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NOTE

SOME APPLICATIONS OF THE KRONECKER INDEX

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The proof by Ames of Jordan's theorem on closed curves with no double points (nos. **306**, **307**) rests upon the consideration of the order of a point, or if one prefers, on the consideration of a variation of the argument.

The generalization to the case where the number of dimensions exceeds two is provided by the *Kronecker index*, which is a notion that is now classical ([1]).

It has received some new applications in several contemporary works. I propose to present some of them here.

All of the arguments that follow can be easily put into a purely-arithmetic form, even though they are not immediately posed in that form, to abbreviate. In order to be valid under the general hypotheses that we shall adopt, they must satisfy the condition that they involve only the continuity of the functions that are employed, moreover.

I. – JORDAN'S THEOREM IN THE PLANE

1. – I shall begin by returning for a moment to the proof of Jordan's theorem in the case of a plane, and in one part of the theorem I will be forced to go a little further than was done in the introduction of the notion of order.

A planar line ([2]) (C) is defined by the two equations:

$$x = x(t), \qquad y = y(t), \tag{1}$$

in which the right-hand sides are continuous functions of t in the interval (t_0, t_1). That curve will be *closed*, i.e., it will be such that one has:

$$x(t_0) = x(t_1), \quad y(t_0) = y(t_1), \tag{2}$$

and it has no *double point*, i.e., the equations in t', t'':

$$x(t') = x(t''), \qquad y(t') = y(t'')$$

([1]) Above all, in the treatise *Traité d'Analyse* by Picard (T. I, pp. 123; T. II, pp. 193).

([2]) It is what was called a *simple closed knot* in the text (no. **290**).

have no other solution than $t' = t_0$, $t'' = t_1$ or $t' = t_1$, $t'' = t_0$ when t' is different from t''.

Jordan's theorem then consists of this:

1. The curve (C) determines *at least* two distinct regions in the plane.

2. The curve (C) determines *only two* distinct regions.

I shall address only the first of those two statements ([1]).

2. – I recall that the *order* of a point P that is not situated on the curve (C) with respect to that curve is defined by means of the continuous variation of the argument of the vector PM when the point M describes the curve C.

That order is zero when one can draw a half-line through the point P that has no point in common with (C).

It is equal to ± 1 when one can draw a half-line through the point P that has one and only one point in common with (C), and is such that if one lets t' denote the value of t that corresponds to that point then the points of (C) that correspond to value of t that are a little smaller than t' and the point of (C) that correspond to values of t that are a little large than t' will be on different sides of the half-line.

3. – Having said that, we shall confirm the existence of a point whose order is 0 and that of a point whose order is ± 1.

First, let $y = y_1$ be a parallel to the x-axis that has some points in common with the curve. Among them, let A (x_1, y_1) be the one that has the smallest abscissa ([2]). If one lets x_1' denote a quantity that is less than x_1 then the point $A'(x_1', y_1)$ will have order 0, since the half-line that starts from that point in the opposite direction to $A'A$ has no point in common with (C).

4. – Now let ξ be a value of x that is included, in the strict sense (i.e., excluding equality), between x_1 and the maximum value of x on (C), in such a way that the curve (C) cuts the line $x = \xi$ at two or more points. Let M, N be the two extreme points of intersection; i.e., the ones that have the smallest and largest ordinate, respectively ([3]), or rather, the ones that are closest to A. By that, I mean the points for which the values of t are the closest to the one that corresponds to A above and below it ([4]). The points M, N divide (C) into two arcs. I shall call the one that contains the point A the *first* of them.

([1]) It was established in the text under the condition that there must exist a vector AB that *cross* the curve (C) (no. **296**).

([2]) The set of values of t (they are infinite in number) that verify the equation $y(t) = y_1$ is closed. The same thing will then be true for the corresponding values of x. The latter set will then contain a maximum element and a minimum element.

([3]) See the preceding footnote.

([4]) That will imply an obvious modification when the point that corresponds to $t = t_0$ or $t = t_1$ is between A and M or between A and N.

Under either of the definitions that I just indicated for M and N, the first arc will contain no point that is located on the prolongation of the segment of the line MN, either when the first arc MN contains no point in common with the line $x = \xi$ besides M, N (first definition) or when the first arc MN has no point in common with it besides M, N (second definition).

Draw a line Aa through A that cuts the line $x = \xi$ at a point a that is located between M and N. That line Aa can meet the first arc MN at only one point (viz., the point A) or at several of them; let B the one of those points that has the largest abscissa ([1]).

Let B' be a point on the line Aa that has an abscissa that is larger than B, but meanwhile there is no point on the second arc MN that is between B and B' [which is possible ([2]) without B itself having to be a point of the second arc, contrary to hypothesis].

The point B' has order ± 1.

In order to see that, join M to N by a path that is composed of two rectilinear segments MP, NQ that are borrowed from the two prolongations of MN, respectively, and are such that aP is equal to aQ, and P and Q are joined by a semi-circle with its center at a, which is located on the line of MN (i.e., on the side that does not have A) and has a radius that large enough that it has no point in common with the first arc ([3]). That new path forms a closed line (C_1) with the first arc MN and a closed line (C_2) with the second arc MN. We agree to choose the sense of traversal along (C) to be the sense that corresponds to increasing t, and (C_1) and (C_2) have a sense such that the parts that are common to (C) will be traversed in the same sense as they are on (C). Under those conditions, the auxiliary path $MPQN$ is traversed in the opposite sense on (C_1) and (C_2).

Upon letting $\Omega_{(C)}(B')$, $\Omega_{(C_1)}(B')$, $\Omega_{(C_2)}(B')$ denote the orders of B' with respect to the closed curves (C), (C_1), (C_2), respectively, one will then have:

$$\Omega_{(C)}(B') = \Omega_{(C_1)}(B') + \Omega_{(C_2)}(B'),$$

since the variations of the argument on $MPQN$ cancel. Now, $\Omega_{(C_2)}(A)$ is zero [for the same reason that $\Omega_{(C)}(A')$ is], so the same thing will be true for $\Omega_{(C_2)}(B')$ (no. **295**), since one can join A to B' by a path that has no point in common with (C_2), namely, the portion of AB of the first arc that follows the segment of the line BB'.

On the other hand, $\Omega_{(C_1)}(B')$ is equal to ± 1, since the half-line that starts from the point B' and moves in the direction of increasing x will meet (C_2) at only one point that is located on the semi-circle, and it will do that under the conditions that were specified above.

Our conclusion is thus proved.

([1]) See footnote ([2]) on the previous page.

([2]) Once more, see footnote ([2]) on the previous page.

([3]) In the second way of defining the points M, N, the path $MPQN$ can be replaced by the line segment MN, so the point B' will be between B and a.

II. – GENERALITIES ON VARIETIES AND SURFACES

5. – Without actually examining whether one can prove Jordan's theorem in the case of more than two dimensions by an analogous method, we shall now suppose that the theorem was proved, and that will be done by specifying just that statement in a manner that will be indicated later.

However, we must define what we mean by surfaces in *n*-dimensional spaces.

In order to do that, we shall first define the *m*-dimensional *tetrahedroid*.

Conforming to the classical definition in elementary geometry, we call the set of points $(x_1, \ldots, x_m)$ in that space that verifies the inequalities:

$$x_1 \geq 0,\ x_2 \geq 0,\ \ldots,\ x_m \geq 0, \qquad x_1 + x_2 + \ldots + x_m \leq 1 \tag{3}$$

a *locus* – or, by extension, any set that is deduced from the first one by a linear substitution that is not necessarily homogeneous (i.e., the intervention of constant terms is possible) and has a non-zero determinant.

6. – Upon replacing one and only one of the $m + 1$ inequalities (3) (or with the ones that correspond to them after substitution, if that has been done) with an equality, to which one continues to append the remaining m inequalities with no modification, one will have one of the $m + 1$ *faces* of the tetrahedroid. Upon similarly replacing two (and only two) of the inequalities in question with equalities, one will have a *primary edge* (which is, by its very definition, common to two faces). Upon writing out three equalities and $m - 2$ inequalities, one will similarly have a *secondary edge*, etc.

Finally, a point that is common to m faces is a summit. If one lets:

$$\xi_1^{(i)}, \ldots, \xi_m^{(i)} \qquad (i = 1, 2, \ldots, m + 1) \tag{4}$$

denote the $m + 1$ summits then one will have:

$$\left\{ \begin{array}{l} \xi_1^{(i)} = 1, \quad \xi_1^{(i)} = 0 \quad (i = 1, \ldots, m;\ j \neq i), \\ \qquad \xi_1^{(m+1)} = \xi_1^{(m+1)} = \cdots = \xi_m^{(m+1)} = 0 \end{array} \right. \tag{5}$$

for the tetrahedroid (3).

7. – An arbitrary point of a tetrahedroid can be represented by the formulas:

$$x_i = \frac{t_1\,\xi_i^{(1)} + t_2\,\xi_i^{(2)} + \cdots + t_{m+1}\,\xi_i^{(m+1)}}{t_1 + t_2 + \cdots + t_{m+1}} \qquad (i = 1, 2, \ldots, m), \tag{6}$$

in which $t_1, t_2, \ldots, t_{m+1}$ are $m + 1$ positive numbers. One sees that immediately for the tetrahedroid (3), and one extends that to the other tetrahedroid by remarking that the linear substitution will not change the relations (6).

