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ON THE NONLINEAR CONTINUUM THEORY OF MOVING DISLOCATIONS

With 1 Figure

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FOREWORD

Through the work of the school of K. KONDO, as well as a B. A. BILBY, R. BULLOUGH, and E. SMITH, continuum mechanics has developed into a theory with far-reaching geometric interpretations. The advantage of a theory like the one here that is, e.g., based upon differential-geometric concepts, is, above all, the fact that in addition to general insights into the structure of the theory, the highly-developed and highly-constructed differential-geometric formalism represents a significant practical tool for the solution of the corresponding problems in physics. In that regard, for the continuum mechanics of crystal defects, it yields far-reaching mathematical, and in a certain sense, also physical, analogies with the theory of relativity, which E. KRÖNER pointed out especially. One can further say that A. EINSTEIN's (1928) theory of teleparallelism, which started from some questions in the general theory of relativity, finds a rigorous physical application in that subject.

We shall present the general nonlinear equations of the continuum theory of moving dislocations on that basis when we associate a crystal that is permeated with moving dislocations with a four-dimensional space with teleparallelism. In conjunction with that, the mathematical similarities with and differences from special, as well as general, relativity, shall be examined thoroughly. In contrast to other nonlinear equations of physics, we will find here that the nonlinearity of the system of equations will imply no restriction on our freedom to specify the dislocations and their currents. Any solution of the linearized system can be extended to a solution of the rigorous solution. That is also true for the dynamical generalization of KRÖNER's theory with foreign atoms and the more general theory that is extended by the introduction of the so-called phenomenological matter tensor. That will be explored as an approximation procedure for the solution of the field equations. At the same time, the procedure will allow us to give the stress field of a moving dislocation explicitly to an arbitrary degree of approximation, up to numerical integration. Furthermore, we shall investigate the non-analytic solutions that are compatible with the equations, and in particular shockwaves in media that propagate by moving dislocations under elastic pre-stresses.

I am thankful to Herrn Prof. Dr. habil H. TREDER for the impetus to address those questions, as well as numerous discussions.

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INTRODUCTION

The first era in the theory of dislocations and internal stresses is naturally characterized by the notion that elementary and detailed physical arguments always defined the starting point and the foundations of the treatment of the most variegated problems.

The start of that development, which is linked with the names of V. VOLTERRA [**1**], U. DEHLINGER [**2**], E. OROWAN [**3**], M. POLANYI [**4**], G. I. TAYLOR [**5**], J. M. BURGERS [**6**], [**7**], *et al.*, and was concerned, above all, with defining the concept of a dislocation and probing its meaning in the context of the plastic and elastic states of crystals, whereby in that quest, as well as in later contributions, many authors put the isolated dislocation in the foreground; cf., e.g., J. D. ESHELBY [**8**], G. LEIBFRIED and K. LÜCKE [**9**], F. C. FRANK [**10**], and many others. Later on, an elementary continuum theory of dislocations was created by E. KRÖNER [**11**] and E. KRÖNER and G. RIEDER [**12**]; see also the review of E. KRÖNER: *"Kontinuumstheorie der Versetzungen und Eigenspannungen"* [**13**], in which one can also find a comprehensive overview of the literature on the dislocation problem that had appeared up to then.

An essential advance came about, as is known, when the theory admitted a geometric interpretation. That means, on the one hand, that one can give the theory of dislocations a unified and compact formulation using the highly-developed mathematical apparatus of differential geometry and tensor analysis, and on the other, the first of the linear equations that follow from elementary arguments would find their rigorous nonlinear generalization in terms of the differential-geometric formulation. That direction of research appeared perhaps simultaneously in Japan and Europe. In Japan, it was through the groundbreaking work of the school that was founded by K. KONDO and his collaborators of "the general differential-geometric treatment of engineering science" (cf., on that, [**14**] and [**15**]), and in Europe, it was by way of B. A. BILBY, R. BULLOUGH, and E. SMITH [**16**], [**17**], and was promoted by E. KRÖNER [**18**] and E. KRÖNER and A. SEEGER [**19**], especially.

In the static theory of dislocations, one finds formal mathematical analogies with the general theory of relativity, as E. KRÖNER [**18**] stressed, in particular. In addition, it has already been known for some time from elementary arguments that the speed of sound in the medium in question sets a definite limit on the velocity with which a dislocation can move in a crystal. Work on that subject goes back to J. I. FRENKEL and T. A. KONTOROVA [**20**], [**21**], [**22**], as well as F. C. FRANK [**23**]. Hence, there also exist parallels with the special theory of relativity under which the speed of light is replaced, so to speak, with the speed of sound.

Now, it is an essential goal of the present work to discuss thoroughly the analogies and differences between the continuum theory of moving dislocations and the special, as well as general, theory of relativity, that are based upon the tensor calculus. In particular, we will go into its relationship to EINSTEIN's theory of teleparallelism in RIEMANNIAN spaces; see A. EINSTEIN [**24**].

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I. – The field equations of moving dislocations

Although the problem of moving dislocations has already been treated many times – cf., S. AMARI [**25**], E. F. HÖLLANDER [**26**], [**27**], H. BROSS [**28**], A. M. KOSEVICH [**29**], T. MURA [**30**], [**31**] (we shall come to speak of those papers later) – we shall next give a derivation of the basic equations of the dynamics of dislocations here that implies the most natural generalization of statics that is possible mathematically, as well as physically. In that way, we will arrive, on the one hand, at a closed presentation of the theory. In addition, we would like to construct the theory in such a way that it will make the discussion of the connection to the theory of relativity that was mentioned in the introduction more accessible.

The object of our considerations is an elastic continuum K that is embedded in our real three-dimensional Euclidian space E_3 . For the sake of simplicity, we shall employ Cartesian coordinates x^i (i = 1, 2, 3) in E_3 . For certain special problems, the conversion of the problem into suitable curvilinear coordinates might be advantageous, so we will then simply have to replace the ordinary derivatives in the basic equations with covariant ones in the well-known way. Since the curvature tensor of E_3 vanishes, that replacement will also be possible for second derivatives in an unambiguous way with no complications. We shall also describe the position of a mass-point of K by its Cartesian coordinates in E_3 . We essentially characterize K by two properties:

1. At each point of K, there exists a set of quantities μ_1 , …, μ_r , … that describes its elastic properties. K will also be characterized as an elastic body in that way. In general, one will have to assume that the elastic quantities are functions of position and time, and therefore $\mu_i = \mu_i\,(x^i, t)$. However, in many practical cases, they will be simply the elastic constants of various orders.

2. At each point of K, there exist three unique non-coplanar directions, namely, the lattice directions of the crystal. K will be characterized as a crystal in that way. K will be found to be permeated by dislocations when it is in any twisted and stretched state.

We shall next sketch out the basic ideas of the statics of dislocations in order to arrive at some viewpoint on the transition to dynamics. For what follows, we will then combine the terminology by referring to a continuum K that is in a state where no sort of crystal defect perturbs the lattice directions as an *ideal crystal*, while in any perturbed state it will be a *real crystal*. For the sake of simplicity, we shall first confine ourselves to the case in which the lattice directions of the ideal crystal define an orthonormal system. We will then show, by an additional consideration, that one can easily generalize to arbitrary ideal crystals.

From the second property above, three distinguished lattice vectors are defined at each point that therefore define a "dreibein." We describe it by the quantities:

$$h_K^i = h_K^i\,(x^r) \qquad \text{with the inverses} \qquad h_i^K = h_i^K\,(x^r)\,, \tag{1}$$

which exist as a result of the fact that the assumed non-coplanarity implies that $h^i_K \cdot h^K_j = \delta^i_j$. (Lowercase Latin indices i, j = 1, 2, 3 refer to tensor components, while K, L, … = 1, 2, 3 are only numbers.)(1) The lattice vectors $h^i_K(x^r)$ come about as a result of the distortion of the ideal crystal into a real crystal. If one describes the lattice of the ideal crystal by the vectors $k^\alpha_K = \delta^\alpha_K$ (2) then when one denotes the distortion quantities by A^i_α, the connection between real and ideal lattices will be given by:

$$h^i_K = A^i_\alpha k^\alpha_K = A^i_\alpha \delta^\alpha_K . \tag{2}$$

It is clear from (2) that the lattice vectors of the real crystal already characterize the distortion completely. Correspondingly, the relation:

$$dx^\alpha = A^\alpha_i dx^i , \tag{3*}$$

which describes the relaxation of two points in a crystal whose mutual separation is dx^i into their mutual position dx^α in the ideal crystal, will be described completely by the projections of dx^i onto the lattice vectors h^K_i :

$$dx^K = h^K_i dx^i , \tag{3}$$

and we can express the distance between the two mass-points in the real crystal by:

$$ds^2 = dx^K dx^L \delta_{KL} . \tag{4}$$

If one substitutes (3) into (4) then:

$$ds^2 = h^K_i h^L_j \delta_{KL} dx^K dx^L = g_{ij} dx^i dx^j, \tag{5}$$

with:

$$g_{ij} = h^K_i h^L_j \delta_{KL} \qquad \text{and inversely} \quad h^i_K h^j_L g_{ij} = \delta_{KL} . \tag{6}$$

We have introduced a metric g_{ij} on K by way of (6), which then makes K into a RIEMANNian space. The physical meaning of that metric consists of the fact that it is immediately obvious from (5) (cf., e.g., [**18**]) that:

$$\varepsilon_{ij} = \tfrac{1}{2}(\delta_{ij} - g_{ij}) \tag{7}$$

means the elastic deformation of the real crystal.

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(1) An overview of the notations employed is compiled once more in the Appendix.

(2) We shall also operate with a Cartesian reference system in an ideal crystal. The Greek index α = 1, 2, 3 appears here to characterize a three-dimensional vector only to distinguish it from the case of a real crystal; otherwise, Greek indices will be employed only for the time-dependent case (cf. *infra*).

Due to the crystalline structure (1), three congruences of curves are defined in that RIEMANNian space in a distinguished way, namely, the lattice lines of the crystal. (The RIEMANNian space is then "fibered.") One can then introduce a teleparallelism in an invariant way when one establishes that a tensor is parallel-translated when the projections of its components onto those curve congruences do not change. The coefficients of the parallel displacement are the coefficients of the integrable displacement that EINSTEIN introduced into field theory:

$$\Gamma^i_{kl} = h^i_K \, h^K_{i,k} , \tag{8}$$

which satisfy the lemma:

$$h^K_{r\|s} \equiv h^K_{r,s} - h^K_n \, \Gamma^n_{sr} = 0, \tag{8a}$$

such that the RICCI lemma:

$$g_{ij\,\|\,k} \equiv g_{ij\,,\,k} - g_{nj}\,\Gamma^n_{ki} - g_{in}\,\Gamma^n_{kj} = 0 \tag{8b}$$

will also be fulfilled (with asymmetric $\Gamma^n_{ij} \neq \Gamma^n_{ji}$).

The essential traits of the real crystal are then mapped to that fibered RIEMANNian space.

From (6), the curve congruences will define an orthonormal system when they are measured with the metric g_{ij} . Naturally, that means nothing but the fact that the lattice lines are orthonormal in the ordinary sense. Furthermore, the metric g_{ij} is defined by (5) in precisely such a manner that those lines will be orthonormal. Naturally, one will get the actual comparison of angles and lengths of the lattice vectors using the metric of Euclidian space δ_{ij} , and one will then have $h^i_K \, h^j_L \, \delta_{ij} \neq \delta_{KL}$.

Due to the existence of a dreibein at each point of *K*, one can now characterize any tensor in two ways: namely, in one case, in the usual way by giving its tensor components $T^{\cdots i\, j \cdots}_{\cdots r\, s \cdots}$, and in the other, by projecting those components onto the congruences:

$$T^{\cdots K\, L \cdots}_{\cdots M\, N \cdots} = \ldots h^K_i \, h^L_j \ldots h^i_K \, h^j_L \ldots T^{\cdots i\, j \cdots}_{\cdots r\, s \cdots} . \tag{9}$$

(9) defines what we would like to call the *dual space* to *K* at every point of *K*. A quantity $T^{K\,L}_{M\,N}$ in that dual space is invariant under coordinate transformations $\bar{x}^i = \bar{x}^i(x^r)$, but will go to $\bar{T}^{K\,L}_{M\,N} = C^K_H \, C^L_J \, C^{-1S}{}_N \, C^{-1R}{}_M \, T^{H\,J}_{R\,S}$ under a linear transformation $d\bar{x}^K = C^K_L dx^L$, which would correspond to the transition to another system of dreibein-congruences, as one can conclude from the invariance of $T^{\,i\,j}_{r\,s}$ under the transformations in (9).

The two spaces then behave as if they were truly dual to each other: Tensors in the dual space are invariants in the RIEMANNian space *K* and conversely (cf., e.g., G. VRANCEANU [**32**]).

The expressions (3) and (6), which characterize the connection between quantities in the ideal and real crystal precisely, are special cases of (9). We can then see the physical meaning of (9) for our case with that: V^r characterizes any physical quantity in the crystal. V^r ($V^K = h_r^K V^r$, resp) is that quantity when it is referred to the real (ideal, resp.) crystal then.

From (4), the dual space will be a Euclidian space whose Cartesian axis directions can be identified with the lattice directions of the ideal crystal. Globally, the Euclidian space is oriented and a complete image of the ideal crystal in that way. The dual space reproduces that behavior point-wise and juxtaposes a small ideal crystal with each point of the real crystal, intuitively speaking.

In summary, we can make the following remarks, which we would like to apply to dynamics (with some meaningful conversion of them):

The "fibered" RIEMANNian space and the dual space that is associated with it point-wise give geometric images of the real and ideal crystal, resp. The metric of the dual space is equal to the metric on our Euclidian space. Its distinguished directions are the directions of the coordinate axes of the Cartesian reference system. The transition from an ideal crystal to a real one consists of twisting and stretching the lattice vectors. [In that way, the presence of dislocations will be characterized precisely by the fact that the law of the transition (3^*) is anholonomic.]

In the event that the lattice vectors of the ideal crystal are not orthonormal, as we have assumed up to now, so in the event that $h_K^\alpha \neq \delta_K^\alpha$, we will alter our definition of the distinguished dreibein (1) by replacing it with:

$$h^{+\alpha}{}_K = h_L^i \, k_\alpha^L \, \delta_K^\alpha = h^{+i}{}_K(x^r)\,, \tag{1^*}$$

in which k_α^L is the inverse of k_L^α. The congruences of curves will then be determined by the lattice of the real crystal as much as by that of the ideal one. (1^*) can then be regarded as a transformation of the congruences of curves with the matrix $C_K^L = k_\alpha^L \, \delta_K^\alpha$, under which h_K^i will go to $h^{+i}{}_K$ and k_K^α will go to δ_K^α, in such a way that all ratios will remain unchanged when one replaces h_K^i with $h^{+i}{}_K$.

In (8), we have defined the coefficients of the teleparallel displacement as EINSTEIN did [**24**]. Due to the integrability condition of the displacement, the curvature tensor that is defined by Γ_{kl}^i must vanish:

$$R_{ijk}{}^{l} \underset{\text{def.}}{=} \Gamma^{l}_{jk,i} - \Gamma^{l}_{ik,j} + \Gamma^{r}_{jk}\,\Gamma^{l}_{ir} - \Gamma^{r}_{ik}\,\Gamma^{l}_{jr} = 0. \tag{10}$$

Equations (10) will become the true determining equations for the internal stresses when one decomposes the Γ-affinity according to:

$$\Gamma_{kl}^i = \left\{ \begin{matrix} i \\ k\,l \end{matrix} \right\} + T_{kl}{}^i + T^i{}_{kl} + T^{\;i}_{l\;k}\,,$$

with

$$T_{kl}{}^i = \Gamma^{\;i}_{<kl>}\,, \tag{11}$$

$$\left\{ \begin{matrix} i \\ k\,l \end{matrix} \right\} = \text{CHRISTOFFEL affinity that is defined by } g_{ij} \,,$$

and the torsion $T_{k\,l}{}^{i}$ can be interpreted physically as the dislocation density and treated as a given quantity. In the statics of dislocations, that is the process that is used to derive the field equations, which we would now like to apply to dynamics, once we have formulated dynamics in a manner that is suitable to our purposes.

We would like to arrive at the equations of dynamics as a four-dimensional generalization of the three-dimensional static theory above. Since we shall stick to classical mechanics, naturally, the speed of light will not come under consideration for us as a limiting velocity. In that way, there will be no true four-dimensional tensors, and time will be forced into the role of a parameter ([3]). We will also see that in the ultimate form of the field equations. On the other hand, there are known effects in the theory of moving dislocations that had previously been known only for LORENTZ-invariant theories. A dislocation will be restricted by the speed of sound just as a mass-point is restricted by the speed of light in relativistic mechanics. The stress field of a uniformly-moving dislocation suffers a contraction in the direction of motion that is similar to the way that the electromagnetic field of a uniformly-moving electron contracts in LORENTZ-MAXWELL electrodynamics.

Since one will consider singular dislocations in that way, along with ones for which a displacement vector field exists, that effect can be derived using elementary methods. However, certain complications will arise in the theory of elasticity due to the fact that one must generally deal with many speeds of sound (cf., on that, J. D. ESHELBY [**33**], A. W. SÁENZ [**34**], and J. WERTMAN [**35**]). Since those effects all relate to the context of a particular mathematical theory – namely, the machinery of special relativity – we would like to incorporate the theory of moving dislocations in an entirely analogous mathematical model that is naturally subject to a physical interpretation that is completely distinct from special relativity.

We therefore extend our three-dimensional Euclidian space to a four-dimensional MINKOWSKI world in which the role of the speed of light will be taken on by the speed of sound in the medium in question. However, the extension is not unique, since the elastic medium can have many speeds of sound, in contrast to the uniquely-determined speed of light. When we restrict ourselves to the isotropic case, we will have to distinguish between only the transversal speed of sound c_T and the longitudinal one c_L. However, one knows from the treatment of moving singular dislocations that the actual limiting speed is already given by c_T.

If one, in fact, considers, e.g., the expressions that J. WERTMAN [**35**] gave for the stress fields of moving dislocations then they will all go to infinity when the speed of the dislocation v reaches the transversal speed of sound c_T. When one exceeds one of the speeds of sound, for physically-reasonable distributions of dislocations, one will indeed expect finite fields again (cf., H. GÜNTHER [**49**] on that), but it is questionable on energetic grounds whether a dislocation can actually exceed c_T (c_L, resp) for the infinite configurations; see J. WERTMAN [**35**]. As a consequence of our analogy with special

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([3]) The space-time pseudo-rotations of the "reference system" will then have no physical sense here. In fact, there is an absolute reference system that is realized by the crystal (cf. *infra*).

relativity, we will then work with c_T from the outset. We will also justify the foregoing later from the dynamical part of the field equations themselves. We will then find that, in addition to that transversal speed of sound, there is also a longitudinal speed of sound c_L (cf., *infra*, Chap. II), namely, the speed of propagation of longitudinal waves. We then introduce:

$$x^0 = c_T\, t$$

as the fourth coordinate of our "acoustic MINKOWSKI space." We extend the Euclidian space metric $a_{ik} = \delta_{ik}$ to the MINKOWSKI metric:

$$a_{\mu\nu} = \eta_{\mu\nu} = (1, 1, 1, -1) \qquad (\mu, \nu = 0, 1, 2, 3),$$

and our line element will take on the form:

$$ds^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu. \tag{12}$$

We shall discuss these expressions later in a broader context.

If we now adapt the static phenomena to our four-dimensional case by way of analogy then we can regard that MINKOWSKI space as the four-dimensional image of an ideal crystal in the large. The directions of the axes in MINKOWSKI space are the crystal directions. Naturally, the spatial coordinate lines once more represent the usual ideal lattice in that. The physical meaning of the fourth coordinate axis – viz., the time axis – will become clear when we consider that the parallels to the time axis can be interpreted kinematically as the world-lines of mass-points at rest. This "four-dimensional ideal crystal" then represents the history of a three-dimensional ideal crystal at rest here. A four-dimensional real crystal will arise from a now-time-dependent anholonomic deformation and twisting of the four-dimensional lattice, which we associate with a four-dimensional RIEMANNian space with teleparallelism, in complete analogy to statics, as follows:

The three spatial vectors of our ideal lattice first go to the deformed vectors:

$$h_K^\alpha = h_K^\alpha(x^i, x^0) \qquad \text{with} \qquad h_K^0 \equiv 0.$$

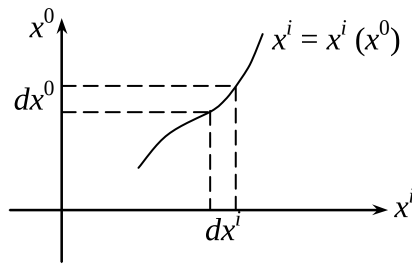
It is clear that these actual lattice vectors cannot take on a temporal component under the deformation; hence, the condition $h_K^0 \equiv 0$. The lattice vectors that are constantly changing in time from the three-dimensional standpoint will then lead us to single out three well-defined fixed congruences of curves in the four-dimensional image. We will get the fourth congruence in such a way that the lines of the mass-points at rest that are parallel to the time axes will go to the world-lines of moving mass-points, which we can describe by the equation:

$$x^i = x^i(x^0)\ .$$

If x^0 is the parameter of that congruence of curves then we will get the following expressions for the components of that congruence:

$$h_0^\alpha = h_0^\alpha(x^i, x^0) \qquad \text{with} \qquad h_0^i \equiv \frac{v^i}{c_T}, \qquad h_0^0 = 1,$$

in which $v^i = v^i\,(x^i,\ x^0)$ means the velocity of the mass-point. We easily convinces ourselves of the validity of that relation by referring to the following sketch:



The projection of a curve segment onto the axes, divided by dx^0, will yield h_0^α, from which, the formula above will arise.

In that way, we have obtained a complete vierbein field, which has the following form:

$$h_\Gamma^\mu = \begin{bmatrix} h_G^m & h_G^0 = 0 \\ h_0^m = \dfrac{v^m}{c_T} & h_0^0 = 1 \end{bmatrix} \quad \text{with the inverse} \quad h_\mu^\Gamma = \begin{bmatrix} h_m^G & h_m^0 = 0 \\ h_0^G & h_0^0 = 1 \end{bmatrix},$$

according to ([4]):

$$h_\Gamma^\mu\, h_\nu^\Gamma = \delta_\nu^\mu \qquad \text{and} \qquad h_\Gamma^\mu\, h_\mu^\Lambda = \delta_\Gamma^\Lambda .$$

The inverses h_ν^Γ exist on the basis of the non-coplanarity of the three spatial lattice vectors that was assumed initially. Due to their special form (13), the individual reciprocity relations will then read:

$$h_G^m\, h_n^G = \delta_n^m \qquad (h_G^m\, h_m^K = \delta_G^K \text{, resp.})$$

and

$$h_0^\mu\, h_\mu^G \equiv h_0^m\, h_m^G + h_0^0\, h_0^G \qquad \Rightarrow \qquad h_0^G = -h_0^m\, h_m^G . \qquad (13^*)$$

Just as we do in statics, we can now introduce the dual space when we characterize a tensor by its tensor components V^α, in one case, or by projecting those components onto the vierbein:

$$V^\Gamma = h_\alpha^\Gamma V^\alpha . \qquad (14)$$

In analogy to statics, we set the metric of the dual space equal to that of MINKOWSKI space, so:

$$g_{\Gamma\Lambda} = \eta_{\Gamma\Lambda} = (1, 1, 1, -1),$$

([4]) Lowercase, as well as uppercase, Greek indices (α, μ, … = tensor indices, Γ, Λ, … = numbers) will always range through the numbers 0, 1, 2, 3. (See Appendix) and with the help of the equation:

$$g_{\mu\nu} = h_\mu^\Gamma\, h_\nu^\Lambda\,, \tag{15}$$

we have endowed our four-dimensional, oriented continuum with a metric, and thus made it into a RIEMANNian space that was the image of the four-dimensional real crystal. The dual space that is associated with it point-wise contrasts with the behavior of the ideal crystal.

Due to the four distinguished congruences of curves that are defined by the vierbeins, we can once more define an EINSTEIN teleparallelism (cf., [**24**]) and the coefficients of the teleparallel displacement are given by ([5]):

$$\Gamma_{\mu\nu}^{\lambda} = h_\Gamma^\lambda\, h_{\nu,\mu}^\Gamma \qquad \text{with} \qquad h^\Gamma_{\alpha\underset{4}{\|}\beta} = 0, \qquad g_{\mu\nu\underset{4}{\|}\lambda} = 0, \tag{16}$$

so one will have, in particular:

$$\Gamma^0_{\alpha\beta} = 0, \tag{17}$$

whereas for all other displacement symbols, one will generally have $\Gamma^{\,i}_{\alpha\beta} \neq 0$, which one can easily believe on the basis of (13). Along with (16), one can define the affinity:

$$\underset{3}{\Gamma}{}^{r}_{ik} = h_G^r\, h_{k,i}^G \tag{18}$$

solely on the basis of the reciprocity relations that exist already between the three spatial lattice vectors $h_G^m = h_G^m\ (x^i, x^0)$ from (13*). (The index "3" in $\underset{3}{\Gamma}{}^{r}_{ik}$ refers to the three-dimensional character of that quantity; cf., the Appendix.)

We then get not only equations (17) for the field equations of moving dislocations (which should give an invariant characterization of the vierbein field (13), geometrically-speaking), but also the integrability conditions for the the affinities (16) and (18), and therefore:

$$\left.\begin{aligned}
&a)\quad R_{\alpha\beta\gamma}{}^{\delta} \equiv \Gamma^\delta_{\beta\gamma,\alpha} - \Gamma^\delta_{\alpha\gamma,\beta} + \Gamma^\kappa_{\beta\gamma}\,\Gamma^\delta_{\alpha\kappa} - \Gamma^\kappa_{\alpha\gamma}\,\Gamma^\delta_{\beta\kappa} = 0,\\
&b)\quad \Gamma^0_{\alpha\beta} = 0,\\
&c)\quad \underset{3}{R}{}_{ijk}{}^{l} \equiv \underset{3}{\Gamma}{}^{\,l}_{jk,i} - \underset{3}{\Gamma}{}^{\,l}_{ik,j} + \underset{3}{\Gamma}{}^{\,r}_{jk}\,\underset{3}{\Gamma}{}^{\,l}_{ir} - \underset{3}{\Gamma}{}^{\,r}_{ik}\,\underset{3}{\Gamma}{}^{\,l}_{jr} = 0.
\end{aligned}\right\} \tag{19}$$

However, not all of equations (19) are algebraically independent of each other, and therefore they are useless in practice. We shall now choose an algebraically-independent system, but we shall first make a few remarks about the geometric interpretation of (19).

With those vectors, (19.*b*) can be written in the form:

([5]) For the meaning of the notation $\underset{4}{\|}$, see the Appendix.

$$\left.\begin{aligned}&\Gamma^0_{\alpha\beta} \equiv h^0_0\, h^0_{\beta,\alpha} + h^0_G\, h^G_{\beta,\alpha} = 0, \quad \text{so}\\ &h^0_{\beta,\alpha} = -\frac{h^0_G}{h^0_0}\, h^G_{\beta,\alpha}.\end{aligned}\right\} \tag{20}$$

We would now like to assume only that the conditions:

$$h^0_\Gamma = \delta^0_\Gamma \tag{21}$$

are valid at some time-point and some well-defined location, and therefore that we also have $h^0_\beta = \delta^0_\beta$, from the reciprocity conditions, and that our vectors are analytic functions of position and time, in addition. If we recall the development of the real crystal from an ideal crystal then naturally that condition will always be fulfilled. We then immediately read off of (26) that the conditions (21) will then be true everywhere and for all times.

