}$, M', $\mathfrak{N}$, N' *shall move on each of four curves (in the plane), whose determination is the objective of the problem, and indeed, in such a way that the points in their motions that are denoted by the same symbols depend on each other according to laws whose determination is the objective of the problem, while the motion of the point-*

pair $\mathfrak{M}M'$ is completely independent of the motion of the point-pair $\mathfrak{N}N'$. It will now be specified that the four curves and the two dependencies that exist between the four points must be so arranged that the distances between the points $\mathfrak{M}\mathfrak{N}$ and $M'N'$ always remains equal to each other.

The **general solution** to this problem is found immediately with the help of the map *B*. Namely, if one maps the ∞^1 point-pairs $\mathfrak{M}M'$ and $\mathfrak{N}N'$, which shall satisfy the problem, then two families of lines will correspond to them that consist of those ∞^1 lines such that every line of one family cuts every line of the other family. However, two such families of lines will be either:

Identical with the two ruled families of a skew, second-order ruled surface (whose ruled families can possibly degenerate into two pairs of ray bundles that lie on two hyperboloids, in which case, line-pairs with common symmetry lines will enter in place of confocal hyperbolas in the corresponding figure). This ruled surface cannot contain the line $\mathfrak{Q}$, since otherwise the points of the one system of point-pairs would not be finite. The two systems of ∞^1 point-pairs that correspond to such a ruled surface will thus give a solution to the problem that we spoke of.

Or:

The two ruled families will belong to one and the same bundle of rays (planar ray field, resp.). In this case, the points $\mathfrak{M}$ and $\mathfrak{N}$ will describe any two completely arbitrary curves in an arbitrary way, while the corresponding points M' and N' will emerge from $\mathfrak{M}$ and $\mathfrak{N}$ by one and the same (arbitrarily-chosen) motion. This latter case thus yields only a trivial solution to the problem.

With that, all possibilities for solving the problem are exhausted.

§ 7. Left (right, resp.) paratactic point-pairs in the plane, left and right paratactic lines in space, resp. Paratactic congruences and ruled surfaces.

Two point-pairs in the plane shall be called $\begin{matrix}\textit{left}-\\ \textit{right}-\end{matrix}$ *paratactic* ([1]) when their $\begin{matrix}\text{left}\\ \text{right}\end{matrix}$ points agree, or at least, when the $\begin{matrix}\text{left}\\ \text{right}\end{matrix}$ point of one of the point-pairs is undetermined. The stated relationship shall also be valid for the associated image lines in space.

The ∞^2 point-pairs in the plane that have one and the same well-defined point $\begin{matrix}\mathfrak{M}\\ M\end{matrix}$ for their $\begin{matrix}\text{left}\\ \text{right}\end{matrix}$ point fill up a "congruence" of point-pairs that are all mutually

([1]) Following an analogous terminology of E. Study in his "Beiträge zur nichteuklidischen Geometrie," in the American Journal of Mathematics, vol. XXIX, no. 2, pp. 131.

$\begin{matrix}\text{left-}\\ \text{right-}\end{matrix}$ paratactic. Such a congruence, and likewise, the associated congruence of lines in image space, shall be referred to as $\begin{matrix}\textit{left}-\\ \textit{right}-\end{matrix}$ *paratactic*. If the point $\begin{matrix}\mathfrak{M}\\ M\end{matrix}$ is finite then the congruence will be called "non-special," and otherwise "special," when that point is an infinitely-distant point that is different from k_1 and k_2 . The $\begin{matrix}\text{left-}\\ \text{right-}\end{matrix}$ paratactic congruences in image space contain the $\begin{matrix}\text{right}\\ \text{left}\end{matrix}$ pairs of ray bundles $\begin{matrix}(k_1I_2)\\ (k_1I_1)\end{matrix}$ and $\begin{matrix}(k_2I_1)\\ (k_2I_2)\end{matrix}$, and are linear.

If the point $\begin{matrix}\mathfrak{M}\\ M\end{matrix}$ is finite then the associated $\begin{matrix}\text{left-}\\ \text{right-}\end{matrix}$ paratactic congruence will have the two lines of the $\begin{matrix}\text{left}\\ \text{right}\end{matrix}$ pair of ray bundles $\begin{matrix}(k_1I_1)\ (k_2I_2)\\ (k_1I_2)\ (k_2I_1)\end{matrix}$ for its guiding lines, which project perpendicularly onto the XY-plane as the minimal line that goes through the point $\begin{matrix}\mathfrak{M}\\ M\end{matrix}$.

If the point $\begin{matrix}\mathfrak{M}\\ M\end{matrix}$ lies on the line at infinity of the XY-plane, without coinciding with k_1 or k_2, then both of the guiding lines of the associated special $\begin{matrix}\text{left-}\\ \text{right-}\end{matrix}$ paratactic congruence will coincide with the infinitely-distant line $\mathfrak{Q}$ in the XY-plane; the congruence will then consists of infinitely many bundles of rays whose carrier point that is incident with $\mathfrak{Q}$ and carrier planes correspond to each other projectively in such a way that the points k_1 and k_2 will be projectively related to the planes $\begin{matrix}I_2 \text{ and } I_1\\ I_1 \text{ and } I_2\end{matrix}$, resp.

Aside from the special and non-special paratactic congruences that were considered, there are four more degenerate paratactic congruences:

1. The left-paratactic congruences that are associated with the point k_1 as left point, which decompose into the ray bundle k_1 and the ray field I_1 .
2. The analogous right-paratactic congruences that are associated with the point k_1 as right point, which decompose into the ray bundle k_1 and the ray field I_2 .
3. The left-paratactic congruences that are associated with the point k_2 as left point, which decompose into the ray bundle k_2 and the ray field I_2 .
4. The right-paratactic congruences that are associated with the point k_2 as right point, which decompose into the ray bundle k_2 and the ray field I_1 .

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A ruled family in image space whose generators are $\begin{matrix}\text{left-}\\ \text{right-}\end{matrix}$ paratactic, and is thus contained in a $\begin{matrix}\text{left-}\\ \text{right-}\end{matrix}$ paratactic congruence, will be referred to as a $\begin{matrix}\textit{left-}\\ \textit{right-}\end{matrix}$ *paratactic* ruled family, and likewise the associated system of ∞^1 point-pairs in the *XY*-plane. The point-pairs of such a system will all have the same $\begin{matrix}\text{left}\\ \text{right}\end{matrix}$ point. If the $\begin{matrix}\text{left-}\\ \text{right-}\end{matrix}$ paratactic congruence in which the ruled family is contained is not special then the stated point will be finite, while the other one will describe a curve; such a ruled family shall be called *non-special* $\begin{matrix}\text{left-}\\ \text{right-}\end{matrix}$ paratactic.

When the aforementioned congruence is special then the ruled family shall also be called *special* $\begin{matrix}\text{left-}\\ \text{right-}\end{matrix}$ paratactic.

The skew, second-order, ruled surfaces, one of whose ruled families is $\begin{matrix}\text{left-}\\ \text{right-}\end{matrix}$ paratactic, always contain a $\begin{matrix}\text{right-}\\ \text{left-}\end{matrix}$ paratactic ruled family for the other one. Such second-order ruled surfaces shall be called *Clifford surfaces*, following a terminology that is used in non-Euclidian geometry. Such a surface might or might not contain the infinitely-distant line $\mathfrak{Q}$ of the *XY*-plane.

In the former case, the ruled family of the surface that does not contain $\mathfrak{Q}$ is special, the one that contains $\mathfrak{Q}$ is non-special, and the Clifford surface shall be called *left-* (*right-*, resp.) *special* when the left- (right-, resp.) paratactic ruled family of the surface is special.

In the latter case, both ruled families of the Clifford surface are non-special, and the surface itself shall be called *non-special*.

The two systems of ∞^1 point-pairs, which correspond to the two ruled families of a non-special Clifford surface in the *XY*-plane, can be represented by the figure of two congruent circles [M] and [$\mathfrak{N}$] whose centers are N and $\mathfrak{M}$, respectively. Thus, the left point of the point-pair of the one system is fixed at $\mathfrak{M}$, while the other one describes the circle [M] around N. For the other system, the right point is fixed at N, while the other one describes the circle [$\mathfrak{N}$] around $\mathfrak{M}$.

The non-special family of a $\begin{matrix}\text{left-}\\ \text{right-}\end{matrix}$ special Clifford surface corresponds to a system of ∞^1 point-pairs whose $\begin{matrix}\text{right}\\ \text{left}\end{matrix}$ point is finite, while the other one – viz., the $\begin{matrix}\text{left}\\ \text{right}\end{matrix}$ one – describes a line.

(The special family corresponds to only imaginary point-pairs.)

