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CHAPTER I

THE AXIOMS OF MECHANICS

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Notations employed

Vectors are denoted by German letters:

$\mathfrak{r}$	position vector
$\dot{\mathfrak{r}} = \mathfrak{v}$	velocity
$\ddot{\mathfrak{r}} = \mathfrak{w}$	acceleration
$\mathfrak{r}\ \mathfrak{v}$	inner (scalar) product
$[\mathfrak{r}\ \mathfrak{v}]$	outer (vectorial) product
$\mathfrak{t}, \mathfrak{n}, \mathfrak{b}$	unit vectors in the natural coordinates of a space curve
μ	mass density
$d\,\mathfrak{f} = \mathfrak{s}\,dF$	surface traction
$d\,\mathfrak{f} = \mathfrak{q}\,dV$	volume force
$d\,\mathfrak{K} = \sum d\,\mathfrak{f}$	sum of the impressed forces that are applied to an element
$d\,\mathfrak{R}$	sum of reaction forces

Dyadics (i.e., second-rank tensors, affinors) are denoted by uppercase Greek symbols: Φ.

S	summation (integration) over regions in space *V* and surfaces *O*.
$\int$	integration over time or length

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I. – Introduction.

1. Historical overview and literature. – The great centuries of mathematics, namely, the Seventeenth and Eighteenth, had also been concerned with mechanics. **Galilei**, **Newton**, **Johann Bernoulli**, **d'Alembert**'s, **Euler**, and **Lagrange** took it to its pinnacle. Above all, **Gauss**, **Hamilton**, **Jacobi**, and the discoverers of non-holonomic systems advanced it to the greatest degree. Moreover, the Nineteenth Century involved a lot of isolated work that split off in various directions, and in so doing, that often diluted the great thoughts and initial criticisms.

Those criticisms – above all, the names of **Mach**, **Kirchhoff**, and **Poincaré** should be mentioned – still did not lead to any new positive constructions, i.e., to axiomatic methods, which were indeed first introduced into geometry at a fundamental level by **Pasch** (1882) and **Hilbert** (1899). As a result, the axiomatic method is even younger in mechanics. Naturally, just as in geometry, the great creators of the pre-critical period were also concerned about the foundations, so that led to axiomatics. However, there was never any ambition to present a definitive purely-logical structure that would express all of its assumptions clearly and examine its dependencies. Since they, as the creators of a new science, wished, above all, to inspire faith in their cause, the sources of that faith were often obscure.

Unwittingly, unproved or unclearly-formulated assumptions were at the basis for arguments. Indeed, **d'Alembert** himself believed that he had proved his principle, whereas today we can show its independence.

The axiomatic method had already been often applied to individual problems with astonishing clarity, e.g., to the investigation of the parallelogram law for forces by **Daniel Bernoulli**, **d'Alembert**, *et al.* The critical epoch of the Nineteenth Century was still far from achieving a systematic examination in the sense of Hilbert's axiomatics. Indeed, it produced some good ideas, but at times it also represented a step backwards. For example, when **Mach** asserted that mechanics was exhausted by Newton's law and that everything else would follow from it, he had undoubtedly been in error. When **Kirchhoff** defined force to be the product of mass and acceleration, that would make mechanics an autonomous science if one were to take his definition literally.

Therefore, the axiomatics of mechanics first arose in this century. As a result, there are not many works to cite on that topic. The textbooks, almost without exception, are pre-axiomatic, indeed pre-critical, without achieving the profundity of the Ancients. One finds a clearly-developed axiomatics of the statics of rigid bodies in **Marcolongo** ([1]). Otherwise, I can, unfortunately, only cite my own works for the most part ([2]). (cited from now on as H. 1, H. 2, H. 3, and H. 4) That is connected with the fact that from now on I will cite very little literature. With regard to the older literature, one might refer to the encyclopedia article by **Voss** ([3]).

([1]) **R. Marcolongo**, *Theoretische Mechanik*, German by **H. Timerding**, Leipzig, 1911.

([2]) **G. Hamel**, Math. Ann. **66** (1909), pp. 350. (cited as H. 1) Jahresber. d. deutsche Math.-Verein. **18** (1909), pp. 357. (cited as H. 2) *Lehrbuch der elementäre Mechanik*, Leipzig, 1st ed., 1912, 2nd ed. 1922. (cited as H. 3) Jahresber. d. deutsche Math.-Verein. **25** (1916), pp. 60. (cited as H. 4)

([3]) *Enzykl. d. math. Wiss.*, Bd. IV, 1, pp. 1.

II. – CLASSICAL MECHANICS

a) Newton's fundamental laws

α) Formulation of the axioms.

2. Axioms I.a to I.f. – Classical mechanics speaks of motion in Euclidian space. A point P in it is established in relation to a point O that is regarded as being "at rest" by the position vector $\mathfrak{r}$. Let x, y, z be its rectangular coordinates. If one considers a point P_0 with the vector $\mathfrak{r}_0$ at "time" t_0 and associates it with the point P with position vector $\mathfrak{r}$ at "time" t then one says that the point has moved from P_0 to P in the time interval $t - t_0$.

Among all of those conceivable motions, there is a distinguished one that we will refer to as the "material" motion. Mechanics deals with it, and we shall express the following axioms about it:

Axiom I.a. – $\mathfrak{r}$ is a function of time that is at least twice continuously differentiable. $\dfrac{d\mathfrak{r}}{dt}$ will be denoted by $\mathfrak{v}$ (velocity) and $\dfrac{d^2\mathfrak{r}}{dt^2}$, by $\mathfrak{w}$ (acceleration).

Axiom I.b. – If the spatial region V_0 goes to the spatial region V under the material motion then the relationship between both of them is generally invertibly single-valued and piecewise continuously differentiable. Discontinuity (e.g., tearing apart, merging together), and therefore multi-valuedness as well, can exist only on isolated two-dimensional surfaces. (*The axiom of impenetrability and unique existence*).

AxiomI.c. – For every spatial region V, there exists a Stieltjes integral ([1]):

$$m = \mathsf{S}\, dm \geq 0 .$$

That integral is constant under material motion, and is called the *mass* of the body that fills up the spatial region at each instant. The mass can then be distributed continuously, as well as discontinuously. (*Axiom of the conservation of mass*)

Axiom I.d. – The mass-elements dm can be assigned one or more, or also infinitely-many, vectors $d\,\mathfrak{k}$ (forces), such that for every spatial region V :

([1]) Which subsumes integrals over continuous distributions, as well as over discrete mass-points. See, perhaps, **Riemann-Weber**, *Differentialgleichungen der Physik*, 7th ed., Braunschweig, 1925, pp. 32.

$$\mathfrak{K} = \mathsf{S} \sum d\,\mathfrak{k}$$

exists. S is a Stieltjes integral over the spatial region, and $\sum$ means a summation of the $d\,\mathfrak{k}$ over the individual volume elements *dm*.

Axiom I.e. – The forces $d\,\mathfrak{k}$ are determined by their *causes*, i.e., by variables that represent the geometric and physical state of the surrounding matter. That dependency is single-valued, and generally continuous and differentiable.

Axiom I.f. Newton's fundamental law. – In the interior of any non-concave spatial region with mass *m*, there is a point whose acceleration obeys **Newton**'s law:

$$m\,\mathfrak{w} = \mathfrak{K}\ .$$

Since $\lim_{\Delta V \to 0} \frac{1}{\Delta V} \mathsf{S}\, dm\,\mathfrak{w} = \mathfrak{w}$ exists, it follows from *I.f* that:

$$I.f^{*} \qquad \mathfrak{w} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \mathsf{S}\, d\mathfrak{k}\,.$$

β) Definition of the concepts and working out the details.

3. Space and time. – According to the current conception of things, which was ultimately expressed by **Hilbert**, axioms define the concepts to the extent that they come under question in the mathematical treatment of things. The freedom that still exists in their realization then serves to give the concepts a physical meaning such that a picture of the world that agrees with experiments will arise. That will imply, first of all, the ultimate determinacy and measurability of the quantities that one speaks of. Space and time also belong to those quantities that are to be first defined and measured, not at the beginning, but only at the end.

As such, every Euclidian space can be used as a basis, and each point *O* of them can be regarded as being at rest, while any coordinate system can serve to establish its directions. Similarly, time can be chosen in many ways, except that the chosen *t* must be invertible single-valued, monotone, at least twice continuously differentiable functions over $-\infty < t < +\infty$. If one initially leaves that choice open then one will have seven undetermined functions of time. They will enter into the $d\,\mathfrak{k}$ as universal functions of time, and in general, that will contradict the axiom *I.e* or the axioms *A*, *B*, *C*, *B* (cf., no. **7**). The assertion is then based upon the fact that those seven functions can be determined in such a way that the system of axioms is fulfilled, with the exception of *I.e.* The space-time manifold thus-determined is called the *absolute* one. The determination is unique, up to the so-called *Galilei transformation:*

$$t' = \alpha\, t + \beta \qquad\qquad (\alpha > 0),$$
$$\mathfrak{r}' = \mathfrak{a}\, t + \mathfrak{b} + \Phi\, \mathfrak{r}\,,$$

in which the scalars *a*, *b*, the vectors $\mathfrak{a}$, $\mathfrak{b}$, and the rotation tensor (versor) Φ are constant (*Galilean relativity principle*). Therefore, we will first know about absolute space and absolute time from a complete theory of mechanics, just as we will also first know the masses of the planets from the theory that determines their motions, but they cannot be measured, *a priori*. Now, there is no contradiction in saying that only relative motion, but not absolute, can be measured by observation. According to **Kant**, whom one must thank for the complete explanation of that notion, absolute space and absolute time are transcendental intuitions, while they had previously been regarded as transcendent realities. According to the ideas of transcendental philosophy, the concept of cause cannot be applied to space and time, but only to events in space and time. As is known, Einstein's theory of relativity has a different conception of things. In it, space and time are again transcendent realities. They are real because they are included in the causal connection. Matter facilitates the measurement of space and time, which again influences the motion essentially. They are transcendent because space and time are not directly objects of experiments (i.e., they are not material) in that theory. A reconciliation of the paradox is then possible when one speaks of the measurement of space and time, instead of space and time. That purely-material approach to the causal connection does not contradict the transcendental philosophy. In Einstein's theory, space and time are also transcendental intuitions in which natural processes play out for us.

4. The static concept of force. – In order to construct this, one requires the special *axiom of gravity.*

Axiom I.g:

α) Among the forces, there is one of them – viz., gravity – for which $d\,\mathfrak{k} = dm\;\mathfrak{g}$.

β) $\mathfrak{g}$ depends upon only the total mass distribution. According to **Newton**, one has:

$$\mathfrak{g}_1 = \gamma\, \mathsf{S}\, \frac{dm}{|\,\mathfrak{r}_1 - \mathfrak{r}\,|}(\mathfrak{r}_1 - \mathfrak{r})$$

at the point $\mathfrak{r}_1$, where the universal constant γ depends upon only the chosen unit of mass.

The integral S extends over all of space.

Only the first half of *I.f* is used initially: In the case of equilibrium (i.e., statics), one must have $\mathfrak{K} = \mathsf{S} \sum d\,\mathfrak{k} = 0$. Forces are then vectors that can be put in equilibrium with weight.

The transition to kinetics is usually formulated as follows then: If $\mathsf{S} \sum d\,\mathfrak{k} \neq 0$ then motion will occur, and *I.f* will be valid. That formulation will then be inadequate, due to the fact that the effect must arise as if *I.g* and the first half of *I.f* defined the forces, and the second half would then define a law of nature. When regarded in that way, the assertion would be directly false since the forces in the course of motion are often different from the forces at rest. For example, if one hangs a body of weight $G = m\,g$ from a spring that is perpendicular to it and waits for equilibrium to occur then the spring must exert a force $Z = G$ that points upward that is a function $Z(x)$ of the extension x of the spring for small G, namely, it is approximately proportional to x : $Z = \lambda\,x$. Now, if one would like to say that under a perturbation of equilibrium by an impulse, $m\frac{d^2x}{dt^2}$ is equal to $G - \lambda\,x$ then that would be only very crudely correct, so strictly speaking false. Namely, while in motion, the spring will impart a force that is less than $-\lambda\,x$ on the body, due to its own inertia.

Just the same, in order to construct Newton's fundamental law, one must place a stronger emphasis upon the concept of force, which will happen later in nos. **38** to **41**.

5. The kinetic concept of force. – From our way of looking at things, the physical sense of *I.f* is the following:

All accelerations $\mathfrak{w}$ are observed and then divided into classes by a grouping of related phenomena relative to a suitable space and a suitable time (**Bacon**: *Dissecare naturam*). For each class of motions, when one eliminates the individual constants that the individual motions are endowed with, one will find a more legitimate expression that satisfies axiom *I.e*, i.e., a function of the geometric and physical variables of the point and its neighborhood (e.g., by going from Kepler's law to the Newtonian law of gravitation). It is an experimental fact that this elimination and classification is achieved by appealing to the picture of acceleration. The forces $d\,\mathfrak{k}$ arise upon multiplying by a suitably-chosen constant dm, each of which belong to a class. The decomposition and recombination will come about according to the parallelogram law of forces that is included in *I.d*. dm can be found in such a way that, except for the law of gravitation *I.g*, all other forces will depend upon dm such that a class of accelerations (an individual force law) will always depend upon the corresponding motions of different masses. We would then like to formulate that concept as:

Axiom I.g.γ : The dm of the mass-element in question does not occur in the other force laws besides the law of gravitation.