8. – Moreover, as one easily convinces oneself, $m + 1$ arbitrary points whose coordinates are given by the formulas (4) can be considered to be the summits of a tetrahedroid, provided that the determinant Δ that is formed by bordering the matrix (4) with a column of non-zero units is non-zero. That tetrahedroid can be considered to be represented by the formulas (6), which are written:

$$(6') \qquad \begin{cases} x_i = t_1\,\xi_i^{(1)} + t_2\,\xi_i^{(2)} + \cdots + t_{m+1}\,\xi_i^{(m+1)}, \\ t_1 + t_2 + \cdots + t_{m+1} = 1 \end{cases}$$

this time. $t_1, t_2, \ldots, t_{m+1}$ will be called the *barycentric coordinates* [*absolute*, in the case of equations (6′) and *homogeneous*, in the case of (6)] of the point $(x_1, \ldots, x_m)$ with respect to the tetrahedroid.

m absolute barycentric coordinates can be considered to define a point. If one, in turn, considers those m quantities $t_1, \ldots, t_m$ to be Cartesian coordinates then the point that has those coordinates will describe the tetrahedroid (3) when the point $(x_1, \ldots, x_m)$ describes the given tetrahedroid.

If the determinant that was just denoted by Δ is zero then we will say that formulas (6) or (6′) define a *degenerate tetrahedroid.*

9. – More generally, the formulas:

$$(7) \qquad \gamma_i = \frac{t_1\,\xi_i^{(1)} + t_2\,\xi_i^{(2)} + \cdots + t_{m+1}\,\xi_i^{(m+1)}}{t_1 + t_2 + \cdots + t_{m+1}},$$

or, what amounts to the same thing:

$$(7') \qquad \begin{cases} y_i = t_1\,\xi_i^{(1)} + t_2\,\xi_i^{(2)} + \cdots + t_{m+1}\,\xi_i^{(m+1)}, \\ t_1 + t_2 + \cdots + t_{m+1} = 1, \end{cases}$$

in which the index i varies, no longer from 1 up to m, but from 1 up to $n \geq m$ ($t_1, \ldots, t_{m+1}$ being positive variables), will define an *m-dimensional tetrahedroid in n-dimensional space*, provided that the determinants that are deduced from the rectangular matrix:

$$\left\| \xi_1^{(i)}, \ldots, \xi_1^{(i)}, 1 \right\| \qquad (i = 1, \ldots, m + 1)$$

are not all zero ([1]). In the contrary case, one will once more be dealing with a *degenerate* tetrahedroid.

([1]) One can also say that such a tetrahedroid is deduced from an m-dimensional tetrahedroid (3) by taking the n variables in it to be linear functions of the m variables x (when at least m of those functions are independent).

The (non-degenerate) tetrahedroid (7) can also be regarded as being defined by $n - m$ first degree equations [that are obtained by eliminating the t from equations (7) or (7′)] and $m + 1$ inequalities.

A face of the m-dimensional tetrahedroid is, in that sense, an $m - 1$-dimensional tetrahedroid. The opposite face to the i^{th} summit of the tetrahedroid (5) is, in fact, represented by the formulas (6) [or (6′)], in which one sets $t_1 = 0$.

A primary edge of an m-dimensional tetrahedroid is likewise an $m - 2$-dimensional tetrahedroid, etc.

10. – The tetrahedroid (3) can be decomposed by the planes:

$$x_i = \frac{k}{p} \qquad (i = 1, 2, \ldots, m\,;\, k = 1, 2, \ldots, p)$$

into convex parts, and they, in turn, can be decomposed into tetrahedroids ([1]) whose dimensions are all less than $1 / p$; consequently, they will be as small as one desires in any sense. The latter property will then extend (by linear substitution) to an arbitrary tetrahedroid.

11. – Having said that, let $u_1, \ldots, u_m$ be m parameters such that the point that they define, and which we will call a *parametric point*, describes an m-dimensional tetrahedroid. Let $x_1, x_2, \ldots, x_n$ be some continuous functions of $u_1, \ldots, u_m$ that are $n \geq m$.

([1]) A *convex polyhedron in m-dimensional space* is the set of points in that space that verify an arbitrary number of inequalities of the first degree:

$$a_1^{(h)} x_1 + a_2^{(h)} x_2 + \cdots + a_m^{(h)} x_m + b^{(h)} \geq 0 \qquad (k = 1, 2, \ldots)\,.$$

Those inequalities are capable of being verified simultaneously, in the strict sense (i.e., excluding equality), and are such that $|x_1|, |x_2|, \ldots, |x_m|$ are necessarily bounded, thanks to those inequalities. A *face* of the polyhedron will again be obtained by replacing one of those inequalities with the corresponding equality, provided that the remaining inequalities can still be verified simultaneously in the strict sense under those conditions.

The proof of the theorem: *Any convex polyhedron in m-dimensional space is decomposable into tetrahedroids* is easy to arithmeticize. One takes an interior point O $(\alpha_1, \ldots, \alpha_m)$ and connects it to each point $(x_1, \ldots, x_m)$ of the frontier S of the polyhedron by a line segment $\left(x_i = \dfrac{\alpha_i + t\, x_i}{1 + t}\right)$, $0 \leq t \leq +\infty$.

When $(x_1, \ldots, x_m)$ describes a face, the point x_i' will describe an *m-dimensional pyramid*, and when all of those pyramids are external to each other, the set of all of them will form a given polyhedron (thanks to the fact that S is cut by an arbitrary half-line that issues from O at one and only one point).

If the theorem is assumed to have been proved for every value of m that is less than the one considered then one can decompose each face into an $m - 1$-dimensional tetrahedroid. The corresponding pyramid will be, at the same stroke, decomposed into parts that will be tetrahedroids [the coordinates x_i' can be easily expressed in the form (6) then].

We say that the point $(x_1, \ldots, x_n)$ (which will be the point *properly speaking*) describes an m-fold extended *element* in the n-dimensional space E_n .

Let ε_1 be such an element that corresponds to a tetrahedroid T_1 that is described by the parametric point $(u_1, \ldots, u_m)$; let ε_2 be an analogous elements that corresponds to a second tetrahedroid T_2 . Suppose that those two elements are linked by the intermediary of a face f_1 of T_1 and a face f_2 of T_2 in the following manner:

If the summits of f_1 (or those of f_2, resp.) have been permuted in a suitable manner, if needed, then any parametric point of f_1 will give the same position for the point $(x_1, \ldots, x_n)$ as the point of f_2 that has the same barycentric coordinates (by means of the permutation that was just spoken of) ([1]).

If that were the case then agree to regard those two parametric points, one of which is taken on f_1 and the other on f_2 , and which we shall call *coupled points*, as being identical to each other (although their coordinates u are different, in general). Consequently, the faces f_1 , f_2 will themselves be regarded as constituting only one face. The elements ε_1, ε_2 will then be called *contiguous* along the unique *common* face that corresponds to f_1 and f_2 (in the space E_n).

A third element ε_3 can be contiguous to ε_2 along a face f_2' of T_2 ; we suppose that it is different from f_2 (the contrary case will be discarded in all of what follows). The primary edge that is common to f_2 , f_2' will give an edge that is common to ε_1 , ε_2 , ε_3 whose points will be the same whether one deduces them from one or the other of the elements considered.

Similarly, several elements can have an edge of higher order in common (or a unique summit). However, one must remark that an arbitrary number of elements can further have the same primary edge in common.

12. – Now consider an arbitrary finite number of m-fold extended elements in n-dimensional space that have contiguity relations between each other of the sort that we have just defined, but in such a manner that any face of one of them can be common with (at most) one other face. We will then have a (finite) m-fold *variety* that is extended in n-dimensional space ([2]).

Conforming to the preceding, two coupled parametric points on the face that is common to two contiguous elements, or more generally, two or more coupled parametric points on an edge (of arbitrary order) that is common to two or more elements (or rather, a common summit to two or more elements) will be considered to give only a single point $(x_1, \ldots, x_n)$ of our variety. In any other situation in which the same point $(x_1, \ldots, x_n)$ in n-dimensional space is found on our variety more than one time [even though it corresponds to two different parametric points of the same element or two different

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([1]) More generally, one can assume there is an *arbitrary* perfect, continuous correspondence between the points f_1 and f_2 such that the parametric points that provide the same point x are the ones that correspond under those new conditions. One shows that this somewhat-more general case can be reduced to the one that is treated in the text.

([2]) The *varieties*, thus-defined, are the ones that we consider in what follows. The question of knowing whether one can give more general definitions to the same word is, of course, entirely reserved for later. It will not be addressed here.

parametric points of elements that are not coupled ([1])], that point will be called *double* or *multiple* according to whether it occurs twice or $m > 2$ times, resp. ([2]).

13. – The same variety can be represented in the form that was just indicated in an infinitude of ways, moreover. Indeed, we shall not consider two varieties V and V' of the preceding type to be distinct if there exists a perfect, continuous correspondence between the parametric points that generate V and the ones that generate V' such that any parametric point of V will give the same point $(x_1, \ldots, x_n)$ as its corresponding one on V'. The continuity and perfection that we just spoke of are supposed to exist only by means of the previously-made convention that we consider coupled points to be identical ([3]), so the decomposition of V' into elements it not at all supposed to correspond to that of V, moreover, and the number itself of those elements can be different.

If the two varieties have no multiple points then if that is to be true, it will suffice that each point of V is also a point of V', and conversely.

In particular, we will not change a variety by subdividing one or more of the tetrahedroids that correspond to its various elements into partial tetrahedroids, as was explained above (no. **10**).

14. – We add that one might have to consider two m-fold varieties V and V' between whose point, there exists a correspondence of the type that was mentioned in that number, but without the values of x being the same at the corresponding points, even if the integer n were the same for V and V'. Two varieties of that type are called *homeomorphic*, and the study of properties that are common to them constitutes *analysis situs*.