§ 8. The groups Γ_6 and Γ_7 of transformations in the domain of point-pairs in the plane and the corresponding groups of collineations in image space. Families of collineations and reciprocities.

If the real point-pairs $\mathfrak{M}M$ in the XY-plane are transformed in such a way that the $\genfrac{}{}{0pt}{}{\text{left}}{\text{right}}$ points remain fixed while the $\genfrac{}{}{0pt}{}{\text{right}}{\text{left}}$ one is subjected to a well-defined – and indeed, the same – rigid transformation then this transformation of point-pairs will correspond to a transformation of lines in image space, under which intersecting lines will again go to other ones, ray bundles, to ray bundles, and planar ray fields, again to planar ray fields, and thus, to a collineation in image space. The ∞^3 collineations in image space thus obtained will define a group $\genfrac{}{}{0pt}{}{\mathfrak{G}_3}{G_3}$ that is characterized by the fact that the collineations that it contains leave each line of the $\genfrac{}{}{0pt}{}{\text{left}}{\text{right}}$ pair of ray bundles $\genfrac{}{}{0pt}{}{(k_1 I_1) \text{ and } (k_2 I_2)}{(k_2 I_2) \text{ and } (k_2 I_1)}$ individually invariant, while the lines of each of the two $\genfrac{}{}{0pt}{}{\text{right}}{\text{left}}$ bundles of rays $\genfrac{}{}{0pt}{}{(k_1 I_2) \text{ and } (k_2 I_1)}{(k_1 I_1) \text{ and } (k_2 I_2)}$ will be permuted projectively amongst themselves within their bundle.

The two groups $\mathfrak{G}_3$ and G_3 of collineations in image space, and likewise the corresponding transformations of point-pairs in the plane, shall be referred to as the group of *left-sided quasi-motions* ($\mathfrak{G}_3$) and the group of *right-sided quasi-motions* (G_3).

Any transformation of $\mathfrak{G}_3$ will commute with any transformation of G_3; i.e., two such transformations will always give the same resulting transformation when they are composed in an arbitrary sequence. The totality of the ∞^6 transformations (collineations in image space) that are thus obtained by composition is a group Γ_6, which will be referred to as the group of *quasi-motions* (Γ_6).

The collineations of Γ_6 in image space leave each individual bundle of rays of the two distinguished pairs of ray bundles $k_1\ I_1$, $k_2\ I_2$; $k_1\ I_2$, $k_2\ I_1$ invariant, as a whole. This property is not characteristic of the group Γ_6, but is peculiar to an enveloping group Γ_7 of ∞^7 transformations that contains Γ_6 as an invariant subgroup.

If the point-pair $\mathfrak{M}M$ in the XY-plane is transformed in such a way that the left point $\mathfrak{M}$ is subjected to a certain direct similarity transformation and the right point M is also subjected to a direct similarity transformation, and indeed of equal modulus (i.e., dilatation factor) then this transformations will correspond, in image space, to a collinear transformation of the associated image lines, under which, each of the four distinguished bundles of rays above will remain invariant. The totality of the ∞^7 transformations (viz., collineations in image space) thus obtained will define a group Γ_7 that shall be called the group of *quasi-similarity transformations* (Γ_7).

Certain families of ∞^6 (∞^7, resp.) transformations are most closely linked with the group Γ_6 (Γ_7, resp.):

If the point-pairs in the *XY*-plane are transformed in such a way that the left points $\mathfrak{M}$ are all subjected to one and the same transfer, and the right points *M*, also to one and the same (different) transfer, then this transformation will correspond to a collineation in image space that permutes the two left ray bundles $k_1\ I_1$ and $k_2\ I_2$ amongst each other, and likewise permutes the two right ray bundles $k_1\ I_2$ and $k_2\ I_1$ amongst each other. The ∞^6 transformations thus obtained will define a family K_6'.

If one composes the transformations of Γ_6 (K_6', resp.) with any simple transformation for which the order of the two points of any point-pair will be simply inverted then the family K_6'' (K_6''', resp.) will arise, which is represented in image space as a family of collineations. The collineations of $\begin{matrix} K_6'' \\ K_6''' \end{matrix}$ permute the $\begin{matrix} \text{planes } I_1 \text{ and } I_2 \\ \text{points } k_1 \text{ and } k_2 \end{matrix}$ amongst themselves, while they leave the $\begin{matrix} \text{points } k_1 \text{ and } k_2 \\ \text{planes } I_1 \text{ and } I_2 \end{matrix}$ individually invariant. If one composes two transformations that belong to that family then a transformation of Γ_6 will arise; if one composes two transformations that belong to different ones of the three families K_6', K_6'', K_6''' then one will obtain a transformation of the third family.

Certain families K_7', K_7'', K_7''' of ∞^7 collineations have the same relationship to Γ_7 that the aforementioned three families have to Γ_6. The former three families can be obtained from the latter by composing with the transformations of Γ_7.

If the $\begin{matrix} \text{left} \\ \text{right} \end{matrix}$ point of the two points of any point-pair is subjected to a certain motion and the $\begin{matrix} \text{right} \\ \text{left} \end{matrix}$ one, to a certain transfer, then this transformations will correspond to a reciprocal transformation in image space, under which, its ray bundle will go to a plane ray field, and conversely; it would permute the ray bundle k_1 with the ray field $\begin{matrix} I_1 \\ I_2 \end{matrix}$, and the ray bundle k_2 with the ray field $\begin{matrix} I_2 \\ I_1 \end{matrix}$. The ∞^6 transformations (viz., reciprocities in image space) thus obtained will define a family Π_6 (Π_6', resp.).

If one composes them with the transformations of Γ_7 then analogous families of ∞^7 transformations Π_7, Π_7' will arise.

Finally, one can derive the families Π_6'', Π_6''' (Π_7'', Π_7''', resp.), which likewise represent families of ∞^6 (∞^7, resp.) reciprocities in image space, by composing the families Π_6, Π_6' (Π_7, Π_7', resp.) with the simple transformation that consists of a mere permutation of the left and right points for any point-pair.

§ 9. Principal meaning of the foregoing developments for the kinematics of a rigid plane that moves within itself.

There is a certain advantage to placing the left points of the point-pair considered in a plane $\mathfrak{E}_0$, and the right points in a plane E_0 that both represent the XY-plane. If one thinks of these planes as being made to move rigidly in the XY-plane then these moving planes shall be denoted by $\mathfrak{E}$ (E, resp.), and the points that correspond to these moving positions of $\mathfrak{E}$ (E, resp.) by $\underline{\mathfrak{M}}$ ($\underline{M}$, resp.), which will lie at the point $\mathfrak{M}$ of the plane $\mathfrak{E}_0$ (the point M of the plane E_0, resp.) in the starting position.

A certain relative position of the moving plane $\mathfrak{E}$ compared to the moving plane E will correspond to a certain congruence relation between its point-fields, under which, the points $\underline{\mathfrak{M}}$ and $\underline{M}$ in the two planes that agree with each other will correspond. These point-fields will remain congruent when they are brought to the starting positions $\mathfrak{E}_0$ and E_0, and the ∞^2 point-pairs that are defined by the corresponding points $\mathfrak{M}$ and M in the XY-plane will define a ray bundle in image space under the map B whose carrier point will be regarded as the image point of the relative position of $\mathfrak{E}$ with respect to E ([1]).

Under this map of the relative positions of $\mathfrak{E}$ with respect to E to the points of space, one must observe that a different choice of starting position $\mathfrak{E}_0$ and E_0 would yield different image points that are, however, equivalent to the previous ones relative to the group Γ_6 .

If one has two systems of equally-many – say, κ – positions of the plane $\mathfrak{E}$ with respect to E then the case can arise in which the figure of all of the copies in the plane E of an (asymmetric) triangle that is fixed in $\mathfrak{E}$ that corresponds to the first system of positions is congruent to the analogous figure of the copies of a different triangle that is fixed in $\mathfrak{E}$ under the corresponding positions of the second system of positions. In this case, the two position systems shall be referred to as "equal."

If two triangles that are fixed in $\mathfrak{E}$ can be chosen such that the copies of the first triangle under the first position system will define a directly-similar figure with the copies of the second triangle under the second position system then the two positions systems will be called *directly similar*.

If one arrives at a situation in which the two systems of copies define oppositely congruent (indirectly simply, resp.) figures then the two position systems shall be called *indirectly equal* (*indirectly similar*, resp.).

If one can choose a triangle in the plane E such that corresponding copy in the plane $\mathfrak{E}$ for the first position system ([2]) defines a figure that is congruent in the same or opposite sense (directly or indirectly similar, resp.) to the figure that is defined by the

([1]) In § 10, it will be shown how one can analogously regard the planes in image space as images of relative positions of the planes $\bar{\mathfrak{E}}$ that are generated from $\mathfrak{E}$ by transfers (which arise by rotating $\mathfrak{G}$ through 180° around its line).