That means that the infinitesimal operators:

$$X_K \equiv h^\mu_K \frac{\partial}{\partial x^\mu} = h^r_K \frac{\partial}{\partial x^r}$$

that one defines with just the three congruences of curves h^μ_K already define a complete system, which is well-known to be the necessary and sufficient condition for those three congruences of curves to define a manifold. (19.*b*) is then the condition for a three-dimensional submanifold to be distinguished in our four-dimensional manifold, namely, our ordinary Euclidian space (endowed with the metric g_{ij}, resp.), which is the three-dimensional RIEMANNian space of the statics of dislocations. (19.*b*) will then define a teleparallelism in that RIEMANNian space with the help of those three lattice vectors, and will therefore be identical with the equations (10) of dislocation statics. They are then preserved here, and they will be extended by only the additional equations (19.*a*) and (19.*b*) under the transition to dynamics. One difference between (10) and (19.*c*) consists of the fact that (19.*c*) is required to be true for all times, while (10) is referred to only one time-point from the outset.

We will introduce special-relativistic behavior with the speed of sound as the limiting velocity when we introduce a four-dimensional line element by way of:

$$ds^2 = \eta_{\mu\nu}\, dx^\mu\, dx^\nu = \delta_{ik}\, dx^i\, dx^k - (dx^0)^2, \tag{12}$$

and in that way, we will construct a theory of moving dislocations. We would now like to make a few remarks about that construction.

In the special theory of relativity, the basis for the introduction of ds^2 is the physical equivalence of all inertial systems. The inertial systems emerge from each other by LORENTZ transformations, and ds^2 is an invariant under the group of Lorentz transformations. However, in our case, the demand that the sound should be produced by a point-like source and propagate like a spherical wave will establish the state of motion of the reference system uniquely; it is the system in which the elastic medium is at rest. The coordinate transformations that mix the space and time coordinates with each other will be excluded then. If we then restrict the allowable coordinate transformations to the orthogonal group of space-space transformations and the identity transformation in time then we will formally justify the condition of invariance of ds^2, which is the centerpiece of the theory of relativity, under this subgroup of "LORENTZ transformations." We will have transferred the same mathematical relationships into continuum dynamics that we find in the general theory of relativity by way of the RIEMANNian space that is defined by:

$$g_{\mu\nu} = h_\mu^\Gamma \, h_\nu^\Lambda \, \eta_{\Gamma\Lambda} \,, \tag{15}$$

and general relativity is also based upon a four-dimensional RIEMANNian space.

The teleparallelism that is introduced by:

$$\Gamma_{\mu\nu}^\lambda = h_\Gamma^\lambda \, h_{\nu,\mu}^\Gamma \,, \qquad h_{\underset{4}{\alpha\|\beta}}^\Gamma = 0, \qquad g_{\underset{4}{\mu\nu\|\lambda}} = 0 \tag{16}$$

corresponds completely to the field theory that EINSTEIN introduced in 1928 (cf., [**24**]). The vierbeins that are defined at each world-point in that theory, which are analogous to our four-dimensional lattice, define local inertial systems. The vierbein components of tensors represent the tensors that are measured in those inertial systems by projection. Just as in continuum dynamics, the dual spaces of the theory of gravitation carry the metric of MINKOWSKI space. However, the vierbeins were not defined uniquely in EINSTEIN's theory of 1915. Furthermore, the general theory of relativity is invariant under the four-dimensional rotations of those vierbeins that vary from point to point – i.e., under local LORENTZ transformations – and according to TREDER [**36**], that is precisely the essence of the principle of general relativity. Such a principle is not true *a priori* in continuum dynamics, since the curve congruences are distinguished uniquely by the lattice structure in that case. The only invariance in that case is under global spatial rotations.

What is the physical meaning of the four-dimensional metric now? With the form (13) for the vierbeins that is guaranteed by the field equations (19), one finds that one has, in detail:

$$\underset{4}{g}_{ik} = h_i^\Gamma \, h_k^\Lambda \, \eta_{\Gamma\Lambda} = h_i^G \, h_k^L \, \delta_{GL} - h_i^0 \, h_k^0 = h_i^G \, h_k^L \, \delta_{GL} = \underset{3}{g}_{ik} \,.$$

The spatial components of the four-dimensional metric then coincide with those of statics and therefore characterize the deformations as they would in the latter theory. It further follows that:

$$g^{0i} = h_\Gamma^0 \, h_\Lambda^i \, \eta^{\Gamma\Lambda} = - h_0^0 \, h_0^i + h_G^0 \, h_L^i \, \delta^{GL} = - h_0^i = - \frac{v^i}{c_T} \,.$$

The contravariant space-time components of the metric then determine the velocity of the matter, and finally:

$$g^{00} = h_\Gamma^0 \, h_\Lambda^0 \, \eta^{\Gamma\Lambda} = - h_0^0 \, h_0^0 + h_G^0 \, h_L^0 \, \delta^{GL} = - h_0^0 \, h_0^0 = -1 \;;$$

i.e., the contravariant zero-zero component is constant and will always be set to minus one. One finds the remaining components from the reciprocity relations:

$$\underset{4}{g}_{\alpha\mu}\,\underset{4}{g}^{\alpha\nu} = \delta_\mu^\nu, \qquad \underset{4}{g}_{ri}\,\underset{4}{g}^{rk} = \delta_i^k,$$

and therefore, in summary, one will have:

$$\left.\begin{array}{ll} \underset{4}{g}_{ik} = \underset{3}{g}_{ik}, & \underset{4}{g}^{ik} = \underset{3}{g}^{ik} - g^{0i} g^{0k}, \\ g^{0i} = -\dfrac{v^i}{c_T}, & g_{0i} = g_{ri} g^{0r}, \\ g^{00} = -1, & g_{00} = -1 + g^{0r} g^{0s} g_{rs}. \end{array}\right\} \tag{25}$$

The condition $g^{00} = -1$, or more generally, the process of fixing some components of the metric to have definite values (restricting them by differential conditions, resp.), is well-known in the general theory of relativity. Due to general covariance, in the general relativity, one can always impose four arbitrary conditions on the metric that correspond to an arbitrary choice of the four coordinates. In that sense, we can now mathematically interpret the requirement $g^{00} = -1$, which comes about as a result of the peculiarities of the RIEMANNian space that is defined here, conversely as a coordinate condition in a general RIEMANNian space.

The three further coordinate conditions that are possible can also find their counterparts in general relativity here. Namely, the deformations are still not established uniquely by (19). As is known, one must specify the external forces that act upon the medium; i.e., one must extend (19) with the equilibrium conditions, which one can write in the form:

$$\sigma^{ri}_{\ ,r} - \rho \frac{d}{dt} v^i = -f^i \tag{26}$$

in the dynamical case, in which σ^{ri} means the stress tensor, ρ means the density of the deformed medium, and f^i means the volume force ([6]). Those equations will give three differential conditions for the metric, which will also establish the metric g_{ik} uniquely, as in general relativity, when one adds a material law:

$$\sigma_{ik} = \sigma_{ik}(\varepsilon_{rs}) = \sigma_{ik}(g_{rs})$$

and introducing the notation:

$$v^i = -c_T\, g^{0i}.$$

Equations (26) can then be interpreted as the remaining three coordinate conditions. Therefore, in contrast to the theory of relativity, changes in the $g_{\alpha\beta}$ that are produced by

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([6]) (26) is an equation in Euclidian space, for which we employ Cartesian coordinates. The position of the indices in (26) is therefore inessential.

coordinate transformations will have an actual physical meaning here. We shall return to that connection and some of its consequences later.

We would now like to choose an algebraically-independent system of field equations from (19). In order to do that, we proceed in such a way that we shall exhibit successive equations that are obviously algebraically-independent of each other and with which, we can go into the still-unconsidered equations and check to see what new requirements that they might then express. As we said, equations (19) will become true determining equations for the metric when we decompose the affinities according to:

$$\left.\begin{aligned} \Gamma^{\mu}_{\alpha\beta} &= \begin{Bmatrix} \mu \\ \alpha\,\beta \end{Bmatrix} + T_{\alpha\beta}{}^{\mu} + T^{\mu}{}_{\alpha\beta} + T^{\mu}{}_{\beta\alpha} = \begin{Bmatrix} \mu \\ \alpha\,\beta \end{Bmatrix} + h_{\alpha\beta}{}^{\mu}, \\ \underset{3}{\Gamma}{}_{ij}^{k} &= \underset{3}{\begin{Bmatrix} k \\ i\,j \end{Bmatrix}} + \underset{3}{T}{}_{ij}{}^{k} + \underset{3}{T}{}^{k}{}_{ij} + \underset{3}{T}{}^{k}{}_{ji} = \underset{3}{\begin{Bmatrix} k \\ i\,j \end{Bmatrix}} + \underset{3}{h}{}_{ij}{}^{k}, \end{aligned}\right\} \tag{27}$$

and therefore into the CHRISTOFFEL symbols and the torsion part, and regards the torsion as a physically-given quantity. $h_{\alpha\beta}{}^{\mu} = h^{\mu}_{\Gamma}\, h^{\Gamma}_{\beta;\alpha}$ is therefore RICCI's rotation tensor for the tensor field h^{μ}_{Γ} (cf., J. A. SCHOUTEN [**37**], G. VRANCEANU [**32**]).

In order to perform that decomposition, we must first find a relation between $\underset{4}{T}{}_{ij}{}^{k}$ and $\underset{3}{T}{}_{ij}{}^{k}$ and then interpret the torsion components with one or more zero indices physically.

We next point out that on the basis of the special form (13) of our congruences of curves, we will have:

$$\underset{3}{\Gamma}{}_{ij}{}^{k} = \underset{4}{\Gamma}{}_{ij}{}^{k}, \tag{28}$$

which we can read off from (16) and (18). With the help of (19.*b*), the torsion components with zero indices can all be brought back to the quantities $T^{0}{}_{ik}$ (vanish, resp.). In that way, we will find the relations:

$$a)\quad T_{\alpha\beta}{}^{0} = 0, \qquad b)\left\{\begin{aligned} T^{0}_{0i} &= g^{0r}\, T^{0}{}_{ri}, \\ T^{0}_{i0} &= g^{0r}\, T^{0}{}_{ri}, \\ T^{0}_{00} &= g^{0r} g^{0s}\, T^{0}{}_{rs}, \end{aligned}\right. \qquad c)\left\{\begin{aligned} \underset{4}{T}{}_{rs}{}^{k} &= \underset{3}{T}{}_{rs}{}^{k} \underset{\text{def.}}{=} T_{rs}{}^{k}, \\ \underset{4}{T}{}^{k}{}_{rs} &= \underset{3}{T}{}^{k}{}_{rs} - g^{0k}\, T^{0}{}_{rs}. \end{aligned}\right. \tag{29}$$

(These relations are derived in the Appendix.) As before, $T_{rs}{}^{k}$ can be interpreted as a dislocation density, while only $T^{0}{}_{ik}$ can be interpreted as new quantities in the equations. We will see that they can describe the dislocation current.

We shall consider what independent demands are still contained in (19.*b*). Due to (29), we only need to consider the symmetric part of it:

$$\Gamma^0_{\underline{\alpha\beta}} = \begin{Bmatrix} 0 \\ \alpha\,\beta \end{Bmatrix} + 2T^{\,0}{}_{\underline{\alpha\beta}} \equiv \tfrac{1}{2} g^{0\kappa} (g_{\alpha\kappa,\,\beta} + g_{\kappa\beta,\,\alpha} - g_{\beta\alpha,\,\kappa}) + T^{\,0}{}_{\alpha\beta} + T^{\,0}{}_{\beta\alpha} = 0.$$

We write out those equations in detail, in which we employ (25) and get:

$$\Gamma^0_{\underline{ik}} \equiv \begin{Bmatrix} 0 \\ i\,k \end{Bmatrix} + 2T^{\,0}{}_{\underline{ik}}$$

$$\equiv \tfrac{1}{2} (g_{ik,\,0} + g_{ik,\,q}\, g^{0q} - g^{0q}{}_{,k}\, g_{iq}) + T^{\,0}{}_{ik} + T^{\,0}{}_{ki} = 0,$$

$$\Gamma^0_{\underline{i0}} \equiv \tfrac{1}{2} g^{0\kappa} (g_{0\kappa,\,i} + g_{\kappa i,\,0} - g_{i0,\,\kappa}) + T^{\,0}{}_{i\,0} + T^{\,0}{}_{0\,\mathrm{i}} = 0$$

$$= g^{0q} \begin{Bmatrix} 0 \\ q\,i \end{Bmatrix} + T^{\,0}{}_{i0} + T^{\,0}{}_{0i} = 0,$$

$$\Gamma^0_{00} \equiv \tfrac{1}{2} g^{0\kappa} (2\, g_{0\kappa,\,0} - \mathrm{g}_{00,\,\kappa}) + 2T^{\,0}{}_{00} = 0$$

$$= g^{0q} g^{0p} \begin{Bmatrix} 0 \\ p\,q \end{Bmatrix} + 2T^{\,0}{}_{00} = 0.$$

If we employ (29.*b*) then it will follow directly that:

$$\Gamma^0_{0i} \equiv g^{0q}\, \Gamma^0_{qi}\,, \qquad\qquad \Gamma^0_{00} \equiv g^{0p}\, g^{0q}\, \Gamma^0_{pq}\,.$$

The only independent requirement that will remain in (19.*b*) is:

$$\Gamma^0_{\underline{ik}} = 0. \tag{30}$$

Now, it is clear that equations (19.*c*) to (30) are algebraically independent. Along with (30), we still have to fulfill:

$$\underset{3}{R}{}_{ijk}{}^{l} = 0, \tag{31}$$

and we still have to investigate which additional requirements must be added to (30) and (31) through equations (19.*a*). The number of those independent equations will be determined from the number of algebraically-independent components of the curvature tensor using (19.*a*). The number of independent components of the curvature tensor is, in turn, given by its symmetry properties. For the algebraic identities in the curvature tensor that will be given in what follows, cf., e.g., J. A. SCHOUTEN [**37**] ([7]). Next, with our convention, we will have antisymmetry in the first two indices from the definition:

$$R_{\alpha\beta\gamma\delta} + R_{\beta\alpha\gamma\delta} = 0. \tag{32}$$

However, the last two indices are also antisymmetric:

$$R_{\alpha\beta\gamma\delta} + R_{\alpha\beta\delta\gamma} = 0. \tag{33}$$

That is a consequence of the special form (27) of our affinity. (33) is the integrability condition for the RICCI Lemma:

$$g_{\alpha\beta\,\|\,\gamma} \equiv g_{\alpha\beta,\,\gamma} - g_{\mu\beta}\,\Gamma^{\mu}_{\gamma\alpha} - g_{\mu\alpha}\,\Gamma^{\mu}_{\gamma\beta} = 0. \tag{34}$$

[Otherwise expressed, (34) is the once-integrated form of (33).] The general solution to (34) is an affinity of the form:

$$\Gamma^{\mu}_{\alpha\beta} = \begin{Bmatrix} \mu \\ \alpha\,\beta \end{Bmatrix} + T_{\alpha\beta}{}^{\mu} + T^{\mu}{}_{\alpha\beta} + T^{\mu}{}_{\beta\alpha} \qquad \text{with} \qquad T_{\alpha\beta}{}^{\mu} = -\,T_{\beta\alpha}{}^{\mu}. \tag{34.a}$$

Therefore, any curvature tensor that is defined by an affinity of the form (34.*a*) will satisfy (33) identically. Since the form of our affinity (27) coincides (34.*a*), (33) will then be fulfilled.

We shall now consider the part of the curvature tensor that is cyclically-symmetric in the first three indices. We can write it in the form:

$$\begin{aligned} R_{\{\alpha\beta\gamma\}\,\delta} &\equiv 2\,[T_{\alpha\beta\gamma,\,\delta} + T_{\beta\gamma\delta,\,\alpha} + T_{\gamma\alpha\delta,\,\beta} + T_{\gamma\beta\,\kappa}\Gamma^{\kappa}_{\alpha\delta} + T_{\alpha\gamma\beta}\,\Gamma^{\kappa}_{\beta\delta} + T_{\beta\alpha\,\kappa}\Gamma^{\kappa}_{\gamma\delta}\,] \\ &\equiv 2\,[T_{\alpha\beta\gamma;\,\delta} + T_{\beta\gamma\delta;\,\alpha} + T_{\gamma\alpha\delta;\,\beta} + T_{\alpha\beta}{}^{\kappa}(T_{\gamma\kappa\delta} + T_{\delta\gamma\,\kappa} + T_{\delta\kappa\gamma}) \\ &\quad + T_{\beta\gamma}{}^{\kappa}(T_{\alpha\kappa\delta} + T_{\delta\alpha\,\kappa} + T_{\delta\kappa\alpha}) + T_{\gamma\alpha}{}^{\kappa}(T_{\beta\kappa\delta} + T_{\delta\beta\,\kappa} + T_{\delta\kappa\beta})]. \end{aligned} \tag{35}$$

That is an expression that does not vanish identically and which includes only the torsion (i.e., physically speaking, the dislocations) in the linear approximation. We will then have:

$$R_{\{\alpha\beta\gamma\}}{}^{\delta} = 0 \tag{36}$$

as our new equations. Now, the cyclically-symmetric part of (31) is likewise an expression that includes only the torsion in the linear approximation (and is known to express the closure condition for the dislocations). In the Appendix, it will be shown that of equations (36), only the conditions:

([7]) In comparing the corresponding expressions, one must observe which affinity that the covariant differentiation is performed with.

$$R_{\{0ij\}}{}^{k} = 0 \tag{37}$$

are algebraically independent of (30) and (31).

From the last of the symmetry properties that characterize the curvature tensor, from formula (27), one will have:

$$R_{\alpha\beta\gamma\delta} - R_{\gamma\delta\alpha\beta} \equiv \tfrac{1}{2}[R_{\{\alpha\beta\gamma\}\ \delta} + R_{\{\beta\alpha\delta\}\ \gamma} + R_{\{\delta\gamma\beta\}\ \alpha} - R_{\{\gamma\delta\alpha\}\ \beta}] \tag{38}$$

for the affinity, as one easily verifies. However, since one already has $R_{(\alpha\beta\gamma)}{}^{\delta} = 0$, on the basis of the field equations (31) and (37), one will also always have:

$$R_{\alpha\beta\gamma\delta} - R_{\gamma\delta\alpha\beta} = 0\ . \tag{39}$$

We have seen that the curvature tensor $R_{\alpha\beta\gamma\delta}$ has all of the symmetry properties of the RIEMANN-CHRISTOFFEL curvature tensor, on the grounds of conditions (36) and the form of the affinity (27). We can then assume those symmetry properties in our search for other possible independent equations. If we employ the triply-covariant, singly-contravariant form:

$$R_{\{\alpha\beta\gamma\}}{}^{\delta} = 0$$

then we can restrict ourselves to:

$$R_{\{\alpha\beta\gamma\}}{}^{k} = 0$$

due to the validity of (19.*b*). Here, we must consider the equations:

$$\underset{4}{R}{}_{ijk}{}^{l} = 0, \tag{40}$$

$$R_{0\,i\,j}{}^{k} = 0, \tag{41}$$

$$R_{0\,i\,0}{}^{k} = 0 \tag{42}$$

separately. All other equations will then follow from (40)-(42) as a result of the symmetry properties. However, we can show that equations (40)-(42) imply no new requirements. Moreover, our use of the expressions (25) for the metric and (29) for the form of the torsion components by itself will suffice to require equations (30), (31), and (37), so the system (40)-(42) will also be fulfilled. (The calculations are given in the Appendix.) When we split off the extra cyclically-symmetric part of (31), we can then write our system of equations for moving dislocations in the following form:

$$\left.\begin{array}{llll} a) & \underset{3}{\bar{R}}{}_{ijkl} = 0, & b) & \underset{3}{R}{}_{\{ijk\}l} = 0, \\ c) & \Gamma^{0}_{\underline{ik}} = 0, & c) & \underset{3}{R}{}_{\{ijk\}l} = 0. \end{array}\right\} \tag{43}$$

It is possible to replace this system of equations with one for which the analogy to the equations of general relativity emerges very sharply (see Appendix). In that way:

$$\underset{3}{R}_{\{0ij\}k} \underset{\text{def}}{=} R_{\{0ij\}}{}^{r} g_{rk}$$

and

$$\underset{3}{\overline{R}}_{ijkl} \underset{\text{def}}{=} \underset{3}{R}_{ijkl} - \underset{3}{R}_{\{ijk\}l}\,, \quad \text{so} \qquad \overline{R}_{\{ijk\}l} \equiv 0.$$

(43.*a*) is equivalent to:

$$\underset{3}{R}_{\underline{ij}} = 0. \tag{43.a'}$$

$\underset{3}{R}_{ij}$ is then the RICCI tensor that is defined by $\underset{3}{R}_{ij} \underset{\text{def}}{=} \underset{3}{R}_{sij}{}^{s}$, so $\underset{3}{R}_{ij}$ has just as many algebraically-independent components as $\underset{3}{R}_{kijl}$.

The explicit form of (43.*c*) is given by:

$$\Gamma^{0}_{\underline{ik}} \equiv \tfrac{1}{2}(g_{ik,\,0} - g_{ik,\,q}\, g^{0q} - g^{0q}{}_{,\,i}\, g_{kq} - g^{0q}{}_{,\,k}\, g_{iq}) + T^{0}{}_{ik} + T^{0}{}_{ki} = 0, \tag{43.c}$$

while (43.*d*) can be brought into the following form:

$$\underset{3}{R}_{\{0ij\}k} \equiv \underset{3}{T}_{ijk,0} - T^{0}{}_{ik,j} + T^{0}{}_{jk,i} - g^{0r} T_{ijk,r} + g^{0r}{}_{,j}\, \underset{3}{T}_{ijk} - g^{0r}{}_{,i}\, \underset{3}{T}_{rjk} - g^{0r}{}_{,k}\, \underset{3}{T}_{ijr}$$

$$+ T^{0}{}_{ir}\, \underset{3}{\Gamma}{}^{r}_{jk} - T^{0}{}_{jr}\, \underset{3}{\Gamma}{}^{r}_{ik} + 2T^{0}{}_{kr}\, \underset{3}{T}_{ij}{}^{r} = 0. \tag{43.d}$$

(43.*d*) can now serve as the basis for the physical interpretation of $T^{0}{}_{ik}$. In order to see that, we consider the linearized equations (43.*d*):

$$T_{ijk,\,0} = T^{0}{}_{ik,\,j} - T^{0}{}_{jk,\,i}\,. \tag{44}$$

Now, the connection between the dislocation density T_{ijk} and the plastic distortion $\beta^{P}{}_{ik}$ is well-known (cf., e.g., E. KRÖNER [**18**]):

$$T_{ijk} = \tfrac{1}{2}(\beta^{P}{}_{jk,\,i} - \beta^{P}{}_{ik,\,j})\,.$$

Differentiating with respect to time will yield:

$$T_{ijk,\,0} = \tfrac{1}{2}(\beta^{P}{}_{jk,\,0\,i} - \beta^{P}{}_{ik,\,0\,j})\,.$$

Furthermore, the dislocation migration tensor N_{ijk} is defined in the literature to be the number of α_{jk}-dislocations per unit length that migrate in the i direction perpendicular to the line of motion of the dislocations and the direction of migration. According to KRÖNER [**13**], its connection to plastic distortion is:

$$\beta^{P}{}_{ij} = -\,b\,\varepsilon_{rsi}\,N_{rsj}\,.$$

(b is the BURGERS vector in this, which is assumed to be constant, for the sake of simplicity.)

Therefore, only the antisymmetric part of the first two indices of N_{ijk} will come under consideration for plastic distortion. One can call the time derivative of this quantity the *dislocation current tensor* that pertains to plastic distortion (cf., E. F. HOLLÄNDER [**26**]). If one denotes that by I_{ij} then one will have:

$$\beta^P{}_{ij,\,0} = -\,I_{ij}\,, \tag{45}$$

and one will have:

$$T_{ijk,\,0} = \tfrac{1}{2}(I_{ik,\,j} - I_{jk,\,i})\ . \tag{46}$$

A comparison of (44), (45), and (46) will then suggest that one might relate the next purely-geometrically-defined quantity $T^{\,0}{}_{ik}$ to the dislocation current (time derivative of the plastic distortion, resp.) according to:

$$\left.\begin{aligned} &2T^{\,0}{}_{ik} = I_{ik}\,,\\ &2T^{\,0}{}_{ik} = -\beta^P{}_{ik,0}\,,\\ &\varepsilon^P{}_{ik,0} \underset{\text{def}}{=} \beta^P{}_{(ik),0} = -(T^{\,0}{}_{ik} + T^{\,0}{}_{ki}). \end{aligned}\right\} \tag{47}$$

We can then regard (43.*d*) as the nonlinear generalization of the elementary connection between the dislocation current and the time derivative of the dislocation density. That also makes the physical meaning of (43.*d*) clear. Those equations give the connection between the dislocation current $T^{\,0}{}_{ik}$, the material velocity $v^{\,i}$, and the time evolution of the elastic strain $g_{ik,\,0}$.

We have therefore found interpretations for all of the quantities that enter into our basic equations (43) as three-dimensional tensors. $T_{ij}^{\;k}$ describes the dislocation density, $T^0{}_{ik}$ is the dislocation current, g_{ik} is the elastic strain, and g^{0i} is the material velocity. The basic equations (43) will then become equations between three-dimensional tensors in which time once more plays the role of a parameter. That must also be true on the grounds of physical reality, as we mentioned already to begin with. In what follows, we will then work with only three-dimensional quantities and drop the numeral 3 to characterize the three-dimensional character of the quantity in question, so we will mean that:

$$g^{ik} = \underset{3}{g}{}^{ik}\,, \qquad T^{\,i}{}_{;\,k} = \underset{3}{T}{}^{\,r}{}_{;s}\,, \quad \text{etc.}$$

(If we would like to employ the four-dimensional notation again specifically then we will characterize it by putting a “4” under the symbol.)

II. COMPARISON WITH THE LITERATURE

We shall now compare the field equations (43) with the earlier work on the dynamics of dislocations. To begin with, we shall go into a little-known paper by AMARI [**25**]. To our knowledge, it was the first (and up to now, the *only*) treatment of moving dislocations that appealed to a geometric procedure that was extended to four dimensions for that purpose. AMARI therefore deserves the credit for having adapted the differential-geometric methods that had proved so fruitful in statics to the dynamical problems. The relationships that AMARI derived also contain the basic equations (43) implicitly, although they were not actually given in the latter form. The author generally treated small perturbations and material speeds that were small in comparison to the speed of sound, such that many of his relationships could be regarded as only linear approximations. However, that came about more on the grounds of simplicity and should not be regarded as an essential restriction here, since the geometric methods are exactly the tool that is suited to the task of obtaining the rigorous, nonlinear equations.