([1]) We then exclude the case in which an element is contiguous to itself (viz., two of its faces are coupled to each other) from what we are expressing. If that case presents itself (which is not at all impossible) then we can easily modify our definition in such a manner as to take it into account. However, one can also discard it by a convenient subdivision (see above) of the element considered, as one can assure oneself with no difficulty.

([2]) One can have $m = \infty$.

([3]) In other words, here, contiguity signifies that:

1. A parametric point of an element ε' and only one point of V' correspond to a point that belongs to an element ε (and only one) of V, so the values of u that relate to one of those points are continuous functions of the ones that relate to the other.

2. If a parametric point P that belongs to just one element ε of V corresponds to a point P' that is common to several elements ε_1', ε_2', …, ε_p' of V', and if one considers a second point Q of ε such that its corresponding Q' belongs to ε_1', for example (while still being able to be common to ε_1' and to one or more of the other element ε_2', …, ε_p'), then the values of u that correspond to Q' *in* ε_1' will be continuous functions of the ones that correspond to Q.

3. One has statements for the entirely general case in which a point that is common to several elements ε_1, …, ε_p of V corresponds to a point that is common to several elements ε_1', …, ε_p' of V' that are analogous to the statements that one makes in the case where a point that is common to several elements of V corresponds to a point that is taken from just one element of V',

15. – The changes of parametric representation that we just indicated are attached to the possibility of making a supplementary hypothesis that one generally makes on the varieties *V* that we have envisioned, and which we shall make in what follows.

No matter what *P* is taken on *V*, one assumes that among the various decompositions of *V* into elements that one can carry out in conformity with the preceding conception, there exists at least one of them in which *P* belongs to only one element.

That hypothesis is distinct from the preceding one: Certain admissible varieties that do not include it are excluded by its intervention ([1]).

16. – The variety *V* will be *a single piece* if one can pass from an arbitrary element to another likewise arbitrary element by a chain of elements, each of which is contiguous to the preceding one along a face.

It is *closed* of every face is common to two elements. In the contrary case, the set of faces, each of which belongs to only one element, constitutes the *frontier S* of *V*.

That frontier, which is an $m - 1$-times extended variety, might not be in just one piece; however, it is necessarily *closed*, as one easily assures oneself ([2]).

17. – In all of the preceding, the order in which one arranged the coordinates of the parametric point in each element (or what amounts to the same thing, the order in which one arranged the summits of the corresponding tetrahedroid) was irrelevant.

We now agree to introduce such an order, but only to the following extent:

We divide the $(m + 1)!$ permutations to which one arrives by arranging the $(m + 1)$ summits of the tetrahedroid in all possible ways into two classes according to the method that is employed in the theory of determinants. In other words, we put all of the ones for which the passage from one to the other is an alternating permutation into the same class.

([1]) For example, that is what happens for the volume of the cone that has a circular ring for its base. Such a volume is decomposable into parts that each have a perfect, continuous correspondence with a tetrahedron, but the summit of the cone must be common to at least two of those parts.

Furthermore, let the torus be represented by the equations:

$$x = \cos\psi\,(r + a\cos\varphi), \qquad y = \sin\psi\,(r + a\cos\varphi), \qquad z = a\sin\varphi,$$

in which φ and ψ are considered (mod 2π). One can deduce the three-fold extended variety in four-dimensional space:

$$x_1 = t\cos\psi\,(r + a\cos\varphi), \quad x_2 = t\sin\psi\,(r + a\cos\varphi), \quad x_3 = t\,a\sin\varphi, \qquad x_4 = t\cos\varphi,$$

in which t varies from 0 to 1. That variety is likewise decomposable into elements, but the point $x_1 = x_2 = x_3 = x_4 = 0$ is forced to belong to several of them.

([2]) If the elements *f* of *S* are faces of *V* then the faces *a* of *S* will be the primary edges of *V*. If one of them belongs to only one element of *V* then the two faces that include *a* will belong to *S* and will be contiguous. If it is common to a series of elements that are contiguous to each other (and the last of which is not contiguous to the first at the moment when *a* on the frontier) then the two exterior faces of the extreme elements will belong to *S* and will be contiguous along *a*.

We shall not choose from the permutations of the same class, but we shall choose from the two classes, and that choice will be what we call the *orientation* of the tetrahedroid.

Furthermore, there is no loss of generality in supposing that the chosen class is the first one (i.e., the one that includes the natural order), and that is what we shall do from now on.

A well-defined orientation of the tetrahedroid T will obviously correspond to a well-defined orientation for each face, which will be called the *resulting* orientation: One obtains it by arranging the m summits of that face into an order such that when they are preceded by the opposite summit, one will have a permutation of the first class of the $(m + 1)$ summits of T. We say that this orientation is the one that *results* from that of the tetrahedron for the face considered.

If two elements of a variety V are contiguous along a face then we will say that the orientations of the two elements are *concordant* if they *do not* imply the same orientation for the common face (in the sense that was just explained).

18. – After having oriented one of the elements of V in an arbitrary way, choose the orientations of the elements that are contiguous to it in such a way that they are concordant with the first one, and then proceed similarly for the elements that are contiguous to the ones whose orientation was just determined, and so on. If V is in one piece then one will arrive at an orientation for each element of V in that way.

Since there generally exist several ways of passing from one element to another by the intermediary of contiguous elements, it can happen that the orientations that are obtained will differ according to the way of making that transition that one adopts. In that case, V will be called *unilateral*. If, on the contrary, one never meets up with such a contradiction then the variety will be called *bilateral*.

From now on, we shall suppose that our variety V is bilateral and oriented in such a manner that the orientations of all elements are concordant. That will be possible in only two different ways (if V is in one piece).

19. – Finally, we call an n-fold extended variety that is supposed to have no double point in n-dimensional space a *volume* in that space.

We call an $n – 1$-fold extended variety in n-dimensional space a *surface*.

20. – The notion of multiple integral, as well as its reduction to simple integrals and the changes of variables that one can perform on it, can likewise be presented in an entirely arithmetic form. I will suppose that those properties have been established ([1]).

([1]) Their proof can present some difficulty for the most general domains. However, in what follows, the domain of integration will also be a tetrahedroid, and the volume of one tetrahedroid is interior to the other (which is a volume that can itself be decomposed into tetrahedroids).

Under those conditions, the formula for changing variables will be established for only linear changes of variables.

Moreover, one can deduce Green's formula from them. For the sake of speed, I will likewise be content to state that formula, which is easy to prove by following the classical route.

In the m-dimensional space that is the locus of the points $(x_1, \ldots, x_m)$, let V be a volume that is bounded by a frontier surface S, and let there be m continuous functions $\psi_1, \ldots, \psi_m$ of $x_1, \ldots, x_m$ in that volume that admit integrable derivatives of first order. The position of a point on an element of S will be defined by $m - 1$ coordinates $v_1, v_2, \ldots, v_m$; for example, they might be the barycentric coordinates of the parametric point with respect to the tetrahedroid that it described. We suppose that those coordinates are arranged in an order such that the determinant:

$$(8) \qquad \begin{vmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_m \\ \dfrac{\partial x_1}{\partial v_1} & \dfrac{\partial x_2}{\partial v_1} & \cdots & \dfrac{\partial x_m}{\partial v_1} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial x_1}{\partial v_{m-1}} & \dfrac{\partial x_2}{\partial v_{m-1}} & \cdots & \dfrac{\partial x_m}{\partial v_{m-1}} \end{vmatrix}$$

is always positive whenever the direction $(a_1, \ldots, a_m)$ is directed to the interior of V (at the point considered). We will then have:

$$(9) \qquad \underbrace{\int \cdots \int}_{m} \left(\frac{\partial \psi_1}{\partial x_1} + \frac{\partial \psi_2}{\partial x_2} + \cdots + \frac{\partial \psi_m}{\partial x_m} \right) dx_1 \cdots dx_m = \underbrace{\int \cdots \int}_{m-1} \begin{vmatrix} \psi_1 & \cdots & \psi_m \\ \dfrac{\partial x_1}{\partial v_1} & \cdots & \dfrac{\partial x_m}{\partial v_1} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial x_1}{\partial v_{m-1}} & \cdots & \dfrac{\partial x_m}{\partial v_{m-1}} \end{vmatrix} dv_1 \cdots dv_{m-1} .$$

In that formula, the m-tuple integral on the left-hand side is extended over the volume V. The $(m - 1)$-uple integral on the right-hand side is extended over all elements of S, in which the v are coordinates that are arranged according to the rule that was just explained in every case.

21. – In the case where V is a tetrahedroid with summits:

$$x_1^{(k)}, \ldots, x_m^{(k)} \qquad (k = 1, 2, \ldots, m + 1),$$

the rule in question can be formulated in the following manner:

Agree to say that the orientation of the tetrahedroid V *conforms to that of the system of coordinates* $x_1, \ldots, x_m$ if the determinant Δ that was considered above, namely:

$$\Delta = \begin{vmatrix} x_1^{(1)} & \cdots & x_m^{(1)} & 1 \\ x_1^{(2)} & \cdots & x_m^{(2)} & 1 \\ \cdots & \cdots & \cdots & \cdots \\ x_1^{(m+1)} & \cdots & x_m^{(m+1)} & 1 \end{vmatrix} \tag{10}$$

is positive.