Force is then a legitimate expression for a class of mass-accelerations (H.1).

Force is therefore not equal to mass times acceleration, as ***Kirchhoff*** *asserted.*

Force equals the cause of motion is not a definition, but at most an approximate rewriting of I.f when one makes the word "cause" suitably precise.

The true causes of motion are other physical or material phenomena. One can combine only the individual groups of them that are in effect by using the concept of force.

6. The parallelogram of forces. – From that, it is clear that the parallelogram of forces has nothing to do with the composition of motions. Indeed, one does not deal with the purely-mathematical decomposition of an individual vector $\mathfrak{w}$, but with the composition of force laws. As **Johann Bernoulli** said: "Peccant qui compositionem virium cum compositione motuum confundut." (†) The question then takes the form: Suppose that geometric and physical data are given that require a force law $d\,\mathfrak{k}_1$, along with ones that require a second force law $d\,\mathfrak{k}_2$. If both groups of geometric and physical data are observed then would they collectively require another force law, and what would it be? It is not even obvious and provable by pure mathematics that they should require one. For that reason, we shall then peel off the following part of *I.d*.

Axiom $I.d'$: If the force laws $d\,\mathfrak{k}_1$, $d\,\mathfrak{k}_2$, … are all given by their effects on a *dm* then they will all be equivalent to a force $d\,\mathfrak{K} = \sum d\,\mathfrak{k}$ for the determination of motion.

Now, $I.d'$ can be decomposed even further. After some earlier attempts ([1]) by **Daniel Bernoulli**, **d'Alembert**, and **Poisson**, **Darboux** essentially solved the problem when he presented the following replacement for $I.d'$:

$I.d'.\alpha$: There is a uniquely-determined resultant.
$I.d'.\beta$: The associative and commutative laws are valid for the composition.
$I.d'.\gamma$: The composition is invariant under the orientation of space.
$I.d'.\delta$: Forces with the same direction can be added.
$I.d'.\varepsilon$: The composition is continuous.

Siacci then examined the influence of the axiom:

$I.d'.\vartheta$: The composition is independent of the units.

If one adds the axioms:

$I.d'.\eta$: The composition formulas are differentiable, and
$I.d'.\iota$: $\mathfrak{k} + 0 = \mathfrak{k}$ and $0 + \mathfrak{k} = \mathfrak{k}$

then α, ε, ϑ, η, ι will already suffice. (H.3) ([2])

(†) Translator: "He sins who confuses the composition of forces with the composition of motions."
([1]) See, *Encykl. d. math. Wiss.*, Bd. IV.1, art. 1 (**Voss**). No 19.
([2]) A very thorough study of this question was made by **Schimmack**, "Axiomatische Untersuchungen über die Vektoraddition," Göttinger Dissertation, Halle 1908. In addition to **Voss**, one can also find a historical overview of the older literature is in the dissertation of **Ernst Georges**, "Die Zusammensetzung der Kräfte," Halle 1908.

7. General axioms of natural philosophy. – Axioms $I.d'.\alpha$ to ι obviously have very epistemological values. $I.d'.\alpha$ makes an autonomous concept out of force. If α were not fulfilled then one would have to dispense with the concept of force. To this day, $I.d'.\varepsilon$ and ι are far-reaching axioms that physics has adopted with the same general character as *I.a* and *b*. $I.d'.\delta$ stands by itself. When the magnitude of the force is already defined mainly by the left-hand side of *I.f*, it becomes an autonomous axiom that is quite plausible, but not obvious (cf., no. **39** on this). $I.d'.\beta$, γ, ϑ have a different character. $I.d'.\beta$ says that the various forces (i.e., groups of causes) have no classification scheme, but rather, they are equivalent and independent of each other. $I.d'.\gamma$ is a statement about space: viz., it is *isotropic*, i.e., it has no distinguished direction, just as one assumes that it is *homogeneous*, i.e., it has no distinguished location. If distinguished locations or directions do occur then they must be required by the properties of matter. Time is also homogeneous in the same sense. These are the *general axioms of natural philosophy* then:

General axiom A: Time and space are homogeneous.
General axiom B: Space is isotropic.

$I.d'.\delta$ can be regarded as part of a third general axiom, namely:

Axiom C of sufficient grounds: All events must have knowable causes by which they can be determined uniquely.

In mechanics, events mean: Changes in the quantities of motion *dm* $\mathfrak{v}$ (the *quantitas motus*, to **Newton**). However, the causes are distributed among groups that are each associated with a force. Nothing beyond those forces is definitive, and in particular, no arrangement or grouping of them. Symmetries in the groups of causes then demand symmetries in the forces, and they must also be expressed in the laws of motion.

Now let us say something about $I.d'.\vartheta$. From *I.f*, in mechanics one has three units that can be chosen independently of each other, namely, the unit of length, that of velocity, and that of mass. The choice of a unit of force remains free as long as there are no distinguished mass-accelerations in the universe. If that were the case then $I.d'.\vartheta$ would be self-evident. The fact that there are no distinguished lengths is connected with the Euclidian character of space. However, the fact that we are not entitled to a distinguished velocity or a distinguished mass that might influence the basic structure mechanics is not an obvious assumption. In the theory of relativity, we do, in fact, have a distinguished velocity, namely, the speed of light. There is also a distinguished mass, namely, the astronomically-measured unit mass. However, since it can be established only when the unit of length and the unit of velocity have been determined, it can imply no well-defined effect as long as no well-defined length has been distinguished, say, in a non-Euclidian space. We then formulate the:

General axiom D: There is no distinguished length, no distinguished velocity, and no distinguished mass that are meaningful for the structure of classical mechanics.

8. Mass. – Mass also takes on a three-fold meaning. First of all, it appears in the left-hand side of *I.f* as inertial mass. However, it then appears twice in the **Newton**'s law of gravitation *I.g* :

$$d\,\mathfrak{k} = dm_1\,\mathfrak{g}_1 = dm_1\,\gamma\,\mathsf{S}\,dm\,\frac{\mathfrak{r}-\mathfrak{r}_1}{|\mathfrak{r}-\mathfrak{r}_1|^3},$$

once as a factor of $\mathfrak{g}_1$, namely, as the attracted mass (i.e., the gravitational mass), and then in the integral S that represents $\mathfrak{g}_1$ (i.e., the attracting mass). The equality of inertial and gravitational mass, i.e., the independence of the gravitational acceleration from the mass of the accelerated body, can be verified experimentally with great precision and also defined an essential foundation of Einsteinian mechanics. The equality of gravitational and attracting mass has been verified experimentally with much less precision. In order to support that assumption, since the time of **Newton**, one has added the generalized principle of the *equality of action and reaction*, which will be discussed more thoroughly later on (nos. **10** and **26**). Therefore, the forces that two masses exert upon each other are equal to each other, and that will imply the equality that we speak of, or at least a proportionality. That is because if dm_1 is the gravitational mass of 1, dm_2 is that of 2, dM_1 is the attracting mass of 1, and dM_2 is that of 2 then $dm_1\,dM_2 = dm_2\,dM_1$, so $\frac{dm_1}{dM_1} = \frac{dm_2}{dM_2}$, which depends upon neither 1 nor 2; hence, it is a universal constant. Mass, which initially appears as an auxiliary mathematical device, can be measured by a series of experiments with an accuracy that is associated with that series of experiments, say, by means of oscillation experiments with a spring (cf., H.1). The formerly-popular controversies regarding whether one must first define the force and then the mass or *vice versa* are untenable in the axiomatic method, in which the logical dependency of the concepts on each other is defined reciprocally ([1]).

9. Galilei's inertia experiment. – This is included in *I.f.* When no forces are acting, one will have $\mathsf{S}\,dm\,\mathfrak{w} = 0$ for any region of space, so each $\mathfrak{w} = 0$, and therefore $\mathfrak{r} = \mathfrak{a}\,t + \mathfrak{b}$, where $\mathfrak{a}$, $\mathfrak{b}$ are constants. The fact that this is never *exactly* the case should not be grounds for objection. No single force law will ever exist alone and exactly. The complete statement is just *I.f*, which can also be formulated thus: One seeks a force law $d\,\mathfrak{k}$ such that when $\mathfrak{w}$ is the actual observed acceleration, one will have:

$$\left|\mathfrak{w} - \lim_{\Delta V\to 0}\mathsf{S}\sum d\,\mathfrak{k}\right| < \varepsilon,$$

in which ε gives the precision of the observation. We will speak later of the *a priori* significance of the law of inertia (no. **38**).

([1]) For what was implemented in this subsection, cf., *Enzykl. d. math. Wiss.*, Bd. V.1, art. 2 (**Zenneck**).

b) System mechanics.

α) Structure of mechanics under the continuity hypothesis.

10. The stress dyadic and the reaction principle. – When we speak of a spatial region V that is filled with matter, it will always have a piecewise-continuous tangent plane, and therefore also a piecewise-continuous exterior normal that we think of as being given by the unit vector $\mathfrak{n}$. For real bodies that are not merely imaginary ideal ones, continuity will even prevail without exception.

Axion II.I.a: One has $dm = \mu\, dV$, where μ is the mass density, which is a finite, piecewise-continuous function of position. (*First continuity hypothesis*).

Axiom II.I.b: The forces $d\,\mathfrak{k}$ split into two groups: Spatially-distributed ones and ones distributed over surfaces. The spatially-distributed ones have the form $dV\,\mathfrak{q}$ (Example *I.g*), in which the $\mathfrak{q}$ are finite vectors that are piecewise-continuous in space and time. The surface forces are associated with a surface element dF with a distinguished exterior normal $\mathfrak{n}$: $d\,\mathfrak{k} = \mathfrak{s}_n\, dF$. The $\mathfrak{s}_n$ are, without exception, continuous and continuously-differentiable functions of position. (*Second continuity hypothesis*)

Axiom *I.f* must now be given the more well-defined form:

Axiom $I.f^*$:

$$m\,\mathfrak{w} = \sum \mathfrak{q} + \lim_{dV\to 0} \frac{1}{dV} \mathsf{S}\, \mathfrak{s}_n\, dF\,,$$

in which the integral S extends over the surface of the volume dV. This axiom, and in particular, the existence of the last limiting value, implies the theorem:

Theorem 1:

The $\mathfrak{s}_n$ are homogeneous linear functions of the components of $\mathfrak{n}$, i.e., they are the products of $\mathfrak{n}$ with a dyadic (viz., a second-rank tensor), namely, the stress tensor T:

$$\mathfrak{s}_n = \mathsf{T}\,\mathfrak{n}\,,$$

or when written out in components ([1]):

$$X_n = X_x \cos(n, x) + X_y \cos(n, y) + X_z \cos(n, z)\,.$$

([1]) Here, and in what follows, only the first of the three equations in Cartesian coordinates will always be written out.

X, Y, Z mean the components along the axes, while the index gives the distinguished normal.
That theorem includes the *reaction principle* for stresses:

Theorem 1.a:

At the same location and the same time, $\mathfrak{s}_n$ *has the opposite sign to* $\mathfrak{n}$. (The proof of this is known, cf., H.1 and H.3)

Theorem 2:

From equation $I.f^*$, *one will have:*

$$\mu\,\mathfrak{w} = \sum \mathfrak{q} + \nabla \mathsf{T}\,,$$

or when written out in components:

$$\mu \frac{d^2 x}{dt^2} = \sum X + \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z}\ .$$

(The explicit form of ***Newton****'s fundamental equation).*

Theorem 3 *(First fundamental law of mechanics: the center of mass theorem):*

$$\underset{V}{\mathsf{S}}\, dm\,\mathfrak{w} = \mathsf{S}\sum_{V} \mathfrak{q}\ dV + \underset{O}{\mathsf{S}}\ \mathfrak{s}_n\, dF\ .$$

The first two integrals extend over any partial volume V, while the last one extends over its surface O. In order to prove this theorem, one needs only to multiply the foregoing equation by dV, integrate it over the volume, and apply Gauss's law to the last term.

Definition: The forces that appear on the right-hand side of this equation, namely, the spatially-distributed $\mathfrak{q}\ dV$ and $\mathfrak{s}_n\ dF$, which acts upon the surface of the spatial region in question, are called the *external forces* on the spatial region, as opposed to the stresses $\mathfrak{s}\ dF$ in the interior of the spatial region, which do not enter into the equation, namely, the *internal forces* (i.e., stresses). Since one defines the *center of mass* (incorrectly also called the center of gravity) by:

$$\mathfrak{r}^*\ \mathsf{S}\ dm = \mathsf{S}\ \mathfrak{r}\ dm\ ,$$

one can write the theorem (with $m = \mathsf{S}\ dm$):

$$m\ \mathfrak{w}^* = \mathfrak{K}_a\ ,$$

in which $\mathfrak{K}_a$ denotes the sum of the external forces.