A first example of such an approximation is that of the expressions for the components of the distortion that AMARI referred to by (1.4) and (1.41), which will read ([8]):

$$\underset{L}{h}{}^{\mu}_{\Gamma} = \begin{bmatrix} \delta^m_G + \beta^m_G & 0 \\ v^m & 1 \end{bmatrix} \qquad \text{with} \qquad \beta^m_G \ll 1, \qquad v^m \ll 1$$

in our notations, and the inverses will then be given by:

$$\underset{L}{h}{}^{\Gamma}_{\mu} = \begin{bmatrix} \delta^G_m + \beta^G_m & 0 \\ v^G & 1 \end{bmatrix} \qquad \text{with} \qquad \begin{aligned} \beta^G_m &= -\delta^G_n\,\delta^K_m\,\beta^n_K, \\ v^G &= -\delta^G_m v^m. \end{aligned}$$

As one sees, those formulas correspond to the linear approximation of (13). The fact that one finds the expression v^m / c_T here in place of v^m is based upon the fact that AMARI chose t to be the fourth coordinate, not $c_T\,t$.

AMARI introduced a metric by way of equations (1.7), and in our notation, it was:

$$g_{\mu\nu} = \underset{L}{h}{}^{\Gamma}_{\mu}\,\underset{L}{h}{}^{\Lambda}_{\nu}\,g_{\Gamma\Lambda}\,. \tag{1.7*}$$

It differs from the metric that we defined by (15) only in that $g_{\Gamma\Lambda}$ does not have the form $\eta_{\Gamma\Lambda}$, but $g_{\Gamma\Lambda}$ = (1, 1, 1, c), with a constant c that is left undetermined ([9]) and is not connected with the speed of sound *a priori*, and can be negative, as well as positive. However, that will not lead to any essential alteration of the theory, since we can again eliminate the speed of sound from the basic equations *a posteriori*.

([8]) AMARI chose a unit of time such that the speed of sound would have an order of magnitude of 1.

([9]) In order to compare (1.7*) with (15), one must observe that we would have to choose $\eta^*_{\Gamma\Lambda}$ = (1, 1, 1, $-\,c_T^2$), instead of $\eta_{\Gamma\Lambda}$ in our formalism if we were to choose t to be the fourth coordinate, instead of $c_T\,t$, as AMARI did.

We would not like to go further into that connection at this point. Namely, we mean that distinguishing the MINKOWSKI signature (1, 1, 1, – 1) and using the transversal speed of sound as the limiting speed is most likely to do justice to a four-dimensional picture of the dynamics of dislocations (if one would like to employ such a thing, at all). In order to do that, we consider our starting equations (19.*a*) in the linear approximation:

$$\underset{L}{\overset{4}{R}}{}_{\alpha\beta\gamma\delta} = 0.$$

We will then regard, above all, the (four-dimensional) RICCI tensor as the components that characterize the dynamical description of the system. In the linear approximation, we can write:

$$\underset{L}{\overset{4}{R}}{}_{\underline{\mu\nu}} \equiv \eta^{\alpha\beta} \underset{L}{\overset{4}{R}}{}_{\alpha\underline{\mu\nu}\beta} = 0.$$

If we consider the i, k-components in this, in turn, then with the use of (25) and the relation $T_{\mu\nu}{}^{0} = 0$, we will find that:

$$\underset{L}{\overset{4}{R}}{}_{ik} = \eta^{\alpha\beta}\,\tfrac{1}{2}(-g_{ik,\,\alpha\beta} - g_{\alpha\beta,\,ik} + g_{i\,\alpha,\,k\beta} + g_{k\alpha,\,i\beta}) + T^{\,\alpha}{}_{ik,\,\alpha} + T^{\,\alpha}{}_{ki,\,\alpha} + T_{i\,\alpha}{}^{\alpha}{}_{,\,k} + T_{k\,\alpha}{}^{\alpha}{}_{,\,i} = 0 \qquad (48^{*})$$

or

$$\tfrac{1}{2}\,[(-g_{ik,\,rr} - g_{rr,\,ik} + g_{ir,\,kr} + g_{kr,\,ir}) + \eta^{00}(-g_{ik,\,00} - g_{i\,0,\,k0} + g_{k\,0,\,i\,0})]$$

$$= -(T^{\,0}{}_{ik,\,0} + T^{\,0}{}_{ki,\,0} + T_{rik,\,r} + T_{rki,\,k} + T_{i\,rr,\,k} + T_{k\,rr,\,i}) \,. \qquad (48)$$

We now consider the linearized equilibrium conditions (26) for the force-free case ($f^{\,i}$ = 0):

$$\sigma_{ri,\,r} - \rho_0 \frac{\partial v^i}{\partial t} = 0 \qquad (\rho_0 = \text{density in the stressed state}), \qquad (49)$$

and with HOOKE's law for isotropic bodies:

$$\sigma_{ik} = 2\mu\,\varepsilon_{ik} + \lambda\,\delta_{ik}\,\varepsilon_{rr}\,. \qquad (50)$$

With:

$$g_{ij} = \delta_{ik} - 2\varepsilon_{ik}\,, \qquad \rho_0\,c_T^2 = \mu\,, \qquad (51)$$

we can combine (49) and (50) into:

$$g_{ir,\,r} + \frac{\lambda}{2\mu}\,g_{rr,\,i} + \frac{1}{c_T^2}\frac{\partial v^i}{\partial t} = 0. \qquad (52)$$

Equation (48) will now go to precisely the (inhomogeneous) wave equation that characterizes dynamical processes when ([10]):

([10]) In order to do that, one must only take care that one has:

$$1. \qquad \eta^{00} = -1, \qquad 2. \qquad x^0 = c_T\, t\,.$$

We will then have $\underset{L}{g}{}_{0i} = g^{0i} = -\, v^i / c_T$, and we can write (52) in the form:

$$g_{ir,\,r} + \frac{\lambda}{2\mu}\, g_{rr,\,i} - g^{0i}{}_{,\,0} = 0. \tag{53}$$

If we then substitute that into (48) then it will follow that:

$$\tfrac{1}{2}\,[(-\, g_{ik,\,rr} - g_{rr,\,ik} + g_{ir,\,kr} + g_{kr,\,ir})$$

$$= -\,(T^0{}_{ik,\,0} + T^0{}_{ki,\,0} + T_{rik,\,r} + T_{rki,\,k} + T_{i\,rr,\,k} + T_{k\,rr,\,i}) \underset{\text{def}}{=} -\, T_{ik}\,. \tag{54}$$

For $\varepsilon_{rr} = 0$ – i.e., for shearing waves – we will get from this that:

$$-\, g_{ik,\,rr} + \frac{1}{c_T^2}\frac{\partial^2 g_{ik}}{\partial t^2} = -\, T_{ik}\,,$$

and thus, the correct description of transversal waves. For $\varepsilon_{rr} \neq 0$, when we contract over i and k, it will follow that:

$$-\, \Delta g_{ik} + \frac{1}{\left(c_T\sqrt{\dfrac{2\mu+\lambda}{\mu}}\right)^2}\frac{\partial^2 g_{rr}}{\partial t^2} = -\, T_{rr}$$

for longitudinal waves, or:

$$-\, \Delta g_{rr} + \frac{1}{c_T^2}\frac{\partial^2 g_{rr}}{\partial t^2} = -\, T_{rr}\,,$$

resp., with:

$$c_L = c_T\sqrt{\frac{2\mu+\lambda}{\mu}}\,,$$

and thus, the correct relationship between c_T and c_L . From that viewpoint, the signature and limiting speed are established. However, the fact that one can also avoid such a way of looking at things without having to decide upon a well-defined signature and a distinguished limiting speed is based upon the fact that equation (54) "degenerates" in continuum mechanics; i.e., it decomposes into two summands that each vanish on the

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$$\eta^{00}\,\frac{\partial^2}{\partial x^{02}} = -\,\frac{1}{c_T^2}\frac{\partial^2}{\partial t^2}\,.$$

Naturally, one can arrive at that by setting $\eta^{00} = -\,1 / c_T^2$, $x^0 = t$ or other corresponding combinations that are less customary and lead to no new statements.

basis of the field equations. Namely, if one observes the linearized form of equations (43.*a′*) and (43.*c*) then one will have:

$$\underset{L}{\overset{4}{R}}{}_{\underline{ik}} = \underset{L}{\overset{3}{R}}{}_{\underline{ik}} - \eta^{00}\, \underset{L}{\Gamma}{}^{0}_{\underline{ik},0} = 0, \tag{55}$$

with

$$a) \qquad \underset{L}{\overset{3}{R}}{}_{\underline{ik}} = 0, \qquad\qquad b) \qquad \underset{L}{\Gamma}{}^{0}_{\underline{ik}} = 0, \qquad (55)[sic]$$

such that η^{00} will then remain undetermined. However, the four-dimensional description will be perturbed in that way, since (43.*a′*) and (43.*c*) are only three-dimensional equations. If one looks closer at the basic equations (43) then it will be clear to begin with that the speed of sound does not enter into equations (*a*) and (*b*), since they are, in fact, equations of statics. However, the speed of sound does not enter into equations (*c*) and (*d*) explicitly either. Namely, if one observes that $x^0 = c_T\, t$, $g^{0i} = -\, v^i \,/\, c_T$, and in addition, that a factor of $1 \,/\, c_T$ likewise enters into those quantities on the basis of the definition of $T^{\,0}{}_{ik}$ in equations (45)-(47), then one will see that when one multiplies (43.*c*) and (43.*d*) by c_T , the speed of sound will also drop out of those equations completely. That corresponds to the possibility of developing a version of the theory of moving dislocations that is free from the introduction of the speed of sound, like the one that AMARI gave. Furthermore, it is clear in this that the basic equations will also be true for the general case of an inhomogeneous and time-varying medium. We shall now return to AMARI's work.

By introducing a four-dimensional teleparallelism, AMARI likewise arrived at the equations that are referred to in (19.*a*):

$$R_{\alpha\beta\gamma}{}^{\delta} \equiv \Gamma^{\delta}_{\beta\gamma,\alpha} - \Gamma^{\delta}_{\alpha\gamma,\beta} + \Gamma^{\kappa}_{\beta\gamma}\Gamma^{\delta}_{\alpha\kappa} - \Gamma^{\kappa}_{\alpha\gamma}\Gamma^{\delta}_{\beta\kappa} = 0.$$

However, since he did not make the distinction between four-dimensional and three-dimensional quantities that is suggested by formula (13) [(1.4) and (1.41), resp.], he did not directly link that with the requirements (19.*b*, *c*) as further geometric conditions. He treated (19.*c*) as the spatial part of (19.*a*), and therefore, in our notation:

$$\underset{4}{R}{}_{ijk}{}^{l} = 0.$$

However, upon restricting to linear quantities, that will be identical to (19.*c*), as one can infer from relations (3.11), (3.14) that are given in the Appendix. AMARI obtained the linearized equations (19.*b*) [more precisely, equations (53) (cf., *infra*)] from an auxiliary consideration that he inferred from a linear relationship. Combining the equations (3.10) and (3.11) that he gave:

$$\frac{D}{Dt}\,ds^2 = 4\;T_{0\,\underline{lk}}\,dx^l\,dx^k,$$

$$\frac{D}{Dt}\,ds^2 = 2\,a_{lk}\,dx^l\,dx^k = \left(\frac{\partial g_{lk}}{\partial t} + v_{l,k} + v_{k,l}\right)\,dx^l\,dx^k,$$

will give (53) precisely:

$$\tfrac{1}{2}(g_{lk,\,0} - g^{0l}{}_{,\,k} - g^{0k}{}_{,\,l}) + T^{\,0}{}_{lk} + T^{\,0}{}_{kl} = 0.$$

Naturally, the relationships between dislocation densities and dislocation currents that AMARI obtained from linear considerations represent the first approximation to (43.*d*), and will then be identical to conditions (44):

$$T_{ijk,\,0} = T^{\,0}{}_{ik,\,j} - T^{\,0}{}_{jk,\,i}\,,$$

which HOLLÄNDER [**26**] already gave [the compatibility conditions (60.*d*) that KOSEVICH [**27**] presented, resp. (cf., *infra*)].

We pointed out that in AMARI's relation (2.2) between dislocation density, dislocation current, and dislocation velocity $v^{\,r}$, which reads:

$$T^{\,0}{}_{ik} = v^{\,r}\,T_{rik}$$

with our notation, v^r will not change with the likewise-denoted particle velocity. Moreover, that relation is true only for the case of constant dislocation speed (cf., the discussion below of the similar problem in HOLLÄNDER).

AMARI did not split off an algebraically-independent system of equations, so some of the relations that he gave are redundant. For example, AMARI's separate consideration of the components of the four-dimensional curvature tensor in equations (3.8) that have one zero index and two zero indices, which is:

$$\underset{L}{\overset{0}{R}}{}_{0mlk} - T_{lkm,0} - T_{0mk,l} + T_{0ml,k} = 0,$$

$$\underset{L}{\overset{0}{R}}{}_{n0l0} - T_{0ln,0} - T_{0nl,0} = 0$$

in our notations, will imply no new statements, which is clear from equations (3.9) in our Appendix, when one considers (44).

E. F. HOLLÄNDER [**26**] undertook the search for a presentation of the linear equations of dislocation dynamics in a series of papers. The meaning of those papers consists of, above all, the fact that it was the first time that the mathematical analogies between dislocation dynamics and special relativity were consulted in order to formula the dynamics of dislocations, and indeed, in a four-dimensional form. We point out that one will already find the linear form of equations (43.*d*) in those papers:

$$T_{ijk,0} - T^{\,0}{}_{ik,j} + T^{0}{}_{jk,i} = 0. \tag{44}$$

It is written in the form:

$$\text{Rot}\,\mathbf{I} + \frac{\partial \boldsymbol{\alpha}}{\partial t} = 0 \tag{44*}$$

in those papers, in which $\boldsymbol{\alpha}$ means the dislocation density, and **I** means the dislocation current tensor. However, in total, the equations that were presented in them still do not lead to a closed physical theory of moving dislocations. We would therefore not like to go further into the detailed results. HOLLÄNDER himself emphasized that fact in a later paper [**27**], which we shall now discuss.

Nonlinear equations for moving dislocations were presented for the first time in that paper. HOLLÄNDER even appealed to a geometric procedure in order to do that, but in three-dimensional form, and time appeared as a parameter. His methods can be regarded as a sort of generalization of the procedure that KRÖNER applied to statics. In that way, he found the following system of equations:

$$\frac{1}{2}(v_{l;k}+v_{k;l})+\frac{1}{2}\frac{\partial g_{kl}}{\partial t} = I_{\underline{kl}} \equiv -2\, v^m T_{m\,\underline{kl}}\,, \tag{56}$$

in which the notations are the ones that we defined, and I_{kl} is the dislocation current tensor. We compare (56) with (43.c):

$$\tfrac{1}{2}(g_{kl,0} - g_{kl,r}\, g^{0r} - g^{0r}{}_{,k}\, g_{lr} - g^{0r}{}_{,l}\, g_{kr}) = -(T^0{}_{kl} + T^0{}_{lk}).$$

If we consider that $g^{0r} = -v^r / c_T$, $x^0 = c_T\, t$ then we can write this in the form:

$$\frac{1}{2}\left(v_{l;k}+v_{k;l}+\frac{\partial g_{kl}}{\partial t}\right) = -c_T\,(T^0{}_{kl} + T^0{}_{lk}), \tag{57}$$

as we can easily check. (56) and (57) [i.e., (43.c)] are therefore entirely identical in form, but the right-hand sides do not agree, since we have:

$$c_T T^0{}_{\underline{kl}} \neq v^m\, T_{m\,\underline{kl}}\ .$$

One has merely:

$$T^0{}_{ik} \equiv g^{0\alpha}\, T_{\alpha ik} = g^{0r}\, T_{rik} + g^{0r}\, T_{rik}\,,$$

$$T^0{}_{ik} = -\frac{v^r}{c_T}T_{rik} - T_{0ik}\ .$$

Initially, the sign in this is the opposite of the one that would make the right-hand sides of (56) and (57) coincide. In addition, from (16), one has:

$$T_{ik}{}^0 \equiv \Gamma^k_{<0i>} \equiv \tfrac{1}{2} h^k_G (h^G_{i,0} - h^G_{0,i})\,.$$

However, that expression, and therefore T_{0ik}, as well, will not vanish unless one makes special assumptions about h^α_Γ. One therefore cannot regard the right-hand side of (56) as the dislocation current tensor ([11]).

([11]) In his first paper [**26**], HOLLÄNDER still did not employ the relation $c_T\, T^0_{kl} = v^m\, T_{mkl}$, so the linear form of equation (48) that he gave there is not correct with no restrictions.

In a paper on the theory of moving dislocations, H. BROSS [**28**] exhibited the following linear system of differential equations for the strain tensor ε_{ij} :

$$\mu\,\Delta\varepsilon_{ij} + (\lambda + \mu)\,\varepsilon_{kk,\,ij} - \rho\,\ddot{\varepsilon}_{ij} = \rho\,\ddot{\varepsilon}_{ij}^{P} + \mu\,(\eta_{ij} - \delta_{ij}\,\eta_{kk}). \tag{58}$$

BROSS based this upon merely the equilibrium conditions when one considers the inertial term in the force-free case [and thus, our equation (26) with $f^{\,i} = 0$], the decomposition of the total distortion β_{ik}^{G} into the elastic and plastic parts β_{ik} and β_{ik}^{P}, resp., according to:

$$\beta_{ik}^{G} = \beta_{ik} + \beta_{ik}^{P}, \qquad \text{with} \quad \beta_{ik} = \varepsilon_{ik}\,,$$

and the connection between plastic distortion and dislocation density (cf., e.g., [**13**]):

$$\alpha_{ik} = -\,\varepsilon_{irl}\,\beta_{ik,r}^{P}\,.$$

Equations (58) refer to the isotropic case, in which μ, λ are the LAMÉ constants, ρ is the density of the medium in the stressed state, and η_{ik} is the incompatibility tensor here, which is connected with the dislocation density in the following known way (cf., e.g., [**18**]):

$$\eta_{ik} = -\,\tfrac{1}{2}\,(\varepsilon_{inl}\,\varepsilon_{rsk}\,T_{rsn,\,l} + \varepsilon_{knl}\,\varepsilon_{rsi}\,T_{rsn,\,l}). \tag{59}$$

In order to compare this with the system (13), we remark that we have derived equations (54) from (48*), which however follows from the system (43), according to (55). If we now observe (47), (51), (59), and the facts that $g^{0i} = -\,v^i / c_T$, $x^0 = c_T\,t$, in addition, then we will see that (58) is identical to (59). H. BROSS's equations (58) are then included in the system (43), and (43) also includes their nonlinear generalization.

T. MURA [**30**, **31**] arrived at the same physical results as the ones that are given in (58) by a different process. MURA started from the *displacement field* of an isolated moving dislocation and then went over to continuous distributions of dislocations. He derived the following relation for the connection between dislocation density and dislocation current:

$$\dot{\alpha}_{ik} = \varepsilon_{hlk}\,(\varepsilon_{mnk}\,V_{mni})_{\,,\,l}\,.$$

As one easily sees, these are equations (44*) [(44), resp.]. As in those equations, α is the dislocation density, while $\varepsilon_{mnk}\,V_{mni}$ is identical to **I**.

We shall now briefly touch upon the work that A. M. KOSEVICH [**29**] did on the problem of moving dislocations. That author considered the linear theory of elasticity and found the following system of equations:

$$\left.\begin{aligned} &a) \quad \rho\frac{\partial v^i}{\partial t} = \lambda_{iklm}\, u_{im,k}\,, \\ &b) \quad \varepsilon_{ilm}\, u_{mk,l} = -D_{ik}\,, \\ &c) \quad \frac{\partial u_{ik}}{\partial t} - v^k{}_{,i} = I_{ik}\,. \end{aligned}\right\} \tag{60}$$

KOSEVICH then imposed the additional requirement that:

$$\left.\begin{aligned} &d) \quad \frac{\partial \mathbf{D}}{\partial t} + \mathrm{Rot}\,\mathbf{I} = 0, \qquad \text{or} \\ &d') \quad \frac{\partial D_{ik}}{\partial t} + \varepsilon_{irs} I_{sk,r} = 0, \quad \text{resp., in tensor notation} \end{aligned}\right\} \tag{60}$$

as the compatibility conditions for (60.*c* and *b*). In these equations, ρ is the mass density, v^i is the material velocity, λ_{iklm} is the tensor of elastic moduli, and u_{kl} is the distortion tensor, whose symmetric part is then the strain tensor:

$$u_{\underline{kl}} = \varepsilon_{kl}\ .$$

D_{ik} is the dislocation density tensor, and I_{ik} is the dislocation current tensor.

We now see that (60.*a*) is equivalent to our condition (53), in which only isotropy was assumed. (60.*b*) are then the equations of dislocation statics that KRÖNER [**13**] discussed, from which one had to go over to equations (55.*a*) in order to determine the internal stresses. Equations (60.*c*), which must be symmetrized in order to determine the internal stresses, correspond to the linearized system (48) that HOLLÄNDER [**26**] gave already, and they are identical to (55.*b*). Finally, as was pointed before in the discussion of HOLLÄNDER's work, (60.*d*) is identical to the linearized form of equations (43.*d*). With that, KOSEVICH then had the complete system (43) for the determination of internal stresses, but in linearized form.

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III. – DIFFERENTIAL IDENTITIES

Equations (61) do, in fact, define an algebraically-independent system, and therefore cannot be reduced to a smaller number of equations. Nevertheless, not all of the equations are mutually independent, since differential identities exist between their left-hand sides. One must then impose them on the right-hand sides, as well, by means of the field equations.

The system (43) was derived on the basis of the teleparallelism that is defined by the (four-dimensional) lattice structure of the medium. We shall now go another step further by allowing that structure to be perturbed at isolated points or also in finite regions. We express that by saying that we have introduced phenomenological matter tensors, as we would like to say; i.e., we go from equations (43) to the following general equations:

$$\left.\begin{array}{llll} a) & \bar{R}_{ijkl} = \bar{M}_{ijkl}, & b) & R_{\{ijk\}\,l} = V_{ijkl}, \\ c) & \Gamma^{0}_{\underline{ik}} = N_{ik}, & d) & R_{\{0ij\}k} = L_{ijk}. \end{array}\right\} \tag{61}$$

We will often also combine (61.*a*) and (61.*b*) into one equation:

$$R_{ijkl} = M_{ijkl}, \quad \text{so} \quad M_{\{ijk\}\,l} = V_{ijkl}, \tag{61.a + b}$$

We shall now discuss the system (61) and consider it to be the first of the differential identities that exist between the field equations.

As is known, the complete three-dimensional curvature tensor fulfills the BIANCHI identity, which can be written in the following form:

$$\overset{*}{H}_{ijklm} \underset{\text{def}}{=} R_{\{ij\,|\,kl\,|\,;\,m\}} - R_{\{ij\,|\,rl\,|}\,h_{m\}\,k}{}^{r} + R_{\{ij\,|\,rk\,|}h_{m\}\,l}{}^{r} \equiv 0 \tag{62}$$

for non-vanishing torsion, with the use of the notation that is explained in the Appendix (cf., J. A. SCHOUTEN [**37**])([12]). When we substitute this in the field equations, we will get a relation that we can write as follows:

$$H_{ijkln} \underset{\text{def}}{=} M_{\{ij\,|\,kl\,|\,;\,n\}} - M_{\{ij\,|\,rl\,|}\,h_{n\}\,k}{}^{r} + M_{\{ij\,|\,rk\,|}h_{n\}\,l}{}^{r} \equiv 0\,. \tag{63}$$

In order to obtain further identities, we recall the starting equations (19) – in particular, (19.*a*) – which will indeed be fulfilled due to the homogeneous system (43). Now, the four-dimensional curvature tensor will likewise fulfill the BIANCHI identity, which reads the same as (62):

$$\underset{4}{R}_{\{\alpha\beta|\mu\nu|\underset{4}{;}\lambda\}} - \underset{4}{R}_{\{\alpha\beta|\sigma\nu|}\,\underset{4}{h}_{\lambda\}\mu}{}^{\sigma} + \underset{4}{R}_{\{\alpha\beta|\sigma\mu|}\,\underset{4}{h}_{\lambda\}\nu}{}^{\sigma} \equiv 0. \tag{64}$$

([12]) When comparing the corresponding expressions, one must observe which affinity was used for the covariant derivation.

However, the four-dimensional curvature tensor $R_{\alpha\beta\mu\nu}$ can be expressed by $\underset{4}{R}_{ijkl}$, Γ^0_{ij}, and $\underset{3}{R}_{\{0ij\}k}$ in our case (see Appendix). Therefore, (64) expresses identities between the left-hand side of (61). It is also clear that all of the identities are contained in (62) and (64). In particular, we consider the identity (64) for $(\alpha, \beta, \mu, \nu, \lambda) = (i, j, k, l, 0)$, so:

$$\underset{4}{R}_{\{ij|\,kl\,|;0\}}{}_{\!4} - \underset{4}{R}_{\{ij|\sigma l|}\,\underset{4}{h}_{0\}k}{}^{\sigma} + \underset{4}{R}_{\{ij|\sigma k|}\,\underset{4}{h}_{0\}k}{}^{\sigma} \equiv 0. \tag{63}$$

By reverting to the three-dimensional quantities, (63) can then be brought into the following form:

$$\begin{aligned}
R_{ijkl,\,0} \equiv\ & \Gamma^0_{\underline{lj},ik} + \Gamma^0_{\underline{ik},lj} - \Gamma^0_{\underline{jk},ik} - \Gamma^0_{\underline{ik},jk} \\
& + \Gamma^r_{jl}[-\Gamma^0_{\underline{ik},r} + \Gamma^0_{\underline{ir},k} + \Gamma^0_{\underline{rk},i}] + \Gamma^r_{ik}[-\Gamma^0_{\underline{lj},r} + \Gamma^0_{\underline{rj},l} + \Gamma^0_{\underline{lr},j}] \\
& - \Gamma^r_{il}[-\Gamma^0_{\underline{jk},r} + \Gamma^0_{\underline{jr},k} + \Gamma^0_{\underline{rk},j}] - \Gamma^r_{jk}[-\Gamma^0_{\underline{il},r} + \Gamma^0_{\underline{ir},l} + \Gamma^0_{\underline{rl},i}] \\
& - \Gamma^0_{\underline{rs}}[\Gamma^r_{ik}\Gamma^s_{jl} - \Gamma^r_{jk}\Gamma^s_{il}] \\
& + g^{0r} R_{ijkl,\,r} + g^{0r}{}_{,\,i} R_{rjkl} + g^{0r}{}_{,\,j} R_{irkl} + g^{0r}{}_{,\,k} R_{ijrl} + g^{0r}{}_{,\,l} R_{ijkr} \\
& + \underset{3}{R}_{(0jk)l,i} + \underset{3}{R}_{\{0lj\}k,i} + \underset{3}{R}_{\{0lk\}j,i} - \underset{3}{R}_{\{0ik\}l,j} - \underset{3}{R}_{\{0li\}k,j} - \underset{3}{R}_{\{0lk\}i,j} \\
& + \Gamma^r_{jk}[\underset{3}{R}_{\{0ir\}l} + \underset{3}{R}_{\{0li\}r} + \underset{3}{R}_{\{0lr\}i}] + \Gamma^r_{jl}[\underset{3}{R}_{\{0ik\}r} + \underset{3}{R}_{\{0ri\}k} + \underset{3}{R}_{\{0rk\}i}] \\
& - \Gamma^r_{ik}[\underset{3}{R}_{\{0jr\}l} + \underset{3}{R}_{\{0lj\}r} + \underset{3}{R}_{\{0lr\}j}] - \Gamma^r_{il}[\underset{3}{R}_{\{0jk\}r} + \underset{3}{R}_{\{0rj\}k} + \underset{3}{R}_{\{0rk\}j}] \\
& - 2\,T^0{}_{lr}\,R_{ijk}{}^r + 2\,T^0{}_{kr}\,R_{ijl}{}^r .
\end{aligned} \tag{66}$$

We can now verify that none of the other index combinations in (64) will yield any independent, non-trivial identities beyond (62) and (66). However, we can avoid that verification, since will see (cf., *infra*) that no further identities will come into play by explicitly integrating (61). A relationship will then exist between the matter tensors that is completely analogous to (66), due to the field equations. We shall cite those equations only once:

$$\begin{aligned}
M_{ijkl,\,0} =\ & N_{lj,\,ik} + N_{ik,\,lj} - N_{jk,\,il} - N_{il,\,jk} \\
& + \Gamma^r_{jl}[-N_{ik,\,r} + \dots] + \dots - 2\,N_{rs}\,[\Gamma^r_{ik}\Gamma^s_{jl} - \Gamma^r_{jk}\Gamma^s_{il}] \\
& + g^{0r} M_{ijkl,\,r} + \dots + L_{jkl,\,i} + \dots \\
& + \Gamma^r_{jk}[N_{irl} + \dots] + \dots + 2\,T^0{}_{kr}\,M_{ijl}{}^r .
\end{aligned} \tag{67}$$

Just as one had with the homogeneous field equations (43), only the dislocation densities enter into $R_{ijkl} = 0$, while the equations $\Gamma^0_{ik} = 0$ include the influence of the motion of the dislocation, when expressed in terms of its current, so one can understand that the $\bar{M}_{ijkl}$ in (61) means the phenomenological matter tensor, while N_{ik} is its current. In an entirely analogous way, V_{ijkl} can be regarded as the hypothetical lack of closure of the dislocations, and its current can once more be expressed by L_{ijk}. (67) will then give the connection between the matter tensor and the currents. The currents determine the time evolution of the matter tensor, as they must. The next question for us to ask is whether the nonlinearity of the basic equations will imply any restrictions on the phenomenological matter tensor (the dislocation currents, resp., in the case of pure dislocations).