In order to obey the rule in question, the coordinates v on each face, which are then supposed to be linear functions of x (for example, $m - 1$ of the absolute barycentric coordinates), must be arranged in some order such that their orientation conforms to the one that results (no. **17**) from that of V itself on the face in question. For example, on the face that is opposite to the index 1, one must have:

$$\begin{vmatrix} v_1^{(1)} & \cdots & v_{m-1}^{(1)} & 1 \\ v_1^{(2)} & \cdots & v_{m-1}^{(2)} & 1 \\ \cdots & \cdots & \cdots & \cdots \\ v_1^{(m+1)} & \cdots & v_{m-1}^{(m+1)} & 1 \end{vmatrix} > 0, \tag{10′}$$

in which $v_1^{(k)}, \ldots, v_{m-1}^{(k)}$ are the values of v at the summit with index k.

22. – A particular case of what we just said is the following one: If a parallel to the x_1-axis (i.e., a line whose equations are x_2 = const., …, x_m = const.) meets a face F of a tetrahedroid T in such a way that x_1 increases when one passes from the exterior to the interior of T upon traversing that face then its orientation (which results from that of T) will or will not conform to that of the coordinate system $x_2, \ldots, x_m$ according to whether the orientation of T itself does or does not conform to that of the coordinate system $x_1, x_2, \ldots, x_m$, resp.

The opposite situation will take place when x_1 increases when one passes from the interior to the exterior.

If the face F is opposite to the summit with index 1 (as one can suppose by means of a permutation of the first class between the summits) then that is what one will get by setting $v_1 = x_2$, $v_2 = x_3$, …, $v_{m-1} = x_m$ in the determinant (10′) and $\alpha_1 = \pm 1$, $\alpha_2 = \ldots = \alpha_m = 0$ in the determinant (8).

III. – ORDER OF A POINT WITH RESPECT TO A SURFACE

23. – Having posed those preliminaries, we can define the *order* of a point with respect to a *closed* surface S in the n-dimensional space E_n.

Let such a surface be defined as was indicated in the preceding (and presenting double points or not, moreover). Suppose, for the time being, that in each element, the coordinates $x_1, \ldots, x_n$ of the space E_n admit continuous partial derivatives of the first order with respect to the parameters $u_1, \ldots, u_{n-1}$. On the contrary, when one passes from one element to another contiguous one, the x will be subject to only being continuous.

Moreover, that will obviously imply the existence and continuity of the (first) derivatives of the x with respect to the parameters v (no. **20**) on the face that is common to those two elements.

Consider the integral then:

$$(11)\qquad I = -\underbrace{\int\cdots\int}_{n-1} \frac{1}{r^n} \begin{vmatrix} x_1 & \cdots & x_n \\ \dfrac{\partial x_1}{\partial u_1} & \cdots & \dfrac{\partial x_n}{\partial u_1} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial x_1}{\partial u_{n-1}} & \cdots & \dfrac{\partial x_n}{\partial u_{n-1}} \end{vmatrix} du_1 \cdots du_{n-1}$$

which extends over the entire surface S, and in which r denotes the positive quantity:

$$r = \sqrt{x_1^2 + \cdots + x_n^2}\,,$$

while $u_1, \ldots, u_{n-1}$ are the coordinates of the parametric point in each element, when they are arranged (no. **21**) in conformity to the orientation of the element ([1]).

That integral will make sense as long as the coordinate origin in E_n does not belong to S. One confirms immediately that it will not change in value:

a) When one multiplies $x_1, \ldots, x_n$ by the same positive quantity λ, *which is constant or varies with the u* (provided that it is continuous and differentiable under the same conditions as the ones on the x).

b) When one performs any sort of orthogonal substitution with constant coefficients and a determinant equal to + 1 on the x.

Furthermore, let S and S' be two closed surfaces that have a certain number of elements in common. Suppose that they are bilateral and oriented in such a manner that the orientation of each of the common elements in question in S is *opposite* to the one in S'. If one then forms a new surface (which is obviously closed and bilateral, like the first one) by suppressing those common elements and combining the remaining elements (which is a surface that we shall call the *resultant* of S and S') then when the integral (11) is taken over that resultant, it will be the sum of the integrals over each of the component surfaces S and S', which is an obvious consequence of the form of that integral.

24. – The fundamental property of the integral (11), when it is extended over the closed surface, is the following one:

One has:

$$(12)\qquad I = \omega \cdot K_n\,,$$

([1]) I. e., the orientation of the tetrahedroid that gave rise to them.

in which K_n is a numerical constant that depends upon only n ([1]), and ω is a (positive, zero, or negative) integer.

In order to establish that, we suppose that the proof has been made for every value of n that is less than the one that one considers.

We then set:

$$r_0 = + \sqrt{x_2^2 + \cdots + x_n^2}\,,$$

and let θ denote the angle that is defined by the two concordant equations:

$$\cos\theta = \frac{x_1}{r}, \qquad \sin\theta = \frac{r_0}{r},$$

where that angle is supposed to be found between 0 and π (which is possible, since its sine is positive). The differential of θ is coupled to that of x by the relation:

$$-\sin\theta\, d\theta = \frac{r_0^2\, dx_1 - x_1(x_2\, dx_2 + \cdots + x_n\, dx_n)}{r^2},$$

which will permit one to put the quantity under the $\int\ldots\int$ sign in I into the form:

$$(13) \qquad -\frac{\sin^{n-2}\theta}{r_0^{n-1}} \begin{vmatrix} 0 & x_2 & \cdots & x_n \\ \dfrac{\partial\theta}{\partial u_1} & \dfrac{\partial x_2}{\partial u_1} & \cdots & \dfrac{\partial x_n}{\partial u_1} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial\theta}{\partial u_{n-1}} & \dfrac{\partial x_2}{\partial u_{n-1}} & \cdots & \dfrac{\partial x_n}{\partial u_{n-1}} \end{vmatrix} = -\frac{\sin^{n-2}\theta}{r_0^{n-1}} \sum x_i \frac{\partial\theta}{\partial u_k} A_{i,k}\,,$$

in which, for example:

$$(13') \qquad A_{21} = -\frac{D(x_2, \ldots, x_n)}{D(u_2, \ldots, u_{n-1})}.$$

Now let $F(\theta) + h$ (where h is an arbitrary constant) be the primitive of $\sin^{n-2}\theta$, in other words, let:

$$F(\theta) = \sin^{n-2}\theta,$$

so the quantity (13) can be written:

$$(14) \qquad -\left(\frac{\partial\psi_1}{\partial u_1} + \frac{\partial\psi_2}{\partial u_2} + \cdots + \frac{\partial\psi_{n-1}}{\partial u_{n-1}}\right);$$

with

([1]) K_n is nothing but the areas of the sphere in n-dimensional space.

$$\psi_k = [F(q) + h]\left(\frac{x_2}{r_0^{n-1}} A_{2,k} + \frac{x_3}{r_0^{n-1}} A_{3,k} + \cdots + \frac{x_n}{r_0^{n-1}} A_{n,k}\right).$$

The set of terms in (14) that do not enter into (13), namely:

$$\sum_{i,k} \frac{\partial}{\partial u_k}\left(\frac{x_i}{r_0^{n-1}} A_{ik}\right),$$

is, in fact, zero (the index i varies from 2 to n, while k varies from 1 to $n - 1$).

That will result from the identity [which is easy to verify ([1])] for $n - 1$ arbitrary (differentiable) functions $X_2, X_3, \ldots, X_n$ of $x_2, \ldots, x_n$:

$$\frac{\partial}{\partial u_1}(A_{2,1} X_2 + A_{3,1} X_3 + \ldots + A_{n,1} X_n) + \frac{\partial}{\partial u_2}(A_{2,2} X_2 + A_{3,2} X_3 + \ldots)$$

$$+ \ldots + \frac{\partial}{\partial u_{n-1}}(A_{2,n-1} X_2 + \ldots)$$

$$= \left(\frac{\partial X_2}{\partial x_2} + \frac{\partial X_3}{\partial x_3} + \cdots + \frac{\partial X_n}{\partial x_n}\right)\frac{D(x_2, \ldots, x_n)}{D(u_2, \ldots, u_{n-1})},$$

when one sets $X_i = x_i / r_0^{n-1}$ and observes that one has:

$$\frac{\partial}{\partial x_2}\left(\frac{x_2}{r_0^{n-1}}\right) + \ldots + \frac{\partial}{\partial x_n}\left(\frac{x_n}{r_0^{n-1}}\right) = 0. \tag{15}$$

The form (14) that is given to the integration element permits one to apply Green's formula, as long as the functions ψ are continuous: With that reservation, and upon transforming the result by the known theorem on the multiplication of determinants, we will have:

$$I = -\sum \underbrace{\int \cdots \int} \frac{F(\theta) + h}{r_0^{n-1}} \begin{vmatrix} x_2 & x_3 & \cdots & x_n \\ \frac{\partial x_2}{\partial v_1} & \frac{\partial x_3}{\partial v_1} & \cdots & \frac{\partial x_n}{\partial v_1} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial x_2}{\partial v_{n-2}} & \frac{\partial x_3}{\partial v_{n-2}} & \cdots & \frac{\partial x_n}{\partial v_{n-2}} \end{vmatrix} dv_1\, dv_2 \ldots dv_{n-2}\,. \tag{16}$$

([1]) That identity, in which the $A_{i,k}$ are the quantities that are defined by (13), (13), is well-known for the case of constant $X_2, \ldots, X_n$: It enters into the theory of the *multiplier* (see Jordan, Goursat, etc., *Traités d'Analyse*).

The summation $\sum$ here relates to all faces of all the elements of S. On each of those faces, the coordinates v must be defined and arranged as was explained in nos. **20**, **21**, with the orientation of each face being the one that results from the orientation of the element from which it was taken.