Now since $I.f^*$ again follows from theorems 1 and 3, the essential consequences of the axioms up to now have been exhausted with the aforementioned theorems.

11. The Boltzmann axiom.

Axiom II.I.c: The stress dyadic is symmetric, i.e., it is equal to its conjugate. In coordinates, that is:

$$X_y = Y_x\,, \qquad Y_z = Z_y\,, \quad Z_x = X_z\,.$$

As a law, those equations are already much older. Their axiomatic character was first known to **Boltzmann** ([1]). It follows from this axiom that:

Theorem 4 *(Second fundamental law of mechanics: the law of moments):*

For any spatial region, one has:

$$\underset{V}{\mathsf{S}}\, dm\,[\mathfrak{r}\,\mathfrak{w}] = \underset{V}{\mathsf{S}}\, dV\,[\mathfrak{r}\,\mathfrak{q}] + \underset{O}{\mathsf{S}}\,[\mathfrak{r}\,\mathfrak{w}]\,dF\,.$$

(The integrals are understood to mean what they did in the first fundamental law. The proof is also the same as in that law.)

Since the left-hand side is equal to $\frac{d\mathfrak{D}}{dt} = \frac{d}{dt}\mathsf{S}\,dm\,[\mathfrak{r}\,\mathfrak{v}]$, one can write the law as:

$$\frac{d\mathfrak{D}}{dt} = \mathfrak{M}_a\,.$$

In words: *The temporal change in velocity of the angular impulse $\mathfrak{D}$ is equal to the moment $\mathfrak{M}_a$ of the external forces.*

Since *Axiom II.I.c* can be deduced from Theorem 4, it represents the full conclusion of that axiom.

One can then divide Axiom *II.I.c* into two parts that one first expresses only in *statics* $(II.I.c')$, and then for *kinetics* $(II.I.c'')$, as well. That fact has some importance. (cf., nos. **17** and **24**)

([1]) **L. Boltzmann**, "Die Grundprinzipien und Grundgleichungen der Mechanik," *Populäre Schriften*, 3rd ed., pp. 253-307. Leipzig, 1925.

12. The law of energy.

Theorem 5 *(third fundamental law: the law of energy):*

Let $E = \frac{1}{2}\mathsf{S}_V \mathfrak{q}\mathfrak{v}\, dV$ *denote the kinetic energy, and let:*

$$L_a = \mathsf{S}_V \mathfrak{q}\mathfrak{v}\, dV + \mathsf{S}_O \mathfrak{s}_n\, \mathfrak{v}\, dF$$

be the power developed by external forces, while L_i *is the power developed by internal stresses. One then has*:

$$\frac{dE}{dt} = L_a + L_i ,$$

in which (1):

$$L_i = -\mathsf{S}_V \mathsf{T}\,\Gamma\, dV = -\mathsf{S}_V \left\{ X_x \frac{\partial v_x}{\partial x} + X_y \left(\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right) + \cdots \right\} dV .$$

The symmetric dyadic Γ with the six components:

$$\varepsilon_x = \frac{\partial v_x}{\partial x}, \qquad \gamma_{xy} = \frac{1}{2}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right), \text{ etc.,}$$

is called the *deformation dyadic* that corresponds to $\mathfrak{v}$ (v_x, v_y, v_z are the three components of $\mathfrak{v}$).

In general, one has:

Theorem 6:

Let $\delta\mathfrak{r}$ *be any continuously-differentiable vector with the components* ξ, η, ζ, *and let* Ψ *be the symmetric dyadic with the six components:*

$$\varepsilon_x = \frac{\partial \xi}{\partial x}, \qquad \gamma_{xy} = \frac{1}{2}\left(\frac{\partial \xi}{\partial y} + \frac{\partial \eta}{\partial x} \right),$$

etc. One will then have:

$$\mathsf{S}_V dm\, \mathfrak{w}\, \delta\mathfrak{r} = \mathsf{S}_V \mathfrak{k}\, \delta\mathfrak{r}\, dV + \mathsf{S}_O \mathfrak{s}_n\, \delta\mathfrak{r}\, dF - \mathsf{S}_V \mathsf{T}\,\Psi\, dV .$$

(1) This is the definition of L_i .

Theorem 5 and Theorem 6 follow from Theorem 2 upon multiplying by $\mathfrak{v}$ ($\delta\,\mathfrak{r}$, resp.) and integrate over V, while using Gauss's theorem, Theorem 1, and the **Boltzmann** axiom.

Conversely, *the first fundamental law, as well as the symmetry of the stress dyadic, follows from Theorem 6, due to the arbitrariness of* $\delta\,\mathfrak{r}$.

Theorem 6 is the principle of virtual work in the mechanics of continua.

If one now lets $\delta\,\mathfrak{r}$ represent an arbitrary, merely imaginary, differential displacement (viz., a virtual displacement) and establishes, as is always permissible, but not necessary ([1]), that: $d\,\delta\,\mathfrak{r} = \delta\,d\,\mathfrak{r}$, and furthermore, that $\delta\,dt = 0$, $\delta t = 0$, then one can convert the left-hand side of 6 into:

$$\frac{d}{dt}\mathsf{S}_V\, dm\,\mathfrak{w}\,\delta\mathfrak{r} - \delta E$$

and obtain the Lagrange *central equation:*

$$\frac{d}{dt}\mathsf{S}_V\, dm\,\mathfrak{w}\,\delta\mathfrak{r} - \delta E = \delta A_a + \delta A_i \equiv \delta A\ ,$$

in which one refers to the first two integrals as the *virtual work δA_a done by external forces*, and the last integral, with its minus sign, as the *virtual work done by internal forces*. δA is the *total virtual work done by all forces*. This (partial) principle of virtual work includes nothing more than the axioms up to now ([2]).

13. The classification of systems by the character of the stress dyadic. First: systems with no internal work. – General mechanics is therefore complete with that. We must further distinguish special systems according to the character of the stress dyadic. We first look for systems with no virtual internal work, so ones for which one always has:

$$\mathsf{T}\,\Psi = 0\ ,$$

and the central equation will then read:

$$\frac{d}{dt}\mathsf{S}_V\, dm\,\mathfrak{w}\,\delta\mathfrak{r} - \delta E = \delta A_a\ .$$

Now, the equation above (when written in coordinates):

$$X_x \frac{\partial \xi}{\partial x} + X_y \left(\frac{\partial \xi}{\partial y} + \frac{\partial \eta}{\partial x}\right) + \cdots = 0$$

([1]) On this, cf., **G. Hamel**, Math. Ann. **59** (1904), pp. 416.

([2]) For this subsection, cf., *Enzykl. d. math. Wiss.*, Bd. IV.4, art. 30 (**Hellinger**).

can be fulfilled:

a) without a relation between the six stress quantities X_x, … having to be prescribed, i.e., for all X_x, X_y, … One must then have $\Psi = 0$, from which it will follow that:

$$\delta \mathfrak{r} = \delta \mathfrak{c} + [\delta \mathfrak{d}\ (\mathfrak{r} - \mathfrak{c})]\ ,$$

in which $\mathfrak{c}$, $\delta \mathfrak{c}$, $\delta \mathfrak{d}$ are vectors that are independent of $\mathfrak{v}$. That is, the virtual displacements of the body consist of translations and rotations, so the *body is rigid* (cf., H.1). *One has* $\delta A_i = 0$ *for them, or the principle of virtual work assumes the form above for them:*

$$\frac{d}{dt} \mathsf{S}_V\, dm\, \mathfrak{w}\, \delta \mathfrak{r} - \delta E = \delta A_a\ .$$

Mechanics shows that these equations (the equivalent fundamental laws, resp.) suffice to determine the motion of the rigid body from the external forces).

b) *One* linear homogeneous relation between the X_x, X_y, … can be prescribed. If we assume that the body is *isotropic*, i.e., it has no distinguished direction, then it will follow from the general axiom of the *isotropy of space* and the principle of the *sufficient grounds* (cf., *B* and *C* in no. **7**) that this relation must be invariant under rotation. Since $X_x + Y_y + Z_z$ is the only linear invariant of T, the relation must read:

$$X_x + Y_y + Z_z = 0\ .$$

Should one always have $\mathsf{T}\ \Psi = 0$ under this auxiliary condition then one must have:

$$\frac{\partial \xi}{\partial x} = \frac{\partial \eta}{\partial y} = \frac{\partial \zeta}{\partial z}\ ,$$

while the other three components of Ψ are once more zero. It then follows that (H.1):

$$\delta \mathfrak{r} = \delta \mathfrak{c} + [\delta \mathfrak{d}\ (\mathfrak{r} - \mathfrak{c})] + \delta\lambda\ (\mathfrak{r} - \mathfrak{c}) - \delta \mathfrak{p}\ (\mathfrak{r} - \mathfrak{c})^2 + 2\ (\mathfrak{r} - \mathfrak{c}) \cdot \delta \mathfrak{p}\ (\mathfrak{r} - \mathfrak{c})\ ,$$

with arbitrary constants:

$$\mathfrak{c}, \quad \delta \mathfrak{c}, \quad \delta \mathfrak{d}, \quad \delta\lambda, \quad \mathfrak{p}, \quad \delta \mathfrak{p}.$$

Those motions represent the ten-parameter group of conformal transformation of space. The physical realization of this case is not known.

c) Two linear homogeneous equations between the X_x, X_y, … can be prescribed. On the same grounds as in *b*), they must read:

$$X_x = Y_y = Z_z = -p\,,$$

from which, it must follow that $X_x = Y_y = Z_z = 0$. For $\delta\mathfrak{r}$, it follows from $\mathsf{T}\,\Psi = 0$ that:

$$\text{div}\;\delta\mathfrak{r} = \frac{\partial\xi}{\partial x} + \frac{\partial\eta}{\partial y} + \frac{\partial\zeta}{\partial z} = 0\,.$$

i.e., the system is incompressible (e.g., an incompressible fluid). p is called the *pressure* in the fluid. Here as well, equations $I.f^*$, together with the so-called continuity equation:

$$\text{div}\;\mathfrak{v} = 0$$

will suffice to determine the motion when the necessary initial and boundary conditions are given (cf., no. **44**).

d) If three linear homogeneous equations between the X_x, … are prescribed then one will have:

$$\mathsf{T} = 0$$

under isotropy, such that $\delta\mathfrak{r}$ will remain arbitrary. The system would then have no internal stresses, which only exists as an ideal (i.e., a cloud of isolated points).

It is not necessary to say that the two cases of rigid bodies and incompressible fluids also represent ideal situations that never occur in reality, but still often represent a good approximation. The simple fact is that *one needs to know nothing further about the internal stresses in order to determine the motion*. They degenerate into auxiliary mathematical tools that one seeks to eliminate, which happens as a result of the fundamental laws for rigid bodies. Later on, we shall generally refer to such forces as *forces of reaction* (cf., no. **15**).

14. Second: systems with internal work. – For these systems, one must know more details about the dyadic T if one is to determine the motions. Based upon prior experience, it depends essentially upon two further dyadics. First of all, one has the velocity dyadic Γ, which has the six components:

$$\frac{\partial v_x}{\partial x}, \quad \frac{1}{2}\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right),$$

etc., and secondly, those of the deformation dyadic, which gives the change in form of a differentially-small volume around the point in question from its normal state. If that normal state is given by $\mathfrak{r}_0$, so $\mathfrak{l} = \mathfrak{r} - \mathfrak{r}_0$ is the displacement with the components u, v, w, then that deformation dyadic will have the six components:

$$e_x = 2\frac{\partial u}{\partial x} - \left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial v}{\partial x}\right)^2 - \left(\frac{\partial w}{\partial x}\right)^2,$$

$$e_y = 2\frac{\partial v}{\partial y} - \left(\frac{\partial u}{\partial y}\right)^2 - \left(\frac{\partial v}{\partial y}\right)^2 - \left(\frac{\partial w}{\partial y}\right)^2,$$

$$g_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial w}{\partial x}\frac{\partial w}{\partial y},$$

etc. [cf., H.3 ([1])].

It is called $2\,\Delta\,(\mathfrak{l})$. For small displacements, one typically neglects the quadratic terms. We then write it as $2\,\Delta'\,(\mathfrak{l})$.

a) *For completely-elastic systems*, which are also only ideal systems, T is merely a function of Δ (Δ', resp.). One assumes, in addition, that a potential exists, i.e., that:

$$dV\,\mathsf{T}\,\Psi = dm\; dU_i\,(\Delta) = dm\frac{\partial U_i}{\partial \Delta}\,\delta\,\Delta(\mathfrak{l}) \;.$$

In that way, Ψ is the dyadic of $\delta\,\mathfrak{r} = \delta\,\mathfrak{l}$. We write that in detail as $\Psi\,(\delta\,\mathfrak{l})$.

Now, for small displacements, when one consistently neglects the second-order terms, one will have:

$$\delta\,\Delta'\,(\mathfrak{l}) = \Psi\,(\delta\,\mathfrak{l})\,,$$

and therefore:

$$\mathsf{T} = \mu\,\frac{\partial U_i}{\partial \Delta'}\;.$$

If one now takes U to be homogeneous of order two and the medium to be homogeneous and isotropic then it will follow that:

$$\mathsf{T} = \kappa\,\Delta' + \lambda\,\text{div}\,\mathfrak{l}\cdot \mathrm{I}\,,$$

in which κ, λ are two material constants, and I means the identity dyadic (i.e., the idemfactor):

$$\begin{matrix} 1\,0\,0 \\ 0\,1\,0 \\ 0\,0\,1 \end{matrix}\;.$$

For *finite* displacements, there exist relations that are essentially more complicated [H.1] ([2]).