We would next like to draw some general conclusions from the identities.

1. Geometrically-speaking, introducing the matter tensor means perturbing the teleparallelism. However, the form of our displacement symbols $\Gamma^r_{ik} = \left\{ \begin{matrix} r \\ i\,k \end{matrix} \right\} + T_{ik}{}^r + T^{\ r}{}_{ik} + T^{\ r}{}_{ki}$ further guarantees the validity of the RICCI lemma:

$$g_{ij\,\|\,r} = 0, \tag{68}$$

which is essentially equivalent to:

$$R_{ij\underline{kl}} = 0. \tag{69}$$

The physical meaning of our parallel displacement is as follows: Suppose that we are given any physical quantity that is expressed as a tensor field $T(P)$. If we now parallel-transports $T(P_1)$ to P_2 using Γ^i_{kl} then we will get $\tilde{T}(P_2)$, and the difference:

$$T(P_2) - \tilde{T}(P_2)$$

will yield the change of $T(P_1)$ relative to $T(P_2)$. If the teleparallelism is perturbed then that will mean that this difference depends upon the direction that we take along the path. In other words, the rotation of the element of matter will no longer be an integrable function then; physically-speaking, it will not be a state function.

However, if one maintains the RICCI lemma then that will mean that this difference does not depend upon the difference between the lengths of the paths. In other words, the relative rotation is an integrable function. The rotation will then be a state function, as it must be for the bodies that we shall consider here (see, e.g., E. KRÖNER [**38**]). However, if one had:

$$M_{ijkl} \neq 0$$

then the rotation would no longer be integrable. The RICCI lemma would no longer be fulfilled then, and instead of (68), we would have:

$$g_{ik\,\|\,r} = f_{ikr}\,.$$

The tensor f_{ikr} has just as many algebraically-independent components as M_{ijkl} (namely, 18) and is therefore equivalent to the latter.

We will always demand that:

$$M_{ij\underline{kl}} = 0.$$

One can now conclude from (66) [(67), resp.] that it suffices for one to demand that the latter condition should be true at one time-point in order for it to remain true at all times, independently of how all remaining matter tensors and currents have been chosen. Namely, one has:

$$R_{ij\underline{kl},0} \equiv g^{0r} R_{ij\underline{kl},r} + g^{0r}{}_{,i}\, R_{rj\underline{kl},r} + g^{0r}{}_{,j} R_{ir\underline{kl}} + g^{0r}{}_{,k} R_{ij\underline{rl}} + g^{0r}{}_{,l} R_{ij\underline{kr}} \,. \tag{70}$$

[From (67), one will have a relationship for the matter tensor $M_{ij\underline{kl}}$ that reads the same]. However, if $M_{ij\underline{kl}} \neq 0$ at some time-point then the further time evolution of those quantities will follow from (70); it will depend upon only g^{0i} and $M_{ij\underline{kl}}$ themselves.

2. In the case of a pure dislocation, the equation:

$$R_{\{ijk\}\,l} = 0$$

will express the fact that a dislocation cannot end in the interior of the medium (viz., the closure condition for dislocations; cf., E. KRÖNER [**18**]) ([13]). However, the treatment of dislocations that are not closed still holds some theoretical interest (see, e.g., E. KRÖNER [**18**]). Nonetheless, a violation of the closure condition will not only be caused by V_{ijkl}, as one might suspect, but can also be a result of all matter tensors and their currents, in general (which will naturally give rise to non-vanishing V_{ijkl} then). Namely, one infers from (66) that:

$$\begin{aligned}
R_{\{ijk\}\,l,\,0} \equiv\; & 2(\underset{3}{R}{}_{\{0jk\}l,i} + \underset{3}{R}{}_{\{0ki\}l,j} + \underset{3}{R}{}_{\{0ij\}l,k}) \\
& + 2T_{jk}{}^{r}\,(\underset{3}{R}{}_{\{0ir\}l} + \underset{3}{R}{}_{\{0li\}\,r} + \underset{3}{R}{}_{\{0lr\}\,i}) - 2\Gamma^{r}_{il}\,\underset{3}{R}{}_{\{0jk\}\,r} \\
& + 2T_{ki}{}^{r}\,(\underset{3}{R}{}_{\{0jr\}l} + \underset{3}{R}{}_{\{0lj\}\,r} + \underset{3}{R}{}_{\{0lr\}\,j}) - 2\Gamma^{r}_{jl}\,\underset{3}{R}{}_{\{0ki\}\,r} \\
& + 2T_{ij}{}^{r}\,(\underset{3}{R}{}_{\{0kr\}l} + \underset{3}{R}{}_{\{0lk\}\,r} + \underset{3}{R}{}_{\{0lr\}\,k}) - 2\Gamma^{r}_{kl}\,\underset{3}{R}{}_{\{0ij\}\,r} \\
& + g^{0r}\, R_{\{ijk\}\,l,\,r} + g^{0r}{}_{,\,i}\, R_{\{rjk\}\,l} + g^{0r}{}_{,\,j}\, R_{\{irk\}\,l} + g^{0r}{}_{,\,k}\, R_{\{ijr\}\,l} + g^{0r}{}_{,\,l}\, R_{\{ijk\}\,r} \\
& - 2\,T^{0}{}_{lr}\, R_{\{ijk\}}{}^{r} + 2\,T^{0}{}_{\{k|r|}\, R_{ij\}\,l}{}^{r} \\
& - 2\,T_{jk}{}^{r}\,(-\Gamma^{0}_{\underline{il},l} + \Gamma^{0}_{\underline{ir},l} + \Gamma^{0}_{\underline{lr},i}) \\
& - 2\,T_{ki}{}^{r}\,(-\Gamma^{0}_{\underline{jl},r} + \Gamma^{0}_{\underline{jr},l} + \Gamma^{0}_{\underline{lr},j}) \\
& - 2\,T_{ij}{}^{r}\,(-\Gamma^{0}_{\underline{kl},l} + \Gamma^{0}_{\underline{kr},l} + \Gamma^{0}_{\underline{lr},k}) \\
& - 4\Gamma^{0}_{\underline{rs}}(T_{jk}{}^{r}\,\Gamma^{s}_{il} + T_{ki}{}^{r}\,\Gamma^{s}_{jl} + T_{ij}{}^{r}\,\Gamma^{s}_{kl})
\end{aligned} \tag{71}$$

[and an analogous equation for the matter tensor, due to (67), resp.].

One learns from (71) that in the theory without matter tensors, the dislocations will always remain closed when they are closed at just one time-point, on the basis of the field

([13]) The formula that is denoted by (45) in [**18**] is the linear approximation to $R_{\{ijk\}\,l} = 0$. One will find the rigorous equations in the form $R_{<ij>} = 0$ in KRÖNER.

equations; that will be true in the linear theory as well as in the nonlinear one. However, if one has non-zero matter tensors then the situations in the linear and nonlinear theories will be different. In the linear theory, only equation (61.*d*) is responsible for determining whether the lack of closure of the dislocations (which should naturally not be confused with the lack of closure of the BURGERS path) changes in time or remains constant (so in particular whether the dislocations remain closed or not). One infers from the linearized form of (71) that the necessary and sufficient condition for the constancy in time of the lack of closure is:

$$L_{\{ij\,|\,k\,|\,l\}} = 0. \tag{72}$$

(This can be satisfied identically by way of, e.g., the Ansatz $L_{ijk} = L_{ik,\,j} - L_{jk,\,i}$.)

If we allow a temporal change in the lack of closure of dislocations in the nonlinear theory then from (71) all matter tensors will contribute to that, in general. The time evolution of the lack closure is also given explicitly by (71). Conversely, the condition for the constancy in time of the lack of closure is a system of differential equations for the matter tensor. We restrict ourselves to the important case in which the dislocations should remain closed! The system of equations will then read:

$$\begin{aligned} L_{jkl,\,i} &+ L_{kil,\,j} + L_{ijl,\,k} + \Gamma^r_{il}\, L_{kjr} + \Gamma^r_{jl}\, L_{ikr} + \Gamma^r_{kl}\, L_{jir} \\ &+ T_{ik}{}^{r}\,(L_{irl} + L_{lir} + L_{lri}) + T_{ki}{}^{r}(L_{jrl} + L_{ljr} + L_{lrj}) \\ &+ T_{ij}{}^{r}\,(L_{krl} + L_{lkr} + L_{lrk}) + T^{\;0}_{(k|r|}M_{ij)\,l}{}^{r} \\ &- T_{jk}{}^{r}\,(-N_{il,r} + N_{ir,l} + N_{rl,i}) \\ &- T_{ki}{}^{r}\,(-N_{jl,r} + N_{jr,l} + N_{rl,j}) \\ &- T_{ij}{}^{r}\,(-N_{kl,r} + N_{kr,l} + N_{rl,k}) \\ &- 2\, N_{rs}\,(T_{jk}{}^{r}\,\Gamma^s_{il} + T_{ki}{}^{r}\,\Gamma^s_{jl} + T_{ij}{}^{r}\,\Gamma^s_{kl}) = 0. \end{aligned} \tag{73}$$

[Like (72), these are three independent equations.]

One can distinguish some different cases here. Consider the simple case in which only M_{ijkl} is non-zero. (73) will then reduce to some algebraic relations between dislocation currents and the matter tensor:

$$T^{\;0}_{\{k\,|\,r\,|}\, M_{ij\}\,lr} = 0. \tag{74}$$

If one continues to allow the dislocation current to be arbitrary then (74) will give three algebraic conditions for the matter tensor. However, since, from (63), it is already subject to three differential conditions in the form of the BIANCHI identities, and due to its symmetry properties, it has only six independent components, in all (one must also observe the closure condition of the dislocations $M_{(ijk)\,l} = 0$ here), the matter tensor itself must vanish for a sufficiently-general motion of the dislocation. Conversely, if one is given the matter tensor then (74) will express three restricting conditions for the dislocation density. M_{ijkl} will then represent something like an obstruction to the free motion of dislocations.

One will get an analogous situation when one also regards the remaining matter tensors L_{ijk} and N_{ij}, as well as the dislocation density, as freely-given. In place of (74), an inhomogeneous system of equations will arise from (73), which one can write in the form:

$$T_{\{k|}{}^{r}{}_{|}M_{ij\}\,lr} = P_{ijkl}\,. \tag{75}$$

M_{ijkl} will then be determined completely for a given P_{ijkl} and a sufficiently-general motion of the dislocation. On the other hand, being given M_{ijkl} and P_{ijkl} will again imply conditions for the motion of the dislocation. Furthermore, one can regard all quantities in (73) up to $T_{ij}{}^{r}$ as being given, and in that way, one will get three algebraic conditions for the dislocation density itself that have the form:

$$T_{\{ij|}{}^{r}{}_{|}V_{k\}\,lr} = W_{ijkl}\,, \tag{76}$$

and the six degrees of freedom of the dislocations (the closure condition will reduce the original nine degrees of freedom to precisely six) will be reduced to only three. Finally, (73) can also be regarded as a system of differential equations for the partial determination of L_{ijr}. All other quantities in (73) can be given freely then, since the integrability conditions for the system (73) are fulfilled identically in our case, where all indices run through only the numbers 1, 2, 3 (see Appendix, section 5).

3. From equations (67), the time evolution of M_{ijkl} will be established by the matter tensors and their currents. On the other hand, one knows from (63) that M_{ijkl} is subject to three differential conditions; no time derivatives enter into those conditions. They will then refer to one time-point (viz., static conditions). One can now show that it will suffice to require the conditions (63) at one time-point in order for them to also be fulfilled at all times on the basis of (67). Namely, if one takes the time derivative of (63) and expresses the time derivative of the matter tensor using (67) and the time derivative of the affinities by means of the field equations then one will find, by direct calculation, that:

$$\begin{aligned} H_{ijkln,\,0} &= g^{0r}\,H_{ijkln,\,r} - 2T^{\,0\,r}_{\;l}\;H_{ijkrn} - 2T^{\,0\,r}_{\;k}\;H_{ijrln} \\ &\quad + g^{0r}{}_{,\,n}\,H_{ijklr} - g^{0r}{}_{,\,l}\,H_{ijkrn} - g^{0r}{}_{,\,k}\,H_{ijrln} + g^{0r}{}_{,\,j}\,H_{irkln} + g^{0r}{}_{,\,i}\,H_{rjkln}\,. \end{aligned} \tag{77}$$

One reads immediately from (77) that (63) is, in fact, a static condition that no longer needs to be observed once it is fulfilled, and is therefore inessential for dynamical problems.

We shall now discuss the question of the solubility of the basic equations (61).

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IV. – INTEGRATING THE BASIC EQUATIONS

We saw above (see also the Appendix, section 4) that the basic equations of the theory of moving dislocations are similar to EINSTEIN's equations of gravitation. The stress field in the former corresponds to the gravitational field in the latter, and both are expressed in terms of a metric tensor. Moreover, the sources of that metric, namely, the dislocations and other lattice fields, in the one theory correspond to the masses in the other. The field equations are nonlinear in both cases. Now, in the theory of relativity, one has the following fundamental state of affairs: If one is given the distribution of sources (say, a mass point) at one time-point, and its internal structure is known, moreover, then its further motion will be established uniquely by the field equations alone. That is essentially a consequence of the nonlinearity of the equations and the BIANCHI identity. That is the content of the celebrated work of A. EINSTEIN, L. INFELD, and B. HOFFMANN [**39**], V. FOCK [**40**], and A. PAPEPETROU [**41**] (also see the monograph of L. INFELD and J. PLEBANSKI [**42**]). One might then suspect that a similar situation prevails in the theory of moving dislocations. However, in contrast to that, we shall discover the following relationships upon integrating the basic equations:

The system of equations (43) can be integrated when one gives the distribution of dislocations at one time-point and the dislocation current for all times, which one has complete freedom to do. Likewise, for the extended system (61), one is, in addition, free to choose the distribution of matter at one time-point and its current at all times. (Here, one needs merely to consider the viewpoint that was discussed in Chap. III.) The same thing is also true for the dynamical generalization of KRÖNER's theory of foreign atoms, to which we shall return. It is therefore true in full generality that any solution of the linearized theory can be extended to a solution of the rigorous theory. That is the dynamical generalization of the theorem that KRÖNER [**18**] presented for statics to the effect that that the essential drift of the physical content of the theory of dislocations is already contained in the linear equations.

We will deal with the proof of those facts in what follows. In order to do that, we shall show how the solutions of (61) can be obtained to an arbitrary degree of approximation by approximation process. In that, we shall appeal to the same process that was developed in the celebrated work of A. EINSTEIN, L. INFELD, and B. HOFFMANN in order to derive the equations of motion for masses from the field equations for gravitation (which shall briefly be called the "EIH method" in what follows). Since we would like to apply the process explicitly, we first need to extend the system (61) by way of the equilibrium condition (26) and the matter equations:

$$a)\quad \sigma_{ri,\,r} - \rho \frac{d}{dt} v^i = - f^{\,i}, \qquad b)\quad \sigma_{ik} = \sigma_{ik}\,(\varepsilon_{rs}) = \sigma_{ik}\,(g_{rs})\,. \tag{78}$$

As we have discussed already, those three equations, together with the algebraic condition $g^{00} = -1$, enter in place of the four coordinate conditions of general relativity. In our RIEMANNian space with the metric that is defined by strain, we can also interpret them geometrically, as such, since the Cartesian coordinate system that we have defined in our EUCLIDIAN space means only an arbitrary coordinate system in it.

It is clear that changes in the conditions (78), and therefore changes in the forces and matter laws (which will then mean only changes in the coordinate conditions), can have no effect upon the behavior of the solutions of the basic equations. It is therefore also permissible for other conditions to enter in place of (78.*a*, *b*), which might come into play, for instance, as a result of a dependency of the stress tensor upon the time derivative of the distortion tensor or by considering moment stresses. In that latter case, the stress tensor will be asymmetric, and (78.*a*) will be replaced with:

$$c) \qquad \sigma_{\underline{ri},r} - \sigma \frac{d}{dt} v^i = -(f^i + \sigma_{\underline{ri},r}). \tag{78}$$

The stress-strain relationship is now true for $\sigma_{\underline{ri}}$, and the antisymmetric part $\sigma_{<ri>}$ is connected with the moment stress $M_{rsi} = - M_{ris}$ by:

$$d) \qquad M_{rst,\,r} + 2\sigma_{<rt>} + m_{st} = 0, \tag{78}$$

in which m_{st} characterizes the external moments in that. If one now connects the moment stresses M_{rst} with the structural curvatures by a material law (cf., e.g., E. KRÖNER [**43**], AMARI [**44**]):

$$f) \qquad M_{rst} = M_{rst}\,(h_{ijk}) \tag{78}$$

then one will once more find “generalized coordinate conditions” when one substitutes the differential equations (78.*d*) in (78.*c*) with the use of (78.*f*).

However, we can restrict ourselves to the force-free, homogeneous case, for the sake of simplicity, and substitute the very special series ([14]):

$$c) \qquad \sigma_{ik} = 2\mu_1\, \varepsilon_{ik} + 2\mu_2\, \varepsilon_{ir}\, \varepsilon_{rk} + 2\mu_3\, \varepsilon_{ir}\, \varepsilon_{rs}\, \varepsilon_{sk} + \dots \tag{78}$$

in (78.*b*). The series (78.*c*) has only a formal significance, since the elastic constants μ_3, … are no longer known. This restriction to second-order elastic constants, which is prescribed in practice, also implies a restriction in the usefulness of the strictly-nonlinear theory, since it makes sense to push the approximation only as far as higher elastic constants can be given. However, since we are interested in purely-mathematical questions of solubility here, in order to be able to make general statements, we must continue up to an arbitrary approximation.

If one now considers:

$$g^{0i} = -\frac{v^i}{c_T}, \qquad x_0 = c_T\, t, \qquad \rho = \rho_0 \sqrt{g} = \frac{\mu_1}{c_T}\sqrt{g} \qquad [g = \det(g_{ik})]$$

([14]) That means we are no longer dealing with the general isotropic case, but that is irrelevant here.

and writes $\nu_i = \dfrac{\mu_{i+1}}{\mu_1}$, $\mathfrak{g}^{0r} = \sqrt{g}\, g^{0r}$ then (78.*a*) and (78.*c*) can be combined into:

$$2\varepsilon_{ir,r} + 2\nu_1(\varepsilon_{is_1}\,\varepsilon_{s_1 r})_{,r} + \ldots + 2\nu_n\,(\varepsilon_{is_1}\,\varepsilon_{s_1 s_2}\cdots\varepsilon_{s_n r})_{,r} + \sqrt{g}\,g^{0i}{}_{,0} - g^{0i}{}_{,r}\,\mathfrak{g}^{0r} = 0. \tag{79}$$

According to the EIH method, we can further restrict ourselves to quasi-stationary motion (i.e., all velocities that enter into consideration shall be small in comparison to the speed of sound), and we shall employ the development Ansatz for the metric that is customary in that case (cf., [**39**]). We shall write it out for the g_{ik} and g^{0i}, since there exists a simple physical interpretation for those components in terms of strain and velocity:

$$\left.\begin{aligned} g_{ik} &= \delta_{ik} + \underset{2}{g}{}_{ik} + \underset{4}{g}{}_{ik} + \underset{6}{g}{}_{ik} + \cdots, \\ g^{0i} &= \qquad\;\; \underset{3}{g}{}^{0i} + \underset{5}{g}{}^{0i} + \underset{7}{g}{}^{0i} + \cdots, \end{aligned}\right\} \tag{80}$$

and in addition, we have $g^{0i} = -1$. We have chosen series of only even (odd, resp.) order to appear in order that radiation terms should not appear. That corresponds to choosing the partial sums of retarded and advanced potential (cf., [**39**]). Since one must have $v \ll c_T$, one must substitute $g^{0i} = -v^i / c_T$ in (80) with one order higher.

(80) corresponds completely with the series Ansatz in the EIH method, up to other coordinate conditions. Here, we have to extend these developments with the series Ansatz for torsion, and indeed:

$$\left.\begin{aligned} T_{irk} &= \underset{2}{T}{}_{ikr} + \underset{4}{T}{}_{ikr} + \underset{6}{T}{}_{ikr} + \cdots, \\ T^{\,0}_{\;ik} &= \quad\;\; \underset{3}{T}{}^{\,0}_{\;ik} + \underset{5}{T}{}^{\,0}_{\;ik} + \underset{7}{T}{}^{\,0}_{\;ik} + \cdots \end{aligned}\right\} \tag{81}$$

We again substitute the series for $T^{\,0}_{\;ik}$ with one order higher, since a factor of $1 / c_T$ will enter into our definition of the dislocation tensor [cf., (45) – (47)]. Since $x^0 = c_T / t$, quasi-stationarity will imply that:

$$\underset{r}{g}{}_{\mu\nu,0} \sim \underset{r+1}{g}{}_{\mu\nu,i}\,, \tag{82}$$

in addition; i.e., taking the derivative with respect to x^0 will raise the order of smallness by one step. The meaning of the developments (80), (81) consists of the fact that the field equations will split into equations for the individual orders of the field quantities that will then be relatively simple to integrate.

We shall now turn to the case of the homogeneous equations (43); i.e., the case of pure dislocations. If we substitute the series (80) in (79) then we will get the following system of equations:

$$\left.\begin{aligned} &a) \quad -\underset{2}{g}{}_{ir,r} = 0, \\ &b) \quad -\underset{4}{g}{}_{ir,r} + \underset{3}{g}{}^{0i}{}_{\underset{1}{,0}} = -\tfrac{1}{2}\nu_1(\underset{2}{g}{}_{is_1}\,\underset{2}{g}{}_{s_1 r})_{,r}\,, \\ &c) \quad -\underset{2n}{g}{}_{ir,r} + \underset{2n-1}{g}{}^{0i}{}_{\underset{1}{,0}} = \underset{2n}{NL_i}\,(g_{rs}\,g_{rs}, g_{rs}\,g^{0n}, g^{0n}\,g^{0r}{}_{,s}), \end{aligned}\right\} \tag{A}$$

in which NL_i means the nonlinear terms. [We point out that the left-hand sides of (A) also agree formally with the coordinate conditions that are employed in the theory of relativity.]

We shall now substitute the developments (80) and (81) into the system (43), in which we shall employ equations (43.a'), instead of (43.a). We will get, individually:

1. For (43.a'):

$$\left.\begin{aligned}
&1) && \tfrac{1}{2}(-\underset{2}{g}_{kl,ss}-\underset{2}{g}_{ss,kl}+\underset{2}{g}_{ks,ls}+\underset{2}{g}_{ls,ks})\\
& &=\;& -(\underset{2}{T}_{skl,s}+\underset{2}{T}_{skl,s}+\underset{2}{T}_{kss,l}+\underset{2}{T}_{lss,k}),\\
&2) && \tfrac{1}{2}(-\underset{4}{g}_{kl,ss}-\underset{4}{g}_{ss,kl}+\underset{4}{g}_{ks,ls}+\underset{4}{g}_{ls,ks})\\
& &=\;& -(\underset{4}{T}_{skl,s}+\underset{4}{T}_{skl,s}+\underset{4}{T}_{kss,l}+\underset{4}{T}_{lss,k})\\
& && +\underset{4}{BL}_{kl}(\underset{2}{g}_{rs}\underset{2}{g}_{mn},\underset{2}{g}_{rs}\underset{2}{T}_{mnt},\underset{2}{T}_{rst}\underset{2}{T}_{mpq}),\\
& &\vdots\;& \\
&3) && \tfrac{1}{2}(-\underset{2n}{g}_{kl,ss}-\underset{2n}{g}_{ss,kl}+\underset{2n}{g}_{ks,ls}+\underset{2n}{g}_{ls,ks})\\
& &=\;& -(\underset{2n}{T}_{skl,s}+\underset{2n}{T}_{skl,s}+\underset{2n}{T}_{kss,l}+\underset{2n}{T}_{lss,k})\\
& && +\underset{2n}{BL}_{kl}(g_{rs}\,g_{mn},g_{rs}\,T_{mnt},T_{rst}\,T_{mpq}),
\end{aligned}\right\}\qquad\text{(B)}$$

in which $\underset{2n}{BL}_{kl}(\dots)$ mean linear forms.

