"Sur l'extension des équations mécaniques de M. Appell à la Physique des milieux continus. Application à la théorie des électrons," C. R. Acad. Sci. Paris **156** (1913), 875-879.

On the extension of Appell's mechanical equations to the physics of continuous media. Application to the theory of electrons.

Note by **ÉDOUARD GUILLAUME**, presented by Appell

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Appell showed ([1]) that the general equations of dynamics for an arbitrary system can be obtained by seeking the minimum of the quadratic function of the accelerations:

(I) $$R = S - \sum_{i=1}^{n} Q_i\, q_i'', \qquad S = \sum \frac{m}{2}\, j^2 .$$

S is the *energy of acceleration*, Q_i is the generalized force, and q_i'' is the second derivative with respect to time of the generalized coordinate q_i. He suggested that the principle could be applied to hydrodynamics and electrodynamics ([2]).

In order to extend the relation (I) to continuous media, we first make the following remarks:

1. If the system possesses a potential energy W then one will have:

$$\sum_{i=1}^{k} Q_i\, q_i'' = W'' + R_1 \qquad (i = 1, 2, \ldots, k),$$

where Q_i denotes the forces that are derived from that potential, and R_1 denotes a term that is independent of the q_i''. The $n - k$ remaining forces are called *forces that are external to the system*, and one sets:

$$E = \sum_{i=k+1}^{n} Q_i\, q_i'' \quad (l = k + 1, \ldots, n) .$$

2. If there are constraint equations of the form:

([1]) P. APPELL, *Traité de Mécanique rationnelle*, t. II.

([2]) P. APPELL, "Aperçu sur l'emploi possible de l'énergie d'accélération dans les équations de l'Électrodynamique," C. R. Acad. Sci. Paris, session on 22 April 1912.

$$J_h = 0$$

then one can introduce ([1]), by a generalization of the method of Lagrange multipliers, some undetermined functions λ_h in such a way that $\sum \lambda_h J_h$ can be considered to be a supplementary potential energy.

3. In the case where the kinetic energy T is expressed in Cartesian coordinates, one has:

$$\frac{1}{2}\frac{\partial T''}{\partial x''} = \frac{\partial S}{\partial x''}.$$

If one makes use of Cartesian coordinates then equation (I) can then be replaced with:

(I′) $$R = \tfrac{1}{2}T'' + W'' + \sum(\lambda_h J_h) - E.$$

If the coordinates are arbitrary then one can put S in place of $\tfrac{1}{2}T''$.

It is easy to write out the function R for continuous media now. In that case, one considers the motion of a volume element $d\tau$ of a certain volume V that is bounded by a surface σ in the medium. The functions S or T and W will become integrals over the volume V. The term that relates to the constraint equations is obtained by multiplying the left-hand sides of those equations by $\lambda_h\, d\tau$, respectively, and then adding them and integrating over the entire volume V. The term E can give both a volume integral and a surface integral. By definition, R is presented in the form:

$$R = \iiint \varphi_0\, d\tau + \iint \psi_0\, d\sigma,$$

in which φ_0 and ψ_0 can contain the accelerations and their partial derivatives. One will then specify the acceleration in such a fashion that one puts R into the form:

$$R = \iiint \varphi_1\, d\tau + \iint \psi_1\, d\sigma,$$

in which φ_1 and ψ_1 are polynomials of degree two or one in the accelerations. That transformation is possible, since the system is supposed to be mechanical. Upon varying the accelerations, one will form the variation δR, which must be zero for any variations of the accelerations. One will obtain the desired relations upon annulling the coefficients of those variations.

Application to the theory of electrons. – Maxwell was the first to establish a mathematical link between mechanics and electrical phenomena. He appealed to the Lagrange equations: He then

([1]) H. POINCARÉ, *Leçons sur la théorie de l'élasticité*, 1892.

supposed that the systems were holonomic. H.-A. Lorentz reprised and generalized Maxwell's ideas ([1]). In particular, he showed that if one considers the energy of the magnetic field:

$$T = \tfrac{1}{2}\iiint \mathfrak{h}^2 d\xi \tag{1}$$

to be a kinetic energy and the energy of the electric field:

$$W = \tfrac{1}{2}\iiint \mathfrak{d}^2 d\xi \tag{2}$$

to be a potential energy then since the vectors $\mathfrak{h}$ and $\mathfrak{d}$ satisfy the constraint equations:

$$c \operatorname{rot} \mathfrak{h} - \mathfrak{v} \operatorname{div} \mathfrak{d} - \mathfrak{d}' = 0, \tag{3}$$

$$\operatorname{div} \mathfrak{h} = 0, \tag{4}$$

($\mathfrak{v}$ is the velocity of the matter, c is the velocity of light), it is possible to establish the fundamental equation:

$$\operatorname{rot} \mathfrak{d} = -\frac{1}{c}\mathfrak{h} \tag{5}$$

by means of d'Alembert's principle.