([2]) The supposed relationship between the two position systems remains preserved when the first and second position are exchanged with each other.

copies in E of a triangle that is fixed in $\mathfrak{E}$ for the second position system ([2]) then the two positions systems will be called *directly or indirectly inverse* (directly or indirectly inverse-similar, resp.).

One then has the fundamental theorem:

Two position systems of $\mathfrak{E}$ *with respect to E are equal (indirectly equal, inverse, indirectly-inverse, resp.) if and only if the corresponding systems of image points in space are equivalent relative to the group* Γ_6 *(the family* K_6', K_6'', K_6'''*, resp.).*

Two position systems of $\mathfrak{E}$ *with respect to E are directly-similar (indirectly-similar, inversely-similar, indirectly inversely-similar, resp.) iff the corresponding systems of image points in space are equivalent relative to the group* Γ_7 *(the family* K_7', K_7'', K_7'''*, resp.).*

The statement remains true when one is dealing, not with a discrete number of positions of $\mathfrak{E}$ relative to E, but with a continuous sequence of ∞^1 positions, and thus, to a continuous motion of $\mathfrak{E}$ relative to the plane E. One of them will be represented in image space by a continuous sequence of ∞^1 points, and thus, by a (spatial) curve, namely, the *image curve of the continuous motion.*

The question of when two continuous motions of $\mathfrak{E}$ with respect to E are equal (similar, etc., resp.) is equivalent to the question of what their associated image curves are relative to the group Γ_6 (Γ_7, etc., resp.).

All properties of a continuous motion of $\mathfrak{E}$ with respect to E find an adequate expression in the properties of the spatial image curve of the motion that are invariant under the group Γ_6.

The kinematics of a rigid plane that moves in itself is, in this sense, equivalent to the kinematics of a point that moves in three-dimensional space.

This shall be the case from now on.

§ 10. The metric in image space that is based upon the group Γ_6. (An apparently still-not-sufficiently noticed limiting case of non-Euclidian geometry.)

The two distinguished pairs of ray bundles – viz., the left ray-bundle-pair (k_1 I_1), (k_2 I_2) and the right one (k_1 I_1), (k_2 I_2) – can be regarded as a degenerate case of the two ruled families of a skew, second-order, ruled surface. The metric that is based upon the group Γ_6, under which the stated four ray bundles individually remain invariant, will thus be capable of being regarded as a limiting case of a non-Euclidian spatial metric, upon which such a skew, second-order, ruled surface will be based as a Cayley metric surface.

In order to arrive at the new metric in image space, it is simplest to start with the representation of the structure of image space by systems of point-pairs in the XY-plane under the map B.

In fact, it allows one to know the quantities of the plane figure thus obtained that are invariant under the group Γ_6 with no further assumptions, and with that the invariants of the spatial figure relative to the group Γ_6 are also found.

Two lines that are skew to the infinitely-distant line $\mathfrak{Q}$ in the *XY*-plane determine two invariants relative to the group Γ_6, namely, the "left" and "right" "distance between the lines, corresponding to the left and right distances between the two lines that correspond to point-pairs in the *XY*-plane under the map *B* (cf., § 3).

For intersecting lines – and only for such lines – the left and right distances will be equal to each other, such that one can speak of a "distance" between intersecting lines, per se.

Two points that do not lie in the distinguished plane pairs I_1, I_2, when regarded in some specific sequence, define an invariant under Γ_6 that is determined modulo π, including the sign, namely, the "angle" between the points.

This *angle between two spatial points* is defined as one-half the angle of the rotation that must be performed on the position of $\mathfrak{E}$ that belongs to the first point in order to obtain the position of $\mathfrak{E}$ that belongs to the second point. (Therefore, the positive sense of the rotation must be taken to be the one that appears to be counter-clockwise to an observer that is found on the positive *Z*-axis.)

Now, in order to also obtain the invariants of two planes that go through either k_1 or k_2, one considers the ∞^2 point-pairs $\mathfrak{M}M$ that belong to one of the two planes (as the carrier of a planar ray field), in the sense of the map *B*, which exhibit an oppositely-congruent relationship between the planes $\mathfrak{E}_0$ and E_0 (as the carrier plane of the points $\mathfrak{M}$ and *M*). There then exists a certain position of the plane $\bar{\mathfrak{E}}$ that is generated by a flip of the plane $\mathfrak{E}$ (i.e., by rotating around one of its lines through 180°) with respect to *E* for which corresponding points of the two oppositely-congruent related point-fields in $\mathfrak{E}_0$ and E_0 coincide. Therefore, any plane in image space that does not go through k_1 or k_2 belongs to a certain relative position of the (flipped) plane $\bar{\mathfrak{E}}$ relative to *E*, and conversely, each such position belongs to a plane in image space as image plane.

The *angle* between two planes (which is understood in the sense of the new metric) in image space that do not go through k_1 or k_2 (when taken in a specified order), which is invariant under Γ_6, is defined to be one-half the angle of the rotation that must be performed on the position of $\bar{\mathfrak{E}}$ that belongs to the first plane in order to obtain the second position of $\bar{\mathfrak{E}}$ that belongs to the second plane. (The positive sense of it shall be taken to be the one that appears to be counter-clockwise to an observer that is found on the negative *Z*-axis.)

The angle between two planes is thus likewise determined modulo π, including the sign.

Two positions 1 and 2 of the plane $\mathfrak{E}$ (relative to E) that are not derived from each other by a translation will determine two image points in space whose connecting line is skew to the infinitely-distant line $\mathfrak{Q}$ in the XY-plane, and will be represented by a point-pair – say, $\mathfrak{A}A$ – in the sense of the map B. The point $\mathfrak{A}$, when regarded as a point of $\mathfrak{E}_0$, and the point A, when regarded as a point of E_0, will then come into coincidence for the two positions 1 and 2 of $\mathfrak{E}$ relative to E above (because they are corresponding points of the congruently-related point-fields in the XY-plane that belong to the first position, and because they are likewise corresponding points in that congruence relation that belongs to the second position).

The two points $\mathfrak{A}$ and A are thus nothing but the *poles* in $\mathfrak{E}$ and E, resp., that correspond to the positions 1 and 2, resp., and indeed in that position that corresponds to the starting positions $\mathfrak{E}_0$ and E_0 , resp., of these planes.

When the planes $\mathfrak{E}$ *and E are brought to the starting positions* $\mathfrak{E}_0$ and E_0, *resp., the poles in* $\mathfrak{E}$ *and E, resp., that correspond to the two positions* 1 *and* 2, *resp., of* $\mathfrak{E}$ *relative to E will define a point-pair* $\mathfrak{A}A$ *in the XY-plane whose image line under the map B will be identical with the connecting line of the image points of* 1 *and* 2.

Analogously, one will have:

When the planes $\mathfrak{E}$ *and E are brought to the starting positions* $\mathfrak{E}_0$ and E_0, *resp., the poles in* $\overline{\mathfrak{E}}$ *and E, resp., that belong to the positions* 1 *and* 2, *resp., of the (transferred) plane* $\overline{\mathfrak{E}}$ *will define a point-pair* $\mathfrak{A}A$ *in the XY-plane whose image line under the map B is identical with the intersecting lines of the image planes of* $\overline{1}$ *and* $\overline{2}$.

Three positions 1, 2, 3 of the plane $\mathfrak{E}$ relative to E, no two of which can be derived from each other by a mere translation, will determine three poles $\mathfrak{A}_{ik}$ in $\mathfrak{E}_0$ when one combines any two of them, i and k, and three corresponding poles A_{ik} in E_0 . The triangle in $\mathfrak{E}_0$ that is constructed from the $\mathfrak{A}_{ik}$ will be the oppositely-congruent to the triangle in E_0 that corresponds to A_{ik} . For the position i of $\mathfrak{E}$ with respect to E, the sides $\mathfrak{A}_{ik}$, $\mathfrak{A}_{il}$ of the first triangle will coincide with the sides A_{ik}, A_{il} of the second one. The image line of the point-pair $\mathfrak{A}_{ik}\, A_{ik}$ in the sense of the map B will be identical with the connecting line of the image points of the positions i and k. Any position $\overline{4}$ of the (transferred) plane $\overline{\mathfrak{E}}$ for which the three points $\mathfrak{A}_{ik}$ coincide with the corresponding points A_{ik} will be represented in image space by the connecting plane of the three image points of the positions 1, 2, 3.

The position $\overline{4}$ of $\overline{\mathfrak{E}}$ can be taken to each of the positions 1, 2, 3 of $\mathfrak{E}$ by mere reflections (i.e., by mere rotations through 180°) around the respective coincident sides of the two polar triangles. In that sense, the position $\overline{4}$ of $\overline{\mathfrak{E}}$ is symmetric with respect to the three positions 1, 2, 3 of $\mathfrak{E}$.

