"Sur la détermination des forces de réaction dans le movement d'un système matériel," C. R. Acad. Sci, Paris **191** (24 November 1930), 1118-1121.

On the determination of reaction forces in the motion of a material system

Note ([1]) by **E. Gugino**

Translated by D. H. Delphenich

The classical Lagrange equations (second form) for holonomic systems, or the Appell equations (or those of Maggi) for non-holonomic systems, completely define the motion under given forces of action when the constraints are supposed to be frictionless.

The reactions $\mathbf{R}_i$ that are produced by the constraints that are exerted on the various material points P_i (i = 1, 2, …, N) of the system do not enter into those equations. It is the principle of virtual work that permits one to eliminate all of them.

Meanwhile, there are some questions that are posed, notably in engineering, for which the greatest interest is directed towards the determination of those reactions, or if one prefers, the dynamical stresses to which the constraints are subjected.

The study of that class of question has not been neglected. In the last half-century, it has even developed into the subject of a special chapter of rational mechanics that one sometimes calls *kinetostatics.*

In order to determine the $\mathbf{R}_i$, one recalls the fundamental equation of dynamics:

$$m_i \, \mathbf{a}_i = \mathbf{F}_i + \mathbf{R}_i , \tag{1}$$

in which $\mathbf{F}_i$, m_i, $\mathbf{a}_i$ represent the force that is applied directly to the point P_i, its mass, and its (absolute) acceleration, respectively. In principle, equations (1) already solve the problem *as soon as one regards the motion as known.* That is what one currently does, and kinetostatics is only, one might say, a set of theorems and remarks that serve to render the consequences of equations (1) more expressive and manageable. However, the effective calculation of the reactions always remains, in a sense, subordinate to the prior determination of the motion of the system, which provides the accelerations, which amounts to the same thing as the inertial forces, as functions of time. Now, that prior determination of the motion is quite difficult, since it depends upon the integration of a differential system whose rank is higher than the level of mobility in the material system.

([1]) Session on 24 November 1930.

It would obviously be desirable to arrive at a characterization of the reactions by avoiding the solution of the problem of motion. I would like to show that one can essentially achieve that in a completely elementary fashion (viz., solving a system of linear equations) as soon as one proposes to determine the $\mathbf{R}_i$ at any instant as functions of the configuration and the state of motion (i.e., the distribution of the velocities) at that instant.

It should be noted that it is in just that form that one ordinarily supposes that the laws of the applied forces are given.

I can summarize, in a few words, the principle of the method by referring to the Lagrange equations of the first kind ([1]), which (1) one provides, as one knows, by attributing expressions of the form:

$$\mathbf{R}_i = \sum_{k=1}^{r} \lambda_k \, \mathbf{b}_{ki} \tag{2}$$

to the constraint forces, in which the λ_k are the Lagrange multipliers, and the $\mathbf{b}_{ki}$ are vectors that depend exclusively on the configuration of a system at each instant according to the equations of constraint (both holonomic and non-holonomic). More precisely, one must represent (as is always legitimate by differentiating the holonomic constraints with respect to time, if there are any) the equations of constraint in the form:

$$\sum_{i=1}^{N} \mathbf{b}_{ki} \times \mathbf{v}_i = b_k \qquad (k = 1, 2, \ldots, r), \tag{3}$$

in which $\mathbf{v}_i$ denotes the velocity of the point P_i, and the b_k are (in the general case) some scalars that depend upon only the position of the system and time. Naturally, one supposes that all of the constraints are independent, which demands that the matrix (B) that is composed of the components b'_{ki}, b''_{ki}, b'''_{ki} of the vectors $\mathbf{b}_{ki}$ must have rank r, which is equal to the number of equations (3).

From (2), the characterization of the reactions amounts to the determination of the r multipliers λ_k as functions of the positions and velocities of the points P_i of the system (and time t).

$$\sum_{i=1}^{N} \mathbf{b}_{ki} \times \mathbf{a}_i = c_k \qquad (k = 1, 2, \ldots, r), \tag{3′}$$

in which the c_k depend upon the positions and velocities (but not the accelerations).

One must only replace the vectors $\mathbf{a}_i$ equations (3) with their expression (1), in which one understands the $\mathbf{R}_i$ to have their Lagrangian form (2), in order to infer the following linear system in the multipliers λ_k :

([1]) See T. LEVI-CIVITA and U. AMALDI, *Lezioni di Mecaanica Razionale*, **1**, pp. 735, Bologna, Zanichelli, 1930.

$$(4) \qquad \sum_{k=1}^{r} \lambda_k \sum_{i=1}^{N} \frac{1}{m_i} \mathbf{b}_{hi} \times \mathbf{b}_{ki} = G_h \qquad (h = 1, 2, \ldots, r),$$

in which the G_h depend upon configuration and the velocities of the points of the systems that are not being acted upon.

Upon setting:

$$\alpha_{hk} = \sum_{i=1}^{N} \frac{1}{m_i} \mathbf{b}_{hi} \times \mathbf{b}_{ki} \, ,$$

with the right-hand side being a function of nothing but the configuration and time, equations (4) can be put into the definitive form:

$$(4) \qquad \sum_{k=1}^{r} \alpha_{hk} \, \lambda_k = G_h \qquad (h = 1, 2, \ldots, r),$$

It will then be easy to recognize that the system (5) can be effectively solved for the unknowns λ_k . Indeed, since the equations of constraint (2) are independent, the matrix (B) will have rank r, as one has seen already. The same thing will be true for the determinant $\| \alpha_{hk} \|$, as one verifies immediately.

In conclusion, we remark that once the multipliers have been obtained, the reactions $\mathbf{R}_i$ will be likewise determined, and the dynamical problem that relates to the motion of a system in question will reduce to the analogous problem that relates to the same system of points when they are supposed to be free.