Since the surface S is closed, each face will then occur twice, and (from the preceding hypotheses and conventions) with the opposite orientations in the two cases.

Therefore, the two terms that relate to that face in the expression (16) must *cancel* if at least the number h has the same value on one part and the other.

25. – However, all of that is subordinate to the legitimacy of formula (16). Up to now, that was established only when r_0 never became zero; i.e., when the line:

$$x_2 = \ldots = x_n = 0 \tag{17}$$

did not cut S.

In that case, when one sets $h = 0$ in all elements, one will see that I is *zero.*

More generally, *the integral* (11) *will be zero when one can draw an arbitrary line D through the origin that has no point in common with S.*

That is because we can reduce D to the x_1-*axis* by means of an orthogonal transformation that does not change I ; i.e., to something that is represented by equations (17).

25 (cont.). – We abandon the first case, and then we shall nonetheless begin to make certain particular hypotheses.

We first suppose that no line that issues from the origin cuts the same element of S at points that are located on one side of the origin and the other. That first hypothesis does not diminish the generality. In order to verify that, it will suffice to make the elements small enough (by subdivision, if that exists) for the distance between two points on the same element to always be less than twice the minimum distance from the origin to S, which is possible by virtue of the continuity of the x.

We then suppose the existence of a line D whose intersection with the surface satisfies the following conditions:

D does not meet any face.

Any element that is met by D is such that the x, y are linear functions of the u (which are generally non-homogeneous). Furthermore, it is met by D only a single point ([1]).

Such an element is obviously itself a tetrahedroid (and not only in its parametric representation), which can nonetheless be degenerate.

The conditions that were just enumerated will be verifiable, in particular, if the x are linear in the u in *any* element of S, in which case, S will be called a closed *polyhedral* surface ([1]).

([1]) Those suppositions are hardly essential. Their only purpose is to avoid certain complications of the general case.

One can even subject D to being as close as one desires to an arbitrarily-given line D_0 that passes through the origin.

26. – We assume, as we just did, that D is nothing but the x_1-axis, which is the locus of points for which r_0 is zero. θ will be equal to zero or π on that line according to the sign of x_1 .

From that, the elements of S can be divided into three categories:

1. The ones on which θ does not become equal to 0 or π. The corresponding value of h will be taken to be equal to zero.

2. The ones where one has (once and only once) $\theta = 0$ (in such a way that one cannot have $\theta = \pi$). We then take $h = - F(0)$.

3. The ones where one has (once and only once) $\theta = \pi$. We then take $h = - F(\pi)$.

The formula (16) will be valid even when S contains elements of the second or third category by means of that choice of h, although the functions ψ would cease to be continuous in those two cases.

In order to prove that, suppose, for example, that all of the elements belong to the first category, except for just one ε_1 , which belongs to the second. ε_1 will then contain a point P_1 such that:

$$x_1 = \ldots = x_n = 0, \qquad x_1 > 0.$$

Let $(u_1^0, \ldots, u_{n-1}^0)$ be the parametric point that corresponds to P_1 , which is interior to the tetrahedroid T_1 that gave rise to ε_1 . The applicability of Green's formula will be assured when one subtracts the interior of a tetrahedroid τ_1 from T_1 that likewise contains the parametric point $(u_1^0, \ldots)$ (in its interior). One can then write down the formula (16) if one assumes that:

1. The integral I does not extend over the entire surface S, but only the (open) surface S' that one deduces from S by suppressing the part of ε_1 that corresponds to τ_1 .

([1]) If the x are linear in the u then the fact that D meets a face or the fact that it has more than one point in common with an element (and consequently, and infinitude of common points) is expressed by a linear equation that is not an identity (or sometimes several equations) and has the form:

$$A_1\,\xi_1 + \ldots + A_n\,\xi_n = 0,$$

in which $\xi_1, \ldots, \xi_n$ are the coordinates of an arbitrary point on the line, and the A are non-zero constants.

One can always choose the ξ (even in the neighborhood of the given quantities) in such a manner that they do not verify any of the relations that are written in that way.

2. One must add terms to the $(n - 2)$-tuple integral on the right-hand side that correspond to the frontier of τ_1 (each of those terms appears only once, contrary to the ones that correspond to the faces of S).

Let i be the set of terms in question. I must show that i will tend to zero when all of the dimensions of the tetrahedroid τ_1 tend to zero without ceasing to include the point P_1.

Indeed, we remark that an arbitrary half-line that issues from the parametric point $(u_1^0, \ldots)$ cuts the frontier of each of the tetrahedroids T_1 and τ_1 at a single point. In the space E_n, those points will give the points of ε_1 that are in a straight line with P_1, and since the last $n - 1$ coordinates of P_1 are zero, they are all deduced by multiplying each of the coordinates in question by the same positive factor. From what we saw before, it will then result that if one ignores the factor $[F(\theta) + h]$ then the corresponding integral elements on the frontier of T_1 and on that of τ_1 will be identical, because (always without the factor $[F(\theta) + h]$) the integral on the right-hand side has the same form as I when one simply changes n into $n - 1$.

Now, the integral that relates to the frontier of T_1 will make sense even when one replaces each of its elements with its absolute value. Let I_0 be the value that is obtained under those conditions. One will have:

$$| i | < I_0\, \alpha,$$

in which α denotes the maximum of $[F(\theta) + h]$ – i.e., of $[F(\theta) - F(0)]$ – on the frontier of τ_1. Now, that maximum will tend to zero with the dimensions of τ_1, since θ is equal to 0 at P_1.

Therefore, the formula (16) will remain valid under those conditions by means of the choice of h that was indicated, and that result will be likewise established in the case of an arbitrary number of elements of the second or third category. It will suffice to perform the construction that we just made on ε_1 for each of them.

27. – On the other hand, we have seen that only the terms in h will persist, while the other will cancel on the closed surface S.

Hence, one will get:

$$I = J_1\, F(0) + J_2\, F(\pi), \tag{18}$$

in which J_1 and J_2 denote the integrals:

$$\underbrace{\int \cdots \int}_{n-2} \frac{1}{r_0^{n-1}} \begin{vmatrix} x_1 & \cdots & x_n \\ \dfrac{\partial x_1}{\partial v_1} & \cdots & \dfrac{\partial x_n}{\partial v_1} \\ \cdots & \cdots & \cdots \\ \dfrac{\partial x_1}{\partial v_{n-1}} & \cdots & \dfrac{\partial x_n}{\partial v_{n-1}} \end{vmatrix} dv_1\, dv_2 \cdots dv_{n-1}, \tag{11′}$$

when they are extended, in the first case, over the set of frontiers of the elements of the second category, and in the second, over the set of frontiers of elements of the third category, respectively. One will have:

$$J_1 + J_2 = 0,$$

moreover, as one sees upon noting that the right-hand side of (18) must be independent of the additive constant that remains arbitrary in F. One can then write:

$$I = -J_1 [F(\pi) - F(0)] = -J_1 \int_0^\pi \sin^{n-2}\theta \, d\theta .$$

Now, as we have pointed out already, the integral (11′) has the same form as I when one changes n into $n - 1$, and that integral is extended over a system of closed surfaces in the space E_{n-1}, namely, the ones that one obtains by *projecting* ([1]) onto E_{n-1} the frontiers of the elements of the second category of S, which are nothing but the frontiers of the (n – 2)-dimensional tetrahedroid, moreover. By virtue of our hypotheses, it will result that:

$$J_1 = K_{n-1} \sum \omega$$

in which the ω are integers that correspond to the various closed surfaces that were just in question, respectively.

That will indeed give a value to I that has the form (12). It will suffice to define K_n by the relation:

$$K_n = K_{n-1} \int_0^\pi \sin^{n-2}\theta \, d\theta .$$

Once the numbers K_n that are defined by that recurrence relation (with $K_1 = 2$), our conclusion will have been proved by the preceding, since for $n = 2$, the expression (11) will reduce to:

$$-\int \frac{1}{x_1^2 + x_2^2}\left(x_2 \frac{\partial x_1}{\partial u} - x_1 \frac{\partial x_2}{\partial u} \right) du = \int d\left(\arctan \frac{x_2}{x_1} \right);$$

i.e., to the *variation of the argument* that was considered previously. Consequently, that will verify our proposition.

28. – The integer ω that appears in the formula (12) is called the *order* of the origin with respect to S.

The order of an arbitrary point ($a_1, \ldots, a_n$) (that is not located on S) is obtained by replacing $x_1, \ldots, x_n$ with $x_1 - a_1, \ldots, x_n - a_n$ in formula (11) .

That order is zero, by virtue of the preceding considerations if one can draw at least one *half-line* through the point envisioned (and no longer even an entire line) that does not meet S.

([1]) In analytical language, "projecting a figure in the space E_n onto the space E_{n-1}" means simply that one drops one of the coordinates (here, x_1).

If S is the frontier of the tetrahedroid to which the origin is interior then the half-line that is defined by the equation $\theta = \pi$ will cut S at one and only one point. The order of the origin will then be + 1 or – 1, according to whether the determinant (10) is positive or negative, resp.

One can verify that with no difficulty in the case of $n = 2$ (where S is the frontier of a triangle). It is easy to pass from that value of n to other ones by recurrence by applying what was said in no. **22**, since, as we have seen, J_1 is always composed of orders that relate some $(n - 1)$-dimensional tetrahedroids (here, there is just one tetrahedroid).