The problem of connecting three points in the image space with a plane (which should be incident with either the point k_1 or the point k_2) is equivalent to the problem of finding the position $\overline{4}$ of the transferred plane $\overline{\mathfrak{E}}$ that is symmetric to each of the three positions 1, 2, 3 of $\mathfrak{E}$ relative to E.

Two positions, one of which is a position of the plane $\mathfrak{E}$ with respect to E and one of which is a position of the transferred plane $\overline{\mathfrak{E}}$ with respect to $\mathfrak{E}$, are symmetric if and only the associated image points are incident with the associated image plane.

The "distances" and "angle," in the sense of the new metric, that appear for a triangle (= trigon) in space whose plane goes through either k_1 or k_2 can, when one regards the three vertices as the images of positions 1, 2, 3 of the plane $\mathfrak{E}$ with respect to E, be chosen to be the distances and angles – in the sense of the ordinary Euclidian metric – that appear for the associated (oppositely-congruent) polar triangle.

Analogously, the "distances" and "angles" that appear for a trihedron in image space whose planes either go through either k_1 or k_2 can be chosen from the associated plane figure when one regards the face-planes of the trihedron as image planes of three positions of the transferred plane $\overline{\mathfrak{E}}$ with respect to E and determines the associated (i.e., congruent in the same sense) polar triangle.

When one combines any two of them, four positions 1, 2, 3, 4 of the plane $\mathfrak{E}$ with respect to E, no two of which go to each other under a translation, will determine six pairs of corresponding poles $\mathfrak{A}_{ik}$ in $\mathfrak{E}_0$ and A_{ik} in E_0, in all. Under the map B, the six point-pairs $\mathfrak{A}_{ik}\,A_{ik}$ will correspond to the edges ik of the tetrahedron that is defined by the image points of the four positions.

The four positions of the (transferred) plane $\overline{\mathfrak{E}}$ that are symmetric to three of the four positions above – say, i, k, and l – have the face-planes i, k, l of the stated tetrahedron for their image planes. If one always connects only those points in each of the two polar figures with lines that correspond to edges of the tetrahedron that meet then eight triangles will arise in each of the two polar figures, in all, of which, four of them – viz., the triangles of the "first kind" – will correspond to the three-edged figures that appear on the tetrahedron, while the other four – viz., the triangles of the "second kind" – will correspond to the three-sided figures that appear on the tetrahedron.

The corresponding triangles "of the first kind" for both polar figures in $\mathfrak{E}_0$ and E_0 are congruent in the same sense, because the point-pairs that are defined by their corresponding points have image lines that all go through one vertex of the tetrahedron.

The corresponding triangles "of the second kind" for two polar figures $\mathfrak{E}_0$ and E_0 are oppositely-congruent, because the point-pairs that are defined by their corresponding vertices will be represented in image space by the edges of the tetrahedron that lie in one face-plane.

Each of the six poles $\mathfrak{A}_{ik}$ (A_{ik} , resp.) is a vertex for three triangles of the first kind, corresponding to the two three-edged figures of the tetrahedron that contain the edge *ik*, and the two triangles of the second kind that correspond to the two three-sided figures of the tetrahedron that contain the edge *ik*.

The two angles that belong to the triangles of the first kind for such a pole $\mathfrak{A}_{ik}$ ($A_{ik,}$ resp.) will correspond to the "angle," which is measured using the new metric, between the vertices *i* and *k* of the tetrahedron, and are thus equal.

The two triangles of the second kind that meet at the pole $\mathfrak{A}_{ik}$ ($A_{ik,}$ resp.) as their common vertex will have an angle there that corresponds to the "angle," which is measured in the sense of the new metric, between the planes that meet along the edge *ik* of the tetrahedron. These angles will thus be likewise equal to each other.

In that way, the spatial representation of the two corresponding polar figures by the tetrahedron makes it possible, to get a brief overview of all the essential properties of the polar figures that are considered.

On these figures, cf., L. Burmester, *Lehrbuch der Kinematik*, pp. 610, Fig. 630.

§ 11. Evaluation of the mapping principle for the examination of real, continuous motions of a rigid plane $\mathfrak{E}$ with respect to *E*.

If one allows the plane $\mathfrak{E}$ to assume, not a discrete number, but an (in general) continuous sequence of ∞^1 positions with respect to *E* then $\mathfrak{E}$ will be perform a continuous motion with respect to *E*, and the locus of the image points of the ∞^1 positions of $\mathfrak{E}$ (with respect to *E*) will be a well-defined (spatial) curve, namely, the *image curve of the continuous motion.*

This curve can possess separate real parts. If it is algebraic then the continuous motion can also be referred to as algebraic.

If one determines the two associated poles to any two infinitely-close positions of the continuous motion in the planes $\mathfrak{E}$, and *E* denotes those positions that correspond to the starting positions $\mathfrak{E}_0$ and E_0 by $\mathfrak{A}$, *A* then the corresponding points $\mathfrak{A}$, *A* in each of the planes $\mathfrak{E}_0$ and E_0 will trace out a curve, namely, the two *pole curves* $\mathfrak{P}$ and *P* of the continuous motion.

The image lines of the point-pair $\mathfrak{A}A$ that is composed of the corresponding points $\mathfrak{A}$ and *A* of the two pole curves under the map *B* will define the image curve in the system of ∞^1 tangents. Conversely, if one represents the ∞^1 tangents to the image curve (under *B*) by point-pairs $\mathfrak{A}A$ in the *XY*-plane then:

The ∞^1 left points $\mathfrak{A}$ will trace out the polar curve $\mathfrak{P}$.
The ∞^1 right points *A* will trace out the polar curve *P*.

In this, one must observe that the polar curve positions that correspond to the starting positions $\mathfrak{E}_0$ and *E* are denoted by $\mathfrak{P}$ and *P*; they will not contact each other at these places (in general), but merely at the places that are assumed in the course of the continuous motion.

The continuous motion can be generated in the well-known way by letting the polar curve $\mathfrak{P}$ that is fixed in $\mathfrak{E}$ roll without slipping on the polar curve *P* that is fixed in *E*.

The figure of the two polar curves that are related to each other by their corresponding points is certainly not as clear as the corresponding figure in image space, which will be defined by the developable ruled family of all tangents to the image curve of the motion. An essential advantage of the mapping principle lies in this fact.

The differential-geometric concepts that appear for the new metric in image space, when applied to the image curve of a continuous motion of $\mathfrak{E}$ with respect to *E*, have a simple connection with the curvature behavior of the two polar curves at their corresponding points:

An infinitely-small advance of the image point along the image curve will correspond to a rolling of the two polar curves through an infinitely-small angle.

The "angle," when measured in the sense of the new metric, between two infinitely-close points of the image curve is equal to one-half the sum of the contingency angles that belong to the arc elements of the polar curves that roll on each other, when – as would happen ordinarily – the positive sense of this contingency angle is measured oppositely in the two planes $\mathfrak{E}$ and *E*.

The "angle," which is understood in the sense of the new metric, between the planes of oscillation of the image curve at the two neighboring points is equal to one-half the difference between the contingency angles of the two polar curves.

The "distance," which is measured in the sense of the new metric, between the tangents to the image curve at the two neighboring points is equal to the length of the arc elements of the polar curves that roll on each other.

One will thus be led to define the *first curvature* of the image curve of the motion, in the sense of the new metric, to be one-half the sum of the curvatures of the two image curves at their corresponding points and the *second curvature* of the image curve to be one-half the difference between the curvatures of the two polar curves at the corresponding points.

The behavior of the two "curvatures" (in the sense of the new metric) differs by the sign that stands before them, which depends upon whether the image curve can be regarded as *right-hand* or *left-hand screw,* in the sense of the new metric, at the place in question: Thus, two real lines in image space that do not intersect $\mathfrak{Q}$ will be defined as *right-wound* (*left-wound*, resp.), according to whether their right distance is greater than their left distance, or conversely, and a space curve will be defined to be *right-wound* or *left-wound* according to whether two infinitely-close tangents are right or left-wound, resp., at the place in question.

The "second curvature" of the image curve is zero at any point when the two polar curves have curvature circles that lie symmetrically with respect to the common tangents at the associated position of $\mathfrak{E}$ with respect to *E*.

If the second curvature is zero along the entire extent of the image curve then the latter will be a plane curve, and the corresponding continuous motion can be generated by the symmetric rolling of two symmetric polar curves (in particular, two symmetric circles, if the "first curvature" is constant throughout).

The "first curvature" can be zero only at isolated places along the image curve ([1]); such planes will correspond to positions of $\mathfrak{E}$ with respect to E for which the polar curves *osculate* each other.