2. For (43.c):

$$\left.\begin{aligned}
&1) && -\tfrac{1}{2}(-\underset{3}{g}{}^{0i}{}_{,j}+\underset{3}{g}{}^{0j}{}_{,i})\\
& &=\;& -\tfrac{1}{2}\underset{2}{g}_{ij,\underset{1}{0}}-\underset{3}{T}{}^{0}{}_{ij}-\underset{3}{T}{}^{0}{}_{ji},\\
&2) && \tfrac{1}{2}(-\underset{5}{g}{}^{0i}{}_{,j}+\underset{5}{g}{}^{0j}{}_{,i})\\
& &=\;& -\tfrac{1}{2}\underset{4}{g}_{ij,\underset{1}{0}}-\underset{5}{T}{}^{0}{}_{ij}-\underset{5}{T}{}^{0}{}_{ji}+\underset{5}{BL}_{ij}(\underset{2}{g}_{rs}\,\underset{3}{g}{}^{0n}),\\
& &\vdots\;& \\
&3) && -\tfrac{1}{2}(-\underset{2n+1}{g}{}^{0i}{}_{,j}+\underset{2n+1}{g}{}^{0j}{}_{,i})\\
& &=\;& -\tfrac{1}{2}\underset{2n}{g}_{ij,\underset{1}{0}}-\underset{2n+1}{T}{}^{0}{}_{ij}-\underset{2n+1}{T}{}^{0}{}_{ji}+\underset{2n+1}{BL}_{ij}(g_{rs}\;g^{0n}),
\end{aligned}\right\}\qquad\text{(C)}$$

3. For (43.b):

$$\left.\begin{aligned}
&1)\quad \underset{2}{T}_{ijk,l} + \underset{2}{T}_{ijk,l} + \underset{2}{T}_{ijk,l} = 0,\\
&2)\quad \underset{4}{T}_{ijk,l} + \underset{4}{T}_{ijk,l} + \underset{4}{T}_{ijk,l} = \underset{4}{BL}_{ijkl}\,(\underset{2}{g}_{rs}\,\underset{2}{T}_{mnq}, \underset{2}{T}_{mnq}\underset{2}{T}_{rst}),\\
&\qquad\vdots\\
&3)\quad \underset{2n}{T}_{ijk,l} + \underset{2n}{T}_{ijk,l} + \underset{2n}{T}_{ijk,l} = \underset{2n}{BL}_{ijkl}\,(g_{rs}\,T_{mnq}, T_{mnq}T_{rst}),
\end{aligned}\right\}\qquad (D)$$

and

4. For (43.*d*):

$$\left.\begin{aligned}
&1)\quad \underset{2}{T}_{ijk,0} + \underset{2}{T}_{ijk,0} + \underset{2}{T}_{ijk,0} = \underset{2}{T}{}^{0}_{ik,j} - \underset{2}{T}{}^{0}_{jk,i},\\
&2)\quad -(\underset{5}{T}{}^{0}_{ik,j} + \underset{5}{T}{}^{0}_{jk,i}) = -\underset{2}{T}_{ijk,\underset{1}{0}} + \underset{5}{BL}_{ijk}\,(\underset{2}{g}{}^{0r}\,\underset{2}{T}_{mnq}, \underset{2}{g}_{rs}\,\underset{2}{T}{}^{0}_{mn}, \underset{2}{T}_{mnq}\underset{2}{T}{}^{0}_{rs}),\\
&\qquad\vdots\\
&3)\quad -(\underset{2n+1}{T}{}^{0}_{ik,j} + \underset{2n+1}{T}{}^{0}_{jk,i}) = -\underset{2n}{T}_{ijk,\underset{1}{0}} + \underset{2n+1}{BL}_{ijk}\,(g^{0r}\,T_{mnq}, g_{rs}\,T^{0}_{mn}, T_{mnq}T^{0}_{mn}).
\end{aligned}\right\}\qquad (E)$$

The reason for the fact that the terms in (E.1) are arranged differently from the ones in the remaining equations of (E) will be explained later.

One succeeds in integrating equations (A)-(E) in the following way: One considers the lowest order of approximation $\underset{2}{T}_{ijk}(x^i, \underset{0}{x}^0)$ and go to a time-point $\underset{0}{x}^0$ at which (D.1) is fulfilled; naturally, that is always possible. On the one hand, that is because physically that means that only closed dislocations are given and mathematically, that the homogeneous equations (D.1) are integrable, as well as the inhomogeneous ones (2), …, since no integrability conditions exist for those systems (see Appendix, section 5). One is then completely free to choose the dislocation currents in the first approximation $\underset{2}{T}_{ijk}(x^i, \underset{0}{x}^0)$ at all times. $\underset{2}{T}_{ijk}(x^i, x^0)$ will then follow from (E.1) in a trivial way. If one considers the identity (71) to the lowest order (and thus in the linear form) and the fact that (E.1) [i.e., the lowest order of approximation for (43.a)] is fulfilled already then it will follow that equations (D.2) [i.e., the lowest order or approximation for (43.b)], which we initially required to be fulfilled for only $\underset{0}{x}^0$, are also fulfilled for all times.

We now go on to equations (A.1) and (B.1). We make the following remarks about equations (B): The curvature tensor R_{ijkl} can be decomposed into the RIEMANN-CHRISTOFFEL curvature tensor and a torsion part according to:

$$R_{ijkl} = \overset{0}{R}_{ijkl} + T_{ijkl}\,, \qquad (82)$$

and in the case of a pure dislocation, the field equations can then be written in the form:

$$\overset{0}{R}_{ijkl} = - T_{ijkl} . \tag{83}$$

Now, for the RIEMANN-CHRISTOFFEL curvature tensor by itself, one has:

$$\overset{0}{R}_{\{ij|kl|;n\}} \equiv 0 ; \tag{84}$$

i.e., from (82), one seems to have the additional demand that:

$$T_{\{ij\,|\,kl\,|;\,n\}} = 0. \tag{85}$$

However, since one has the BIANCHI identity in the form (62) for the complete curvature tensor:

$$R_{\{ij\,|\,kl\,|;\,n\}} - R_{\{ij\,|\,rl\,|}\, h_{n\}k}{}^{r} + R_{\{ij\,|\,rk\,|}\, h_{n\}l}{}^{r} \equiv 0 , \tag{86}$$

when one substitutes (82) in (86) and considers (84), one will have:

$$T_{\{ij\,|\,kl\,|;\,n\}} - R_{\{ij\,|\,rl\,|}\, h_{n\}k}{}^{r} + R_{\{ij\,|\,rk\,|}\, h_{n\}l}{}^{r} \equiv 0 , \tag{87}$$

in addition. The conditions (85) will then be fulfilled already to an arbitrary degree of approximation when the field equations $R_{ijkl} = 0$ are already fulfilled to the foregoing degree of approximation [since the identity (87) differs from equations (85) only by a bilinear tensor]. However, that is precisely the case in the approximation process.

We now substitute (A.1) in (B.1). We give the general solution to those equations, which fulfills the conditions (A.1), in addition, at the n^{th} step in the approximation. Since the right-hand side of (B.1) is time-dependent, we will then get $\underset{2}{g}_{kl}(x^i, x^0)$ from those equations, and (43.a') is already fulfilled at all times.

The left-hand side of (C.1) has the form of a deformation. There are then integrability conditions for the right-hand side. If one writes the right-hand side in the form:

$$\underset{3}{K}_{ij} = -\tfrac{1}{2}(\underset{3}{g}{}^{0i}{}_{,j} + \underset{3}{g}{}^{0j}{}_{,i}) - \underset{2}{\Gamma}{}^{0}_{\underline{ij}}$$

then that will read:

$$\underset{3}{K}_{ij,kl} + \underset{3}{K}_{ki,ij} - \underset{3}{K}_{ik,jl} - \underset{3}{K}_{jl,ik} \equiv -(\Gamma^{0}_{\underline{ij},kl} + \Gamma^{0}_{\underline{kl},ij} - \Gamma^{0}_{\underline{ik},jl} - \Gamma^{0}_{\underline{jl},ik}) = 0. \tag{88}$$

If we observe the identity (66) in the lowest order of approximation (viz., the linear form), as well as the fact that equations (43.a) and (43.b) R_{ijkl} $(x^i,\, x^0) = 0$ are fulfilled already in this approximation, along with (43.d), viz., $\underset{3}{R}_{\{0ij\}k} = 0$, then we will see that (88) is, in fact, true. We can then get the functions $\underset{3}{g}{}^{0i}(x^i, x^0)$, as well. We shall also give them in the n^{th} order of approximation.

We now assume that we have solved the field equations up to and including order ($2n$ – 1) and then determine the solution for the next order from that. We again begin with equations (D) and (E). According to our procedure, which is based upon a step-wise specification of the local dislocations and their currents that corresponds to the development of those quantities in the reciprocal speed of sound, we must then give $\underset{2n}{T}_{ijk}$ in (D.3) in such a way that these equations will be fulfilled. Since those equations have no integrability conditions, as we remarked above, that will be possible with no further assumptions. However, the right-hand side of (D.3), which is known from the lower approximations, is time-dependent, so we will also have time-dependent quantities in $\underset{2n}{T}_{ijk}(x^i, x^0)$. We employ those quantities in (E.3) in order to determine $\underset{2n+1}{T}{}^0{}_{ik}$.

However, the dislocations do not determine the currents uniquely, since there can be dislocation currents without any free dislocations appearing, just as there are electric currents without free charges. That corresponds to the fact that only the rotation expression $\underset{2n+1}{T}{}^0{}_{ik,j} - \underset{2n+1}{T}{}^0{}_{jk,i}$ will follow from (E.3). One is still free to choose a gradient expression of the form $t^0{}_{k,\,i}$ when one establishes $T^0{}_{ik}$. [The full expression for $T^0{}_{ik}$ is required in (E).]

Naturally, one can start with the first level of approximation, just as one does here. However, one will better see that the dislocations at a time-point and the current at all times can be given independently of each other in a different way. Since the left-hand side of (E.3) has the form of a rotation, there will be integrability conditions for the right-hand side. If one sets:

$$\underset{2n+1}{S}_{kij} \underset{\text{def}}{=} -(\underset{2n+1}{T}{}^0{}_{ik,j} - \underset{2n+1}{T}{}^0{}_{jk,i})$$

then the right-hand side of (E.3) will read:

$$\underset{2n+1}{S}_{kij} - \underset{2n+1}{\underset{3}{R}}_{\{0ij\}k},$$

and the integrability conditions will take on the form:

$$\underset{2n+1}{S}_{kij,n} + \underset{2n+1}{S}_{kjn,i} + \underset{2n+1}{S}_{kni,j} - (\underset{2n+1}{\underset{3}{R}}_{\{0ij\}k,n} + \underset{2n+1}{\underset{3}{R}}_{\{0jn\}k,i} + \underset{2n+1}{\underset{3}{R}}_{\{0ni\}k,j})$$

$$\equiv (\underset{2n+1}{\underset{3}{R}}_{\{0ij\}k,n} + \underset{2n+1}{\underset{3}{R}}_{\{0jn\}k,i} + \underset{2n+1}{\underset{3}{R}}_{\{0ni\}k,j}) = 0. \tag{89}$$

If one now observes the identity (71) and considers that (D.3) is already fulfilled at all times so $\underset{2n}{R}_{\{ijk\}l,0} = 0$, and furthermore that the field equations of the foregoing order should be fulfilled, by assumption, then the validity of (89) will follow immediately.

The right-hand sides of equations (A.3) and (B.3) are known with that. If one now substitutes (A.3) and (B. 3) and isolates the known functions from the unknown ones then one will get a system of equations in the form:

$$\Delta \underset{2n}{g}_{kl} + \underset{2n}{g}_{ss,kl} = \underset{2n}{F}_{kl},$$

in which $\underset{2n}{F}{}_{kl}$ combines all of the known functions. Contraction will yield:

$$\Delta \underset{2n}{g}{}_{ss}(x^i, x^0) = \tfrac{1}{2}\underset{2n}{F}{}_{ss}(x^i, x^0), \tag{90}$$

and one writes:

$$\Delta \underset{2n}{g}{}_{ss}(x^i, x^0) = \underset{2n}{F}{}_{ss}(x^i, x^0) - \underset{2n}{g}{}_{ss,kl}(x^i, x^0)\,. \tag{91}$$

If one first integrates (90) according to:

$$\underset{2n}{g}{}_{ss}(x^i, x^0) = \int G(x^i - x^{i'})\,\tfrac{1}{2}\underset{2n}{F}{}_{ss}(x^{i'}, x^0)\,d^3x'$$

and then substitutes that in (91) then it will follow that:

$$\underset{2n}{g}{}_{kl}(x^i, x^0)$$
$$= \int G(x^i - x^{i'})\left[\underset{2n}{F}{}_{kl}(x^{i'}, x^0) - \frac{\partial^2}{\partial x^{k'}\partial x^{l'}}\left(\int G(x^{i'} - x^{i''})\,\tfrac{1}{2}\underset{2n}{F}{}_{ss}(x^{i''}, x^0)\right)d^3x''\right]d^3x'\,. \tag{92}$$

If one constructs the expression $\underset{2n}{g}{}_{kl,l}$ from (92) then one will see by partial integration that equation (A.3) is also fulfilled, due to the property of the GREEN function that:

$$\Delta G\,(x^i - x^{i'}) = \delta^3\,(x^i - x^{i'})\,.$$

(92) is then the solution of (A.3) and (B.3) for all times. The right-hand side of (C.3) is also known with that. However, as we have established before, (C) has integrability equations, and with:

$$\underset{2n+1}{K}{}_{ij} \underset{\text{def}}{=} -\tfrac{1}{2}(\underset{2n+1}{g}{}^{0i}{}_{,j} + \underset{2n+1}{g}{}^{0j}{}_{,i}) - \underset{2n+1}{\Gamma}{}^{\,0}_{\underline{ij}}\,,$$

they can be written in the form (88) for the order (2n + 1), so:

$$-\left(\underset{2n+1}{\Gamma}{}^{\,0}_{\underline{ij},kl} + \underset{2n+1}{\Gamma}{}^{\,0}_{\underline{kl},ss} - \underset{2n+1}{\Gamma}{}^{\,0}_{\underline{ik},jl} - \underset{2n+1}{\Gamma}{}^{\,0}_{\underline{jl},ik}\right) = 0. \tag{93}$$

However, (93) is true, due to the identity (66), on the basis of the fact that we have already fulfilled $\underset{2n}{R}{}_{ijkl}(x^i, x^0) = 0$ and $\underset{\underset{2n}{3}}{R}{}_{\{0ij\}k} = 0$, and all field equations of lower order of approximation should be fulfilled by assumption. We can then integrate (C.3). We then form the equations:

$$-\tfrac{1}{2}(\underset{2n+1}{g}{}^{0i}{}_{,ji} + \underset{2n+1}{g}{}^{0j}{}_{,ii}) = -\tfrac{1}{2}\underset{2n}{g}{}_{ij,i\underset{1}{0}} - \underset{2n+1}{T}{}^{\,0}{}_{ij,i} - \underset{2n+1}{T}{}^{\,0}{}_{ji,i} + \underset{2n+1}{BL}{}_{ij,i}\,. \tag{94}$$

Repeated differentiation will yield the equations:

$$\Delta \underset{2n+1}{g}{}^{0i}{}_{,i} = \tfrac{1}{2} \underset{2n}{g}{}_{ij,i\underset{1}{0}} - \underset{2n+1}{T}{}^{0}{}_{ij,i} - \underset{2n+1}{T}{}^{0}{}_{ji,i} + \underset{2n+1}{BL}{}_{ij,i}\,, \tag{95}$$

whose solution is given by:

$$\underset{2n+1}{g}{}^{0i}{}_{,i}(x^i, x^0) = \int G(x^i - x^{i'}) \left[\tfrac{1}{2} \underset{2n}{g}{}_{ij,ij\underset{1}{0}} + 2 \underset{2n+1}{T}{}^{0}{}_{ij,ij}(x^{i'}, x^0) - \underset{2n+1}{BL}{}_{ij,ij}(x^{i'}, x^0) \right] d^3x'\,. \tag{96}$$

One then substitutes that in (94) and obtains:

$$\Delta \underset{2n+1}{g}{}^{0j} = \underset{2n}{g}{}_{rj,r\underset{1}{0}} + 2 \underset{2n+1}{T}{}^{0}{}_{jr,r} + 2 \underset{2n+1}{T}{}^{0}{}_{rj,r} - 2 \underset{2n+1}{BL}{}_{rj,r} - \underset{2n+1}{g}{}^{0r}{}_{,rj}\,. \tag{97}$$

The solution (97) is then:

$$\begin{aligned} \underset{2n+1}{g}{}^{0i}(x^i, x^0) = \int G(x^i - x^{i'}) \Big[& \underset{2n}{g}{}_{ij,ij\underset{1}{0}}(x^{i'}, x^0) + 2 \underset{2n+1}{T}{}^{0}{}_{jr,r}(x^{i'}, x^0) - 2 \underset{2n+1}{T}{}^{0}{}_{rj,r}(x^{i'}, x^0) \\ & - \underset{2n+1}{BL}{}_{rj,r}(x^{i'}, x^0) - \underset{2n+1}{g}{}^{0r}{}_{,rj}(x^{i'}, x^0) \Big] d^3x'\,, \end{aligned} \tag{98}$$

in which the term $\underset{2n+1}{g}{}^{0r}{}_{,rj}(x^{i'}, x^0)$ under the integral is defined by (96). One can verify that (98) is, in fact, a solution of (C.3) with the use of the identity (96).

In contrast to general relativity, in which the equations of the second stage of approximation can first be integrated when a well-defined law of motion is assumed for the sources in the first approximation, etc. (cf., [**39**]), here we have to show that the approximation can be continued up to an arbitrary degree with no restricting conditions on the sources and their currents. We can generally say nothing about the convergence of the series that are obtained in that way. However, we might assume that this is also guaranteed for properly-posed physical problems. In all of the actual physical examples, one must truncate the series after a few terms anyway, since higher elastic constants will no longer be known.

Before we go into the integrability of the basic equations with matter (61), we would first like to turn to a discussion of the dynamical extension of the continuum theory with foreign atoms, namely, KRÖNER's general continuum theory (cf., [**18**]). The homogeneous field equations (43) were constructed upon the assumption that a lattice orientation is defined uniquely at each point of a crystal. The perturbed lattice directions of the real crystal therefore emerge from that of the ideal crystal by way of the distribution of dislocations and their migration. According to KRÖNER's argument, the orientations will still remain preserved in the crystal when one assumes that intermediate lattice atoms are introduced into the crystal, along with the dislocations. They will likewise alter the lattice structure and lead to internal stresses. We can describe the alterations that are due to the foreign atoms by a quasi-plastic deformation $\varepsilon^{Q}{}_{kl}$ that will still remain when we cut the mass elements and allow them to relax (cf., [**18**]). Indeed, we now have:

$$\text{a)}\quad h_i^K h_j^L\, \delta_{KL} = c_{ij} \qquad \text{with} \qquad \text{b)}\quad c_{ij} = g_{ij} - 2\, \varepsilon^{Q}{}_{kl}\,, \tag{99}$$

instead of (6), as before, in which g_{ij} again characterizes the elastic deformations. The dislocation density is further given by the expression:

$$\text{c)} \qquad T_{kl}{}^{i} = h_K^i h_{l,k}^K . \tag{99}$$

The basic equations of the theory with foreign atoms then remain formally the same. One merely has to replace g_{ij} with c_{ij} and decompose c_{ij} according to (99.c). In this, $\varepsilon^{Q}{}_{kl}$ is then the given quasi-plastic deformation that originates in the foreign atoms and determines the internal stresses, along with the dislocation density. If one adds the world-lines as the fourth curve congruence in the transition to the time-dependent theory then when one proceeds in a manner that is completely parallel to the development in the beginning, one will get equations (19) as the starting equations for the theory with foreign atoms, but in which $c_{\mu\nu}$ now appears in place of $g_{\mu\nu}$. In this, c_{ik} is connected with g_{ik} by (99.b), and we will once more have:

$$c^{00} = h_\Gamma^0 h_\Lambda^0 \eta^{\Gamma\Lambda} = -1,$$

$$c^{0\,i} = h_\Gamma^0 h_\Lambda^i \eta^{\Gamma\Lambda} = - h_0^i = - \frac{v^i}{c_T} = c^{0\,i}$$

for the remaining components.

One must then regard the system (43) as the basic equations for the theory of foreign atoms in the time-dependent case, in which one merely replaces g_{ik} with c_{ik} using (99.*b*); $c^{0\,i}$ appears in place of $g^{0\,i}$, but it still has the same meaning, namely, $-v^{i} / c_T$. When one expresses the appearance of $c_{\mu\nu}$ in place of $g_{\mu\nu}$ by means of a "*c*" that is placed beneath the symbol, the equations of the theory with foreign atoms will then read:

$$\left.\begin{aligned} &a)\quad \underset{c}{R}_{\underline{ij}} = 0, && b)\quad \underset{c}{R}_{\{ijk\}l} = 0, \\ &c)\quad \underset{c}{\Gamma}{}^{0}_{\underline{ij}} = 0, && d)\quad \underset{c}{\underset{3}{R}}_{\{0ij\}k} = 0. \end{aligned}\right\} \tag{100}$$

(100.*a*) and (100.*b*) can be once more combined into:

$$\underset{c}{R}_{ijkl} = 0. \tag{100.a + b}$$

When one decomposes all terms that are connected with $\varepsilon^{Q}{}_{kl}$, one can also give (100) in the form:

$$\left.\begin{aligned} &a)\quad \bar{R}_{ijkl} = \underset{c}{\bar{M}}_{ijkl}, && b)\quad R_{\{ijk\}l} = \underset{c}{V}_{ijkl}, \\ &c)\quad \Gamma^{0}_{\underline{ij}} = \underset{c}{N}_{ij}, && d)\quad \underset{3}{R}_{\{0ij\}k} = \underset{3}{L}_{ijk}, \end{aligned}\right\} \tag{101}$$

$$(a) + (b) \qquad R_{ijkl} = \underset{c}{M}_{ijkl} \quad \text{with} \quad \underset{c}{V}_{ijkl} = \underset{c}{M}_{\{ijk\}l} , \qquad \underset{c}{\bar{M}}_{ijkl} = \underset{c}{M}_{ijkl} - \underset{c}{V}_{ijkl} ,$$

which formally coincide with (61). [The left-hand sides of (61) and (101) are the same.]

However, it should be emphasized that there is an essential difference in the meaning of the matter tensors here and in the other theory. Namely, whereas the matter tensors in (61) mean the dislocations of the lattice structure, the closure conditions for the dislocations, and their time evolution, that is not the case here. Due to (99.*a*) and (99.*c*), equations (100.*a*) and (100.*b*) now insure the existence of a teleparallelism and the closure condition for the dislocations, while (100.*c*) and (100.*d*) guarantee their temporal conservation.

We have shown that the differential identities (62) and (66) are fulfilled for the left-hand sides of (101). That must then be true for the right-hand sides, as well, and thus, for the tensors $\underset{c}{\bar{M}}_{ijkl}$, $\underset{c}{V}_{ijkl}$, $\underset{c}{N}_{ij}$, and $\underset{c}{L}_{ijk}$. However, that implies no restricting conditions on those tensors, since analogous differential identities are also true for the left-hand sides of (100). We shall shortly show that explicitly for the case of BIANCHI identity (62). Here, we shall denote the covariant tensor that is defined with the CHRISTOFFEL symbols for c_{ij} by "‖" and the one that is defined by the CHRISTOFFEL symbols for g_{ij} by ";". Along with:

$$R_{\{ij\,|\,kl\,|\,;\,n\}} - R_{\{ij\,|\,rl\,|}h_{n\}\,k}{}^{r} + R_{\{ij\,|\,rk\,|}h_{n\}\,l}{}^{r} \equiv 0, \tag{62}$$

one has the identity:

$$R_{\{ij\,|\,kl\,|\,\|\,n\}} - R_{\{ij\,|\,rl\,|}h_{n\}\,k}{}^{r} + R_{\{ij\,|\,rk\,|}h_{n\}\,l}{}^{r} \equiv 0\,. \tag{102}$$

Due to the field equations (101), one must then have:

$$\underset{c}{M}_{\{ij\,|\,kl\,|\,;\,n\}} - \underset{c}{M}_{\{ij\,|\,rl\,|}h_{n\}\,k}{}^{r} + \underset{c}{M}_{\{ij\,|\,rk\,|}h_{n\}\,l}{}^{r} \equiv 0\,. \tag{103}$$

We now set:

$$\underset{c}{R}_{ijkl} = R_{ijkl} - \underset{c}{M}_{ijkl}\,, \qquad \underset{c}{\begin{Bmatrix} r \\ i\,k \end{Bmatrix}} = \begin{Bmatrix} r \\ i\,k \end{Bmatrix} + \Gamma^{r}_{ik}\,. \tag{104}$$

With that, the covariant derivatives can be expressed in terms of each other:

$$\underset{c}{R}_{ijkl\|n} \equiv \underset{c}{R}_{ijkl;n} + (\Pi^{r}_{m\nu}\,\underset{c}{R}_{pqst})\,,$$

and finally:

$$\underset{c}{R}_{ijkl\|n} \equiv R_{ijkl\,;\,n} - \underset{c}{M}_{ijkl;n} + (\Pi^{r}_{m\nu}\,\underset{c}{R}_{pqst})\,. \tag{105}$$

(The expressions in parentheses shall then stand for four terms of that kind, for the sake of brevity.) If one substitutes (104) and (105) into (102) then that will imply:

$$R_{\{ij\,|\,kl\,|\,;\,n\}} - R_{\{ij\,|\,rl\,|}h_{n\}\,k}{}^{r} + R_{\{ij\,|\,rk\,|}h_{n\}\,l}{}^{r} - \underset{c}{M}_{\{ij|\,kl\,|;n\}} + \underset{c}{M}_{\{ij|rl|}h_{n\}k}{}^{r} - \underset{c}{M}_{\{ij|rk|}h_{n\}l}{}^{r} + (\Pi^{r}_{m\nu}\,\underset{c}{R}_{pqst}) \equiv 0. \tag{106}$$

If one observes (62) then it will follow that:

$$\underset{c}{M}_{\{ij|kl|;n\}} - \underset{c}{M}_{\{ij|rl|}h_{n\}k}^{\;\;r} + \underset{c}{M}_{\{ij|rk|}h_{n\}l}^{\;\;r} - (\Pi^{r}_{mv}\,\underset{c}{R}_{pqst}) = 0. \tag{107}$$

(107) differs from (103) only by bilinear terms and is therefore always fulfilled already with our approximation procedure by an argument that is used repeatedly at each step of the approximation. One shows in precisely the same way that no conditions on the tensors $\underset{c}{M}_{ijkl}$, $\underset{c}{N}_{ij}$, $\underset{c}{V}_{ijkl}$, and $\underset{c}{V}_{ijk}$ will follow from the identity (66).

The integration of equations (100) [(101), resp.] now proceeds in complete analogy to what was done in the foregoing case with no foreign atoms. We therefore do not need to go into the details of that procedure.