The proof demands certain restrictions that are due to the use of the quantities of electricity as coordinates and the introduction of their virtual displacements. Lorentz was then led to define a new class of systems that he called *quasi-holonomic*. He supposed that a system of electrons belonged to that class.

Upon starting from the expression (I′), *if one is given equations* (1), (2), (3), *and* (4) *then one can establish* (5) *upon supposing, in a general fashion, that the system is non-holonomic.* Indeed, conforming to the meanings of T and W, the magnetic field $\mathfrak{h}$ is the analogue of a velocity and $\mathfrak{h}'$ is that of an acceleration. The electric field $\mathfrak{d}$ measures the deformation that produces the potential energy, so $\mathfrak{d}'$ will be the velocity of variation of that deformation and $\mathfrak{d}''$ will be its acceleration. Equation (3) will permit one to immediately express $\mathfrak{d}''$ as a function of $\mathfrak{h}'$, in such a way that we will have only one constraint equation to consider, namely, equation (4).

Let $\mathfrak{f}\, d\sigma$ denote the forces that acts upon the element $d\sigma$, so one will have:

$$\begin{aligned} R &= \iiint \left(\tfrac{1}{2}\mathfrak{h}'^2 + c\,\mathfrak{d}\operatorname{rot}\mathfrak{h} - 2\lambda \operatorname{div}\mathfrak{h}\right) d\tau - \iint \mathfrak{f}\mathfrak{h}' d\sigma + \dots \\ &= \iiint \left(\tfrac{1}{2}\mathfrak{h}'^2 + c\,\mathfrak{h}'\operatorname{rot}\mathfrak{d} + 2\mathfrak{h}'\operatorname{div}\lambda\right) d\tau - \iint (c[\mathfrak{d}\,\mathfrak{h}']_n + \lambda'\mathfrak{h}' + \mathfrak{f}\mathfrak{h}') d\sigma + \dots \end{aligned}$$

([1]) H.-A. LORENTZ, Archives néerlandaises **25** (1892), and *Encykl. der math. Wissenschaften*, v. 2, 1904.

One infers from the volume integral that:

$$\mathfrak{h}' = -\frac{1}{c}\ \text{rot}\ \mathfrak{d} - 2\ \text{grad}\ \lambda' \ . \tag{6}$$

In order to determine λ', it suffices to form div $\mathfrak{h}'$, while taking into account equation (4).

One will then find that λ' must be constant. Its gradient will then be zero, and equation (6) will reduce to the desired equation (5). The surface integral permits one to calculate the force $\mathfrak{f}$. In order to find its significance, it suffices to look for the work done per unit time. Upon taking the constant λ' equal to zero, one will find that:

$$\mathfrak{s}\ \mathfrak{h} = -c\ [\mathfrak{d}\ \mathfrak{h}]_n \ ;$$

i.e., the Poynting energy flux.

If one remains in the ether and starts from equations (1), (2), (4), and (5) then the expression (I′) will permit one to determine equation (3), without the term that relates to matter. One thus exhibits the duality that is often observed in electricity in a striking manner.

The fecundity of the method that was proposed here comes from the fact that one substitutes *virtual accelerations* for virtual displacements. There is no need to go deeper into the mechanism of the phenomenon.

The possibility of giving a mechanical interpretation for the theory of electrons follows from the possibility of establishing the expressions φ_1 and ψ_1 for it.

In addition to d'Alembert's principle, one attempts to extend Hamilton's principle to all of physics, and above all, since the time of Helmholtz. Now, those principles do not apply to Maxwell's theory and the theory of electrons very well. One will then have the right to think that Appell's principle, thus-generalized, might substitute for them advantageously in a number of cases, more or less.

One can see that the considerations above extend to Einstein's mechanics. In it, one introduces the function ([1]):

$$H = -m_0\, c \sqrt{1 - \frac{\mathfrak{v}^2}{c^2}}$$

in order to form the equations of Lagrange and Hamilton in his mechanics. It is easy to see that H is the analogue of T in ordinary mechanics. Indeed, one has:

$$\frac{1}{2}\frac{\partial H''}{\partial \mathfrak{v}'} = \mathfrak{f} \ ,$$

in which $\mathfrak{f}$ denotes a force. It is the fundamental equation of motion in the new mechanics. The function R is obtained by replacing T'' with H'' in the expression (I′).

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([1]) A. EINSTEIN, Jahrbuch der Radioaktivität und Elektronik, Bd. IV, Heft 4.