Moreover, one obviously deduces an expression for the order that relates to an arbitrary closed polyhedral surface from that. One has:

$$\omega = N_1 - N_2 , \tag{19}$$

in which $N_1 + N_2$ is the total number of points where a half-line that issues from the point $(a_1 , \ldots, a_n)$ cuts S, and any of those point will be counted in N_1 or N_2 according to whether the n summits of the element that contains that point, after they have been arranged in the order that results from the orientation of that element and is preceded by a summit that is located at the origin, does nor does not give a tetrahedroid whose orientation conforms to that of the coordinate system.

29. – If the x are simply differentiable with respect to u in each element without verifying the particular hypotheses that we just made then the value of the integral I will once more have the form (12).

As we just saw, that fact is not entirely indispensible to what follows, so we shall simply summarize the proof by saying that any portion T of S that is close ([1]) to D can be replaced with another T' that is slightly different from the first one, and is such that in any element of T' that is met by D, the x are linear functions ([2]) of the u. The integral I, when taken on the surface thus-modified S', will have the same value that it has on S. Indeed, the set of T and T' forms a closed surface σ such that S can be considered to be the

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([1]) To be precise: One can introduce any element ε into T that has a point in common with D and any element ε' that is contiguous to an element ε.

([2]) Replace each element ε or ε' with the tetrahedroid e or e' that has the same summit. In other words, replace the coordinates x of such an element with the quantities x' that are defined by formulas (20) (see below, no. **30**): We will get a new open surface T_1'. On the other hand, join each point on the frontier of T (which is composed of the faces F that are borrowed from the elements ε') to the corresponding point of the frontier of T_1'. The line segments thus-traced will form a new portion of the surface T_2'. T' will be the union of T_1' and T_2'.

If the subdivision of the elements is pushed much further then x_1 will not be annulled on either T or T'. Moreover, let ρ be the minimum of r_0 on the frontier of T. If (after forming T, but before deducing T') one subdivides the elements ε' in such a manner that the distance between any two points in each face F of the frontier is less than ρ then the part T_2' will have no point in common with D.

As for the condition that D must not meet the faces, as we saw, it will be realized by giving a very small displacement to D, if needed.

resultant of S' and the surface or surfaces σ, and the integral that relates to σ will be zero, because σ has no point in common with any line D' that has $x_1 = 0$ for one of its equations.

S' satisfies the conditions by means of which the relation (12) was proved just now. That relation is also true for S then.

30. – However, it is not necessary to insist upon evaluating the order in the case of differentiable coordinates. Indeed, we can pass directly from the case that was the subject of no. **25** (cont.) to the completely general case in which the x are arbitrary continuous functions of the parameters.

In order to do that, it will suffice to once more replace the functions $x_1, \ldots, x_n$ with some other ones $x'_1, \ldots, x'_n$ that are close to them and are linear functions of the u in each of the elements (after a convenient subdivision of them).

In order to define those x', we suppose that the elements are made sufficiently small by subdivision that in each of them, the gap between each of the continuous functions x will be smaller than a certain number η.

Having done that, we replace each element by the tetrahedroid (which is or is not degenerate) that has the same summits; in other words, if:

$$x_1^{(k)}, \ldots, x_n^{(k)} \qquad (k = 1, 2, \ldots, n)$$

denote the coordinates of the n summits of an element then we will take:

$$x'_i = t_1\, x_i^{(1)} + t_2\, x_i^{(2)} + \cdots + t_n\, x_i^{(n)} \qquad (i = 1, 2, \ldots, n) \tag{20}$$

for an arbitrary point of the element, when one calls the (absolute) barycentric coordinates of the corresponding parametric point with respect to the tetrahedroid that it describes $t_1, t_2, \ldots, t_n$.

It is clear (by virtue of the relation $t_1 + \ldots + t_n = 1$) that x'_i is found between the smallest and largest of the quantities $x_i^{(1)}, \ldots, x_i^{(n)}$.

It obviously results from this that the absolute value of the difference $x_i - x'_i$ is less than 2η in the element considered.

In order to calculate the order, we substitute the x'_i for the x_i, which will bring us back to the case that was treated previously. In other words, we substitute an *approximating polyhedron* for S, and we define the order with respect to S to be the order with respect to that polyhedron ([1]).

In order to legitimize that definition, we shall show that *the value of ω is the same no matter how one chooses the functions x' that are substituted for the x,* provided that η is sufficiently small.

([1]) For example, in ordinary space, one replaces S with an inscribed polyhedron with sufficiently-small faces.

More precisely, it will suffice that η is less than the smallest reduced distance (see no. **271** in the text) from the point $(a_1, \ldots, a_n)$ to S.

Indeed, consider a second approximating polyhedron – in other words, a second system of functions x_1'', x_2'', …, x_n'' that are close to the x and are defined in a manner that is analogous to the x'. Suppose that the absolute values of the differences $x_i'' - x_i$ are likewise all less than 2η.

The second approximating polyhedron will be deduced by a decomposition of the elements of S that is different from the one that is provided by the first one. For example, let τ_1', … be the tetrahedroids into which T_1 was decomposed in the first case, and let τ_1'', … be the ones into which it was decomposed in the second case. One can find a third decomposition into tetrahedroids t that is a subdivision with an axis in common with the other of the first two ([1]).

In each tetrahedroid τ'', the x_i', like the x_i'', are linear functions of u, and consequently, the same thing will be true for each of the quantities:

$$\xi_i = \frac{x_i' + \mu\, x_i''}{1+\mu},$$

in which μ is an arbitrary positive parameter.

The locus of the point $(\xi_1, \xi_2, \ldots, \xi_n)$ is a new closed polyhedral surface Σ that depends upon the parameter u.

Due to the hypothesis that was made on η, Σ will not pass through the point $(a_1, \ldots, a_n)$ for any positive value of μ.

Indeed, x_i is found between x_i' and x_i'' (for $m > 0$), and consequently, between $x_i - 2\eta$ and $x_i + 2\eta$. Now, by hypothesis, at least one of the quantities $|x_i - a_i|$ is greater than 2η at each point of S.

Moreover, the order of our point with respect to Σ is defined for every positive value of μ, and will vary continuously with μ [as one sees from the expression (11)].

Since that order is, in essence, a whole number, it will necessarily remain constant. It will then have the same value for $\mu = 0$ and $\mu = \infty$; i.e., for the two polyhedral surfaces envisioned.

It likewise results from this that the result obtained will be the same for another closed surface whose points correspond to those of the first one in such a manner that the reduced distance between two corresponding points will constantly be less than η' (η' denotes no particular quantity that is found between 0 and η, excluding the limits).

One then sees that the order ω is *continuous of order zero* with the respect to the expressions for x as functions of the u; i.e., that it will be altered very little (and even not at all) when one alters the x by quantities that are everywhere very small.

([1]) If the two tetrahedroids $-\tau_1'$, τ_2', for example – have a common region (i.e., they have some non-frontier points in common) then that region will be a convex polyhedron (cf., page 6, note 2) that one can decompose into tetrahedroids, and one proceeds similarly for each tetrahedroid τ', combined with each tetrahedroid τ''.

The expression (11) exhibits only continuity of order *one*; in other words, in order to affirm that ω is altered very little, it will seem necessary to assure that not only the functions x, but also their derivatives with respect to u, experience very small variations.

That is true thanks to the continuity of order 0 that we could define for ω by means of an approximating polyhedral surface ([1]).

31. – ω would not change if we were to employ another mode of parametric representation for S either, since its definition by formula (19), when applied to a polyhedral surface that is close to S does not depend upon that representation.

In a word, ω will indeed be a completely well-defined quantity when one is given the surface S and the point $(a_1, a_2, \ldots, a_n)$ that is not located on that surface.

It will remain constant while the point in question varies continually without crossing the surface.

One can easily assure, in a general manner, that it possesses the properties that we confirmed in the case where the x are differentiable even when they are merely continuous, and those are the properties that we must invoke in what follows.

31 (cont.). – We add that the combination of formula (19) with the results of no. **30** will permit us to evaluate the order in the simplest case.

That order is obviously equal to ± 1 when S is a convex polyhedron and $(a_1, a_2, \ldots, a_n)$ is an interior point [since one of the numbers N_1, N_2 in formula (19) would then be equal to 1 and the other to zero].

It is likewise equal to ± 1 for the n-dimensional sphere whose center is $(a_1, a_2, \ldots, a_n)$, as one will see upon replacing a portion of it with a portion of the plane.

IV. – THE KRONECKER INDEX

32. – One now passes from the definition of the order of a point with respect to a closed surface to that of the index of a system of functions on that surface.

Once more, let the surface S be the locus of points $(x_1, \ldots, x_n)$ in n-dimensional space. On the other hand, let $f_1, f_2, \ldots, f_n$ be a system of n continuous functions of $x_1, \ldots, x_n$ that have needed to be defined only on S, up to now.

The *index* of the system of functions $f_1, \ldots, f_n$ is, by definition, the order of the origin of the coordinates with respect to the surface that is generated by the points whose coordinates are $f_1, \ldots, f_n$.

From the foregoing, that definition supposes that $f_1, \ldots, f_n$ are not simultaneously zero at any point of S.

$f_1, f_2, \ldots, f_n$ can be regarded as the components of a vector, and when one indicates their values at each point of S, one will define a continuous *vectorial distribution* that is attached to that surface.

([1]) The area of a curved surface in ordinary space is continuous of order 1 only with respect to the form of the surface. One cannot define it by an approximating polyhedron without special precautions, either.

Such a distribution will have a well-defined index on a closed surface as long as the vector is not zero at any point of the surface.

33. – The fundamental property of the index, thus-defined, relates to the case in which S is the frontier of a volume V in n-dimensional space.

This time, suppose that the functions $f_1, \ldots, f_n$ are defined and continuous over all of that volume, and not only on S.