We shall now apply our approximation procedure to equations (61). In order to do that, we must first extend the series Ansatz to the tensors **M**, **N**, **V**, **L**. When we assume series of the following forms, the field equations will split into ones for the individual levels of approximation, as before:

$$\left.\begin{aligned} \bar{M}_{ijkl} &= \underset{2}{\bar{M}}_{ijkl} + \underset{4}{\bar{M}}_{ijkl} + \cdots, \\ V_{ijkl} &= \underset{2}{V}_{ijkl} + \underset{4}{V}_{ijkl} + \cdots, \\ N_{ij} &= \underset{3}{N}_{ij} + \underset{5}{N}_{ij} + \cdots, \\ L_{ijk} &= \underset{3}{L}_{ijk} + \underset{5}{L}_{ijk} + \cdots \end{aligned}\right\} \tag{108}$$

In order to avoid unnecessary repetitions, we restrict ourselves to just the case in which the dislocations and their currents vanish, and therefore only the tensors $\bar{M}_{ijkl}$ and N_{ij} are non-zero. Only three groups of equations will remain in the approximation procedure, namely:

1. The "coordinate conditions" [i.e., equations (A) on page 35]:

$$\left.\begin{aligned} &1)\quad -\underset{2}{g}_{ir,r} = 0, \\ &2)\quad -\underset{4}{g}_{ir,r} + \underset{3}{g}^{0i}_{\;;\underset{1}{0}} = -\tfrac{1}{2}\nu_1(\underset{2}{g}_{is_1}\,\underset{2}{g}_{s_1 r})_{,r}, \\ &\quad\vdots \\ &3)\quad -\underset{2n}{g}_{ir,r} + \underset{2n+1}{g}^{0i}_{\;;\underset{1}{0}} = \underset{2n}{NL_i}(g_{rs}\,g_{mn}, g_{rs}\,g^{0n}, g^{0n}g^{0n}_{\;\;,r}, \ldots), \end{aligned}\right\} \tag{A}$$

2. (61.*a*):

$$\left.\begin{aligned}
&1)\quad \tfrac{1}{2}(-\underset{2}{g}_{ir,ss}-\underset{2}{g}_{ss,kl}+\underset{2}{g}_{ks,sl}+\underset{2}{g}_{ls,ks})=\underset{2}{M}_{kl}\,,\\
&2)\quad \tfrac{1}{2}(-\underset{4}{g}_{kl,ss}-\underset{4}{g}_{ss,kl}+\underset{4}{g}_{ks,sl}+\underset{4}{g}_{ls,sk})=\underset{4}{BL_i}(\underset{2}{g}_{rs}\,\underset{2}{g}_{mn})+\underset{4}{M}_{kl}\,,\\
&\quad\vdots\\
&3)\quad \tfrac{1}{2}(-\underset{2n}{g}_{kl,ss}-\underset{2n}{g}_{ss,kl}+\underset{2n}{g}_{ks,sl}+\underset{2n}{g}_{ls,sk})=\underset{2n}{BL_i}(g_{rs}\,g_{mn})+\underset{2n}{M}_{kl}\,,
\end{aligned}\right\}\qquad (F)$$

and

3. (61.*c*):

$$\left.\begin{aligned}
&1)\quad -\tfrac{1}{2}(\underset{2}{g}{}^{0i}{}_{,j}+\underset{2}{g}{}^{0j}{}_{,i})=-\tfrac{1}{2}\underset{2}{g}_{ij,\underset{1}{0}}+\underset{3}{N}_{ij}\,,\\
&2)\quad -\tfrac{1}{2}(\underset{2}{g}{}^{0i}{}_{,j}+\underset{2}{g}{}^{0j}{}_{,i})=-\tfrac{1}{2}\underset{4}{g}_{ij,\underset{1}{0}}+\underset{4}{BL_i}(\underset{2}{g}_{rs}\,\underset{3}{g}{}^{0n})+\underset{5}{N}_{ij}\,,\\
&\quad\vdots\\
&3)\quad \tfrac{1}{2}(\underset{2n+1}{g}{}^{0i}{}_{,j}+\underset{2n+1}{g}{}^{0j}{}_{,i})=-\tfrac{1}{2}\underset{2n}{g}_{ij,\underset{1}{0}}+\underset{2n+1}{BL_i}(g_{rs}\,g^{0n})+\underset{2n+1}{N}_{ij}\,.
\end{aligned}\right\}\qquad (G)$$

If one starts from the first approximation then one must give $\underset{2}{M}_{kl}$ at a time-point $\underset{0}{x}{}^{0}$ such that the integrability condition of (F.1) (i.e., the BIANCHI identity in the linear approximation) is fulfilled. The conditions for $\underset{2}{M}_{kl}$:

$$\underset{2}{M}_{kl,l}-\tfrac{1}{2}\underset{2}{M}_{rr,k}=0 \qquad (\underset{2}{\overset{*}{M}}_{kl,l}=0\text{, resp., with } \overset{*}{M}_{kl}=M_{ik}-\tfrac{1}{2}g_{kl}\,M_r{}^r) \tag{109}$$

must then be fulfilled at the time point $\underset{0}{x}{}^{0}$. In addition, one can give $\underset{3}{N}_{ij}(x^i,\ x^0)$ arbitrarily and arrive at $\underset{2}{M}_{ij}(x^i,\ x^0)$ by simply integrating (67). Equation (109), which is the linearized form of (63) (for vanishing dislocation density), is therefore preserved for all times, on the basis of (77). $\underset{2}{g}_{kl}(x^i,\ x^0)$ then follows by integrating (A.1) and (F.1), which takes exactly the same form as the integration of (A.1) and (B.1). We then substitute that into (G.1). (G.1) has an integrability condition. If one writes the right-hand side of (G.1) in the form:

$$\underset{3}{\tilde{K}}_{ij}\underset{\text{def}}{=}-\tfrac{1}{2}(\underset{3}{g}{}^{0i}{}_{,j}+\underset{3}{g}{}^{0j}{}_{,i})-\underset{3}{\Gamma}{}^{0}_{\underline{ij}}+\underset{3}{N}_{ij}$$

then that condition will read:

$$\begin{aligned}
&\underset{3}{\tilde{K}}_{kl,ij}+\underset{3}{\tilde{K}}_{ij,kl}-\underset{3}{\tilde{K}}_{ki,lj}-\underset{3}{\tilde{K}}_{lj,ki}\\
&\qquad\equiv(\underset{3}{N}_{kl}-\underset{3}{\Gamma}{}^{0}_{\underline{kl}})_{,ij}+(\underset{3}{N}_{ij}-\underset{3}{\Gamma}{}^{0}_{\underline{ij}})_{,kl}+(\underset{3}{N}_{ki}-\underset{3}{\Gamma}{}^{0}_{\underline{ki}})_{,lj}+(\underset{3}{N}_{lj}-\underset{3}{\Gamma}{}^{0}_{\underline{lj}})_{,ki}=0.
\end{aligned}\tag{110}$$

If one considers the fact that (F.1) is already fulfilled at all times and combines the identity (66) with equation (67), both of which are taken in the linear approximation, then one will see that (110) is fulfilled. One can therefore determine $\underset{3}{g}{}^{0i}(x^i, x^0)$ from (G.1) with no further discussion. [That happens in precisely the same way as it does with (C.1).]

We shall now further assume that equations (A), (F), and (G) are fulfilled up to an including order $(2n - 1)$, and we would like to show how one can then arrive at the solution to the next-higher order. One must therefore once more demand the BIANCHI identity for $\underset{2n}{M}{}_{kl}$ ($\underset{2n}{\overset{*}{M}}{}_{kl}$, resp.), into which bilinear terms of the lower orders also enter now:

$$\underset{2n}{\overset{*}{M}}{}_{k;l}^{\;l} \equiv \underset{2n}{\overset{*}{M}}{}_{k,l}^{\;l} - \underbrace{\overset{*}{M}{}_r^{\;l}\begin{Bmatrix} r \\ k\,l \end{Bmatrix}}_{2n} + \underbrace{\overset{*}{M}{}_k^{\;r}\begin{Bmatrix} l \\ r\,l \end{Bmatrix}}_{2n} = 0,$$

so

$$\underset{2n}{\overset{*}{M}}{}_{k,l}^{\;l} = \underbrace{\overset{*}{M}{}_r^{\;l}\begin{Bmatrix} r \\ k\,l \end{Bmatrix}}_{2n} - \underbrace{\overset{*}{M}{}_k^{\;r}\begin{Bmatrix} l \\ r\,l \end{Bmatrix}}_{2n}, \tag{111}$$

in which the right-hand side of (111) is known and time-dependent now. However, at first, we shall fulfill (111) only at an arbitrary, but fixed, time-point $\underset{0}{x}{}^0$. In that way, one can then freely choose three suitable functions of $\underset{2n}{\overset{*}{M}}{}_k^{\;l}$, while the remaining ones are then fixed by (111). In addition, we freely give the currents $\underset{2n+1}{N}{}_{kl}$ at all times and then determine $\underset{2n}{\overset{*}{M}}{}_k^{\;l}(x^i,\; x^0)$ at all times from them by using equations (67) in the approximation considered and simple integration. From Chapter III.3, equation (77), that is sufficient for (111) to be also true for all times. Equations (A.3) and (F.3) can now be integrated in the known way and will yield $\underset{2n}{g}{}_{kl}(x^i, x^0)$. We then substitute that into (G.3). The integrability conditions of (G.3) will now read:

$$(\underset{2n+1}{N}{}_{kl} - \underset{2n+1}{\Gamma}{}^{0}_{\underline{kl}})_{,ij} + (\underset{2n+1}{N}{}_{ij} - \underset{2n+1}{\Gamma}{}^{0}_{\underline{ij}})_{,kl} + (\underset{2n+1}{N}{}_{ki} - \underset{2n+1}{\Gamma}{}^{0}_{\underline{ki}})_{,lj} + (\underset{2n+1}{N}{}_{lj} - \underset{2n+1}{\Gamma}{}^{0}_{\underline{lj}})_{,ki} = 0. \tag{112}$$

If one observes that (F.3) is already fulfilled at all times and that, by assumption, all field equations of the foregoing orders are true, moreover, then upon combining the identity (66) with equation (67), one will find that (112) is confirmed. One then determines $\underset{2n+1}{g}{}^{0i}(x^i, x^0)$ from (G.3). The approximation can then be continued arbitrarily with no restrictions, which was to be shown.

V. – NON-ANALYTIC SOLUTIONS

We would now like to consider solutions that do not behave in an analytic way on a well-defined hypersurface in our four-dimensional space-time continuum. In order to do that, we give ourselves a family of hypersurfaces:

$$a)\quad z = z\,(x^i, x^0), \qquad \text{and further let} \qquad p_i \underset{\text{def}}{=} \frac{\partial z}{\partial x^i}\,, \qquad p_0 \underset{\text{def}}{=} \frac{\partial z}{\partial x^0}\,, \tag{112}$$

and let the jump surface be given by:

$$b)\quad z = 0\,. \tag{113}$$

We write out the basic equations explicitly once more in the following form:

$$\begin{aligned} R_{\{ijk\}\,l} \equiv 2\,[&T_{ijl,\,k} + T_{jkl,\,i} + T_{kil,\,i} \\ &- T_{ijr}\left(\begin{Bmatrix} r \\ k\,l \end{Bmatrix} + T_{kl}^{\ \ r} + T^{\ r}_{\ \ kl} + T^{\ r}_{\ \ lk}\right) \\ &- T_{jkr}\left(\begin{Bmatrix} r \\ i\,l \end{Bmatrix} + T_{il}^{\ \ r} + T^{\ r}_{\ \ il} + T^{\ r}_{\ \ li}\right) \\ &- T_{kir}\left(\begin{Bmatrix} r \\ j\,l \end{Bmatrix} + T_{kl}^{\ \ r} + T^{\ r}_{\ \ jl} + T^{\ r}_{\ \ lj}\right)\Bigg], \end{aligned} \tag{I}$$

$$\begin{aligned} \underset{3}{R}_{\{0ij\}k} \equiv\; &T_{ijk,\,0} + T^{\ 0}_{\ \ ik,\,j} + T^{\ 0}_{\ \ jk,\,i} \\ &- g^{0r}\,T_{ijk,\,r} + g^{0r}_{\ \ ,\,j}\,T_{rik} - g^{0r}_{\ \ ,\,i}\,T_{rjk} - g^{0r}_{\ \ ,\,k}\,T_{ijr} \\ &+ T^{\ 0}_{\ \ ir}\,\Gamma^r_{jk} - T^{\ 0}_{\ \ jr}\,\Gamma^r_{ik} + 2T^{\ 0}_{\ \ kr}\,T_{ij}^{\ \ r} = 0, \end{aligned} \tag{II}$$

$$\begin{aligned} \overset{0}{R}_{\underline{ik}} \equiv\; &\tfrac{1}{2}g^{rs}\,(g_{si,\,rk} + g_{rk,\,si} - g_{ik,\,rs} - g_{rs,\,ik}) + g^{rs}\,g_{pq}\left(\begin{Bmatrix} p \\ r\,k \end{Bmatrix}\begin{Bmatrix} q \\ i\,s \end{Bmatrix} - \begin{Bmatrix} p \\ i\,k \end{Bmatrix}\begin{Bmatrix} p \\ r\,s \end{Bmatrix}\right) \\ = &- g^{rs}\,(T_{sik,\,r} + T_{ski,\,r} + T_{krs,\,i} + T_{irs,\,k}) \\ &- g^{rs}\,g_{pq}\left[\begin{Bmatrix} p \\ r\,k \end{Bmatrix}(T^{\ q}_{\ \ is} + T^{\ q}_{\ \ si}) + \begin{Bmatrix} q \\ i\,s \end{Bmatrix}(T^{\ p}_{\ \ rk} + T^{\ p}_{\ \ kr}) + \begin{Bmatrix} p \\ i\,k \end{Bmatrix}(T^{\ q}_{\ \ rs} + T^{\ q}_{\ \ sr}) - \begin{Bmatrix} q \\ r\,s \end{Bmatrix}(T^{\ p}_{\ \ ik} + T^{\ p}_{\ \ ki})\right] \\ &- g^{rs}\,g_{pq}\,[T_{rk}^{\ \ p}\,T_{is}^{\ \ q} + (T^p_{\ rk} + T^p_{\ kr})\,(T^q_{\ is} + T^q_{\ si}) - (T^p_{\ ik} + T^p_{\ ki})\,(T^q_{\ i\,k} + T^q_{\ k\,i})], \end{aligned} \tag{III}$$

$$\overset{0}{\Gamma}{}^{\,0}_{\underline{ik}} \equiv \tfrac{1}{2}(g_{ik,\,0} - g_{ik,\,r}\,g^{\,0r} - g^{\,0r}_{\ \ ,\,i}\,g_{kr} - g^{\,0r}_{\ \ ,\,k}\,g_{ir}) = -\,(T^0_{\ ik} + T^0_{\ ki}). \tag{IV}$$

We add the equilibrium conditions as the fifth system of equations, which we can write in the form:

$$\sigma_{ri,\,r} + \mu\sqrt{g}\,(g^{0i}{}_{,\,0} - g^{0i}{}_{,\,r}\, g^{0r}) = 0, \qquad \mathrm{V}$$

with $\mu = \rho\, c^2$, due to the absence of forces. In that equation, stress is assumed to be an algebraic function of g_{ik} , namely, $\sigma_{ik} = \sigma_{ik}\,(g_{rs})$.

We first ask what sort of discontinuities the g_{rs} , g^{0i} (their derivatives, resp.) might admit when crossing the surface $z = 0$ as a result of the basic equations. The answer depends essentially upon the assumptions that we made about the right-hand sides of III and IV. If we were to demand that they should vanish then we would be dealing with the homogeneous, "matter-free" case, just as in, e.g., MAXWELL's electrodynamics or EINSTEIN's theory of gravitation (cf., H. TREDER [**45**], for that). Only the homogeneous equations III and IV will then remain, which will be satisfied identically by the Ansatz that g_{rs} can be derived from a displacement field ([15]), and g^{0i} (v^s, resp.) is the material derivative of the displacement vector with respect to time.

There are no other solutions, and the investigation of shock waves will reduce to the investigation of jumps in the derivatives of the displacement vector with the help of only the equilibrium condition. That is well-known in the theory of elasticity, and we shall refer to, say, the work of C. TRUESDELL [**46**]. TRUESDELL's study of wave propagation for finite stresses included not only the general theory, but also a thorough examination of the special features of waves propagating in materials with specific properties; here, we are interested in only the former theory. In contrast to our presentation, in which mass-points are described by their coordinates x^r *after* the distortion (as one usually does in the theory of dislocations), TRUESDELL characterizes them by their X^α *before* the deformation, which is expressed by:

$$x^r = x^r\,(X^\alpha, t)\;.$$

We consider jumps in the derivatives of the x^r across a two-dimensional surface, so:

$$[x^r{}_{,\,\alpha\beta}] \neq 0 \qquad \text{with} \qquad x^r{}_{,\,\alpha} = \frac{\partial x^r(X^\alpha, t)}{\partial X^\alpha}\,, \qquad \dot{x}^r = \frac{\partial x^r(X^\alpha, t)}{\partial t}\,,$$

which can be characterized by an "amplitude vector" a^k and a propagation speed U.

We can regard the family of two-dimensional surfaces that is defined by the wave front with the propagation speed U as a hypersurface in the space-time continuum that we consider. The jump conditions will then follow from the equilibrium conditions. Corresponding to our choice of starting coordinates for the variable, they can be written in the form ([16]):

([15]) I. e., for the quantities ε_{rs} in $g_{rs} = \delta_{rs} - 2\varepsilon_{rs}$, one will have $\varepsilon_{rs} = s_{\underline{r,s}} - \frac{1}{2}\, s_{n,\,r}\, s_{n,\,s}$, with the displacement vector s_r .

([16]) In this:

$$T_k{}^\alpha \underset{\text{def}}{=} I\,\sigma_k{}^m\, X^\alpha{}_{;\,m}$$

$$T_{k\;;\alpha}^{\alpha} = \tilde{\rho}\,\ddot{x}_k$$

(we have dropped the external forces), and applying the jump will yield the relation:

$$Q_{km}(n)\,a^m = \tilde{\rho}\,U^2\,a_k\,. \tag{114}$$

Q_{km} is the so-called acoustic tensor, which depends upon the pre-stresses and the direction n of propagation of the waves. The basic equation (114) characterizes the amplitude vector a_k on the right as the eigenvector of the acoustic tensor with $\tilde{\rho}\,U^2$ as its corresponding eigenvalue. We will encounter this equation again in our description below.

One can allow the right-hand sides of III and IV to be non-vanishing, so one must consider equations I and II, in addition, and the derivatives of the dislocations and their currents to likewise exhibit jumps. One generally understands the jump problem of order n to mean that all derivatives of $g_{\mu\nu}$ up to and including order $(n-1)$ should be continuous upon crossing the jump surface, while the n^{th}-order derivatives, and therefore all higher-order derivatives, in general, will suffer a jump upon crossing that surface (cf., TREDER [**45**]). Now, it is clear from the basic equations that in the jump problem of order n, discontinuities in the n^{th} derivatives of the g_{rs} , g^{0r} are coupled with those of order $(n-1)$ of the T_{rsk} , $T^{0}{}_{,\,ik}$. However, if one maintains the demand that the derivatives of order $(n-1)$ of the dislocations and their currents should be continuous in the jump problem of order n then the jumps in the n^{th} derivatives of the g_{rs} , g^{0r} that are allowed by the field equations will have precisely the form that would be produced by a displacement vector field. Those jumps will be either excluded by the equilibrium conditions or they will satisfy certain propagation conditions on the hypersurfaces that the equilibrium conditions define, namely, the propagation surfaces of the sound. Those jumps can then be given freely on a two-dimensional surface that belongs to the hypersurface. We shall then deal with acoustic shock waves in a medium that contains dislocations and their currents with elastic pre-stresses.

In order to do that, we consider the n^{th}-order shock waves, as they were introduced into the theory of gravitation by TREDER [**45**]:

$$\left.\begin{aligned} g_{ik} &= g_{ik}^{-} + \underset{n}{\gamma}_{ik}\,\underset{n}{h}(z) + \underset{n+1}{\gamma}_{ik}\,\underset{n+1}{h}(z) + \cdots, \\ g^{0i} &= g^{0i-} + \underset{n}{\gamma}^{0i}\,\underset{n}{h}(z) + \underset{n+1}{\gamma}^{0i}\,\underset{n+1}{h}(z) + \cdots \end{aligned}\right\} \tag{115}$$

The $\underset{n}{h}(z)$ in this are the so-called "jump functions":

---

is the so-called PIOLA-KIRCHHOFF stress tensor, I is the JACOBIan determinant of the deformation, and with our choice of Cartesian coordinates in the initial state, as well as in the deformed state, I will coincide with $\sqrt{g}$, and furthermore, $\tilde{\rho} = \sqrt{g}\,\rho$. ";" means a generalized covariant differentiation here. Cf., C. TRUESDELL, R. A. TOUPIN [**47**].

$$\underset{n}{h}(z) = \begin{cases} 0 & \text{for} \quad z < 0 \\ \dfrac{1}{n!}z^n & \text{for} \quad z \geq 0 \end{cases}, \qquad \text{so one then has} \qquad \frac{d}{dz}\underset{n}{h}(z) = \underset{n+1}{h}(z), \tag{116}$$

and g_{ik}^{-}, g^{0i-} mean the analytic continuation of the quantities g_{ik}, g^{0i} that were defined for $z < 0$ across the jump surface $z = 0$. We employ the usual notation:

$$[A] = \lim_{z\to 0^+} A - \lim_{z\to 0^-} A \tag{117}$$

for a jump in a field quantity.

Since the equations III have order two, it would seem reasonable to restrict ourselves to the cases of $n = 2, 3, \ldots$ We will discuss the cases of $n = 0$ and $n = 1$ later on. Here, we shall treat the case of $n = 2$, for the sake of simplicity, since it exhibits all that is essential. Since we then have:

$$[T_{ijk,\,\mu}] = [T^{0}{}_{ik,\,\mu}] = [T_{ijk}] = [T^{0}{}_{ik}] = 0, \tag{118}$$

by assumption, when we apply the jump condition to III, we will get the condition:

$$\tfrac{1}{2} g^{rs} (\underset{2}{\gamma}_{si}\, p_r\, p_k + \underset{2}{\gamma}_{rk}\, p_s\, p_i - \underset{2}{\gamma}_{ik}\, p_r\, p_s - \underset{2}{\gamma}_{rs}\, p_i\, p_k) = 0. \tag{119}$$

Due to (118), imposing jumps in I, II, and IV will not produce any condition, but when one differentiates IV once and then applies a jump, one will get the jump relation:

$$\underset{2}{\gamma}_{ij}\, p_0\, p_\mu + \underset{2}{\gamma}_{ij}\, p_q\, p_\mu\, g^{0q} - \underset{2}{\gamma}^{0q}\, p_i\, p_\mu\, g_{jq} - \underset{2}{\gamma}^{0q}\, p_j\, p_\mu\, g_{iq} = 0. \tag{120}$$

Since it is possible to have $p_r\, p^r = 0$ only when p_r itself vanishes, the general solution of (119) will be:

$$\underset{2}{\gamma}_{ij} = \underset{2}{a}_i\, p_j + \underset{2}{a}_j\, p_i, \tag{121}$$

and when one writes $\underset{2}{\gamma}_{0i} = \underset{2}{\gamma}^{0r} g_{ri}$, it will then follow from (120) that:

$$\underset{2}{\gamma}_{0i} = (p_0 - g_{0q}\, p_q)\, \underset{2}{a}_i\,. \tag{122}$$

With that, we immediately recognize the following special cases:

1. $$p_r = 0.$$

The hypersurface is then our three-dimensional space at a well-defined time-point; it will then follow from (121) that:

$$\underset{2}{\gamma}_{ij} = 0,$$

and when we differentiate V and apply the jump, we will also get:

$$\underset{2}{\gamma}^{0i} = 0.$$

2. $$p_0 - g^{0q}\, p_q = 0.$$

The hypersurfaces that are defined in that way are the generalizations to the case of a moving medium of the two-dimensional spatial surfaces of the static case that are established for all times. When one observes that $g^{0q} = -v^q / c$ (so $\frac{\partial}{\partial x^0} - g^{0q}\frac{\partial}{\partial x^q}$ represents the material derivative), one will see that (2.) represents surfaces that are constantly swept out by the same particles. It follows directly from (122) that:

$$\underset{2}{\gamma}^{0i} = 0,$$

while one infers from V by differentiating and applying the jump that the divergence of the stress tensor has no second-order jump. There will also be no jump in $\underset{2}{\gamma}_{ij}$ for a physically-reasonable stress-strain relationship that is one-to-one. No jumps will be allowed by both of the hypersurfaces that are characterized by (1.) and (2.).

Before we go into the general form of the hypersurface, we would like to briefly consider the higher-order jumps that are permitted by the field equations I – IV alone. For the sake of simplicity, we shall pursue the case of $n = 2$ further.