Complete fluids (also called *ideal* or *inviscid*) are defined dynamically by the fact that one has:

([1]) Also **E.** and **F. Cosserat**, *Théorie des corps déformables*, pp. 123, *et seq.*
([2]) Cf., also *Enzykl. d. math. Wiss.*, Bd. IV.4, art. 2 (**Müller-Timpe**).

$$X_x = Y_y = Z_z = -p \qquad (p > 0).$$

$$X_y = Y_z = Z_x = 0 .$$

Under the assumption of isotropy, one must then have (*equation of state*):

$$p = p(\mu) .$$

Other physical variables (e.g., temperature or entropy) can enter into that, as in the more general relations above.

b) For s*ystems*, in particular, *fluids with internal friction*, one assumes that the stress dyadic splits into two parts. The first part T_1 corresponds to the assumptions above (possibly modified), while the second one T_2 depends linearly upon Γ. For isotropy, that will give:

$$\mathsf{T}_2 = \kappa \Gamma + \lambda \operatorname{div} \mathfrak{v} \cdot \mathrm{I} .$$

κ and λ are material constants that are, above all, subject to the condition that the power developed by T_2 must always be negative (dissipation, conversion of energy into heat).

c) *For general systems, in which elastic aftereffects, plasticity, etc., must be considered*, one must assume that T depends upon Γ and Δ (*t*), but also even on Δ ($t - \tau$) for all $\tau > 0$ (aftereffects), and perhaps also upon $d\,\Delta / dt$, etc. (**Jaumann**). For elastic aftereffects, **Boltzmann** made the Ansatz:

$$\mathsf{T} = \int_0^\infty \mathfrak{L}^{(4)}(\tau)\,\Delta'(t-\tau)\,d\tau .$$

That $\mathfrak{L}^{(4)}$ will generally be a linear operator (tensor) of rank four with 36 components that is especially large for $\tau = 0$, and then drops off sharply to zero ([1]).

β) The structure of mechanics when one starts from rigid bodies.

15. Statics of isolated rigid bodies. – We shall address the idealized object of a rigid body, which is defined kinematically by the fact that it remains congruent to itself in the course of its motion. For each of its motions, we will then have:

$$\delta \mathfrak{r} = \delta \mathfrak{r} + [\delta \mathfrak{d}\,(\mathfrak{r} - \mathfrak{c})] ,$$

and in particular:

([1]) Literature is in the cited encyclopedia article by **Hellinger**.

$$\mathfrak{v} = \dot{\mathfrak{c}} + [\dot{\mathfrak{d}}(\mathfrak{r} - \mathfrak{c})]$$

[**Euler**], in which $\mathfrak{c}$ and $\dot{\mathfrak{d}}$ are independent of $\mathfrak{r}$. Initially, only *statics* is considered.

Definition 1: For every system that is subject to well-defined constraints (e.g., a rigid body), the forces split into two types:

First of all, the reaction forces, whose causes are to be sought in the prescribed constraints alone, so they exist only to maintain those constraints.

Secondly, the impressed forces, at least some of which are due to other causes.

The sum of the former, which is associated with a *dm*, is briefly written $d\,\mathfrak{R}$, and the sum of the latter is $d\,\mathfrak{K}$, such that axiom *I.f* will assume the form:

$$dm\,\mathfrak{w} = d\,\mathfrak{R} + d\,\mathfrak{K}\,,$$

and in the case of statics (a body at rest):

$$0 = d\,\mathfrak{R} + d\,\mathfrak{K}$$

(d'Alembert's Ansatz).

Definition 2: One force system on a rigid body (just as for any system) is *equivalent* to another one when they impart the same acceleration to the body when it has the same position and velocity in both cases. The force system is said to be equal to 0 or in equilibrium with the system when it has the same effect on the body as if no impressed forces were acting at all. By contrast, the system itself is said to be in equilibrium when it remains continually at rest.

Axioms *II.I.a-c* will not be assumed now.

Axiom II.2.a: If the line of action of several forces that act upon a rigid body goes through a point then they will be collectively equivalent to a force that acts upon that point and is equal to the geometric sum of the forces, independently of whether other forces are or are not present (cf., *C* in no. **7**).

That axiom includes the known *law of displacement:* One can displace a force that acts upon a rigid body arbitrarily along its line of action.

One adds the following converse to the parallelogram rule of forces: $d\,\mathfrak{K}$ is equivalent to $\sum d\,\mathfrak{k}$ when $d\,\mathfrak{K} = \sum d\,\mathfrak{k}$ and all forces act upon the same point. It will then follow from our axiom that:

Theorem:

One can reduce the forces that act upon a rigid body to three isolated forces that go through three given points of it (H.3).

In a known way, one can then further reduce those three forces to a force $\mathfrak{K}$ that acts upon an arbitrarily-chosen point *O* and a force-couple, i.e., two equal and opposite forces along different, but parallel, lines of action. One has:

$$\mathfrak{K} = \mathsf{S}\, d\,\mathfrak{K}, \qquad \text{which is the so-called resultant,}$$

and:

$$\mathfrak{M} = \mathsf{S}\, [\mathfrak{r}\; d\,\mathfrak{K}], \quad \text{the moment of the force-couple,}$$

in which $\mathfrak{r}$ is the vector from *O* to the point of application of the force.

Fundamental theorem:

Force-systems with equal $\mathfrak{K}$ *and* $\mathfrak{M}$ *are equivalent to each other.*

Axiom II.2.b: If $\mathfrak{K}$ or $\mathfrak{M}$, or both of them, are not equal to zero then the force-system will certainly not be equivalent to 0.

Theorem:

$\mathfrak{K}$ *and* $\mathfrak{M}$ *both being equal to* 0 *are the necessary and sufficient conditions for equilibrium of the force-system that acts upon an isolated rigid body.*

Axiom II.2.c: If $\mathfrak{K}$ and $\mathfrak{M}$ are both equal to 0 then the body will remain at rest when it starts out at rest, but otherwise, it will not.

Theorem:

$\mathfrak{K} = 0$ *and* $\mathfrak{M} = 0$ *are also the necessary and sufficient conditions for equilibrium in a rigid body that is assumed to be initially at rest.*

Axiom *II.2* will follow naturally from the two fundamental laws in $b.\alpha$ (nos. **10**,**11**) when one also knows that the reaction forces on individual rigid bodies are identical to the internal stresses, but without axiom *II.1* in $b.\alpha$ being independent of the axioms *I*. We will discuss this identity in the following subsection.

16. The statics of free systems of rigid bodies. – The fact that the internal forces are equal to the reaction forces for individual rigid bodies is clear from nos. **14** and **15**. That is because since Γ and Δ lose their meaning for rigid bodies, the internal stresses in the rigid body will forfeit all causes, except for the condition of rigidity. However, we must also probably formulate the idea that for individual rigid bodies, the reaction forces are also *only* the internal forces, i.e., the internal stresses that are distributed on surfaces. We formulate that in general as:

Axiom II.2.d: If two bodies or parts of bodies (which do not need to be rigid) contact along a surface, a curve, or at isolated points then they will exert forces on each other there that only serve to maintain the condition of contact, and possible further restrictions on the motion at the contact locations (e.g., *support reactions*). According to which of the three cases is relevant, those forces will be distributed over surfaces (stresses $\mathfrak{s}\, dF$), lines, or isolated points, resp. The last two cases are only idealizations for rigid bodies and otherwise do not occur.

Axiom II.2.e: The support reactions in the previous axiom fulfill the principle of the equality of action and reaction.

It follows from *II.2.d* that for isolated rigid bodies, the forces of reaction are only internal forces, because they maintain the contact along an imaginary section through the body and are then stresses on that surface, according to *II.2.d.*

If the bodies that contact at a point are rigid then this ideal case will necessitate a further assumption that would be unnecessary in the more general cases:

Axiom II.2.f: If two *rigid* bodies contact at an isolated point then they can also exert finite moments on each other at that point that will be equal and opposite to each other in any case.

Since we can always regard any part of a rigid body as another such thing, it will follow from Axiom *II.2.d* that there exist internal stresses $\mathfrak{s}$, that those $\mathfrak{s}$ alone will maintain the connection, and finally that:

$$\frac{d\mathfrak{R}}{dV} = \frac{\partial \mathfrak{s}_x}{\partial x} + \frac{\partial \mathfrak{s}_y}{\partial y} + \frac{\partial \mathfrak{s}_z}{\partial z}.$$

If we now consider a system of *rigid bodies* that contact each other in any way then we will have:

Theorem:

For any system of rigid bodies, the sum of the external force and the sum of their moments must be equal to zero if equilibrium is to prevail in the system of bodies or forces.

External forces are therefore the ones that are external to the individual rigid bodies, with the exception of the support forces and support moments that the rigid bodies exert upon each other (axioms *II.2.d* and *2.f*).

That theorem will follow for individual rigid bodies with no further assumptions when one adds $\mathfrak{M} = 0$ and $\mathfrak{K} = 0$ and observes axioms *II.2.e* and *2.f*.

Conversely, if one assumes that theorem as an axiom then axioms *II.2.e* and *2.f* will follow insofar as they say that for the total action of two partial bodies on each other, the sums of the forces and the sums of the moments must be equal and opposite to each other.

17. The statics of arbitrary systems. The rigidification principle.

Axiom II.2.g: For *every* subsystem, in the case of equilibrium, the sum of the external forces and the sum of their moments must be zero, or otherwise stated, one will not perturb the equilibrium when one imagines that the part has been rigidified in the position that it is found in. (The meaning of rigid bodies for general mechanics is based upon that notion.)

If one applies that axiom to an infinitesimally-small piece of matter then the symmetry of the stress dyadic will follow for the case of statics ($II.I.c'$, no. **11**) in the event that one assumes that the continuity axiom of matter (*II.I.b*) is also true for spatially-distributed forces or that those forces are stronger than at least second order in the linear dimensions of the volume element.

Since we will verify the independence of those axioms from the axioms *I.a-f* and *II.a*, *b* (cf., no. **42**), the independence of the rigidification principle will also follow from that.

18. The statics of constrained systems of rigid bodies. – We further constrain the system of rigid bodies in no. **16** by assuming that they are individually at rest or moving in some given way. We would *not* like to include those bodies in the system now, but refer to them as external support bodies (e.g., a ball rolling on the Earth, when the Earth is regarded as at being rest or moving in a known way and as being rigid). If we apply the statics of rigid bodies to the *n* bodies that now belong to the system then we will get $2n$ vectorial or $6n$ scalar equations. We imagine that we have eliminated the reaction forces form them, i.e., from the definition in nos. **15** and **16**, all forces between two contacting bodies, as long as their causes consist of only the prescribed restrictions on the motion. We will then obtain a certain number of equations and inequalities between general impressed forces that we would like to call *general equilibrium conditions for the forces on the given system*. If they are fulfilled then the force-system is said to be a system in equilibrium. The inequalities arise from the inequalities in reaction forces.

Axiom II.2.h: If two bodies contact at a point then for the three components of the support force, there will be as many reaction forces as there are restrictions on the motion.

Therefore, if no sliding takes place then three restrictions on the motion will exist, and the total support force will then be a reaction force (normal pressure and static friction). However, if sliding is present then only one restriction on the motion will be found (viz., the non-penetration of one body into the other), and therefore only one component of the support force will be a reaction force. It is determined by the single direction that is distinguished by the contact condition, so it will be perpendicular to the common tangent plane (from the axiom of sufficient grounds *C* in no. **7**); it will be a pressure.

Static friction and normal pressure are then reaction forces, while sliding friction is an impressed force. Indeed, it also depends upon not only the contact condition, but also the physical properties of the bodies. (The fact that static friction is restricted by a physically-required *inequality* does not come under consideration for the distinction that we are discussing.)

Axiom II.2.i: Analogous statements are true for the moments that can appear at isolated contact points between rigid bodies.

If boring motions, i.e., rotation around a common normal, are excluded then the corresponding components of the moment will be moments of reaction. If rolling, i.e., rotating around any axis that lies in the tangent plane, is excluded then the components of the moment in the tangent plane will be moments of reaction, and in any other case, they will be impressed moments.

13. The principle of virtual work.

Definition: Let a well-defined mechanical system be given, along with its kinematical constitution, i.e., with precise data about possible restrictions on its motion. Each point will be assigned an infinitesimal vector $\delta \mathfrak{r}$, namely, its virtual displacement. Let it be independent of time and compatible with the kinematical constitution, but otherwise arbitrary. Independent of time means: If the supporting surfaces are in motion then they remain where they are instantaneously, or $\delta t =$ and $\delta\, dt = 0$. Full regularity is assumed here: The contacting boundary surfaces for the bodies, etc., possess tangent planes without exception ([1]).