Finally, suppose that the equations:

$$f_1 = 0, \ldots, f_n = 0 \tag{21}$$

have no common solution inside of V.

If that were true then *the index of the system of functions considered on S would be zero.*

In order to see that, we begin by pointing out that if we let F denote the largest of the n absolute values $|f_1|, |f_2|, \ldots, |f_n|$ at an arbitrary point of V then the quantity F will be itself a continuous function that is never zero, and in turn, will admit a positive minimum g.

Then subdivide the elements of V in such a manner that the oscillation of each of the functions f in each of them is less than g' ($g' < g$). Now let ε be one of those elements, and let s be its frontier. One can make each point of the latter correspond to a new point whose coordinates are $f_1, \ldots, f_n$. Let σ be the closed surface that is described by that second point. One can always make a half-line that does not meet σ start from the coordinate origin, because if f_1, for example, is the largest in absolute value of the n functions at a well-defined point of ε and it is positive (which will imply that $f_1 > g$) then (since $g' < g$) the function f_1 will be likewise positive at any point of ε or s, in such a way that the half-line $f_2 = \ldots = f_n = 0$, $f_1 < 0$ will possess the stated property.

It will result from this that the index of the system $f_1, \ldots, f_n$ is zero on s, since that is true on each frontier of each element of V, and S can be considered to be the resultant (no. **23**) of all those frontiers, so the theorem is proved.

34. – On the contrary, the index will no longer be necessarily zero when equations (21) have common solutions inside of V.

By contrast, when one supposes that there are a finite number of them, to fix ideas, and one lets $P_1, \ldots, P_p$ denote the corresponding points of V, the index that relates to S will be the algebraic sum of the partial indices, each of which is characteristic of one of the solution points.

Indeed, if one decomposes V into elements in such a manner that P_1, for example, is in the interior (and not on the frontier) of an element ε_1 that does not contain any other points P, and that the other points are similarly strictly interior to $p - 1$ other distinct elements $\varepsilon_1, \ldots, \varepsilon_p$ then the index with respect to S will be the sum of the indices with respect to the frontiers $s_1, s_2, \ldots, s_p$ of the $\varepsilon_1, \ldots, \varepsilon_p$ (all other elements will give zero indices, from the foregoing). It is obvious, moreover, from this that the index with respect to s_1, for example, will be independent of the form and dimensions of the element

ε_1 under the single condition that it must contain the solution point P_1 in its interior, but none of the other ones (and in particular, no matter how small ε_1 is). That index will then depend upon only the point P_1 and the way that the functions f behave in the neighborhood of that point.

If the f have derivatives at P_1, and their functional determinant is non-zero, then one can show ([1]) that the partial index will be equal to + 1 or – 1 according to the sign of that functional determinant.

However, we reach the following conclusion, above all: *If ω denotes the index of the system $f_1, \ldots, f_n$ on S then the inequality $\omega \neq 0$ will imply the existence of a common solution to equations* (21) *in V.*

35. – The calculation of the index of a system of functions on a surface is often facilitated by the following theorem in a remarkable way:

Poincaré-Bohl theorem:

Let two systems of n functions $f_1, f_2, \ldots, f_n$; $g_1, g_2, \ldots, g_n$ be given on the same closed surface S, such that the elements of one system are not simultaneously zero at any point on S.

If those two systems do not have the same index then there will exist at least one point on S such that one has:

$$\frac{f_1}{g_1} = \frac{f_2}{g_2} = \ldots = \frac{f_n}{g_n} < 0, \tag{22}$$

and if the ratios of their indices are not $(-1)^n$ *then there will exist at least one point such that one has:*

$$\frac{f_1}{g_1} = \frac{f_2}{g_2} = \ldots = \frac{f_n}{g_n} > 0\,. \tag{22′}$$

That theorem was established by Poincaré in 1886 ([2]) and obtained once more independently by Bohl ([3]) in 1904.

In order to prove it, it will suffice to consider the two auxiliary systems:

$$\frac{f_1 + \mu\, g_1}{1+\mu}, \qquad \frac{f_2 + \mu\, g_2}{1+\mu}, \qquad \ldots, \qquad \frac{f_n + \mu\, g_n}{1+\mu} \tag{23}$$

and

([1]) See Picard, *Traité d'analyse*, volume II.

([2]) POINCARÉ – "Sur les courbes définies par les équations différentielles," J. de Math. (4) **2** (1886), pp. 177.

([3]) BOHL, J. für Math. **127** (1904). – The statements that the two cited authors gave differ from each other, as well the present text, in that they specialized the choice of the functions g (and each, in a different way). However, their arguments (which are likewise different, moreover) will extend to the case in which those functions are taken arbitrarily. The proof that we give here is that of Poincaré.

$$(23') \qquad \frac{f_1 - \mu g_1}{1+\mu}, \qquad \frac{f_2 - \mu g_2}{1+\mu}, \qquad \ldots, \qquad \frac{f_n - \mu g_n}{1+\mu},$$

in which μ is a positive parameter.

If the system of relations (22) is not verified at any point of S then the system of functions (23) will have an index for any positive value of μ, and from an argument that was already presented before, that index will be constant for any μ. Upon successively setting $\mu = 0, +\infty$, one will see that $(f_1, f_2, \ldots, f_n)$ has the same index as $(g_1, g_2, \ldots, g_n)$.

On the other hand, if the relations (22′) are not verified simultaneously at any point of S then it will be the index of the system (23′) that remains constant for any value of μ, in such a way that $f_1, f_2, \ldots, f_n$ will give the same index as $-g_1, -g_2, \ldots, -g_n$. It will then result [as one will see from formula (11) by inspection] that the ratio of the indices of f_1, $f_2, \ldots, f_n$ and $g_1, g_2, \ldots, g_n$ will be $(-1)^n$.

The theorem is thus proved.

If the two indices considered are not equal in absolute value then it will result from the relations (22), (22′) that the quantity:

$$f_1 g_1 + f_2 g_2 + \ldots + f_n g_n$$

cannot have a constant sign on S.

V. – APPLICATIONS.

36. The Schoenflies theorem. – The proof that was just given of Jordan's theorem (see nos. **306** and **307** in the text and the beginning of this note) tells us that not only does a closed curve with no double point divide the plane into an exterior region and an interior one, but also that the order of a point with respect to the curve is equal to 0 in the former region and ± 1 in the latter.

As one sees, that will suffice to prove an important theorem by Schoenflies, which is stated thus:

Let:

$$(24) \qquad X = f(x, y), \qquad Y = g(x, y)$$

be two functions of x and y that are indeed different and continuous on the interior and circumference of the circle:

$$(25) \qquad x^2 + y^2 \leq 1.$$

Suppose that the equalities:

$$f(x, y) = f(x', y'), \qquad g(x, y) = g(x', y'),$$

cannot be verified simultaneously in the circle in question unless one has $x = x'$, $y = y'$.

Let C be the curve that is described by the point (X, Y) when (x, y) describes the circumference of the circle.

One then has:

The equations (24), *in which X, Y are considered to be given, will have a solution* (which is unique, from the hypothesis) *whenever the point* (X, Y) *is taken inside of C.*

That theorem will result immediately from some properties of the order and index that were established in the preceding discussion.

Indeed, a point (X, Y) that is interior to C is characterized by the fact that its order with respect to C is equal to ± 1.

That amounts to saying that the index of the system $(f(x, y) - X, g(x, y) - Y)$ has that same value, and is consequently non-zero.

Hence, the equations (24) have a common solution in the interior of the circle (25).

Q. E. D.

37. – As I said before, I will now address Jordan's theorem in a space with more than two dimensions. I will even suppose that the theorem has been proved, along with a complement that is natural to give it, from the preceding considerations. I shall not only assume that a closed surface with no double point always divides n-dimensional space into an exterior region and an interior region, but also that the order of any interior point will be equal to ± 1.

Under those conditions, *the Schoenflies theorem is proved for an arbitrary number of variables.*

In other words, let:

$$(24') \qquad \begin{cases} f_1(x_1, \ldots, x_n) = X_1, \\ f_2(x_1, \ldots, x_n) = X_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f_n(x_1, \ldots, x_n) = X_n \end{cases}$$

be a system of relations that define the quantities X as continuous functions of the x in a certain volume v in n-dimensional space. I suppose, to simplify (although this hypothesis is not essential), that this volume is bounded by a surface with just one piece.

Suppose that one cannot have:

$$(26) \qquad f_i(x_1, \ldots, x_n) = f_i(x'_1, \ldots, x'_n) \qquad (i = 1, 2, \ldots, n)$$

in the volume in question, unless one has $x_1 = x'_1$, $x_2 = x'_2$, …, $x_n = x'_n$.

Now let S be the surface (which is necessarily closed and has no double point) that is described by the point X when the point x describes the frontier s of v. *The equations* (24′) *have a solution whenever the point X is in the interior of S.*

The proof that was given for the case of two variables in fact persists without modification once one assumes the propositions that were cited above.

One can even point out that the proof of the existence of at least one solution supposes only that the relations (26) are impossible for two distinct points on the *frontier S.*

One sees from what was just said that the Schoenflies theorem is not fundamentally distinct from that of Jordan.

38. – The importance of that theorem is very great, moreover, which will emerge when one compares that theorem to the classical results from the theory of equations of degree one.

Consider a system of linear equations for which the number of unknowns is equal to the number of the equations.

Although such a system will generally admit one and only one solution, it can happen that such a solution is impossible or even indeterminate.

However, the condition for the system to admit one solution, no matter what values are attributed to the right-hand sides, is not distinct from the condition that expresses the idea that it cannot have more than one.