If one goes on to calculate the third-order jumps then they will be composed of the quantities $\underset{3}{\gamma}_{ik}$, $\underset{3}{\gamma}^{0i}$ as well as $\underset{2}{\gamma}_{ik}$, $\underset{2}{\gamma}^{0i}$. When one observes (115) and (117), one will find that:

$$\left.\begin{aligned}
[g_{rs,kln}] &= \underset{2}{\gamma}_{rs,k}\, p_l\, p_n + \underset{2}{\gamma}_{rs,l}\, p_k\, p_n + \underset{2}{\gamma}_{rs,n}\, p_k\, p_n \\
&\quad + \underset{2}{\gamma}_{rs}\,(p_{k,n}\, p_l + p_{n,l}\, p_k + p_{l,k}\, p_n) + \underset{2}{\gamma}_{rs}\, p_k\, p_l\, p_n, \\
[g^{0r}{}_{,kln}] &= \underset{2}{\gamma}^{0r}{}_{,k}\, p_l\, p_n + \underset{2}{\gamma}^{0r}{}_{,l}\, p_n\, p_k + \underset{2}{\gamma}^{0r}{}_{,n}\, p_k\, p_l \\
&\quad + \underset{2}{\gamma}^{0r}\,(p_{k,n}\, p_l + p_{n,l}\, p_k + p_{l,k}\, p_n) + \underset{2}{\gamma}^{0r}\, p_k\, p_l\, p_n.
\end{aligned}\right\} \tag{124}$$

When one differentiates III with respect to x^n and apply the ancillary jump that will then yield:

$$g^{rs}\,(\underset{3}{\gamma}_{si}\, p_r\, p_k + \underset{3}{\gamma}_{rk}\, p_s\, p_i - \underset{3}{\gamma}_{ik}\, p_r\, p_s - \underset{3}{\gamma}_{rs}\, p_i\, p_k)\, p_n$$

$$= -\, g^{rs}\,[\underset{2}{\gamma}_{si,k}\, p_r\, p_n + \underset{2}{\gamma}_{si,r}\, p_k\, p_n - \underset{2}{\gamma}_{si,n}\, p_r\, p_k - \underset{2}{\gamma}_{si}\,(p_{k,r}\, p_n + p_{r,n}\, p_k + p_{n,k}\, p_r)$$

+ three analogous terms]

$$
\begin{aligned}
&- g^{rs} g^{pq} \underset{2}{a}_q \, p_n \, [p_k \, p_r \, (- g_{is,p} + g_{sp,i} + g_{pi,s} + 2\, T_{psi} + 2\, T_{pis}) \\
&+ p_i \, p_s \, (- g_{kr,p} + g_{rp,k} + g_{pk,r} + 2\, T_{pkr} + 2\, T_{prk}) \\
&- p_i \, p_k \, (- g_{rs,p} + g_{sp,r} + g_{pr,s} + 2\, T_{prs} + 2\, T_{psr}) \\
&- p_r \, p_s \, (- g_{ik,p} + g_{kp,i} + g_{pi,k} + 2\, T_{pik} + 2\, T_{pki})] \underset{\text{def}}{=} - \underset{3}{I}_{ik} \, .
\end{aligned} \tag{125}
$$

Since we have already treated the case of $p_r = 0$, in particular, we can exclude it here and then find that the general solution of (125) is:

$$
\underset{3}{\gamma}_{ij} = \underset{3}{a}_i \, p_j + \underset{3}{a}_j \, p_i + \frac{1}{p_r \, p_s \, g^{rs}} \underset{3}{I}_{ij} \, , \tag{126}
$$

with a new jump vector $\underset{3}{a}_i$ that is initially arbitrary. We must substitute (121) for $\underset{2}{\gamma}_{ik}$. In order for (126) to actually be a solution of (125), $\underset{3}{I}_{ij}$ must fulfill the compatibility equations:

$$
\left[\underset{2}{I}_{ij} \, p_r - \tfrac{1}{2} \underset{3}{I}_{rs} \, p_i \right] g^{rs} = 0. \tag{127}
$$

That is a consequence of the BIANCHI identity (54). Namely, if one applies a jump to (54) and observes that the equation $[R_{ijkl}] = 0$ is fulfilled already due to (118), (119), and (120) then (127) will follow immediately. When one now sets:

$$
[T^0{}_{ij,\mu\nu}] = 0, \tag{128}
$$

in addition to (118), one will find by differentiating II and applying the jump that:

$$
\underset{2}{t}_{ijk} \, (p_0 - g^{0r} \, p_r) = p_k \, g^{sn} \, \underset{2}{a}_n \, (T^0{}_{js} \, p_i - T^0{}_{is} \, p_j), \tag{129}
$$

in which one has set $[T_{ijk,\mu\nu}] = \underset{2}{t}_{ijk} \, p_\mu \, p_\nu$. Since the case of $p_0 - g^{0r} \, p_r = 0$ was treated above, it can be excluded here, such that $\underset{2}{t}_{ijk}$ will actually be determined by (129). One sees here that jumps in the n^{th} derivatives of the g_{ik} will also induce jumps in the n^{th} derivatives of the local dislocation densities. That is understandable, since the metric enters into the definition of T_{ijk} implicitly. The equation that arises from I by derivation and applying a jump is fulfilled along with (129). If one now observes (128) then one will get the jump relation for $\underset{3}{\gamma}^{0i}$ ($\underset{3}{\gamma}_{0i} = \underset{3}{\gamma}^{0r} \, g_{ri}$, resp.) from IV, which one can write in the form:

$$
\begin{aligned}
\underset{3}{\gamma}_{0i} \, p_j \, p_\mu \, p_\nu + \underset{3}{\gamma}_{0j} \, p_i \, p_\mu \, p_\nu &= \underset{3}{\gamma}_{0i} \, (p_0 - g^{0r} \, p_r) \, p_\mu \, p_\nu + \underset{3}{\Gamma}_{ij\mu\nu}(\underset{2}{\gamma}_{rs}, \underset{2}{\gamma}_{0r}) \\
&\underset{\text{def}}{=} K_{ij\mu\nu} + p_\mu \, p_\nu \, (p_0 - g^{0r} \, p_r) (\underset{3}{a}_i \, p_j + \underset{3}{a}_j \, p_i) \, ,
\end{aligned} \tag{130}
$$

in which $\underset{3}{\Gamma}_{ij\mu\nu}$ summarizes the terms in $\underset{2}{\gamma}_{ij}$ and $\underset{2}{\gamma}^{0i}$. The solubility conditions for (130), which can be written in the form:

$$K_{ij\mu\nu}\, p_k\, p_l + K_{kl\mu\nu}\, p_i\, p_j - K_{kj\mu\nu}\, p_j\, p_l - K_{jl\mu\nu}\, p_i\, p_k = 0, \tag{131}$$

are fulfilled here on the grounds of the identity (66), which one can easily verify by applying a jump to (66) and considering the fact that one now has $[\underset{3}{R}_{(0ij)k,l}] = [R_{ijkl,\, s}] = 0$ and the lower-order jump relations are fulfilled already. One can write the solution of (130) in the form:

$$\underset{2}{\gamma}_{0i} = (p_0 - g^{0r}\, p_r)\ \underset{3}{a}_i + b_i\,. \tag{132}$$

The vector b_i can be calculated simply by contracting (130). One can calculate all of the quantities $\underset{r}{\gamma}_{ik}$, $\underset{r}{\gamma}^{0i}$ that appear in the jump series in that way with no difficulty. For the higher-order jumps, one must merely observe the product rule for jumps:

$$[A\, B] = A^-\, [B] + [A]\, B^- + [A]\, [B]\,. \tag{133}$$

The first-order jump problem can be treated similarly. In order to calculate the first-order jump relation, one must require the equations:

$$\lim_{z\to 0} \int_{z=-\varepsilon}^{z=+\varepsilon} R_{\underline{ij}} = 0 \tag{134}$$

for III, in place of the simpler jump condition, in analogy with the integral form of the gravitational equations according to A. PAPAPETROU and H. TREDER [**48**]. Here, that condition excludes jumps in the dislocation densities, which led to δ-functions in III. The first-order jump relations that follow from (134) are completely analogous to (119). All of the other jump relations can be obtained precisely as above and will lead to completely analogous results.

We make the following remark about the jump problem of order zero: It is clear from the jump relations (121), (122), (126), (132), etc., that the jumps that are allowed by the basic equations have precisely the same form as the jumps that are produced in a displacement vector. In general relativity, they correspond to the jumps that can be transformed away. In fact, the form of the jumps that are described by (121), (122), etc., in general relativity is precisely the same as the one that can be specified in the corresponding derivatives of a discontinuous coordinate transformation that will make the transformed metric no longer exhibit jumps (cf., TREDER [**45**]). However, such jumps are also permissible in the $g_{\mu\nu}$ themselves. We must also reckon with zero-order jumps in the theory of elasticity from the outset then. The only jumps that are relevant to general relativity are the ones that cannot be transformed away. Only the jumps that are independent of a choice of coordinates can reproduce physical phenomena. However, in contrast to that, in continuum mechanics, as we have emphasized since the beginning (see

Chap. I), the conditions that correspond to a coordinate transformation express a physical law, namely, the equilibrium conditions. Therefore, in contrast to general relativity, the a_i also have a genuine physical meaning here. However, zero-order jumps in the theory of elasticity would also imply jumps in the velocities themselves, which would then require infinitely-large forces. We shall therefore exclude those jumps here on physical grounds. The further treatment of the jump problem will then reduce to a discussion of the vector a_i that is left arbitrary in equations I – IV with the help of only the equilibrium conditions. We then find ourselves in the same situation as in the aforementioned "vacuum waves," except that the quantities g_{ij} , g^{0i} that enter into equations V cannot be constructed from just a displacement vector field, but must be determined from equations I – IV. We once more arrive at the relations of the elementary theory of elasticity as the limiting case of vanishing dislocation density and dislocation current. If we go on to the second-order stress problem here then when we differentiate equations V and apply the jump, that will yield:

$$-\Sigma^i{}_{jmn} \underset{2}{\gamma}{}_{mn}\, p_j + \mu\sqrt{g}\ \underset{2}{\gamma}{}^{0i}\, (p_0 - g^{0r}\, p_r) = 0, \tag{135}$$

with ([17]):

$$\sigma^i{}_j = \sigma^i{}_j\, (-g_{mn}) \qquad \text{and} \qquad -\Sigma^i{}_{jmn} = \frac{\partial \sigma^i{}_j}{\partial g_{mn}},$$

so:

$$[\sigma^i{}_{j,\,rs}] = -\Sigma^i{}_{jmn}\, \underset{2}{\gamma}{}_{mn}\, p_r\, p_s .$$

If we substitute the solutions (121), (122) for $\underset{2}{\gamma}{}_{mn}$, $\underset{2}{\gamma}{}^{0i}$ and define:

$$c\,(p_0 - g^{0r}\, p_r) \underset{\text{def}}{=} \tilde{p}_0 \qquad \text{and further} \qquad g^{rs}\, \underset{2}{a}{}_s = \underset{2}{a}{}^r$$

then (135) will take on the following form:

$$-\Sigma^i{}_r\, \underset{2}{a}{}^r + \rho\sqrt{g}\ \tilde{p}_0^2\, \underset{2}{a}{}^i = 0, \tag{136.a}$$

with

$$\bar{K}^i{}_r = \Sigma^i{}_{jmn}\, p_j\, (p_m\, g_{nr} + p_n\, g_{mr}) .$$

On grounds that will become clear immediately, we divide (136) by $p_r\, p_r$, set:

$$\frac{\tilde{K}^i{}_r}{p_r\, p_r} = K^i{}_r\,, \qquad \frac{\tilde{p}_0^2}{p_r\, p_r} = U^2,$$

and get:

$$-\Sigma^i{}_r\, \underset{2}{a}{}^r + \rho\sqrt{g}\ U^2\, \underset{2}{a}{}^i = 0. \tag{136.b}$$

([17]) We can write $\sigma^i{}_j = \sigma^i{}_j\,(-g_{mn})$, since $g_{mn} = \delta_{mn} - 2\,\varepsilon_{mn}$ and the matter equations take the form $\sigma^i{}_j = \sigma^i{}_j\,(-\varepsilon_{mn})$.

In that form, (136.*b*) agrees completely with equation (114) in TRUESDELL [**46**] ([18]), except that all quantities in (136) refer to the final state, while the ones in (114) refer to the initial state. The quantity:

$$U^2 = \frac{\tilde{p}_0^2}{p_r p_r}$$

has the meaning of the square of the speed of sound for us, as well. Namely, if one writes the equation of the hypersurface, e.g., in the form:

$$Z(x^i, x^0) \equiv x^3 - h(x^1, x^2, t) = 0,$$

then:

$$p_r = (-h_{,1}, -h_{,2}, 1), \qquad p_0 = -\frac{1}{c}\frac{\partial h}{\partial t}.$$

The two-dimensional surfaces:

$$x^3 = h(x^1, x^2, t)$$

run through the family $z = 0$ in the direction of the normal p_r with the speed:

$$u = \frac{\partial h / \partial t}{|p|} = -c\,\frac{p_0}{|p_r|}.$$

In our elastic medium, the speed of propagation of the matter current overlaps with the velocity $v^i = -c\,g^{0i}$. The speed of sound in that medium is then:

$$U = u - \frac{v^r p_r}{|p_r|} = -\frac{c}{|p_r|}(p_0 - g^{0r} p_r),$$

so

$$U^2 = \frac{\tilde{p}_0^2}{p_r\,p_r},$$

which shall be shown.

(136.*b*) is a linear system of equations in $\underset{2}{a}^i$ whose non-trivial solubility for a given stress-strain relation and the quantities g_{ij}, g^{0i} depend upon only the choice of the hypersurface, which is characterized by p_0, p_r. The coefficient matrix will be non-degenerate, in general, so no non-zero solutions will be allowed. Conversely, when one demands that the determinant must vanish, one will get a defining equation for p_0, p_r ($\tilde{p}_0$, p_r, resp.), and the degree of vanishing will dictate how many combinations of $\underset{2}{a}^i$ one is free to choose. Due to the special form of equation (136), that will be an eigenvalue problem, whereby $\tilde{p}_0$ will be connected to the eigenvalue Λ directly by way of:

([18]) The fact that our equation relates to the second-order jump problem for $g_{\mu\nu}$, while (114) is written down for first-order jumps in $g_{\mu\nu}$ is irrelevant, since the form of the latter equation does not depend upon the order of the jump problem (cf., *infra*).

$$\rho\sqrt{g}\,\frac{\tilde{p}_0^{\,2}}{p_r\,p_r} = \Lambda. \tag{137}$$

We remark that the hypersurfaces that are determined in that way are independent of the order of *n*, since we will always get the same equations for the lowest-order jump relations ([19]).

Intuitively, one must introduce the hypersurfaces that follow from (136) as moving two-dimensional surfaces that represent the fronts of the sound waves. It is therefore clear that one cannot give the components of $\underset{2}{a}_i$ that one is free to choose on the total hypersurface – i.e., the totality of all two-dimensional surfaces – since being given the $\underset{2}{a}_i$ on a single two-dimensional surface will determine the further evolution of the $\underset{2}{a}_i$. The $\underset{2}{a}_i$ must then satisfy certain propagation conditions on the hypersurfaces. As in the theory of gravitation, one gets the conditions for the n^{th}-order jumps from the equations for the $(n+1)^{\text{th}}$-order jumps (cf., TREDER [**45**]), and here they will be a consequence of the equilibrium conditions V. There exist far-reaching analogies with the relationships in the theory of gravitation when one fixes the coordinate conditions in the latter (which correspond to the equilibrium conditions here). If one defines the corresponding jump conditions from V by differentiating twice then the relations between the second and third-order jumps that one obtains can be written in the following form:

$$-\,K^i_{\ r}\,\underset{3}{a}^r + \rho\sqrt{g}\;\tilde{p}_0^{\,2}\,\underset{3}{a}^i \;= H^{\,i}\,(2). \tag{138}$$

In that, one has set:

$$[\sigma^i_{j,\,kln}] = -\,\Sigma^{\;i}_{\ jrs}\;p_j\,(p_j\,g_{sm} + p_s\,g_{rm})\;p_k\,p_l\,p_n\,\underset{3}{a}^m + \underset{3}{S}{}^{\,i}_{\ jkln}\,.$$

Here, as with the second-order jumps, the first terms will lead to the tensor $K^{\,i}_{\ r}$, whereas the combinations of lower-order jumps (so second-order jumps) are summarized in $S^i_{\ jkln}$; one must consider (126) here. As one sees from (132), applying the jump to the expression:

$$\left(\sqrt{g}\;g^{0i}_{\ \ ,0} - g^{0i}_{\ \ ,r}\,\mathfrak{g}^{0r}\right)_{,ln}$$

will first lead to a term of the form $\rho\sqrt{g}\;\tilde{p}_0^{\,2}\,\underset{3}{a}^i$, while all of the other terms will contain only lower-order jumps. They are written, along with $\underset{3}{S}{}^{\,i}_{\ jkln}$, on the right-hand side of (138) as $H^{\,i}\,(2)$. As a defining equation for $\underset{3}{a}^i$, (138) has the same form as (136) then, except that the right-hand side is non-vanishing. However, since the hypersurface was chosen in precisely such a way that the homogeneous equations would be soluble, one

([19]) That corresponds to C. TRUESDELL's first equivalence theorem [**46**].

will get different conditions for the inhomogeneity H_i (2) according to the rank of the system of equations. Since H_i (2) is, in turn, a differential expression in $\underset{3}{a_i}$, that expression will give the equations of propagation for the second-order jumps. If they are fulfilled then once more one will be free to choose just as many combinations of the $\underset{3}{a_i}$ on the hypersurface as one previously would with the $\underset{2}{a_i}$, and one will obtain the propagation conditions for the $\underset{3}{a_i}$ from a consideration of the jumps of order four, etc.

We remark that the mathematical basis for the fact that the equilibrium conditions, and with them, the matter equations, play such a central role in a discussion of waves naturally consists of the fact that a characterization of the field equations is first determined by them, in contrast to the theory of gravitation, in which the characteristics are known to be determined by the equation $p_\mu\, p^\mu = 0$. However, a similar relation ($p_\mu\, p_\nu\, g^{\mu\nu} = 0$) is not true in the theory of elasticity.

Here, from (137), one has:

$$\tilde{p}_0^{\,2} = \frac{\Lambda}{\rho\sqrt{g}}\, p_r\, p_r = U^2\, p_r\, p_r\,, \tag{137*}$$

in which Λ is the eigenvalue of (136.*b*), so it will generally be a complicated function of g_{ik}, as well as p_r. Therefore, since Λ generally depends upon direction, (137*) *cannot* be written in the form:

$$g^{*\mu\nu}\, p_\mu\, p_\nu = 0,$$

with suitable $g^{*\mu\nu} = g^{*\mu\nu}(x^\alpha)$. It is only in the special case where the medium is isotropic that one will have:

$$U = U(x^\alpha)$$

(the speed of sound does not depend upon direction then), such that one can define a suitable $g^{*\mu\nu}$ with which one can describe the characteristics by using (137*). As one easily calculates, one will have:

$$g^{*00} = g^{00} = -1,$$
$$g^{*0r} = g^{0r},$$
$$g^{*rs} = \frac{U^2}{c^2}\,\delta^{rs} - g^{0r}\, g^{0s}.$$

The effect of the stress field on the speed of sound is therefore different from the effect of the gravitational field on the speed of light.

We shall now briefly describe the case in which one couples n^{th}-order jumps in the g_{rs}, g^{0r} with $(n + 1)^{\text{th}}$-order jumps in the dislocations and their currents. Here, one must distinguish two ways of posing the question:

1. Are there always distributions of dislocations that are discontinuous in the $(n-1)^{\text{th}}$ derivatives in such a way that arbitrary jumps in the n^{th} derivatives of the g_{rs}, g^{0r} that are given on any sort of hypersurface will be compatible with the field equations?

2. The converse problem: Are discontinuities in the $(n - 1)^{\text{th}}$ derivatives of the dislocations and their currents that are given arbitrarily on any sort of hypersurface $z = 0$ compatible with the field equations?

In the second case, one is free to give the jumps in the derivatives of the dislocations and their currents as long as equations I and II are fulfilled. The corresponding jumps in the g_{rs}, g^{0r} can then be calculated with no difficulty using III and IV. However, in the case where the hypersurface is a characteristic of the basic equations that are defined by the equilibrium conditions, that will imply restrictions on the jumps in the motion of dislocations that represent generalizations of the restrictions on the motion of dislocations that result of the speed of sound.

In question (1.), we again restrict ourselves to the case of $n = 2$. We get the jump relations:

$$\left.\begin{aligned}
a)\quad & g^{rs}(\underset{2}{\gamma}_{si}\, p_i\, p_k + \underset{2}{\gamma}_{rk}\, p_s\, p_i - \underset{2}{\gamma}_{ik}\, p_r\, p_s - \underset{2}{\gamma}_{rs}\, p_i\, p_k) \\
& \qquad = -2g^{rs}(\underset{2}{t}_{sik}\, p_r + \underset{2}{t}_{ski}\, p_r + \underset{2}{t}_{krs}\, p_i + \underset{2}{t}_{irs}\, p_k), \\
b)\quad & \underset{2}{t}_{iks}\, p_l + \underset{2}{t}_{kls}\, p_i + \underset{2}{t}_{lis}\, p_k = 0, \\
c)\quad & \underset{2}{\gamma}_{ij}\,(p_0 - g^{0r}\, p_r) - \underset{2}{\gamma}_{0i}\, p_j - \underset{2}{\gamma}_{0j}\, p_i = -2(\underset{1}{t}{}^{0}_{ij} + \underset{1}{t}{}^{0}_{ji}), \\
d)\quad & \underset{1}{t}_{ijk}\,(p_0 - g^{0r}\, p_r) = \underset{1}{t}{}^{0}_{ik}\, p_j - \underset{1}{t}{}^{0}_{jk}\, p_i .
\end{aligned}\right\} \tag{139}$$

In the case where $p_0 - g^{0r}\, p_r \neq 0$, $\underset{2}{t}_{iks}$ is determined by d) and satisfies b), and in the other case, one satisfies b) identically by a similar Ansatz:

$$t_{ijk} = n_{ik}\, p_j - n_{ik}\, p_j . \tag{140}$$

If we once more consider the case that was connected with the treatment of waves in which the right-hand sides of (139.a, c) vanish then we can infer that:

$$t^{0}{}_{ik} = \tau_{ik} , \tag{141}$$

with an arbitrary choice of $\tau_{ik} = -\tau_{ki}$. The corresponding jump in the derivative of the dislocation then reads:

$$\underset{1}{t}_{ijk} = (p_0 - g^{0r}\, p_r)^{-1}\,(\tau_{ik}\, p_j - \tau_{jk}\, p_i) . \tag{142}$$

Those special jumps in the motion of a dislocation are also allowable in shock waves then. The basis for that fact consists of the fact that those are precisely the parts of the dislocation that will lead to the so-called "stress-free structural curvatures." Namely, that part of the structural curvatures will be produced by precisely the combination (cf., KRÖNER [**18**]):

$$T_{ijk} + T_{kij} + T_{kji} \, .$$

If one forms the first-order jump in this then it will follow that:

$$\underset{1}{t}_{ijk} + \underset{1}{t}_{kij} + \underset{1}{t}_{kji} = 2\,(p_0 - g^{0r}\,p_r)^{-1}\,\tau_{kj}\,p_i \, .$$

However, if one allows the right-hand sides of (139.*a*) and (139.*c*) to become non-vanishing then one can give the quantities $\underset{2}{\gamma}_{ik}$, as well as the $\underset{2}{\gamma}_{0i}$, freely with the single restriction that the equilibrium condition V must be satisfied. If one then substitutes (139.*d*) in (139.*a*) then it will follow that:

$$\begin{aligned} & g^{rs}(\underset{2}{\gamma}_{si}\,p_r\,p_k + \underset{2}{\gamma}_{rk}\,p_s\,p_i - \underset{2}{\gamma}_{ik}\,p_r\,p_s - \underset{2}{\gamma}_{rs}\,p_i\,p_k) \\ & \quad = -\,4\,g^{rs}\,(p_0 - g^{0n}\,p_n)^{-1}\,(\underset{1}{t}{}^0{}_{\underline{si}}\,p_r\,p_r + \underset{1}{t}{}^0{}_{\underline{rk}}\,p_s\,p_i - \underset{1}{t}{}^0{}_{\underline{ik}}\,p_r\,p_s - \underset{1}{t}{}^0{}_{\underline{rs}}\,p_i\,p_k) \, . \end{aligned} \tag{143}$$

The general solution to (143) is:

$$\underset{1}{t}{}^0{}_{ik} = -\tfrac{1}{4}(p_0 - g^{0r}\,p_r)\,\underset{2}{\gamma}_{ik} + \tfrac{1}{4}(\tau_i\,p_k + \tau_k\,p_i) \, . \tag{144}$$

The quantities τ_i , which are still undetermined here, will then be determined by (139.*c*), since, along with (144), it will follow that:

$$\underset{2}{\gamma}_{0i}\,p_k + \underset{2}{\gamma}_{0k}\,p_i = \tau_i\,p_k + \tau_k\,p_i \, , \tag{145}$$

so

$$\tau_i = \underset{2}{\gamma}_{0i} \, . \tag{146}$$

The part $t^0{}_{<ik>}$ still remains undetermined here. [One can construct the solution similarly in the case where $p_0 - g^{0r}\,p_r = 0$ with the use of (140).]

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APPENDIX

1. – Notations.

It will generally be true that the following notations will be employed, unless specifically stated to the contrary:

Uppercase and lowercase Greek indices Γ, Λ, Θ, … and μ, τ, λ, …, resp., will always run through the numbers 0, 1, 2, 3, while uppercase and lowercase Latin indices K, L, M, … and i, k, l, …, resp., will always run through just 1, 2, 3. In that way, lowercase Latin (Greek, resp.) indices i, k, l, … (μ, τ, λ, …, resp.) will each be tensor indices in three- (four-, resp.) dimensional RIEMANNian spaces, while the uppercase indices K, L, M, … (Γ, Λ, Θ, …, resp.) will be just numbers in RIEMANNian spaces. Precisely the opposite will be true for the dual spaces that are explained in the text. Indices that appear twice will always be summed over. In order to distinguish between three- (four-, resp.) dimensional quantities that read the same, when any doubt might arise, the numeral 3 (4, resp.) will be placed under the symbol; e.g.:

$$\underset{3}{g}^{ik}, \quad \text{but} \quad \underset{4}{g}^{ik} \equiv \underset{3}{g}^{ik} - g^{0i}\,g^{0k},$$

and likewise for the covariant derivatives (cf., *infra*):

$$T_{\underset{3}{r\,;\,s}}, \quad \text{but} \quad T_{\underset{4}{r\,;\,s}}.$$

However, in approximation procedures, numbers under the field quantities mean the order in a development in some parameter, and in the jump relations, the numbers stand for the orders of the jumps. No confusion should be possible then.

Symmetrization (antisymmetrization, resp.) will be characterized by an underbar beneath the indices (angle brackets that enclose them, resp.), while curly brackets will stand for cyclic permutation (indices that are not included in that will be separated by vertical lines):

$$T_{\underline{ik}} \underset{\text{def}}{=} \tfrac{1}{2}\,(T_{ik} + T_{ki}), \qquad T_{<ik>} \underset{\text{def}}{=} \tfrac{1}{2}\,(T_{ik} - T_{ki}),$$

$$T_{\{ikl\}} \underset{\text{def}}{=} T_{ikl} + T_{kli} + T_{lki}\,, \qquad T_{\{ik\,|\,n\,|\,l\}} \underset{\text{def}}{=} T_{iknl} + T_{klni} + T_{lkni}\,.$$

δ_{ik} stands for the KRONECKER symbol, and ε_{ikl} will stand for the LEVI-CIVITA symbol. We will employ the symbols:

$$\Delta = \frac{\partial^2}{\partial(x^1)^2} + \frac{\partial^2}{\partial(x^2)^2} + \frac{\partial^2}{\partial(x^3)^2}, \qquad \dot{f} = \frac{\partial f}{\partial t}.$$

In order to characterize the linearized form of equations, an L will be placed under the symbol; e.g. [cf., (43.c) for this]:

$$\underset{L}{\Gamma}{}^{0}_{\underline{ij}} \underset{\text{def}}{=} \tfrac{1}{2}(g_{ij,0} - g^{0i}{}_{,j} - g^{0i}{}_{,j}) + T^{0}{}_{ij} + T^{0}{}_{ji}\,.$$

Boldfaced symbols will characterize tensors without giving their indices:

$$\boldsymbol{\alpha} \,\hat{=}\, \alpha_{ik}\,.$$

The curvature tensor is defined by:

$$R_{\alpha\beta\gamma}{}^{\delta} \underset{\text{def}}{=} \Gamma^{\delta}_{\beta\gamma,\alpha} - \Gamma^{\delta}_{\alpha\gamma,\beta} + \Gamma^{\kappa}_{\beta\gamma}\Gamma^{\delta}_{\alpha\kappa} - \Gamma^{\kappa}_{\alpha\gamma}\Gamma^{\delta}_{\beta\kappa}\,,$$

$$\underset{3}{R}{}_{ijk}{}^{l} \underset{\text{def}}{=} \underset{3}{\Gamma}{}^{l}_{jk,i} - \underset{3}{\Gamma}{}^{l}_{ik,j} + \underset{3}{\Gamma}{}^{r}_{jk}\underset{3}{\Gamma}{}^{l}_{ir} - \underset{3}{\Gamma}{}^{r}_{ik}\underset{3}{\Gamma}{}^{l}_{jr}\,,$$

and the RICCI tensor by:

$$R_{\alpha\beta} \underset{\text{def}}{=} R_{\mu\alpha\beta}{}^{\mu}, \qquad \underset{3}{R}{}_{ik} \underset{\text{def}}{=} \underset{3}{R}{}_{rik}{}^{r}\,.$$

The affinity in this is given by:

$$\Gamma^{\alpha}_{\beta\gamma} = \begin{Bmatrix}\delta\\ \beta\,\gamma\end{Bmatrix} + h_{\beta\gamma}{}^{\delta}, \qquad \underset{3}{\Gamma}{}^{l}_{ik} = \underset{3}{\begin{Bmatrix}i\\ i\,k\end{Bmatrix}} + \underset{3}{h}{}_{ik}{}^{l}\,,$$

with the CHRISTOFFEL affinity:

$$\begin{Bmatrix}\delta\\ \beta\,\gamma\end{Bmatrix} \equiv \tfrac{1}{2}g^{\delta\kappa}(g_{\beta\kappa,\gamma} + g_{\kappa\gamma,\beta} - g_{\gamma\beta,\kappa}), \qquad \underset{3}{\begin{Bmatrix}i\\ i\,k\end{Bmatrix}} \equiv \tfrac{1}{2}\underset{3}{g}{}^{ir}(g_{ir,k} + g_{rk,i} - g_{ki,r})$$

and the RICCI rotation tensors:

$$h^{\delta}_{\Gamma}\, h^{\Gamma}_{\underset{4}{\gamma;\beta}} \equiv h_{\beta\gamma}{}^{\delta} \equiv T_{\beta\gamma}{}^{\delta} + T^{\delta}{}_{\beta\gamma} + T^{\delta}{}_{\gamma\beta}, \qquad h^{i}_{G}\, h^{G}_{\underset{3}{k;i}} \equiv \underset{3}{h}{}_{ik}{}^{l} \equiv \underset{3}{T}{}_{ik}{}^{l} + \underset{3}{T}{}^{l}{}_{ik} + \underset{3}{T}{}^{l}{}_{ki}\,,$$

and for the torsion in this, one will have:

$$T_{\beta\gamma}{}^{\delta} = -\,T_{\gamma\beta}{}^{\delta}, \qquad \underset{3}{T}{}_{ik}{}^{l} = -\,\underset{3}{T}{}_{ki}{}^{l}\,.$$

Furthermore, let:

$$\overset{0}{R}{}_{ijk}{}^{l} \underset{\text{def}}{=} \begin{Bmatrix}l\\ j\,k\end{Bmatrix}_{,i} - \begin{Bmatrix}l\\ i\,k\end{Bmatrix}_{,j} + \begin{Bmatrix}r\\ j\,k\end{Bmatrix}\begin{Bmatrix}l\\ i\,r\end{Bmatrix} - \begin{Bmatrix}r\\ i\,k\end{Bmatrix}\begin{Bmatrix}l\\ r\,j\end{Bmatrix},$$

$$\overline{R}{}_{ijkl} \underset{\text{def}}{=} R_{ijkl} - R_{\{ijk\}\,l}\,.$$

The EINSTEIN tensor will be introduced by:

$$E_{\alpha\beta} = R_{\alpha\beta} - \tfrac{1}{2} g_{\alpha\beta} R, \qquad \underset{4}{R} = R_{\alpha\beta}\, g^{\alpha\beta},$$

$$\underset{3}{E}_{ik} = \underset{3}{R}_{ik} - \tfrac{1}{2} g_{ik} \underset{3}{R}, \qquad \underset{3}{R} = \underset{3}{R}_{ik}\, \underset{3}{g}^{ik}.$$

The covariant derivatives that one forms with the CHRISTOFFEL affinity will be expressed by way of the semi-colon:

$$T^r{}_{;s} = T^r{}_{,s} - T^n \begin{Bmatrix} r \\ s\,n \end{Bmatrix},$$

and the ones that are formed with the full affinity will be indicated by two vertical lines:

$$T^r{}_{\| s} = T^r{}_{,s} - T^n\, \Gamma^r_{sn},$$

while a comma means the partial derivative:

$$T^r{}_{,s} = \frac{\partial T^r}{\partial x^s}.$$

2. – On the derivation of the relations (29).