Axiom II.2.k: For all possible virtual displacements, $\mathsf{S}\, d\,\mathfrak{K}\, \delta \mathfrak{r} \leq 0$ is the necessary and sufficient equilibrium condition for the impressed forces.

Remark 1: If only invertible constraints are present or we find ourselves in the interior of regions of admissible motions then instead of taking each $\delta \mathfrak{r}$, we can also take $-\delta \mathfrak{r}$, and as a result, the equilibrium condition will read $\mathsf{S}\, d\,\mathfrak{K}\, \delta \mathfrak{r} = 0$ in that case (**Johann Bernoulli**), so the less than sign is valid for only one-sided constraints.

Remark 2: The principle can be proved for all idealized systems with constraints are known up to now (cf., no. **13**). For systems of rigid bodies, it follows from axioms *II.2* and *I* (a partial

([1]) For the irregular cases, cf., **P. Stäckel**, "Bemerkungen zum Prinzip des kleinsten Zwanges," Sitzungsber. Heidelberg Akad. (1919).

proof is in **Appell**'s *Mécanique rationelle*, a complete one in H.1), and it is known already for incompressible fluids (cf., no. **13**). The internal pressure p is the reaction force here. Since μ is constant, the equation $p = p(\mu)$ will be irrelevant. The pressure will cease to have a cause. The principle is also true for flexible or inflexible inextensible cables (cf., the following subsection for this).

One can rise from such provable cases to the more general cases by the following:

Axiom II.2.k^*: Systems with the same virtual displacements are statically equivalent.

This axiom defines the gist of **Lagrange**'s known proof for the pulley and other proofs ([1]).

Remark 3: Here, we might point out a fact that seems almost paradoxical. From no. **16**, axiom *II.2.d*, the support forces that two bodies exert upon each other along a real or imaginary separation surface will be reaction forces when no relative motion takes place at the separation surface. It is also clear that those forces collectively do no work. Nevertheless, it would be false if we were to conclude that all internal stresses on a body are reaction forces and thus do no work (cf., nos. **13** and **14**) That internal work, which was denoted by dA_i there, is then very likely to be observed in non-rigid bodies. However, one still does not need to add a special work term for an individual internal real or imaginary separation surface when no relative motion takes place. One must then be careful about how to make the transition from the individual separation surfaces to the set of all internal stresses when it comes to the work that is done. *Therefore, one cannot define the internal work done to be the sum of all works done on all conceivable separation surfaces in the interior of the body*, but one must make a special definition of it (cf., no. **12**).

20. Bodies with distinguished centerlines (cables and beams). – The unit vectors $\mathfrak{t}$, $\mathfrak{n}$, $\mathfrak{b}$ give the moving triad along the centerline. In a section that is perpendicular to the centerline, we reduce the stresses $\mathfrak{s}_1\, dF$ relative to the point of intersection with the centerline and thus get the resultant tension $Z\,\mathfrak{t}$, the resultant shearing stress $S\,\mathfrak{n} + S'\mathfrak{b}$, the resultant torsional moment $M\,\mathfrak{t}$, and the resultant bending moment $B\,\mathfrak{n} + B'\mathfrak{b}$.

We place two sections that cut out the element ds along the centerline. When reduced to the intersection point O in the first plane, the external forces on the body, which act along the piece of length ds, might have the resultant force $\mathfrak{g}\,ds$ and moment $\mathfrak{G}\,ds$. From the rigidification principle, one will then have the following two equilibrium conditions:

$$\frac{d}{dt}(Z\,\mathfrak{t} + S\,\mathfrak{n} + S'\mathfrak{b}) + \mathfrak{g} = 0\,,$$
$$\frac{d}{dt}(M\,\mathfrak{t} + B\mathfrak{n} + B'\mathfrak{b}) + S[\mathfrak{t}\,\mathfrak{n}] + S'[\mathfrak{t}\,\mathfrak{n}] + \mathfrak{G} = 0\,.$$

([1]) Cf., *Enzykl. d. math. Wiss.*, Bd. IV.1, art. 1 (**Voss**), no. 32.

Due to the fact that:

$$[\mathfrak{t}\ \mathfrak{n}] = \mathfrak{b}\ , \quad [\mathfrak{t}\ \mathfrak{b}] = -\ \mathfrak{b}\ ,$$

from the Frenet-Serret formulas ($\rho,\ \rho'$ are the radii of curvature and torsion, resp.):

$$\frac{d\mathfrak{t}}{ds} = \frac{1}{\rho}\mathfrak{n}\ , \qquad \frac{d\mathfrak{b}}{ds} = \frac{1}{\rho'}\mathfrak{n}\ , \qquad \frac{d\mathfrak{n}}{ds} = -\frac{1}{\rho}\mathfrak{t} - \frac{1}{\rho'}\mathfrak{b}\ ,$$

the fundamental formulas will assume the form:

$$\mathfrak{t}\left(\frac{dZ}{ds} - \frac{S}{\rho}\right) + \mathfrak{n}\left(\frac{dS}{ds} + \frac{1}{\rho}Z + \frac{1}{\rho'}S'\right) + \mathfrak{b}\left(\frac{dS'}{ds} - \frac{1}{\rho'}S\right) + \mathfrak{g} = 0\ ,$$

$$\mathfrak{t}\left(\frac{dM}{ds} - \frac{1}{\rho}B\right) + \mathfrak{n}\left(\frac{dB}{ds} + \frac{1}{\rho}M + \frac{1}{\rho'}B' - S'\right) + \mathfrak{b}\left(\frac{dB'}{ds} - \frac{1}{\rho'}B + S\right) + \mathfrak{G} = 0\ .$$

When decomposed into components, that will give six differential equations for Z, S, S', M, B, B', ρ, ρ'.

The cable is now called *inextensible* when the length of the centerline cannot change. From the axiom of sufficient grounds (C in no. **7**), Z will then be a force of reaction. *It will do no work in that case.*

Proof: In order to calculate the virtual work, we must multiply the left-hand side of the first equation by $\delta\,\mathfrak{r}\ ds$, and analogously multiply the second one by $\delta\,\mathfrak{d}\ ds$, add both equations, and then integrate over the length of the centerline. The $\delta\,\mathfrak{d}$ in that means a virtual rotation of the element of the cable around O. *We assume the motion of a part of it between neighboring separation surfaces is a rigid motion, at least to a sufficient approximation* (1). The integrals:

$$\int_0^l \mathfrak{g}\,\delta\mathfrak{r}\,ds \qquad \text{and} \qquad \int_0^l \mathfrak{G}\,\delta\mathfrak{d}\,ds$$

will yield the components of δA_a, which is the virtual work of the external forces. After partial integration, the other integrals will give:

$$(Z\,\mathfrak{t} + S\,\mathfrak{n} + S'\mathfrak{b})\,\delta\mathfrak{r} + (M\,\mathfrak{t} + B\,\mathfrak{n} + B'\mathfrak{b})\,\delta\mathfrak{r}\,\Big|_0^l$$

(1) All equations will always be *exact*, while only the *meaning* of the following for the energy principle possibly just an approximate one, but it will become all the more exact the more that one can regard the element as rigid (e.g., so-called infinitely-thin wires).

$$-\int_0^l (Z\,\mathfrak{t} + S\,\mathfrak{n} + S'\mathfrak{b})\frac{d\,\delta\mathfrak{r}}{ds}\,ds - \int_0^l (M\,\mathfrak{t} + B\,\mathfrak{n} + B'\mathfrak{b})\frac{d\,\delta\mathfrak{d}}{ds}\,ds + \int_0^l (S'\mathfrak{b} - S\,\mathfrak{n})\,\delta\mathfrak{d}\,ds\ .$$

The expressions that appear before the integrals represent the virtual works at the ends of the cable. Either they will vanish when $\delta\mathfrak{r} = 0$ ($\delta\mathfrak{d} = 0$, resp.) at the ends (i.e., the ends are fixed or clamped, and therefore Z, S, S' or M, N, B', resp., are reaction forces) or $\delta\,\mathfrak{r}$ and $\delta\,\mathfrak{d}$ will not be zero, and Z, …, M, … will then be impressed force magnitudes at the ends, so the terms in question will then be counted as the work δA_e, which will also include the works $\mathfrak{g}\,ds$ ($\mathfrak{G}\,ds$, resp.). The three remaining integrals define the internal work δA_e. Due to the fact that $d\,\delta\mathfrak{r} = \delta\,d\,\mathfrak{r}$ and $\mathfrak{t}\,ds = d\,\mathfrak{r}$ the single term that occurs in that with the tension Z can also be written:

$$-\tfrac{1}{2}\int_0^l Z\,\frac{\delta\,d\mathfrak{r}^2}{ds}\ .$$

It will vanish then when one has $\delta\,d\,\mathfrak{r}^2 = 0$ everywhere, i.e., the cable is inextensible.

The principle of virtual work is also proved here.

21. Continuation – *It will also be assumed that the distinguished centerline always consists of the same material points under its motion.*

The virtual $\delta\,\mathfrak{d}$ will then consist of two parts $\delta\,\mathfrak{d}_1 + \delta\,\mathfrak{d}_2$. The first of them is produced by virtual rotation of the natural coordinate system, while the second one represents the rotation of matter with respect to that system. Since the mass-element is to remain rigid, due to the first assumption, the material points that define a plane perpendicular to the distinguished line will remain perpendicular to it, such that the entire plane can only experience a material rotation into itself, i.e., one has $\delta\,\mathfrak{d}_2 = \delta\,\chi\,\mathfrak{t}$ (e.g., twisting the beam or wire).

Under this further assumption, S and S' will also be reaction forces, and indeed even when the cable is extensible. That associated motion, namely, a shear, does not take place. S and S' will do no virtual work.

Proof: In the expression for the work, S is multiplied with $-\,\mathfrak{n}\,d\,\delta\,\mathfrak{r} + \mathfrak{b}\,\delta\,\mathfrak{v}\,ds$. It is equal to:

$$-\,\mathfrak{n}\,\delta\,d\,\mathfrak{r} + \mathfrak{b}\,(\delta\mathfrak{d}_1 + \mathfrak{t}\,\delta\chi)\,ds = -\,\mathfrak{n}\,\delta(\mathfrak{r}\,ds) + \mathfrak{b}\,\delta\mathfrak{d}_1\,ds = -\,\mathfrak{n}\,ds\,\delta\mathfrak{t} + \mathfrak{b}\,\delta\mathfrak{d}_1\,ds\ .$$

However, if one has:

$$\delta\,\mathfrak{t} = [\delta\,\mathfrak{d}_1\ \mathfrak{t}]$$

then the expression will become:

$$-\mathfrak{n}\,ds\,[\delta\mathfrak{v}_1\,\mathfrak{t}] + \mathfrak{b}\,\delta\mathfrak{v}_1\,ds = ds\,(-\,\delta\mathfrak{v}_1\,[\mathfrak{t}\,\mathfrak{n}] + \mathfrak{b}\,\delta\mathfrak{v}_1) = 0\,.$$

One proves the theorem for S' in exactly the same way.

It then remains for us to examine the *internal work done by moments:*

$$-\int_0^l (M\,\mathfrak{t} + B\,\mathfrak{n} + B'\,\mathfrak{b})\,d\,\delta\mathfrak{v},$$

in which:

$$\delta\mathfrak{v} = \delta\mathfrak{v}_1 + \mathfrak{t}\,\delta\chi\,.$$

One will find the following three relations from a simple calculation:

$$\mathfrak{t}\,d\,\delta\mathfrak{v}_1 = -\,ds\,\delta\frac{1}{\rho'} - \frac{1}{\rho'}\,\delta\,ds\,, \qquad \mathfrak{n}\,d\,\delta\mathfrak{v}_1 = 0\,, \qquad \mathfrak{n}\,d\,\delta\mathfrak{v}_1 = ds\,\delta\frac{1}{\rho} + \frac{1}{\rho}\,\delta\,ds\,.$$

As a result, the internal work done by moments:

$$-\int_0^l M\,\mathfrak{t}\,\delta\chi - \int_0^l B\frac{1}{\rho}\,\delta\chi\cdot ds + \int_0^l M\left(ds\,\delta\frac{1}{\rho'} + \frac{1}{\rho'}\,\delta\,ds\right) - \int_0^l B'\left(ds\,\delta\frac{1}{\rho} + \frac{1}{\rho}\,\delta\,ds\right),$$

or when one introduces the angles of contingency and torsion $d\alpha$ and $d\beta$:

$$-\int_0^l M\,\delta\,d\chi - \int_0^l B\,\delta\chi\,d\alpha + \int_0^l M\,\delta\,d\beta - \int_0^l B'\,\delta\,d\alpha\,.$$

If one introduces a system $\mathfrak{t}$, $\mathfrak{x}$, $\mathfrak{y}$ that is comoving with the matter that is therefore rotated with respect to the $\mathfrak{t}$, $\mathfrak{n}$, $\mathfrak{b}$ system through the angle χ around the $\mathfrak{t}$-axis then one will first have to introduce the small rotations:

$$d\kappa = \sin\chi\,d\alpha\,, \qquad d\lambda = \cos\chi\,d\alpha\,, \qquad d\mu = d\chi - d\beta \tag{1}$$

($d\alpha$ is a rotation around the $\mathfrak{b}$-axis, and $-\,d\beta$ is one around the $\mathfrak{t}$-axis), and furthermore, the corresponding moments:

$$K = B\cos\chi + B'\sin\chi\,, \qquad L = B'\cos\chi - B\sin\chi\,, \qquad M\,. \tag{2}$$

It follows from (1) that:

$$\delta\,d\alpha = \delta\,d\kappa\cos\chi - \delta\,d\lambda\sin\chi\,, \qquad d\alpha\,\delta\chi = -\,\delta\,d\kappa\sin\chi - \delta\,d\lambda\cos\chi\,.$$

Therefore, one will get the following expression for the work done by the moments:

$$-\int (K\,\delta\,d\kappa + L\,\delta\,d\lambda + M\,\delta\,d\mu) \;,$$

which is well-known ([1]).