The Schoenflies theorem tells us that the latter condition again implies the former one, at least on a conveniently-chosen portion of space when one envisions completely-general equations of the form (24′) instead of linear equations.

39. – The following remark results immediately from the Schoenflies theorem.

Suppose that a perfect, continuous correspondence exists between two volumes V, V' in n-dimensional space (each of which is bounded by a surface with only one piece). I say that if that were true then *interior point of V will correspond to interior points of V′, and frontier points of V will correspond to frontier points of V′.*

Indeed, suppose that the point P, which is interior to V, corresponds to a point P' that is located on the frontier S' of V'. If that were true then the frontier S of V would correspond to a surface S' (which is closed and has no double point) to which P would not belong. From Jordan's theorem, S' will bound a volume V' that is completely interior to V and to which P' will consequently be exterior ([1]), in such a way that the points P that are interior to S will not correspond to any point P' that is interior to S'.

That therefore contradicts the Schoenflies theorem, and our conclusion is proved.

40. – The preceding considerations likewise permit us to prove *Brouwer's theorem* for the volume of a sphere any dimension.

The volume of the n-dimensional sphere that has its center at the origin and a unit radius is the set V of points that verify the inequality:

([1]) The fact that a point P' of S' is necessarily exterior to any surface S that is interior to V' results from the fact that one can join P' to infinity by a continuous path that has no point in common with V' or S' besides P' (which must, in turn, be considered to belong to it, from Jordan's theorem).

(27) $$x_1^2 + x_2^2 + \cdots + x_n^2 \leq A .$$

The theorem that we shall prove is the following one:

Any perfect, continuous transformation of the volume V into itself will leave at least one point invariant, either in its interior or on its frontier.

Indeed, suppose that $x_1, x_2, \ldots, x_n$; $x_1', x_2', \ldots, x_n'$ are the coordinates of two corresponding points. Consider the n functions:

(28) $$x_1' - x_1,\ x_2' - x_2,\ \ldots,\ x_n' - x_n,$$

while first supposing that the point $(x_1, \ldots, x_n)$ describes the frontier of the sphere.

If those n functions are annulled simultaneously then the theorem will be proved. Otherwise, the system of functions that we just wrote down will have the same index as the system $(x_1, \ldots, x_n)$, because we always has ([1]):

$$x_1(x_1 - x_1') + x_2(x_2 - x_2') + \cdots + x_n(x_n - x_n') > 0.$$

Since the latter index (i.e., the order of the center with respect to the surface of our sphere) is non-zero, there has to exist an interior point where the functions (28) are simultaneously zero.

Q. E. D.

41. – Note further that the consideration of the index permits one to generalize a classical notion that relates to the *sense* of areas to an arbitrary bijective and continuous transformation.

When the f in equations (24′) have derivatives, one knows that such a transformation will preserve the senses of areas if its functional determinant is positive and will change them if it is negative.

While confining ourselves to equations (24), let (x, y) be a point on a closed curve c in the plane that has no double point and bounds the area s, so the points that are interior to s will have an order equal to + 1 or – 1 with respect to c.

c will correspond [if we suppose that the correspondence that is defined by equations (24) is perfect and continuous] to a closed curve C with no double point that bounds an area S. The points interior to it will have order ± 1 with respect to C.

One can say that a curve like c or C is described in the *direct* sense if the order of the interior points with respect to that curve is + 1 and in the *retrograde* sense if that order is equal to – 1.

([1]) By virtue of Lagrange's well-known identity, that inequality is a consequence of the equations:

$$x_1^2 + x_2^2 + \cdots + x_n^2 = x_1'^2 + x_2'^2 + \cdots + x_n'^2 = 1.$$

Now, the transformation (24) can preserve or change the sense thus-defined according to the case. However, that will depend upon only the transformation itself (at the moment when it is supposed to be perfect) and not upon the choice of the particular closed curve *c*.

In order to see that, it will suffice to deform the latter in a continuous manner.

In the case of the plane, one easily passes to an oriented, bilateral surface in the way that was explained above.

42. – I will now point out, in summary form, another series of applications of the same principle ([1]).

Consider a variety V that is closed and m-fold extended in n-dimensional space ($n > m$). That variety will be supposed to admit a well-defined *tangent plane* at each of its points that varies continuously with the position of the point.

The simplest case (which is, above all, the one that we have in mind to fix ideas) is that of the surface of a sphere in ordinary space.

Imagine a *tangent vectorial distribution* on V; i.e., make each point of V correspond to a vector that is tangent to V at that point. The components of that vector will be supposed to vary continuously, but not necessarily admit partial derivatives with respect to the coordinates of their origin.

Above all, only its direction will be relevant. It is essential to note that the direction will be indeterminate when and only when the vector is zero.

The direction parameters can be considered to be proportional to a system of differentials $dx_1, \ldots, dx_n$ of the x that correspond (in each element) to well-defined differentials $du_1, \ldots, du_m$ of the parameters u (while supposing that the x are differentiable with respect to the u, which is legitimate, from the hypothesis that was made on V).

Those differentials $du_1, \ldots, du_m$, or rather, of other (finite) quantities $\lambda_1, \ldots, \lambda_m$ that are proportional to them with a *positive* proportionality factor, can, in turn, be regarded as defining a direction vector in the space that is the locus of parametric points. That direction vector will be indeterminate only if the direction is. We assume that this is true only at a finite number of points, and we suppose that the decomposition of V into elements has been done in such a way that those points are not on any face.

Under those conditions, the corresponding system ($\lambda_1, \ldots, \lambda_m$) will admit a well-defined index. *Consider the sum σ of all those indices.*

One proves ([2]) that it does not depend upon either the parametric representation that is adopted for V or the choice of the tangent distribution.

When the number m is odd, the sum in question will be zero, because from what we just said, it must not change when one simultaneously changes the signs of all the λ, which will be multiplied by $(-1)^m$, as we saw.

Things are different for the even values of m. The integer σ will then be one of the quantities that characterize V from the standpoint of *analysis situs.* It will have the same

([1]) On that subject, see the papers of Poincaré on the curves that are defined by differential equations in J. de Math. (3) t. **7**, **8**; *ibid.* (4), t. **1**; those of Dyck in the Berichte der Ges. Wiss. Leipzig **37** (1885), pp. 314; *ibid.* **38** (1886), pp. 53, and several recent works by Brouwer that were included in the minutes of the Royal Academy of Sciences in Amsterdam.

([2]) See the works that were cited above.

value for two arbitrary homeomorphic varieties, even when they are located in spaces with different dimensions.

In order for σ to have a value that is not zero, it will suffice to take V to be the surface of a sphere in three-dimensional space (or, more generally $2p + 1$). For the sphere in ordinary space, one can, for example, take the tangent direction to be that of the tangent to a meridian, which is supposed to be traversed towards a well-defined pole. Such a distribution will be indeterminate at the two poles. In the tetrahedroid (here, the triangle) that surrounds one of them, one can take the parameters to be the rectangular coordinates of the point that is projected onto the equator, and one will then find that a sum of the indices is:

$$\sigma = 2. \tag{29}$$

43. – A truly-remarkable consequence results immediately from the fact that the integer s is non-zero:

It is impossible to make each point of V correspond to a direction tangent to V at that point without that direction being indeterminate at one or more points of V.

That is because the index that relates to each element will be zero, as we proved in no. **33**.

Consequently, such an impossibility will be true on the of the sphere in ordinary space.

44. – One can deduce *Brouwer's theorem* on that surface from that result. In that case, the theorem is confined to transformations that preserve the sense of orientation (no. **41**), moreover. It is stated as follows:

If any perfect, continuous transformation of the surface of a sphere preserves the sense of orientation then it will leave at least one point invariant.

In order to prove that ([1]), let M be an arbitrary point of the surface, let M' be its homologue, and let A be a fixed point. Take the tangent direction at M to be the tangent to the circle $MM'A$; more precisely, the tangent to that one of the two arcs of that circle that are determined by the points A and M that contains the point M'. Such a direction will become indeterminate only in the following three cases:

If M coincides with A.
If M coincides with A.
If M coincides with M'.

([1]) The method of proof was communicated to me by Brouwer.

Each of the first two situations is realized once and only once. First, consider an element of the sphere that surrounds the point A. If the homologue A' of the point A coincides with A then the theorem will have been proved. Otherwise, the circle AMM' will be (for M close to A) very close to the circle AMA'. It will then result that it will not be very small at any moment. Since AM is, on the contrary, a small arc, it will make a very small angle with the great circle AM at M. Under those conditions, the tangents to those two arcs will have the same total rotation when M turns around A. It will result from this that the index of the variable direction in question along the element that contains A will be equal to + 1.

If, on the contrary, it is the point M' that coincides with A then the point M will become a point B that is distinct from A (since otherwise the theorem would have been proved).

When M turns around B, M' will turn around A, and in the same sense, by virtue of the hypothesis that was made on our transformation. The circle $AM'M$ will be very close to the circle $AM'B$, anyway. Now, the tangent to the latter at B or M will have a sense of rotation that is inverse to that of the tangent at A. (Those two lines are symmetric to each other with respect to the plane that is perpendicular to the midpoint of AB.) That is, the sense is inverse to the sense of rotation of M' around B.

Therefore, the index along a small element that surrounds the point B will be equal to – 1, and will give a sum of zero with the first one.

Since, on the contrary, the total sum is non-zero, the third hypothesis must be verified at one or more points on the surface, as well.

Q. E. D.

As for the transformations that change the orientation, they cannot admit any invariant point at all. That is the case in which one replaces each point of the surface with the diametrically-opposite one.

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