One addresses the equations:

$$\left.\begin{array}{ll} a) & T_{\alpha\beta}{}^0 = 0, \\ b) & T^0{}_{0i} = g^{0r}\, T^0{}_{ri}, \\ c) & T^0{}_{i0} = g^{0r}\, T^0{}_{ir}, \\ d) & T^0{}_{00} = g^{0r} g^{0s}\, T^0{}_{rs}, \\ e) & \underset{4}{T}{}_{rs}{}^{k} = \underset{3}{T}{}_{rs}{}^{k}, \\ f) & \underset{4}{T}{}^0{}_{rs} = \underset{3}{T}{}^k{}_{rs} - g^{0k} T^0{}_{rs}. \end{array}\right\} \tag{2.1}$$

(2.1*a*) follows immediately from antisymmetrizing (19.*b*):

$$\Gamma^0_{<\alpha\beta>} \equiv T_{\alpha\beta}{}^0 = 0, \qquad \text{so one will also have} \quad T^{\alpha}{}_{\beta}{}^0 = T_{\alpha}{}^{\beta 0} = T^{\alpha\beta 0} = 0. \tag{2.2}$$

Due to the facts that:

$$\Gamma^0_{\underline{ik}} \equiv \begin{Bmatrix} 0 \\ i\,k \end{Bmatrix} + T^0{}_{ik} + T^0{}_{ki} = 0, \tag{2.3}$$

$$\Gamma^0_{\underline{0i}} \equiv g^{0r}\left\{\begin{matrix}0\\ r\;i\end{matrix}\right\} + T^{\,0}{}_{0i} + T^0{}_{i\,0} = 0, \tag{2.4}$$

$$\Gamma^0_{00} \equiv g^{0r} g^{0s}\left\{\begin{matrix}0\\ r\;s\end{matrix}\right\} + 2\,T^{\,0}{}_{00} = 0, \tag{2.5}$$

(19.*b*) will further imply the relation:

$$T^{\,0}{}_{00} = g^{0r}\, g^{0s}\, T^{\,0}{}_{rs}\,; \quad \text{i.e., equation (2.1}d\text{)},$$

and further:

$$T^{\,0}{}_{i\,0} + T^{\,0}{}_{0i} = g^{0r}\,(T^{\,0}{}_{ri} + T^{\,0}{}_{ir}). \tag{2.6}$$

However, the use of the expression (25) on page 12, when one employs (2.1), will yield:

$$\begin{aligned}
T^{\,0}{}_{i\,0} &\equiv g^{0\alpha}\, T^{\,0}{}_i{}^{\alpha} = g_{0r}\, T^{\,0}{}_i{}^{r} \\
&= g_{0r}\, \underset{4}{g}{}^{r\alpha}\, T^{0}{}_{i\alpha} = g_{0r}\, \underset{4}{g}{}^{rs}\, T^{0}{}_{is} + g_{0r}\, g^{0r}\, T^{0}{}_{i\,0} \\
&= g_{0r}\,(\underset{4}{g}{}^{rs} - g^{0r}\, g^{0s})\, T^{0}{}_{is} + g_{0r}\, g^{0r}\, T^{0}{}_{i\,0} \\
&= g^{0s}\, T^{0}{}_{is} - g_{0r}\, g^{0r}\, g^{0s}\, T^{0}{}_{is} + g_{0r}\, g^{0r}\, T^{0}{}_{i\,0} \\
&= g^{0s}\, T^{0}{}_{is} - g_{0r}\, g^{0r}\, g^{0\alpha}\, T^{0}{}_{i\alpha}\,,
\end{aligned}$$

and therefore:

$$T^{\,0}{}_{i\,0} = g^{0s}\, T^{0}{}_{is}\,;$$

i.e., one has (2.1*c*). However, if one substitutes (2.1*c*) in (2.6) then (2.1*b*) will also follow. Equation (2.1*c*) is an immediate consequence of (28), since it is precisely the antisymmetric part of it. Finally, (2.1*f*) will follow from:

$$\begin{aligned}
\underset{4}{T}{}^{k}{}_{rs} &\equiv \underset{4}{g}{}^{k\alpha}\, g_{s\beta}\, T_{\alpha r}{}^{\beta} = \underset{4}{g}{}^{k\alpha}\, g_{sp}\, T_{\alpha r}{}^{p} \\
&= \underset{4}{g}{}^{kq}\, g_{sp}\, \underset{4}{T}{}_{qr}{}^{p} + g^{0q}\, g_{sp}\, T_{0\,r}{}^{p} \\
&= (\underset{4}{g}{}^{kq} - g^{0k}\, g^{0q})\, g_{sp}\, \underset{4}{T}{}_{qr}{}^{p} + g^{0k}\, g_{sp}\, T_{0\,r}{}^{p} \\
&= \underset{3}{T}{}^{k}{}_{rs} - g^{0k}\, g_{sp}\, T^{0}{}_{r}{}^{p} \\
&= \underset{3}{T}{}^{k}{}_{rs} - g^{0k}\, g_{s\alpha}\, T^{0}{}_{r}{}^{\alpha}, \\
\underset{4}{T}{}^{k}{}_{rs} &= \underset{3}{T}{}^{k}{}_{rs} - g^{0k}\, T^{0}{}_{r\,s}\,; \qquad \text{i.e.,} \qquad (2.1f).
\end{aligned}$$

3. – On systems of independent equations.

We next go on to some formulas that we will need in the following calculations and which can be easily verified by substituting (25) into the CHRISTOFFEL symbols:

$$\left.\begin{aligned}
&a)\quad \begin{Bmatrix} 0 \\ i\ j \end{Bmatrix} = \tfrac{1}{2}(g_{ij,0} - g_{ij,q}\, g^{0q} - g^{0q}{}_{,i}\, g_{jq} - g^{0q}{}_{,j}\, g_{iq}),\\
&b)\quad \begin{Bmatrix} 0 \\ 0\ j \end{Bmatrix} = g^{0q} \begin{Bmatrix} 0 \\ q\ j \end{Bmatrix},\\
&c)\quad \begin{Bmatrix} 0 \\ 0\ 0 \end{Bmatrix} = g^{0p} g^{0q} \begin{Bmatrix} 0 \\ p\ q \end{Bmatrix},\\
&d)\quad \begin{Bmatrix} k \\ 0\ 0 \end{Bmatrix} = g^{0k}{}_{,0} + g^{0k}{}_{,q}\, g^{0q} + g^{0r} g^{0s} \underset{3}{\begin{Bmatrix} k \\ r\ s \end{Bmatrix}} + 2 g^{kr} g^{0s} \begin{Bmatrix} 0 \\ r\ s \end{Bmatrix} - g^{0k} g^{0r} g^{0s} \begin{Bmatrix} 0 \\ r\ s \end{Bmatrix},\\
&e)\quad \begin{Bmatrix} k \\ 0\ j \end{Bmatrix} = g^{0k}{}_{,j} + g^{0q} \underset{3}{\begin{Bmatrix} k \\ q\ j \end{Bmatrix}} + \underset{3}{g}{}^{kq} \begin{Bmatrix} 0 \\ q\ j \end{Bmatrix} - g^{0k} g^{0q} \begin{Bmatrix} 0 \\ q\ j \end{Bmatrix},\\
&f)\quad \underset{4}{\begin{Bmatrix} k \\ i\ j \end{Bmatrix}} = \underset{3}{\begin{Bmatrix} k \\ i\ j \end{Bmatrix}} - g^{0k} \begin{Bmatrix} 0 \\ i\ j \end{Bmatrix}.
\end{aligned}\right\} \tag{3.1}$$

We must now show that of equations (36), only the part (37) is independent of the system (30), (31). Along with (37), (36) also include the equations:

$$\begin{aligned}\underset{4}{R}_{\{ijk\}}{}^{l} \equiv 2[&\underset{4}{T}_{ij}{}^{l}{}_{\underset{4}{;}k} + \underset{4}{T}_{jk}{}^{l}{}_{\underset{4}{;}i} + \underset{4}{T}_{ki}{}^{l}{}_{\underset{4}{;}j} + \underset{4}{T}_{ij}{}^{\nu}(\underset{4}{T}_{k\nu}{}^{l} + \underset{4}{T}{}^{l}{}_{k\nu} + \underset{4}{T}{}^{l}{}_{\nu k})\\
&+ \underset{4}{T}_{jk}{}^{\nu}(\underset{4}{T}_{i\nu}{}^{l} + \underset{4}{T}{}^{l}{}_{i\nu} + \underset{4}{T}{}^{l}{}_{\nu i}) + \underset{4}{T}_{ki}{}^{\nu}(\underset{4}{T}_{j\nu}{}^{l} + \underset{4}{T}{}^{l}{}_{j\nu} + \underset{4}{T}{}^{l}{}_{\nu j})] = 0.\end{aligned} \tag{3.2}$$

In this, one has:

$$\underset{4}{T}_{ij}{}^{l}{}_{\underset{4}{;}k} \equiv \underset{4}{T}_{ij}{}^{l}{}_{,k} - T_{\alpha j}{}^{l} \begin{Bmatrix} \alpha \\ k\ i \end{Bmatrix} - T_{i\alpha}{}^{l} \begin{Bmatrix} \alpha \\ k\ j \end{Bmatrix} + T_{ij}{}^{l} \begin{Bmatrix} l \\ k\ \alpha \end{Bmatrix}. \tag{3.3}$$

The cyclically-symmetric part of (31) has the completely-analogous form:

$$\begin{aligned}\underset{3}{R}_{\{ijk\}}{}^{l} \equiv 2[&\underset{3}{T}_{ij}{}^{l}{}_{\underset{3}{;}k} + \underset{3}{T}_{jk}{}^{l}{}_{\underset{3}{;}i} + \underset{3}{T}_{ki}{}^{l}{}_{\underset{3}{;}j} + \underset{3}{T}_{ij}{}^{n}(\underset{3}{T}_{kn}{}^{l} + \underset{3}{T}{}^{l}{}_{kn} + \underset{3}{T}{}^{l}{}_{nk})\\
&+ \underset{3}{T}_{jk}{}^{n}(\underset{3}{T}_{in}{}^{l} + \underset{3}{T}{}^{l}{}_{in} + \underset{3}{T}{}^{l}{}_{ni}) + \underset{3}{T}_{ki}{}^{n}(\underset{3}{T}_{jn}{}^{l} + \underset{3}{T}{}^{l}{}_{jn} + \underset{3}{T}{}^{l}{}_{nj})] = 0.\end{aligned} \tag{3.4}$$

In this:

$$\underset{3}{T}_{ij}{}^{l}{}_{\underset{3}{;}k} \equiv \underset{3}{T}_{ij}{}^{l}{}_{,k} - \underset{3}{T}_{rj}{}^{l} \underset{3}{\begin{Bmatrix} r \\ k\ i \end{Bmatrix}} - \underset{3}{T}_{ir}{}^{l} \underset{3}{\begin{Bmatrix} r \\ k\ j \end{Bmatrix}} + \underset{3}{T}_{ij}{}^{l} \underset{3}{\begin{Bmatrix} r \\ k\ r \end{Bmatrix}}. \tag{3.5}$$

If one now substitutes (3.3) in (3.2) and (3.5) in (3.4) and employs formulas (3.1) and (2.1) then one can easily calculate that the right-hand sides of (3.2) and (3.4) differ merely by terms that can be combined in such a way that they will each contain a factor of Γ^{0}_{ik}. However, they will vanish on the basis of the field equations (30), and (3.2) will, in fact, express no new requirements. However, due to the fact that (37) actually does yield a new requirement in comparison to (30) and (31), it will be clear that the latter are equations in the torsion components $T^{\,0}{}_{ik}$, which do not appear in the remaining equations at all. One must now show that equations (40) – (42):

$$\underset{4}{R}{}_{ijk}{}^{l} \equiv \underset{4}{\Gamma}{}^{l}_{jk,i} - \underset{4}{\Gamma}{}^{l}_{ik,j} + \underset{4}{\Gamma}{}^{\kappa}_{jk}\underset{4}{\Gamma}{}^{l}_{j\kappa} - \underset{4}{\Gamma}{}^{\kappa}_{ik}\underset{4}{\Gamma}{}^{l}_{j\kappa}, \tag{3.6}$$

$$R_{0ij}{}^{k} \equiv \underset{4}{\Gamma}{}^{k}_{ij,0} - \underset{4}{\Gamma}{}^{k}_{0j,i} + \underset{4}{\Gamma}{}^{\kappa}_{jk}\Gamma^{k}_{0\kappa} - \Gamma^{\kappa}_{0j}\underset{4}{\Gamma}{}^{k}_{i\kappa}, \tag{3.7}$$

$$R_{0i0}{}^{k} \equiv \Gamma^{k}_{i0,0} - \Gamma^{k}_{00,i} + \Gamma^{\kappa}_{0i}\Gamma^{l}_{0\kappa} - \Gamma^{\kappa}_{00}\underset{4}{\Gamma}{}^{k}_{0\kappa} \tag{3.8}$$

are fulfilled already due to the system (43). For (3.6), that is trivial, since one needs only to consider (28), $\underset{4}{\Gamma}{}^{i}_{kl} = \underset{3}{\Gamma}{}^{i}_{kl}$, and (43.*c*) in order to get (43.*c*) and (43.*b*). However, for the sake of better clarity, we would like to carry out the proof for all three cases together. In order to do that we decompose the curvature tensor into its RIEMANN-CHRISTOFFEL and torsion parts according to:

$$\begin{aligned} R_{\alpha\beta\gamma}{}^{\delta} \equiv\; & \overset{0}{R}{}_{\alpha\beta\gamma}{}^{\delta} + T_{\beta\gamma}{}^{\delta}{}_{\underset{4}{;}\alpha} + T^{\delta}{}_{\beta\gamma\underset{4}{;}\alpha} + T^{\delta}{}_{\gamma\beta\underset{4}{;}\alpha} - T_{\alpha\gamma}{}^{\delta}{}_{\underset{4}{;}\beta} - T^{\delta}{}_{\alpha\gamma\underset{4}{;}\beta} - T^{\delta}{}_{\gamma\alpha\underset{4}{;}\beta} \\ & + (T_{\beta\gamma}{}^{\kappa} + T^{\kappa}{}_{\beta\gamma} + T^{\kappa}{}_{\gamma\beta})\,(T_{\alpha\kappa}{}^{\delta} + T^{\delta}{}_{\alpha\kappa} + T^{\delta}{}_{\kappa\alpha}) \\ & - (T_{\alpha\gamma}{}^{\kappa} + T^{\kappa}{}_{\alpha\gamma} + T^{\kappa}{}_{\gamma\alpha})\,(T_{\beta\kappa}{}^{\delta} + T^{\delta}{}_{\beta\kappa} + T^{\delta}{}_{\kappa\beta}) \end{aligned} \tag{3.9}$$

with

$$\overset{0}{R}{}_{\alpha\beta\gamma}{}^{\delta} \equiv \begin{Bmatrix}\delta\\ \beta\,\gamma\end{Bmatrix}_{,\alpha} - \begin{Bmatrix}\delta\\ \alpha\,\gamma\end{Bmatrix}_{,\beta} + \begin{Bmatrix}\kappa\\ \beta\,\gamma\end{Bmatrix}\begin{Bmatrix}\delta\\ \alpha\,\kappa\end{Bmatrix} - \begin{Bmatrix}\kappa\\ \alpha\,\gamma\end{Bmatrix}\begin{Bmatrix}\delta\\ \beta\,\kappa\end{Bmatrix}. \tag{3.10}$$

By an identical conversion and the use of the expression (3.1) [the temporal derivative of the metric will then be expressed by, e.g., (3.1*a*)]:

$$\underset{4}{\overset{0}{R}}{}_{ijk}{}^{l} \equiv \underset{3}{\overset{0}{R}}{}_{ijk}{}^{l} - g^{0l}\,\overset{0}{R}{}_{ijk}{}^{0} + \underset{3}{g}{}^{lr}\left(\begin{Bmatrix}0\\ j\,k\end{Bmatrix}\begin{Bmatrix}0\\ i\,r\end{Bmatrix} - \begin{Bmatrix}0\\ i\,k\end{Bmatrix}\begin{Bmatrix}0\\ j\,r\end{Bmatrix}\right), \tag{3.11}$$

$$\underset{4}{\overset{0}{R}}{}_{0ij}{}^{k} \equiv g^{0s}\,\underset{4}{\overset{0}{R}}{}_{sij}{}^{k} + g^{0k}\,\overset{0}{R}{}^{0}{}_{ij}{}^{0} + \underset{3}{g}{}^{sk}\,\overset{0}{R}{}_{jsi}{}^{0}, \tag{3.12}$$

$$\overset{0}{R}{}_{0i0}{}^{k} \equiv g^{0r}\,\overset{0}{R}{}_{0ir}{}^{k} + \underset{3}{g}{}^{kr}\,\overset{0}{R}{}_{0ir}{}^{0}. \tag{3.13}$$

With the use of (2.1), as well as a consideration of the equations:

$$T_{jk}{}^{0}{}_{\underset{4}{;}i} = T_{jk}{}^{r}\begin{Bmatrix}0\\ i\,r\end{Bmatrix},$$

$$T^{0}{}_{jk\underset{4}{;}i} = T^{0}{}_{jk,i} - T^{0}{}_{rk}\underset{3}{\begin{Bmatrix} r \\ i\,j \end{Bmatrix}} - T^{l}{}_{jr}\underset{3}{\begin{Bmatrix} r \\ i\,k \end{Bmatrix}} + \underset{3}{T}{}^{r}{}_{jk}\begin{Bmatrix} 0 \\ i\,r \end{Bmatrix},$$

$$\underset{3}{T}{}^{l}{}_{jk\underset{3}{;}i} = \underset{3}{T}{}^{l}{}_{jk,i} - \underset{3}{T}{}^{r}{}_{jk}\underset{3}{\begin{Bmatrix} l \\ i\,r \end{Bmatrix}} - \underset{3}{T}{}^{l}{}_{rk}\underset{3}{\begin{Bmatrix} r \\ i\,j \end{Bmatrix}} - \underset{3}{T}{}^{l}{}_{jr}\underset{3}{\begin{Bmatrix} r \\ i\,k \end{Bmatrix}},$$

$$\underset{4}{T}{}^{l}{}_{jk\underset{4}{;}i} = \underset{3}{T}{}^{l}{}_{jk\underset{3}{;}i} - g^{0l}T^{0}{}_{jk\underset{4}{;}i} + \underset{3}{g}{}^{rl}\begin{Bmatrix} 0 \\ i\,r \end{Bmatrix}T^{0}{}_{jk} - \underset{3}{\begin{Bmatrix} r \\ i\,j \end{Bmatrix}}T^{0l}{}_{k},$$

$$T_{jk\underset{4}{;}i}{}^{l} = T_{jk\underset{3}{;}i}{}^{l} - g^{0l}T^{0}{}_{jk\underset{4}{;}i} + T^{0\,l}{}_{k}\begin{Bmatrix} 0 \\ i\,j \end{Bmatrix} - T^{0\,l}{}_{j}\begin{Bmatrix} 0 \\ i\,k \end{Bmatrix},$$

one can now check that the torsion part of the curvature tensors $\underset{4}{R}{}_{ijk}{}^{l}$, $R_{0ij}{}^{k}$, $R_{0i0}{}^{k}$ can be converted in such a way that they will extend RIEMANN-CHRISTOFFEL curvature tensors on the right-hand sides of (3.11) [(3.13), resp.] to the complete curvature tensors that are defined by $\Gamma^{\alpha}_{\mu\nu}$ according to (3.9) precisely. One must add equation (37) to (3.12), as well. Equations (3.6) – (3.8) can then be written in the form:

$$\underset{4}{R}{}_{ijk}{}^{l} = \underset{3}{R}{}_{ijk}{}^{l} - g^{0l}R_{ijk}{}^{0} + \underset{3}{g}{}^{lr}(\Gamma^{0}_{jk}\,\Gamma^{0}_{ir} - \Gamma^{0}_{ik}\,\Gamma^{0}_{jr}) = 0, \tag{3.14}$$

$$R_{0ij}{}^{k} = g^{0r}\,\underset{4}{R}{}_{rij}{}^{k} + g^{0k}R^{0\,0}_{ij} + \underset{3}{g}{}^{kr}R_{jri}{}^{0} = 0, \tag{3.15}$$

$$R_{0i0}{}^{k} = g^{0r}R_{0ir}{}^{k} + \underset{3}{g}{}^{kr}R_{0ir}{}^{0} = 0. \tag{3.16}$$

However, the right-hand sides of (3.14) – (3.16) vanish term-wise, on the grounds of (43), and will therefore represent no new requirements, which was to be shown.

4.

At this point, let us add a remark in regard to the choice of algebraically-independent systems (43) from the starting equations (19). It is clear from (3.14) that in order to fulfill:

$$\underset{4}{R}{}_{\underline{ik}} \equiv \underset{4}{R}{}_{\alpha\underline{ik}}{}^{\alpha} = 0, \tag{4.1}$$

the equations:

$$\underset{3}{R}{}_{ijk}{}^{l} = 0, \tag{4.2}$$

$$\Gamma^0_{ik} = 0 \tag{4.3}$$

are all that will be necessary. In addition, the system (4.1) contains just as many algebraically-independent equations as (4.3). Algebraically, one can replace (4.3) with (4.1), such that instead of (43), one will then have the following system:

$$\left.\begin{array}{llll} a) & \underset{3}{\bar{R}}{}_{ijkl} = 0, & b) & \underset{3}{R}{}_{\{ijk\}l} = 0, \\ c) & \underset{4}{\bar{R}}{}_{\underline{ij}} = 0, & d) & \underset{3}{R}{}_{\{0ij\}k} = 0. \end{array}\right\} \tag{4.4}$$

However, the combination of equations (4.1) and (4.2) is only a system of differential equations in Γ^0_{ik} that does not have just the trivial solution $\Gamma^0_{ik} = 0$. Therefore, in order for (4.4) to be identical to (43), one must pose some addition conditions that we would not like to go into here. A further algebraically-independent system is:

$$\left.\begin{array}{llll} a) & \underset{4}{R}{}_{\underline{ij}} = 0, & b) & \underset{3}{R}{}_{\{ijk\}l} = 0, \\ c) & \Gamma^0_{\underline{ij}} = 0, & d) & \underset{3}{R}{}_{\{0ij\}k} = 0. \end{array}\right\} \tag{4.5}$$

We now remark that equations (4.1), especially when we separate the RIEMANN-CHRISTOFFEL part from the torsion part of the curvature tensor and go over to the EINSTEIN tensor, when it is written in the form:

$$\overset{0}{E}{}_{ik} = T_{ik} , \tag{4.6}$$

will coincide precisely with the spatial components of the equations of general relativity:

$$\overset{0}{E}{}_{\alpha\beta} = -\,\kappa T_{\alpha\beta} , \tag{4.7}$$

so the expressions for T_{ik} in (4.6) and (4.7) will naturally be different.

5.

Let:

$$G_{ijk} \underset{\text{def}}{=} F_{ij,\,k} + F_{jk,\,i} + F_{ki,\,j} , \tag{5.1}$$

with

$$F_{ij} = -\,F_{ji} . \tag{5.2}$$

One will then have:

$$F_{ijkr} \underset{\text{def}}{=} G_{ijk,\,r} - G_{rjk,\,i} - G_{irk,\,j} - G_{ijr,\,k} \equiv 0. \tag{5.3}$$

The integrability conditions for the system of equations for F_{ij} that takes the form:

$$G_{ijk} = g_{ijk} \,, \tag{5.4}$$

with any sort of functions g_{ijk} , will then read:

$$f_{ijkr} \underset{\text{def}}{=} g_{ijk,\,r} - g_{rjk,\,i} - g_{irk,\,j} - g_{ijr,\,k} = 0. \tag{5.5}$$

When the system (73) is interpreted as a system of differential equations for the partial determination of L_{ikr} , while the other functions are regarded as givens, it will have precisely the form (5.4) when one makes the association $L_{ikr} \to F_{ij}$, so it must fulfill the integrability conditions (5.5). However, due to (5.2), G_{ijk} is totally-antisymmetric, and from (5.3), F_{ijkr} (i.e., , f_{ijkr} , as well) will also be totally-antisymmetric then. However, f_{ijkr} must vanish identically then, since our indices run through just 1, 2, 3. Therefore, (73) has no integrability conditions.

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