If the "first" and "second" curvatures of the image curve of the motion are constant then the polar curves will be circles, for which a line can appear in place of one of the two circles. Since the two "curvatures" will remain constant for them, the image curves of such "cycloidal" – or "trochoidal" – motions will be regarded as *helices*, in the sense of the new metric; when the constant "second curvature" is zero, the image curve will then be plane curves, which will be regarded as *circles*, in the sense of the new metric, respectively.

[Any motion for which the two symmetric (ordinary) circles roll symmetrically without slipping on each will belong to such "circles."]

§ 12. Derivation of the properties of (real) algebraic, planar, continuous motions from the properties of their image curves that are invariant under the group Γ_6.

The character of the polar curves $\mathfrak{P}$ and P (as well as also the relationship between them that is exhibited by their corresponding points) must depend upon the properties of the spatial image curve of the continuous motion that are invariant under the group Γ_6.

For algebraic motions, one has the theorem:

The order of the polar curve P at E is equal to the rank of the image curve (i.e., the order of its developable tangent surface) minus the number of tangents to the to image curve of the motion that are contained in the "left" pair of ray-bundles $(k_1\, I_1)$, $(k_2\, I_2)$.

The order of the polar curve $\mathfrak{P}$ *at* $\mathfrak{E}$ *is equal to the rank of the image curve minus the number of tangents to the image curve of the motion that are contained in the "right" pair of ray-bundles* $(k_1\, I_1)$, $(k_2\, I_2)$.

Thus, if any point of a polar curve is always associated with several – say, ν – points on the other polar curve as corresponding points then the former polar curve must be ν-fold counted.

The multiplicity of the absolute circle points k_1, k_2 on the polar curve $\begin{matrix} P \\ \mathfrak{P} \end{matrix}$ at $\begin{matrix} E \\ \mathfrak{E} \end{matrix}$ is equal to one-half the number of tangents to the image curve that are incident with k_1, k_2, I_1, I_2 minus one-half the number of tangents to it that contained in the $\begin{matrix} \text{left} \\ \text{right} \end{matrix}$ pair of ray bundles $\begin{matrix} (k_1 I_1), (k_2 I_2) \\ (k_1 I_2), (k_2 I_1) \end{matrix}$.

([1]) If the "first curvature" is zero along the entire extent of the image curve then the latter will be a line, and the continuous motion will be a continuous rotation around a point that remains fixed.

Any continuous motion of the rigid plane $\mathfrak{E}$ over E will raise the question of which paths are distinguished by the points and lines in the one plane.

The *order* of these point-paths ([1]), and likewise the *class* of these line-paths ([1]), have a simple relationship with the image curve of the continuous (assumed algebraic) motion:

If one makes the image curve intersect the ∞^4 left-special Clifford surfaces then the number of resulting moving intersection points (i.e., ones that depend upon the respective Clifford surfaces that are singled out) will provide:

1. The order of the point-paths in the plane E that are distinguished by fixed points in $\mathfrak{E}$, and likewise:

2. The class of the line-paths in the plane $\mathfrak{E}$ that are distinguished by planes that are fixed in E.

Analogously, they will yield:

3. The order of the point-paths in the plane $\mathfrak{E}$ that are distinguished by fixed points in E, and likewise:

4. The class of the line-paths in the plane E that are distinguished by planes that are fixed in $\mathfrak{E}$

when one makes the image curve of the continuous motion cut the ∞^4 right-special Clifford surfaces and determines the number of moving intersection points.

The equations of all these point-paths and line-paths can be presented all at once when the equation of the image curve of the continuous motion in line coordinates is given. If:

$$F[q_0, q_1, q_2, q_3, q_4, q_5] = 0$$

is the equation of the image curve of the motion in the Q-coordinates, which were introduced in § 3, and are represented as homogeneous, linear functions of the ordinary Plückerian coordinates p_{ik}, then one can obtain the equation of all point-paths from this equation by certain substitutions. As we did before in § 9, let $(\mathfrak{x}, \mathfrak{y})$ be the coordinates of a point $\underline{\mathfrak{M}}$ of the plane $\mathfrak{E}$, relative to a coordinate system on the plane $\mathfrak{E}$ that agrees with the coordinate system OXY for the starting position $\mathfrak{E}_0$, and analogously, let (x, y) be the coordinates of a point $\underline{M}$ in the plane E relative to a coordinate system in the plane E that coincides with the coordinate system OXY for the starting position E_0; for fixed $\mathfrak{x}$, $\mathfrak{y}$, the equation:

$$F\left[1, \mathfrak{x}, \mathfrak{y}, x, y, \tfrac{1}{2}(\mathfrak{x}^2 + \mathfrak{y}^2 - x^2 - y^2)\right] = 0$$

will then represent the point-path that is described in E by the point $\mathfrak{M}(\mathfrak{x}, \mathfrak{y})$ in the plane $\mathfrak{E}$, while for fixed (x, y) the point $M(x, y)$ in the plane E will describe a distinguished point-pair in $\mathfrak{E}$.

Similarly, the equation of all line-paths:

([1]) In general. The numbers that were given above can be reduced for special positions of the points (lines, resp.) that describe them.

can be written down immediately. Let $u_1x + u_2y + u_0 = 0$ be the equation of a line that lies in E (relative to the coordinate system in E that was just defined), and let:

$$u^* = \sqrt{u_1^2 + u_2^2}$$

be the square root whose adjunction brings about the orientation of the line in question. The homogeneous quantities:

$$u_0,\ u_1,\ u_2,\ u^*,$$

between which the homogeneous, quadratic relation exists:

$$u_1^2 + u_2^2 - u^{*2} = 0,$$

are then to be regarded as the homogeneous coordinates of the oriented line m in E that is in question.

In an analogous sense, the homogeneous coordinates of an oriented line $\mathfrak{m}$ in the plane $\mathfrak{E}$ (relative to the coordinate system that introduced into $\mathfrak{E}$ above) might be denoted by:

$$\mathfrak{u}_0,\ \mathfrak{u}_1,\ \mathfrak{u}_2,\ \mathfrak{u}^*.$$

For fixed $\mathfrak{u}$, the equation:

$$F\left[0, \frac{\mathfrak{u}_1}{\mathfrak{u}^*}, \frac{\mathfrak{u}_2}{\mathfrak{u}^*}, \frac{u_1}{u^*}, \frac{u_2}{u^*}, \left(\frac{u_0}{u^*} - \frac{\mathfrak{u}_0}{\mathfrak{u}^*}\right)\right] = 0$$

will then represent the line-path in E that is enveloped by the lines $\mathfrak{m}$ in the plane $\mathfrak{E}$, while for fixed u, it will represent the line-path in $\mathfrak{E}$ that is distinguished by the lines m in the plane E.

In this, we must remark that the line-path will not always be represented purely by this equation, while the latter might possibly represent certain bundles of parallel rays.

In order to be able to evaluate the properties of the individual point-paths with the help of the spatial map, it is necessary to represent the various points of the planes $\mathfrak{E}$ and E by corresponding figures in image space.

A point $\begin{matrix}\mathfrak{M}\\ \underline{M}\end{matrix}$ in the plane $\begin{matrix}\mathfrak{E}\\ E\end{matrix}$ that has the position $\begin{matrix}\mathfrak{M}\\ M\end{matrix}$ in the XY-plane for the starting position $\begin{matrix}\mathfrak{E}_0\\ E_0\end{matrix}$ of $\begin{matrix}\mathfrak{E}\\ E\end{matrix}$ will determine a certain non-special $\begin{matrix}\text{left -}\\ \text{right -}\end{matrix}$ paratactic congruence of point-pairs in the XY-plane that have them for their $\begin{matrix}\text{left}\\ \text{right}\end{matrix}$ points; the image lines in space that correspond to the point-pairs of this congruence will fill up a non-special

$\genfrac{}{}{0pt}{}{\text{left-}}{\text{right-}}$ paratactic congruence whose guiding lines belong to the $\genfrac{}{}{0pt}{}{\text{left}}{\text{right}}$ pair of ray bundles $\genfrac{}{}{0pt}{}{(k_1 I_1), (k_2 I_2)}{(k_1 I_2), (k_2 I_1)}$. This congruence is to be regarded as the image of the point $\genfrac{}{}{0pt}{}{\mathfrak{M}}{M}$ that is fixed in $\genfrac{}{}{0pt}{}{\mathfrak{E}}{E}$.

The properties of the point-path that is described in the plane $\genfrac{}{}{0pt}{}{\mathfrak{E}}{E}$ by the point $\genfrac{}{}{0pt}{}{\mathfrak{M}}{M}$ in the plane $\genfrac{}{}{0pt}{}{\mathfrak{E}}{E}$ depend upon the properties of any spatial figure that is defined by the image congruence of the respective describing points and the image curve of the motion that are invariant under the group Γ_6.