One can derive it more simply: From the known transition formulas that are already found in **Lagrange**, one has:

$$d\,\delta\mathfrak{d} = \delta\,d\,\mathfrak{d} + [d\,\mathfrak{d}\;\delta\mathfrak{d}]\;.$$

Furthermore, when δ' means the change relative to the system that is fixed in the body, which experiences the rotation $\delta\mathfrak{d}$ in its own right, from the rules of the relative motion, one will have:

$$d\,\delta\mathfrak{d} = \delta'\,d\,\mathfrak{d} + [d\,\mathfrak{d}\;\delta\mathfrak{d}]\;.$$

Thus ([2]):

$$d\,\delta\mathfrak{d} = \delta'\,d\,\mathfrak{d}\;.$$

However, since the axes $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{t}$, which are fixed in the body, will remain unchanged under δ', that will equal:

$$\mathfrak{x}\;\delta\,d\kappa + \mathfrak{y}\;\delta\,d\lambda + \mathfrak{t}\;\delta(d\chi - d\beta)\;.$$

The formula above will then follow immediately from:

$$-\int (M\,\mathfrak{t} + B\,\mathfrak{n} + B'\mathfrak{b})\,d\,\delta\mathfrak{d}\;.$$

22. Completely-flexible and stiff cables. – The completely-flexible cable is defined by $B = 0$, $B' = 0$. We would also like to take it to be inextensible, so Z, S, S' will be reaction forces. Ordinarily, one also takes the cable to be completely-twistable ($M = 0$) and $\mathfrak{G} = 0$, moreover. The last three equilibrium conditions in no. **20** then demand that $S = 0$, $S' = 0$, and after eliminating Z, the first three will give two equations for ρ and ρ', i.e., for the form of the string.

In order to treat the *stiff cable*, two theories have been proposed up to now:

First theory: Bending stiffness has the same character as static (sliding, resp.) friction: No bending at all takes place along pieces, and B and B' will be reaction moments (corresponding to $\delta\,d\alpha = 0$, $\delta\chi = 0$), just like M. However, that will be true only up to certain limits, e.g., $|\,B\,| < r\,Z$.

([1]) E. g.: **W. Thomson** and **P. G. Tait**, *Treatise on Natural Philosophy*, v. II, art. 594, Cambridge. **E.** and **F. Cosserat**, *Théorie des corps deformable*, pp. 10, *et seq.*

([2]) See **Palatini**, Annali di mat. (3) **27** (1918). A thorough presentation of the topic appeared in yearly issue 25 of the Sitz.-Ber. d. Berl. Math. Ges.

If those limits are attained, or just one of them, then bending or twisting would occur, and its sign would be determined in such a way that the internal work would be negative (H.3).

Second theory: Bending stiffness has the same character as the friction in viscous fluids. Thus, for plane motion, one will set:

$$B' = \kappa \frac{d}{dt}\frac{1}{\rho}, \qquad \kappa > 0 ,$$

so the internal power, which must be intrinsically negative, will be ([1]):

$$-\kappa \int_0^l \left(\frac{d}{dt}\frac{1}{\rho}\right)^2 ds .$$

23. Another way of constructing stereostatics. The lever law of Archimedes ([2]).

Definition: A *lever* is a rigid body that can rotate around a fixed axis and displace along it.

Axiom II.2.l:

α) Forces that intersect the axis and are perpendicular to it will not perturb the equilibrium (which follows from Axiom *C*, no. **7**).

β) Forces that are parallel to the axis will be in equilibrium when their sum is zero.

γ) One can add or remove forces on any rigid body that are in equilibrium by themselves.

As a result of that axiom, one needs to consider only forces that are perpendicular to their lever arms, i.e., the altitudes from their points of application to the axis.

Axiom II.2.l:

δ) Two forces $d\mathfrak{k}_1$, $d\mathfrak{k}_2$ that are equal to each other, have equal lever arms and opposite sense of rotation will be equivalent to zero. (This axiom will also follow from the axiom of sufficient grounds, viz., Axiom *C* in no. **7**.)

ε) (The converse of α): Forces that are in equilibrium are equivalent to a single force that intersects the axis and is perpendicular to it (equivalent in the sense of free rigid bodies).

([1]) Details are in **G. Hamel**, Zeit. angew. Math. Mech. (1927).
([2]) Compare **Duhem**, *Les origins de la statique*, esp., pp. 10, *et seq.*, Paris, 1905.

If one does not include friction among the impressed forces then this axiom would be incorrect.

Axiom II.2.l:

η) The rigidification principle: Equilibrium in any rigid body will not be perturbed by the addition of constraints.

It follows from this that: Two equal and equally-directed forces on any rigid body are equivalent to a single force that is parallel to both of them, is twice as large as them, and lies at the midpoint between them. Proof: One first assumes that one has a rotational axis that intersects that centerline perpendicularly, so the two forces will then be in equilibrium, and therefore equivalent to a force that intersects the axis perpendicularly. That will establish the line of action of the resultant. Its magnitude will be determined by β) when one assumes that one has an axis of rotation that is parallel to given forces.

If one now replaces one of the two forces in Axiom δ) with two forces with half its magnitude, one which intersects the axis, while the other possesses twice the lever arm, then one will get a lever law for two forces that have a ratio of 1 : 2.

From that, one again derives the law that every force is equivalent to two parallels, one of which has 1/3 of its magnitude, while the other has 2/3 that magnitude, and whose distances from that force have a ratio of 1 : 2.

One can proceed analogously and obtain the lever law for rational ratios.

Axiom II.2.l:

ϑ) A limiting case of an equilibrium position will also be an equilibrium position.

From that continuity axiom, one will get the lever law for irrational ratios of the forces.

One will get the general equilibrium conditions for a free rigid body from the lever law when one derives the equilibrium conditions that follow from the lever law according to the rigidification principle by introducing fixed axes for all axes, and those conditions are known to be collectively equivalent to the equilibrium conditions for free rigid bodies.

In **Marcolongo** ([1]), one will find a system of axioms for rigid bodies whose statics is likewise based upon the parallelogram law. Naturally, one can be spared a summary of the details of the latter, since some of the same ideas appear in both groups of axioms. One can also confer the work of **Georges** that was cited in no. **6**.

24. Transition to kinetics. D'Alembert's principle. – Let any idealized system be given, i.e., a system with well-defined constraints. For example, it might consist of rigid bodies with supporting surfaces, as in no. **18**, or an incompressible fluid, as in no. **13**, or an inextensible completely-flexible string or a combination of such systems.

([1]) **R. Marcolongo**, *Theoretische Mechanik*, Bd. I, German by **H. Timerding**, Leipzig, 1911.

When we imagine that the reaction forces have been eliminated from the equilibrium conditions, as was shown in no. **18**, we will get a certain system of equations and inequalities for the forces, which are generally thought to be impressed. If they are fulfilled then we will say that the force system puts the given system in equilibrium. We shall initially overlook the *inequality constraints*; we shall speak of them in the next subsection.

Now, **d'Alembert** succeeded in determining the *equations of motion* for one such system from the following principle, which is named after him:

Axiom II.2.m: While in motion, the system of so-called *lost* forces $d\,\mathfrak{K} - dm\,\mathfrak{w}$ will be in equilibrium, in the foregoing sense. (Remark: From the d'Alembertian Ansatz, the lost forces are the negative reactions $-\,d\,\mathfrak{R}$.)

Contrary to **d'Alembert**'s assumption, this axiom is not provable from Axioms *I.1* and *II.2.a-l*, but can probably be traceable to the following one (see H.3):

Axiom $II.2.m^*$: If a motion that is produced by the impressed force $d\,\mathfrak{K}$ according to the law $dm\,\mathfrak{w} = d\mathfrak{K}$ is compatible with the conditions then it will also actually occur (passivity of the reaction forces: they do not affect the constraints unnecessarily).

The proof of independence will follow later (no. **42**). The center of mass and moment laws will follow from that for isolated free bodies.

However, those theorems *will not* follow for completely-arbitrary systems when the impressed forces are internal forces since d'Alembert's principle says nothing about impressed forces.

Therefore, Boltzmann's Axiom *II.1.c* follows from d'Alembert's only for the interiors of rigid bodies, but not in general (the first misunderstanding about d'Alembert's principle). On the other hand, d'Alembert's principle is not exhausted by its application to the individual rigid bodies (second widespread misunderstanding), or what amounts to the same thing, the equilibrium conditions *do not always* read: The sum of the forces and the sum of the moments of *impressed* forces equals zero.

It is even less common that it is restricted to free point-systems, where it becomes identical to Newton's fundamental law, so it would be trivial (third misunderstanding). In and of itself, it also has nothing to do with the principle of virtual work (fourth misunderstanding).

The equilibrium conditions in question can probably be brought into the form $\mathsf{S}\; d\,\mathfrak{K}\;\delta\mathfrak{r} = 0$ using the principle of virtual work. From that, one will get the union of the two principles into $\mathsf{S}(dm\mathfrak{w} - d\,\mathfrak{K})\;\delta\mathfrak{r} = 0$ that **Lagrange** completed (once again: inequalities will be initially overlooked here).

The structure of mechanics that concerned the rigid body up to now will then give Boltzmann's axiom in general only in statics and in kinetics only for the interiors of rigid bodies. For the kinetics of more general systems, one must state specifically.

In its first beginnings [by **Jakob Bernoulli** and **de l'Hospital** ([1])], as well as in many newer textbooks, one finds a fallacy regarding d'Alembert's principle that, in reality, reduces the principle to the following two axioms:

$Axiom\, II.2m^{**}.\alpha)$: Any mechanical system will move in precisely the same way as if its points were individually attached to a massless system ("ghost") that is subject to the same kinematical conditions as the system itself. The reaction principle is valid for the action of the force between the points and the ghost.

$Axiom\, II.2m^{**}.\beta)$: Such a ghost will also satisfy the dynamical laws of statics during the motion.

Those two axioms go even further than d'Alembert's principle since they imply the full Boltzmann axiom. That is because, from α), one has for each mass-point that:

$$\mu\, \mathfrak{w} = \frac{d\mathfrak{K}}{dV} + \frac{d\mathfrak{N}}{dV},$$

in which $d\,\mathfrak{N}$ is the reaction on the part of the ghost. However, once we have introduced stresses into the ghost, we will have:

$$-\frac{d\mathfrak{K}}{dV} + \frac{d\mathfrak{N}}{dV} + \frac{\partial \mathfrak{s}_y}{\partial y} + \frac{\partial \mathfrak{s}_z}{\partial z} = 0,$$

and we will get the symmetry of the stress dyadic (see no. **17**) from the rigidification principle since the laws of statics must indeed be valid.

25. D'Alembert's principle for inequalities. – It has often been asserted that **d'Alembert**'s principle will not reach its goal for inequalities (i.e., one-sided constraints) ([2]).

Perhaps one takes the following example: A point (i.e., a rigid body when we ignore its rotational motion) is at rest on a planar surface $z = 0$. Let the half-space $z \geq 0$ be allowed. The equilibrium conditions then read:

$$X = 0, \quad Y = 0, \quad Z \leq 0.$$

With a reasonable extension of d'Alembert's principle, as **Fourier** had implemented, one will get:

$$X - m\ddot{x} = 0, \quad Y - m\ddot{y} = 0, \quad Z - m\ddot{z} \leq 0.$$

([1]) Precise information is in **Heun**, *Formeln und Lehrsätze der allgemeinen Mechanik*, Göschen, 1902, in a historical Appendix.

([2]) Literature and a more detailed exposition are in **P. Stäckel**, "Bemerkungen zum Prinzip des kleinsten Zwanges," Sitzungsber. Heidelberger Akad. (1919). Also in the book by **Brill** that will be cited in no. **33**.

It is clear that events are not determined completely by the last inequality.

Axiom.II.2.m (Fourier's extension of d'Alembert's principle): It is also true when one considers inequalities that the difference between the impressed forces and the mass-acceleration is in equilibrium, or when one adds the principle of virtual work:

$$\mathsf{S}\,(d\,\mathfrak{K} - dm\,\mathfrak{w})\,\delta\mathfrak{r} \leq 0\,.$$

Axiom.II.2.n : Upon removing one one-sided constraint, the reaction forces will become continuous functions of time, or otherwise stated: The equations that follow from d'Alembert's principle in the *interior* of the admissible region can also be applied when one begins to leave the boundary.