If one would like to assess, for example, whether the order (that was determined above) of the point-path that is associated with certain positions of the describing point $\genfrac{}{}{0pt}{}{\mathfrak{M}}{M}$ is reduced (by omitting certain components of the point-path) then one will need only to see whether the number of moving intersection points of the ∞^2 $\genfrac{}{}{0pt}{}{\text{left-}}{\text{right-}}$ special Clifford surfaces such that the image congruence contains precisely the point $\genfrac{}{}{0pt}{}{\mathfrak{M}}{M}$ in question with the image curve of the motion is less than the number of moving intersection points of an arbitrary variable $\genfrac{}{}{0pt}{}{\text{left-}}{\text{right-}}$ special Clifford surface with the stated image curve.

The number that gives the reduction of this intersection point number will likewise be the number by which the order of the special point-path in question is reduced from that of the general case.

In order to be able to assess the properties of the individual line-paths with the help of the spatial map, it is convenient to represent the unoriented bundle of parallel rays in the planes $\mathfrak{E}$ and E by corresponding figures in image space.

Such a bundle of parallel rays in $\genfrac{}{}{0pt}{}{\mathfrak{E}}{E}$ defines a certain infinitely-distant point $\genfrac{}{}{0pt}{}{\underline{\mathfrak{M}}_\infty}{\underline{M}_\infty}$ whose direction is perpendicular to the direction of the bundle. If $\genfrac{}{}{0pt}{}{\mathfrak{M}_\infty}{M_\infty}$ is the position of $\genfrac{}{}{0pt}{}{\underline{\mathfrak{M}}_\infty}{\underline{M}_\infty}$ in the *XY*-plane that corresponds to the starting position $\genfrac{}{}{0pt}{}{\mathfrak{E}_0}{E_0}$ of $\genfrac{}{}{0pt}{}{\mathfrak{E}}{E}$ then the point $\genfrac{}{}{0pt}{}{\mathfrak{M}_\infty}{M_\infty}$ will define a $\genfrac{}{}{0pt}{}{\text{left-}}{\text{right-}}$ paratactic special congruence of point-pairs in the *XY*-plane that has it for their $\genfrac{}{}{0pt}{}{\text{left}}{\text{right}}$ point; the image line in space that corresponds to the point-pairs of this congruence trace out a $\genfrac{}{}{0pt}{}{\text{left-}}{\text{right-}}$ paratactic special congruence that is to be regarded as the image congruence of the bundle of parallel rays that is chosen in $\genfrac{}{}{0pt}{}{\mathfrak{E}}{E}$.

The properties of line-paths that belong to the lines of the bundle of parallel rays that is chosen in $\genfrac{}{}{0pt}{}{\mathfrak{E}}{E}$ (describing lines) depend upon the properties of any spatial figure that is constructed from the image congruence of the parallel ray bundle in $\genfrac{}{}{0pt}{}{\mathfrak{E}}{E}$ in question and the image curve of the motion.

If one would like to assess whether the class (as defined above) of the line-path that is associated with certain positions of the describing lines in $\genfrac{}{}{0pt}{}{\mathfrak{E}}{E}$ is reduced (omitting certain components of the line-path) then one will need only to check whether the number of moving intersection points of the ∞^2 $\genfrac{}{}{0pt}{}{\text{left-}}{\text{right-}}$ special Clifford surfaces that are contained in the image congruence of the parallel ray bundle that is determined by the line in question with the image curve of the motion is less than the number of moving intersection points of an arbitrary, variable $\genfrac{}{}{0pt}{}{\text{left-}}{\text{right-}}$ special Clifford surface with the stated image curve.

The number that gives the reduction of this intersection point number will likewise be the number by which the class of the special line-path considered is reduced from the general case.

If one determines the ruled family in image space that the image congruence of a point $\genfrac{}{}{0pt}{}{\mathfrak{M}}{\underline{M}}$ in the plane $\genfrac{}{}{0pt}{}{\mathfrak{E}}{E}$ has in common with the complex of secants to the image curve, and if the generators of this ruled family are represented by ∞^1 point-pairs in the XY-plane, in the sense of the map B, then the ∞^1 $\genfrac{}{}{0pt}{}{\text{right}}{\text{left}}$ points of these point-pairs will trace out the point-path that is described by the points $\genfrac{}{}{0pt}{}{\mathfrak{M}}{\underline{M}}$ in $\genfrac{}{}{0pt}{}{\mathfrak{E}}{E}$, and indeed they will give the place in this point-path that corresponds to the starting position $\genfrac{}{}{0pt}{}{\mathfrak{E}_0}{E_0}$.

Similarly, the line-path in $\genfrac{}{}{0pt}{}{E}{\mathfrak{E}}$ that is described by a line $\genfrac{}{}{0pt}{}{\mathfrak{m}}{\underline{m}}$ in the plane $\genfrac{}{}{0pt}{}{\mathfrak{E}}{E}$ can be obtained by means of that ruled family that the image congruence of the bundle of parallel rays that is determined by $\frac{\mathfrak{m}}{m}$ has in common with the complex of secants of the image curve. The generators of this ruled family will be represented in the *XY*-plane by ∞^1 imaginary point-pairs.

Each of these are associated with the figure of two congruently-related bundles of parallel rays, one of which – namely, the $\genfrac{}{}{0pt}{}{\text{left}}{\text{right}}$ one – will contains the line $\genfrac{}{}{0pt}{}{\mathfrak{m}}{m}$ ($\genfrac{}{}{0pt}{}{\mathfrak{m}}{m}$ is the position of $\frac{\mathfrak{m}}{m}$ that corresponds to the starting position $\genfrac{}{}{0pt}{}{\mathfrak{E}_0}{E_0}$); the position in the respective other (of the ∞^1) bundles of parallel rays under the congruence relation will belong to the desired line-path. The orientations of the tangents to the line-path thus-obtained (when the describing tangents are oriented) will be obtained by means of the congruence relation of the bundle of parallel rays above.

§ 13. Relationship between the point-paths and line-paths of a continuous, algebraic, planar motion to the circle points and the infinitely-distant line. Examples.

The multiplicity of the absolute circle points k_1 and k_2 on the point-path that is described in $\genfrac{}{}{0pt}{}{E}{\mathfrak{E}}$ by a point $\frac{\mathfrak{M}}{M}$ in $\genfrac{}{}{0pt}{}{\mathfrak{E}}{E}$, and the position of the tangents to the circle points, and thus the position of the associated exceptional focal points, can be deduced from the associated spatial figure. One draws a bundle of planes through one of the two guiding lines of the image congruence of the describing point $\frac{\mathfrak{M}}{M}$ and determines the number of moving intersection points of a variable plane of this bundle with the image curve of the motion. The number that says by how much this number is smaller than the order (as was determined in the previous § 11) of the point-path in question will give the desired multiplicity of the circle points k_1, k_2 on the point-path. Now, as far as the exceptional focal points of the individual point-planes are concerned, this will show that two different types of them must be distinguished, namely:

1. *General* e. o. (viz., extraordinary) focal points. These are ones that are independent of the position of the describing point in its plane, and

2. *Isolated* e. o. focal points, which do depend upon that position, and will be different for different describing points in that plane.

If the image curve of the motion meets the plane I_1 at a point φ that does not lie on the infinitely-distant line $\mathfrak{Q}$ in the *XY*-plane then it will correspond to (each) one **general** e. o. focal point, and indeed this will yield the position of this focal point that corresponds to the starting position $\frac{E_0}{\mathfrak{E}_0}$ when one projects the line $\frac{\varphi k_2}{\varphi k_1}$ onto the XY-plane from the infinitely-distant point of the Z-axis, and determines the real point of the projection.

The general e. o. focal points thus obtained of the point-path that is described by $\frac{\underline{\mathfrak{M}}}{\underline{M}}$ are likewise also general e. o. focal points of the line-path that is described by the lines $\frac{\underline{\mathfrak{m}}}{\underline{m}}$ of that same plane, for which, isolated lines will generally occur in each parallel ray bundle of that plane ([1]), for which, this focal point will not be an extraordinary focal point, but an ordinary one. (In the latter case, lines that connect it to the absolute circle points k_1, k_2 will then be *stationary* tangents to the line-path in question.)

If the image curve of the motion has a branch that goes through k_1 whose tangent lies in either I_1 or I_2 then this branch will correspond to an isolated e. o. focal point for each point-path that is described by a point $\frac{\underline{\mathfrak{M}}}{\underline{M}}$ in the plane $\frac{\mathfrak{E}}{E}$ in the other plane $\frac{E}{\mathfrak{E}}$.

A *direct similarity transformation* exists between the respective describing points $\frac{\underline{\mathfrak{M}}}{\underline{M}}$ in the plane $\frac{\mathfrak{E}}{E}$ and the associated isolated e. o. focal points of its point-path in the other plane $\frac{E}{\mathfrak{E}}$.

The points in the two planes $\mathfrak{E}$ and E that are associated with each other under this transformation will define point-pairs in the starting positions of these planes in the XY-plane whose image lines will cut the tangents to the image curve of the motion at the point k_1 and its conjugate-imaginary, at the point k_2 .

A remarkable reciprocity exists here, in that under the continuous motion in the plane $\mathfrak{E}$, the e. o. focal point $\underline{M}$ of the point-path that is described by the point $\underline{\mathfrak{M}}$ of the plane $\mathfrak{E}$ will describe a point-path in E with the e. o. focal point $\underline{\mathfrak{M}}$.

One must also distinguish two types of asymptotic directions for the point-paths:

1. *General* ones, which are independent of the position of the describing point in its plane, and

2. *Isolated* ones, which vary with the position of the describing point in its plane.

([1]) In general, these lines will be imaginary. It is only for special directions of the describing line that they can be real.

If the image curve of the motion has a branch that cuts the line $\mathfrak{Q}$, and which contacts no line of the $\begin{matrix}\text{right}\\ \text{left}\end{matrix}$ pair of ray bundles $\begin{matrix}(k_1 I_2), (k_2 I_1)\\ (k_1 I_1), (k_2 I_2)\end{matrix}$ at its intersection point with $\mathfrak{Q}$, then this branch of the point-path that is described by the points $\frac{\mathfrak{M}}{\underline{M}}$ will correspond to a **general** asymptotic direction that is linked to the tangent of the branch of the image curve at its intersection point with $\mathfrak{Q}$ quite simply.

Isolated (imaginary) asymptotic directions can appear for the points of the point-paths in $\begin{matrix}E\\ \mathfrak{E}\end{matrix}$ that are distinguished by the points $\frac{\mathfrak{M}}{\underline{M}}$ of the plane $\begin{matrix}\mathfrak{E}\\ E\end{matrix}$, when the image curve of the motion possesses a branch through k_1 that contacts a line of the $\begin{matrix}\text{right}\\ \text{left}\end{matrix}$ pair of ray bundles there that differs from $\mathfrak{Q}$, while the plane of oscillation in k_1 does not go through $\mathfrak{Q}$.

If it is assumed that the branch considered of the image curve at k_1 exhibits no singularity (i.e., it possesses no stationary point, stationary tangent, or plane of oscillation) then, relative to the asymptotic directions that are associated with the individual points $\frac{\mathfrak{M}}{\underline{M}}$ of the point-paths, which correspond to the approach of a variable point of the image curve to the point k_1, the motion that we spoke of can be replaced by a continuous motion of the simplest kind, which is such that the same relationship exists for the latter between the describing points and the associated asymptotic directions. The cubic space curve, which osculates the image curve of the continuous motion considered at k_1 and k_2 , is the image curve of this simple motion:

The latter motion consists of an:

$$\left\{\begin{matrix}\text{ordinary ellipsograph motion that is inverse to an ordinary ellipsograph motion,}\\ \text{or more briefly: an inverse ellipsograph motion}\end{matrix}\right\}$$

of $\mathfrak{E}$ compared to E for which a certain circle $\begin{matrix}\mathfrak{K}\\ K\end{matrix}$ in the plane $\begin{matrix}\mathfrak{E}\\ E\end{matrix}$ rolls on a circle $\begin{matrix}K\\ \mathfrak{K}\end{matrix}$ in the other plane $\begin{matrix}E\\ \mathfrak{E}\end{matrix}$ that is twice as large at the point of contact. The point-paths that are distinguished by the points $\frac{\mathfrak{M}}{\underline{M}}$ in the plane $\begin{matrix}\mathfrak{E}\\ E\end{matrix}$ in the other plane by this motion are, as is known, ellipses whose imaginary asymptotic directions that correspond to different positions of the describing points will be different.

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If the image curve of the motion has a branch through k_1 of the kind considered then the line-paths in $\genfrac{}{}{0pt}{}{\mathfrak{E}}{E}$ that are distinguished by the lines $\frac{m}{\underline{\mathfrak{m}}}$ in the plane $\genfrac{}{}{0pt}{}{E}{\mathfrak{E}}$ will possess *isolated, extraordinary, focal points*, and indeed, the dependency between the describing lines and the respective associated extraordinary focal points will be precisely the same as it was for the $\genfrac{}{}{0pt}{}{\text{ordinary}}{\text{inverse}}$ ellipsograph motion of $\mathfrak{E}$ relative to E that was applied already (for which, however, the describing lines in the respective other of the two planes are chosen to be the describing points in the case above, and thus not the ones in the plane whose points trace out ellipses in the other plane).

In fact, for the $\genfrac{}{}{0pt}{}{\text{ordinary}}{\text{inverse}}$ ellipsograph motion of $\mathfrak{E}$ relative to E, isolated e. o. focal points will appear on the line-paths in $\genfrac{}{}{0pt}{}{\mathfrak{E}}{E}$ that are distinguished by the lines $\frac{m}{\underline{\mathfrak{m}}}$ in the plane $\genfrac{}{}{0pt}{}{E}{\mathfrak{E}}$, namely, the centers of the line-paths, which are circular here. They will trace out the smaller of the two aforementioned polar circles, namely, the circle $\genfrac{}{}{0pt}{}{\mathfrak{K}}{K}$ in the plane $\genfrac{}{}{0pt}{}{\mathfrak{E}}{E}$.

The same e. o. focal point belongs to parallel positions of the describing lines.

If the direction of the describing line changes (in its plane) then the center of the circular enveloping path on the circle $\genfrac{}{}{0pt}{}{\mathfrak{K}}{K}$, which is regarded as an e. o. focal point, will change. Thus, the magnitude of the aforementioned change in direction will be equal to the peripheral angle that arises from the arcs of the circle $\genfrac{}{}{0pt}{}{\mathfrak{K}}{K}$, as described from the e. o. focal point.

Precisely the same relationship between the direction of the describing lines and the respective associated e. o. focal points of its enveloping curve will exist for the general motion that was considered earlier.

The line-paths are, in general, oriented curves whose tangents are oriented in a well-defined way. What sense that the tangents to the line-path are oriented with will depend upon the orientation of the describing lines. (Cf., § 11, last paragraph.)

It is only for certain describing lines that the line-path can be an unoriented curve, namely, when it is doubly-enveloped by the describing lines in the course of the continuous motion, and indeed, with the opposite orientations.

The number of moving intersection points of the image curve of the motion with a variable plane through the infinitely-distant line $\mathfrak{Q}$ in the *XY*-plane gives the number of oriented tangents to an arbitrary line-path that are parallel – in the same sense – to an arbitrarily-oriented line.

Naturally, the number of oppositely-parallel tangents to the line-path is just as large, such that twice the number of moving intersection points above will give the number of finite tangents to the line-path that will be drawn to it from any infinitely-distant point that is different from k_1 and k_2:

The value of the difference by which the stated doubled number is less than the class of the line-path (as determined in § 12) will give the *multiplicity of the infinitely-distant line as a tangent to the line-path.*

The foregoing serves to show how important the method that was employed here is for a *systematic treatment of plane kinematics.* Some examples might now be considered.

Examples.

Especially suitable for this purpose are the special continuous motions that were considered by S. Roberts ([1]), among others, which are defined when one demands that two points in the one plane must describe two well-defined curves in the other plane, etc.

Here, only the case for which the curves in question are circles shall be considered. Such a motion can always be obtained by connecting the two planes $\mathfrak{E}$ and E by two rigid rods whose endpoints are each fixed in one of the two planes:

Two-bar motion = linked quadrangle motion.

In order to obtain the image curve of this motion, one next imagines removing one rod. The image points of the ∞^2 positions of the plane $\mathfrak{E}$ relative to E that are then possible will trace out a non-special, skew, Clifford surface that can be regarded as the image of the stress that will be produced by the rod-connection of the plane $\mathfrak{E}$ in question: viz., the *stress surface*.

Analogously, another non-special Clifford surface will belong to the other rod connection as the second *stress surface*.

The intersection curve of these two stress surfaces is the image curve of the continuous motion that is compatible with the two-rod connection of the planes $\mathfrak{E}$ and E.

Naturally, the image curve of the motion is independent ([2]) of the way that the starting positions $\frac{\mathfrak{E}_0}{E_0}$ are chosen. If one chooses this starting position, in particular, such

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([1]) S. Roberts, "On the motion of a plane under certain conditions," Proceedings of the London Mathematical Society, 8 June 1871, vol. III, pp. 286.