In the example above, that principle will obviously imply:

a) $Z - m\ddot{z} = 0$ when one leaves it since that equation will be valid in the interior of $z > 0$.

b) When one does not leave it ($z = 0$), since $\ddot{z} = 0$, one will have:

$$Z - m\ddot{z} \leq 0\,.$$

One easily overlooks the fact that the behavior is the same in all cases. Either one constraint remains justified, and d'Alembert's principle is then satisfied, or the constraint breaks down, and the continuity axiom *II.2.n* will then be in force.

If one would like to determine the initial separation without investigating the further motions then d'Alembert's principle, which is applied at only that moment, will generally not suffice (see also no. **45** on this).

γ) Structure of mechanics when one starts from points.

26. Axioms of point mechanics.

Axiom II.3.a: Any mechanical system consists of a finite number of discrete mathematical points that are each endowed with a finite mass m_i .

The Stieltjes integral $\mathsf{S}\,dm\,\mathfrak{w}$ will then become the finite sum:

$$\sum_i m_i\,\mathfrak{w}_i\,.$$

Axiom II.3.b: Likewise, the forces are assigned to the mass-points as finite vectors $\mathfrak{k}_{i,l}$ (they are applied to those points). The fundamental Newtonian equation then reads:

$$m_i \, \mathfrak{w}_i = \sum_i \mathfrak{k}_{i,l} ,$$

in which the sum on the right-hand side extends over the forces that act upon the i^{th} point.

Axiom II.3.c: The causes of the $\mathfrak{k}_{i,l}$ lie in the *other* points m_i .

The sum above will then extend over all mass-points, with the exception of $l = i$.

Axiom II.3.d: The first full reaction principle:

$$\mathfrak{k}_{i,l} = - \mathfrak{k}_{l,i} .$$

Axiom II.3.e: The second full reaction principle:

$$[\mathfrak{r}_i \, \mathfrak{k}_{i,l}] + [\mathfrak{r}_l \, \mathfrak{k}_{l,i}] = 0 ,$$

or: The forces $\mathfrak{k}_{i,l}$ and $\mathfrak{k}_{l,i}$ lie along the connecting line between the two points.

Definition: For a system of points, $\mathfrak{k}_{i,l}$ is called an *internal* force when it is attached to the i^{th} point of the system, as well as the l^{th} one, while the other forces are called *external* forces.

One immediately obtains the *center of mass* and *moment* laws from those axioms and the definition in a known way, so the two fundamental laws, along with the third one, namely, the *law of energy*. However, one needs the reaction principle in a formulation that includes distant forces.

However, the development of the concept of the stress tensor from this picture has still not been achieved satisfactorily. One finds an earnest attempt at doing that in **Love** ([1]).

The idea is perhaps the following one: One imagines the number of points being as being very large and the points being very dense, such that there will still be a very large number of points in a volume ΔV that is very small, in practice. The forces that go though a small surface patch ΔF in the volume element ΔV will combine into a resultant $\mathfrak{s}\, \Delta F$. The stress $\mathfrak{s}$ is then determined for ΔF. Although finite ΔF and ΔV are required for the construction, afterwards one treats both of them as differentials, i.e., one passes to the limit, or in other words, one employs the point picture in order to conveniently arrive at the fundamental laws and then abandon that picture.

([1]) **Love**, *Theoretical Mechanics*, 1st ed., Cambridge. Older literature is in *Enzykl. d. math. Wiss.*, Bd. IV, art. 23 (**Müller-Timpe**), and in particular nos. 2.a, 4.b, 4.c, 5.a.

One will then get the Boltzmann axiom by applying the moment theorem to such a differential dV, whose mass is also to be treated as differentially small.

Obviously, the distinction between forces at a distance and contact forces (surface forces) really should be dropped. However, since one cannot do without those forces, one must proceed consistently in such a way that one resolves the point-clouds into individual ΔV and tries to derive the contact action from the action of forces on neighboring ΔV alone. One would then get difference equations for continuum mechanics, rather than differential equations. However, an *exact* implementation of that line of reasoning probably does not exist yet.

δ) The Lagrangian structure of mechanics.

27. The liberation principle. – We shall assume Axioms *I* here. Furthermore, we shall assume the definition of reaction forces in no. **15**, and therefore the so-called d'Alembertian Ansatz:

$$dm\,\mathfrak{w} = d\,\mathfrak{K} + d\,\mathfrak{R}\,.$$

However, our goal now is to gradually gain some insight into the reaction forces, rather than making intuitive statements about them from the outset, as we did before (as in *II.1.b* or *II.2.d.e*, *f*).

However, the further and more important goal is to gain some information about certain impressed forces from a liberation principle, and to then develop the concept of the stress tensor from that, instead of placing it at the forefront, as in no. **10**.

Moreover, the *principle of virtual work* and d'Alembert's principle will be assumed in combination:

$$\mathsf{S}\,dm\,\mathfrak{w}\,\delta\mathfrak{r} = \mathsf{S}\,d\mathfrak{K}\,\delta\mathfrak{r}\,, \qquad \delta t = 0\,, \qquad \delta\,dt = 0$$

(we shall overlook inequalities).

A finite or even infinite number of parameters q_1, q_2, … might give the general configuration of the system. Moreover, a finite or infinite number of constraint equations might exist:

$$\sum_i f_{i,l}\,dq_i + g_l\,dt = 0\,.$$

As is known, one must add the sum:

$$\sum_l \lambda_l \sum_i f_{i,l}\,\delta q_i$$

to the principle of virtual work and then treat the δq_i as arbitrary. The λ_l are initially just purely-mathematical devices.

Now, the liberation principle says:

Axiom II.d: If one juxtaposes the constrained system with a free one, in which one drops all constraints, then formally the same equations of motion will be valid, except that now the *l* will be magnitudes of impressed forces, or more precisely, they will be determined in such a way that their virtual work is:

$$\sum_{i,l} \lambda_l \, f_{i,l} \, \delta q_i \, .$$

Up until then, they would have been reaction forces (H.4).

28. Special cases. First of all: The completely-flexible, inextensible string. – This is a system with a distinguished centerline and forces that are distributed along it. It can move freely, and the length of each piece of it is unvarying.

Let $dm = \mu \, ds$, where ds is the arc-length element, and $d\,\mathfrak{k} = \mathfrak{g}\, ds$ is the force. One will then have to integrate over the length l :

$$\int_0^l \mu\, ds\, \mathfrak{w}\, \delta \mathfrak{r} = \int_0^l \mathfrak{g}\, \delta \mathfrak{r}\, ds \, ,$$

with $\delta\, ds = 0$. If the ends are fixed then one will also have:

$$\delta \mathfrak{r}_0 = 0 \, , \qquad \delta \mathfrak{r}_l = 0 \, .$$

One must then set:

$$\int_0^l \mu\, ds\, \mathfrak{w}\, \delta \mathfrak{r} - \int_0^l \mathfrak{g}\, \delta \mathfrak{r}\, ds + \int_0^l \lambda\, \delta\, ds + \mathfrak{L}_0\, \delta \mathfrak{r}_0 + \mathfrak{L}_l\, \delta \mathfrak{r}_l = 0 \, ,$$

and now and then treat all $\delta\,\mathfrak{r}$ as if they were free. However, if $\delta s^2 = d\,\mathfrak{r}^2$, so $ds\ \delta\, dt = d\,\mathfrak{r}\ \delta\, d\,\mathfrak{r}$, and if one assumes that the unit vector t points in the direction of increasing arc-length, $\delta\, ds = \mathfrak{t}\, \delta\, d\mathfrak{r}$ then:

$$\int_0^l \lambda\, \delta\, ds = \int_0^l \lambda\, \mathfrak{t}\, \delta\, d\mathfrak{r} = \int_0^l \lambda\, \mathfrak{t}\, d\, \delta \mathfrak{r} = \lambda\, \mathfrak{t}\, \delta \mathfrak{r}\Big|_0^l - \int_0^l \frac{d(\lambda\, \mathfrak{t})}{ds}\, \delta \mathfrak{r}\, ds \, .$$

One will then get the equations:

$$\mu\, \mathfrak{w} = \mathfrak{g} + \frac{d}{ds}(\lambda\, \mathfrak{t}) \, , \qquad \mathfrak{L}_0 = (\lambda\, \mathfrak{t})_0 \, , \qquad \mathfrak{L}_l = -\,(\lambda\, \mathfrak{t})_l \, .$$

Thus, for extensible strings, from the liberation principle, l t will be an impressed force, namely, the tangential tension. $-\,\mathfrak{L}_0$ and $-\,\mathfrak{L}_l$ are the external forces of tension at the ends that fix (move, resp.) the string.

29. Secondly: The infinitely-thin wire. – From now on, we shall deal with a body with a distinguished centerline. However, let it now be initially regarded as rigid, which can be characterized most simply by:

$$\delta\, ds = 0\,, \qquad \delta \frac{1}{\rho} = 0\,, \qquad \delta \frac{1}{\rho'} = 0\,. \tag{1}$$

It might be further assumed that the elements of the system that are cut out by planes perpendicular to the centerline move only like rigid bodies and can also rotate about the centerline such that they will always consist of the same material points.

[D.H.D.: (2) seems to appear out of nowhere here, but with no associated equation.] (2)

Finally, a rotation of the element around the centerline with respect to the natural system along the centerline shall also be excluded:

$$\delta \chi = 0\,, \tag{3}$$

such that the system is initially completely-rigid.

The mathematical formulation of (2) can be presented independently of (1) and (3) as follows: The virtual rotation of an element is:

$$\delta \mathfrak{d} = \delta \mathfrak{d}_1 + \mathfrak{t}\, \delta \chi\,,$$

in which $\delta \mathfrak{d}_1$ is the rotation of the natural coordinate system under any displacement $\delta \mathfrak{r}$ of the centerline. However, as one easily calculates, one now has:

$$\delta \mathfrak{d}_1 = \mathfrak{t}\, \rho \cdot \mathfrak{b} \frac{d^2 \delta \mathfrak{r}}{ds^2} - \mathfrak{n} \cdot \mathfrak{b} \frac{d\, \delta \mathfrak{r}}{ds} + \mathfrak{b} \cdot \mathfrak{n} \frac{d\, \delta \mathfrak{r}}{ds}\,.$$

Thus, (2) demands that:

$$\mathfrak{n}\, \delta \mathfrak{d} = \mathfrak{n}\, \delta \mathfrak{d}_1 = -\, \mathfrak{b} \frac{d\, \delta \mathfrak{r}}{ds}\,, \tag{2.a}$$

$$\mathfrak{b}\, \delta \mathfrak{d} = \mathfrak{b}\, \delta \mathfrak{d}_1 = \mathfrak{n} \frac{d\, \delta \mathfrak{r}}{ds}\,. \tag{2.b}$$

The external force $\mathfrak{g}\, ds$ and the external moment $\mathfrak{G}\, ds$ might act on the element *ds*. The Lagrangian factors for (1) should be:

$$-\, Z - \frac{1}{\rho} B' + \frac{1}{\rho'} M\,, \quad -B'\, ds\,, \quad \text{and} \quad M\, ds\,,$$

while the Lagrangian factor for (3) should be:

$$-\frac{1}{\rho}B'\,ds + \frac{dM}{ds}\,ds\,,$$

but the factors for (2.a) [(2.b), resp.] are:

$$-\,S'\,ds \qquad [S\,ds, \text{resp.}].$$

The principle of virtual work for statics will then imply that:

$$\int \mathfrak{g}\,\delta\mathfrak{r}\,ds + \int \mathfrak{G}\,\delta\mathfrak{d}\,ds + \int\left(-Z - \frac{1}{\rho}B' + \frac{1}{\rho'}M\right)\delta\,ds + \int -B'\,\delta\frac{1}{\rho}\,ds + \int M\,\delta\frac{1}{\rho'}\,ds$$
$$+\int\left(-\frac{1}{\rho}B' + \frac{dM}{ds}\right)\delta\chi\,ds + \int -S'\left(\mathfrak{n}\,\delta\mathfrak{d} + \mathfrak{b}\frac{d\,\delta\mathfrak{r}}{ds}\right)ds + \int S\left(\mathfrak{b}\,\delta\mathfrak{d} - \mathfrak{n}\frac{d\,\delta\mathfrak{r}}{ds}\right)ds = 0\,.$$

From no. **21**, one has:

$$\delta\frac{1}{\rho}ds = 0 = \mathfrak{b}\;d\;\delta\mathfrak{d}_1 - \frac{1}{\rho}\delta\,ds = \mathfrak{b}\;d\;\delta\mathfrak{d} - \frac{1}{\rho}\delta\,ds\,,$$
$$\delta\frac{1}{\rho'}ds = 0 = -\,\mathfrak{t}\;d\;\delta\mathfrak{d}_1 + \frac{1}{\rho'}\delta\,ds = -\,\mathfrak{t}\;d\;\delta\mathfrak{d} + \frac{1}{\rho'}\delta\,ds\,,$$
$$\mathfrak{n}\;d\;\delta\mathfrak{d}_1 = 0\,,$$

so due to the fact that:

$$d\;\delta\mathfrak{d} = d\;\delta\mathfrak{d}_1 + \mathfrak{t}\;d\;\delta\chi + \frac{1}{\rho}\,\mathfrak{n}\,ds\,\delta\chi\,,$$

one will have:

$$\frac{1}{\rho}\;\delta\chi\,ds = \mathfrak{n}\;d\;\delta\mathfrak{d}\,.$$

Finally, one has:

$$\int \mathfrak{g}\,\delta\mathfrak{r}\,ds + \int \mathfrak{G}\,\delta\mathfrak{d}\,ds - \int Z\,\mathfrak{t}\,d\,\delta s - \int B'\,\mathfrak{b}\,d\,\delta\mathfrak{d} + \int\frac{dM}{ds}\,\delta\chi\,ds + \int M\,(-\,\mathfrak{t}\,d\,\delta\mathfrak{d} + d\delta\chi)$$
$$-\int B\,\mathfrak{n}\,d\,\delta\mathfrak{d} + \int S'\left(\mathfrak{n}\,\delta\mathfrak{d} + \mathfrak{b}\frac{d\,\delta\mathfrak{r}}{ds}\right)ds - \int S\left(\mathfrak{b}\,\delta\mathfrak{d} - \mathfrak{n}\frac{d\,\delta\mathfrak{r}}{ds}\right)ds = 0\,.$$

Partial integration and treating $\delta\,\mathfrak{r}$ and $\delta\,\mathfrak{d}$ as arbitrary quantities will yield:

$$\mathfrak{g} + \frac{d}{ds}(Z\,\mathfrak{t} + S'\,\mathfrak{b} + S\,\mathfrak{n}) = 0\,,$$

$$\mathfrak{G} + \frac{d}{ds}(B'\mathfrak{b} + M\,\mathfrak{t} + B\mathfrak{n}) + S\mathfrak{b} - S'\mathfrak{n} = 0 .$$

However, those are the same equations as in no. **20**.