([2]) From the remark that was made at the beginning of § 9, this dependency consists in the fact that for any change in the starting position, an image curve will appear that is equivalent to the original one under the group Γ_6 .

that it also belongs to one of the positions that are allowed by the two-rod linkage then both stress surfaces will contain the infinitely-distant point Z_∞ of the Z-axis, in the ordinary sense.

They are determined completely as follows:

If one defines a point-pair $\mathfrak{M}M$ from the starting point and endpoint of one of the two rods in the starting position then the spatial image line of this point-pair will meet four lines:

$$L_{11}, L_{22}, L_{12}, L_{21}$$

of the four distinguished ray bundles. A skew, second-order surface (i.e., a Clifford surface) will be determined by the spatial quadrilateral that is defined by these lines and the infinitely-distant point Z_∞ on the Z-axis, and it will be one "stress surface." Analogously, the second "stress surface" will be provided by the second rod. The image curve of the two-rod motion will be the intersection line of the two stress surfaces, and thus a *fourth-order curve of the first kind* that contains the point Z_∞ , and thus projects (perpendicularly) from it onto a third-order curve the XY-plane, which is "circular," since it – just like the image curve of the motion itself – must contain the points k_1, k_2 .

The points of this circular third-order curve are nothing but the rotational centers around which the plane $\mathfrak{E}$ must be rotated from its starting position $\mathfrak{E}_0$ in order to obtain the ∞^1 positions that the plane $\mathfrak{E}$ will assume in the course of its continuous motion when E remains in the starting position E_0 .

Therefore, the distance [measured in the sense of the ordinary metric ([1])] from the point of the image curve of the motion that lies perpendicular over the center of rotation in question to the XY-plane will yield the cotangent of one-half the angle of rotation.

However, not any fourth-order curve of the first kind that contains the infinitely-distant point Z_∞ on the Z-axis ([2]) and the circle points in the XY-plane can be regarded as the image curve of a two-rod motion in the given way. In order for that to be true, it will be, moreover, necessary (and also sufficient) that the 2×2 points:

$$\varphi_1,\ \psi_1\ ;\ \varphi_2,\ \psi_2$$

that the image curve possesses in the plane I_1 and I_2, except for k_1 and k_2 – i.e., outside of $\mathfrak{Q}$ – must have a special position relative to the tangents of the image curve at the points k_1 and k_2 , namely:

Any plane that links one of the two stated tangents with one of the points φ_1, ψ_1 must contain one of the two points φ_2, ψ_2 (and conversely).

The rank of the image curve is 8. Since no tangents to the image curves are contained in the four distinguished ray bundles, the order of the two polar curves will likewise be 8.

([1]) In the sense of the new metric, the "angle" between the stated points of the image curve relative to the stated center of rotation would be equal to simply one-half the rotational angle that was cited in the text.

([2]) Naturally, whether this point lies in the curve is completely inessential since that can always be arranged by a transformation of the group Γ_6 (corresponding to an altered choice of starting position).

Since two infinitesimally-following tangents to the image curve go through the point k_1, and likewise through the point k_2, the polar curves will have two infinitesimally-following points at the circle points k_1 and k_2, and thus vertices.

The number of moving intersection points of the image curve with the ∞^4 left-special, and likewise with the ∞^4 right-special Clifford surfaces, is six, because two eight intersection points are fixed at k_1 and k_2 . If follows from this that all points of the one plane will describe point-paths of order 6 in the other plane, and all lines of the plane will envelop *line-paths of class 6* in the other plane.

A reduction of the order of the point-paths occurs for the point-paths that are described by the endpoints of the rods: They will be doubly-counted circles, and thus, of order 4.

The general point-paths are tri-circular, because any plane bundle through a line in one of the four distinguished ray bundles has three moving intersection points in common with the image curve, and this number is 3 less than the order of the general point-path.

One has two general and one isolated e. o. focal point for the point-paths to distinguish:

The two general e. o. focal points are the starting points of the rods in the planes in question. The isolated e. o. focal point is connected with the respective describing point of the other plane by a similarity transformation in a known way.

Relative to the line-paths that, from what was said, are of class 6, one finds that two parallel (in the same sense) tangents to a line-path belong to every oriented line, because the plane bundle through $\mathfrak{Q}$ has two moving intersection points in common with the image curve of the motion. Thus, $2\times 2 = 4$ finite tangents to the line-path go through an infinitely-distant point. Since this number is 2 less than the class of the line-path, *the infinitely-distant line must be* **doubly-tangent** *to all line-paths.* It is an *ideal* double tangent, because it contacts the line-paths at imaginary points.

As *degenerate* and *limiting* cases of the two-rod motion [i.e., crankcase motions (*Kurbelgetriebes*)], we mention:

1. The case of the deltoid angle joint, for which the image curve of the motion decomposes into a cubic space curve that goes through the circle points k_1, k_2 in the *XY*-plane and a one-point secant to them. For the choice of $\frac{\mathfrak{E}_0}{E_0}$ that was made above, this image curve will project perpendicularly to the *XY*-plane onto a circle.

2. The case of an anti-parallelogram (parallelogram, resp.) as a four-bar linkage. Here, the image curve of the motion decomposes into two conic sections that have two points in common with each other, and one of which (conic section) contains the points k_1, k_2, while the other one has no points in common with the line $\mathfrak{Q}$.

The four-bar linkage will remain a parallelogram for the partial motion that corresponds to the first conic section.

It will remain an anti- parallelogram for the partial motion that corresponds to the second conic section.

3. The case of the thrust shaft motion (*Schubkurbelbewegung*), for which the stress that is required to keep a point that is fixed in the plane $\mathfrak{E}$ on a line that is fixed in E will appear in place of one of the two rods.

The image curve of the motion is a fourth-order curve of the first kind that has lines of the left pair of ray bundles for its tangents at the points k_1 and k_2, so the fourth intersection point of the plane of oscillation at each of these points with the image curve will lie in the plane I_2 (I_1, resp.).

4. The case that is inverse to the previous one – i.e., the inverse thrust shaft motion – is likewise of the fourth order and the first kind, and has the same relationship to the right pair of ray bundles as the previous one did to the left pair.

5. The ordinary ellipsograph motion. The image curve of the motion is a cubic curve that has lines of the left pair of ray bundles for tangents at k_1 and k_2 .

6. The motion that is inverse to the latter motion – i.e., the inverse ellipsograph motion. Its image curve is a cubic curve that contacts lines of the right pair of ray bundles at k_1 and k_2 .

7. The motion of the (asymmetric) sliding with friction gearbox (*Scheifschiebergetriebes*) (in the terminology of L. Burmester), for which:

A point of the plane $\mathfrak{E}$ *will be compelled to describe a line in E,* and likewise

A point in the plane E will be compelled to describe a line in $\mathfrak{E}$.

The image curve of the motion will be a cubic curve that cuts the line $\mathfrak{Q}$ at two points that are harmonically separated from k_1 and k_2 , so the tangents at these points will be harmonically separated by the planes I_1 and I_2 .

A degenerate case will occur here when the line that is fixed in the plane $\mathfrak{E}$ has the same distance from the point that is fixed in it that the line that is fixed in E has from the point that is fixed in it. The asymmetric motion would then give way to:

8. The motion of the symmetric sliding-with-friction gearbox.

The cubic curve above then decomposes into:

A line that cuts $\mathfrak{Q}$ (which is the image of a continuous, rectilinear, translational motion) and a conic section that cuts $\mathfrak{Q}$ at one point whose plane (therefore) does not go through $\mathfrak{Q}$ (which is the image of the interesting partial motion in this case).

The peculiarities of all these motions can be derived from the properties of the associated image curves that are invariant under the group Γ_6 .

Without going into the details, let it be remarked that one gets an insight in this way, for example, into how for each of the motions that were defined in 4, 6, 7, 8 *the evolutes of the* ∞^2 *line-paths in the plane E are similar to each other,* and indeed they are *cata-caustics of conic sections* under parallel illumination perpendicular to the major (real, resp.) axis of the conic section in question, so the ∞^1 conic sections that appear in the line-paths of such a motion can be obtained from the largest of them by rotation through an angle α around a focal point (that is identical with the point that is fixed in E) with a simultaneous dilatation (viz., shortening) with a ratio cos α about the stated focal point.

Kinematics on the sphere can be treated similarly, except that a group of ∞^6 collineations will appear in place of the group Γ_6, which can be regarded as motions in the sense of ordinary elliptic, non-Euclidian geometry.

(There is no group that would correspond to the group Γ_7.)

One sees clearly, in this way, how ordinary, elliptic, non-Euclidian geometry in space can be of service to kinematics on the sphere, and one recognizes that the methods that were proposed here are capable of even further development.