We will now liberate the rigid line, i.e., remove (1) and (3), while (2) still remains. B, B', M, and Z will then be force magnitudes whose meanings are clear, while S and S' remain reaction forces. For that reason, it will be obvious that when one eliminates S and S', one will need to pose only four equilibrium conditions, rather than six. That can be found in no. 21 of the cited book by **E.** and **F. Cosserat**, pp. 41.

Lagrange made the same considerations as above, but without observing the possibility of twisting $\delta\chi$. Moreover, he replaced the conditions (1) with the equivalent ones:

$$\delta\, d\,\mathfrak{r}^2 = 0 , \qquad \delta\, d^2\,\mathfrak{r}^2 = 0 , \qquad \delta\, d^3\,\mathfrak{r}^2 = 0 .$$

Finally, he exhibited the concluding equation by eliminating S and S'. We will get the same thing when we replace $\delta\mathfrak{d}_1$ with the expression above in terms of $\delta\mathfrak{r}$ everywhere without introducing S and S'.

Our considerations will require some changes for the instantaneously-straight line ($1 / \rho = 0$) since $1/\rho'$ will loses its meaning then (1).

30. Remark concerning two-dimensional bodies. – Just as we sketched out a theory in nos. **20**, **21**, **22**, and **28**, **29** of bodies with distinguished centerlines, and therefore arrived at the foundations for a mechanics of strings, cable, and wires, we can also devise a theory of bodies with a distinguished middle surface. That has led us to the mechanics of membranes, shells, and capillarity (2).

We would like to content ourselves with that reference to it.

31. Thirdly: The ideal incompressible fluid. – This is a three-dimensional system with the restriction on its motion that:

$$\text{div}\,\mathfrak{v} = 0 , \quad \text{so also} \quad \text{div}\,\delta\mathfrak{r} = 0 .$$

Let a normally-directed velocity be made impossible on the boundary by way of fixed walls:

$$\mathfrak{n}\,\mathfrak{v} = 0 , \qquad (\mathfrak{n}\,\delta\mathfrak{r} = 0, \text{ resp.}).$$

Thus, one will have:

(1) A somewhat-different presentation of that appeared in the treatise that was announced in no. **21**.

(2) For this, one might confer the known textbooks on the theory of elasticity (**Love**, **Föppl**), and furthermore the cited book by **E.** and **F. Cosserat**, as well as *Enzykl. d. math. Wiss.*, Bd. V, art. 9 (**Minkowski**), and *ibidem*, Bd. IV. IV, art 30 (**Hellinger**), no. 12.

$$\underset{V}{\mathsf{S}}\, dm\, \mathfrak{w}\, \delta\mathfrak{r} = \underset{V}{\mathsf{S}}\, \mathfrak{q}\, dV\, \delta\mathfrak{r} + \underset{V}{\mathsf{S}}\, \lambda\, \mathrm{div}\, \delta\mathfrak{r}\, dV + \underset{O}{\mathsf{S}}\, \lambda'\, \mathfrak{n}\, \delta\mathfrak{r}\, dF\,,$$

or since:

$$\underset{V}{\mathsf{S}}\, \lambda\, \mathrm{div}\, \delta\mathfrak{r}\, dV = \underset{O}{\mathsf{S}}\, \lambda\, \delta\mathfrak{r}\, \mathfrak{n}\, dF - \underset{V}{\mathsf{S}}\, \mathrm{grad}\, \lambda\, \delta\mathfrak{r}\, dV\,,$$

$$\underset{V}{\mathsf{S}}\, dV\, (\mu\, \mathfrak{w} - \mathfrak{q} + \mathrm{grad}\, \lambda)\, \delta\mathfrak{r} - \underset{O}{\mathsf{S}}\, dF\, (\lambda_0 + \lambda')\, \mathfrak{n}\, \delta\mathfrak{r} = 0\,.$$

It will then follow from the interior equations that:

$$\mu\, \mathfrak{w} = -\, \mathfrak{q} - \mathrm{grad}\, \lambda\,,$$

and on the outer surface:

$$\lambda_0 + \lambda' = 0\,.$$

From the liberation principle, since $\delta A = \lambda'\, \mathfrak{n}\, \delta\mathfrak{r}\, dF$ is a type of work that is > 0 when $\lambda' > 0$ and under outward motion, $-\lambda'$ will be the external pressure, so it will also be λ_0, and λ will be internal pressure.

32. Fourth: The rigid body and the general system. – From no. **13**, the following equations are true for the virtual displacements of the rigid body:

$$\frac{\partial\xi}{\partial x} = 0\,, \qquad \frac{\partial\eta}{\partial y} = 0\,, \qquad \frac{\partial\zeta}{\partial z} = 0\,,$$

$$\frac{\partial\xi}{\partial y} + \frac{\partial\eta}{\partial x} = 0\,, \qquad \frac{\partial\eta}{\partial z} + \frac{\partial\zeta}{\partial y} = 0\,, \qquad \frac{\partial\zeta}{\partial x} + \frac{\partial\xi}{\partial z} = 0\,.$$

Let the body be fixed on the surface, so $(\delta\mathfrak{r})_0 = 0$. One will then have:

$$\underset{V}{\mathsf{S}}\, dm\, \mathfrak{w}\, \delta\mathfrak{r} = \underset{V}{\mathsf{S}}\, \mathfrak{q}\, dV\, \delta\mathfrak{r} - \underset{V}{\mathsf{S}}\left[\lambda_{xx}\frac{\partial\xi}{\partial x} + \lambda_{xy}\left(\frac{\partial\eta}{\partial x} + \frac{\partial\xi}{\partial y}\right) + \cdots\right] dV + \underset{O}{\mathsf{S}}\, \mathfrak{L}\, \delta\mathfrak{r} = 0$$

(by definition, let $\lambda_{xy} = \lambda_{yx}$, etc.), or since one has:

$$\underset{V}{\mathsf{S}}\left(\lambda_{xx}\frac{\partial\xi}{\partial x} + \cdots\right) dV = \underset{O}{\mathsf{S}}\, \{[\lambda_{xx}\cos(n, x) + \lambda_{xy}\cos(n, y) + \lambda_{xz}\cos(n, z)]\}$$

$$-\, \underset{V}{\mathsf{S}}\left\{\xi\left(\frac{\partial\lambda_{xx}}{\partial x} + \frac{\partial\lambda_{xy}}{\partial y} + \frac{\partial\lambda_{xz}}{\partial z}\right) + \cdots\right\} dV\,,$$

the equations will follow:

$$\mu \frac{d^2 x}{dt^2} = X + \frac{\partial \lambda_{xx}}{\partial x} + \frac{\partial \lambda_{xy}}{\partial y} + \frac{\partial \lambda_{xz}}{\partial z}, \qquad \text{etc.,}$$

$$L_x = [\lambda_{xx} \cos(n, x) + \lambda_{xy} \cos(n, y) + \lambda_{xz} \cos(n, z)]_0, \qquad \text{etc.}$$

With that, one has clearly introduced the surface stresses $\mathfrak{L} = L_x\,\mathfrak{i} + L_y\,\mathfrak{j} + L_z\,\mathfrak{k}$ and the internal stresses λ. From the outset, one has $\lambda_{xy} = \lambda_{yx}$, so the Boltzmann axiom is fulfilled. *The mechanics of general systems is thus achieved by means of the liberation principle*. The Lagrangian path then *leads to the goal entirely.* It is equivalent to the first path. However, the consideration in no. **32** no longer goes back to **Lagrange**, but to **G. Piola** (1845) (¹).

ε) Energetic structure of mechanics.

33. The Hertzian and Gaussian principles for special systems. – Of the axioms that were posed up to now, we assume that only Axioms *I.a* to *c* are fulfilled, but not Newton's fundamental law *I.f*, nor *I.d* or *I.e*, which speak of forces. We add the definition of kinetic energy E and the Stieltjes integral $E = \frac{1}{2}\mathsf{S}\, dm\, \mathfrak{v}^2$, and the acceleration function $S = \frac{1}{2}\mathsf{S}\, dm\, \mathfrak{w}^2$. Constraints might exist. We also set:

$$E = \tfrac{1}{2} m \left(\frac{ds}{dt}\right)^2, \tag{1}$$

with $m = \mathsf{S}\ dm$, which defines ds.

Since:

$$\mathfrak{v} = \frac{d\mathfrak{r}}{ds}\frac{ds}{dt}$$

and

$$\mathfrak{w} = \frac{d^2\mathfrak{r}}{ds^2}\left(\frac{ds}{dt}\right)^2 + \frac{d\mathfrak{r}}{ds}\frac{d^2 s}{dt^2},$$

one will have:

$$S = \tfrac{1}{2}\,\mathsf{S}\, dm \left[\frac{d^2\mathfrak{r}}{ds^2}\left(\frac{ds}{dt}\right)^2 + \frac{d\mathfrak{r}}{ds}\frac{d^2 s}{dt^2}\right]^2$$

$$= \tfrac{1}{2}\left(\frac{ds}{dt}\right)^4 \mathsf{S}\, dm \left(\frac{d^2\mathfrak{r}}{ds^2}\right)^2 + \frac{d^2 s}{dt^2}\left(\frac{ds}{dt}\right)^2 \mathsf{S}\, dm\, \frac{d^2\mathfrak{r}}{ds^2}\frac{d\mathfrak{r}}{ds} + \tfrac{1}{2}\left(\frac{d^2 s}{dt^2}\right)^2 \mathsf{S}\, dm \left(\frac{d\mathfrak{r}}{ds}\right)^2.$$

However, from (1), one has:

$$\tfrac{1}{2}\,\mathsf{S}\, dm \left(\frac{d\mathfrak{r}}{ds}\right)^2 = m,$$

(¹) Cf., *Enzykl. d. math. Wiss.*, Bd. IV, art. 23 (**Müller-Timpe**), pp. 23, as well as *ibidem*, Bd. IV, art. 30 (**Hellinger**), pp. 620.

and therefore:

$$\mathsf{S}\, dm\, \frac{d\mathfrak{r}}{ds}\frac{d^2\mathfrak{r}}{ds^2} = 0\,.$$

Thus:

$$S = \tfrac{1}{2}\left(\frac{ds}{dt}\right)^4 \mathsf{S}\, dm\left(\frac{d^2\mathfrak{r}}{ds^2}\right)^2 + \tfrac{1}{2}\, m\left(\frac{d^2 s}{dt^2}\right)^2.$$

We imagine that the motions take place in a finite or also infinite-dimensional space, call ds the *arc-length*, $\mathfrak{r} = \mathfrak{r}\,(s)$, the *path*, $\frac{d^2 s}{dt^2}$, the *path acceleration*, and set:

$$\mathsf{S}\, dm\left(\frac{d^2\mathfrak{r}}{ds^2}\right)^2 = \frac{m}{\rho^2}\,.$$

ρ is called the *radius of curvature* of the path. Obviously, ρ and s depend upon only the paths of motion, but not upon the time interval in which the motion plays out. Space and time then appear separately in the formulas:

$$E = \tfrac{1}{2}\, m\left(\frac{ds}{dt}\right)^2,$$

$$S = \tfrac{1}{2}\, m\left[\frac{1}{\rho^2}\left(\frac{ds}{dt}\right)^4 + \left(\frac{d^2 s}{dt^2}\right)^2\right].$$

Axiom II.5.a.α: For a free, purely-mechanical, closed system, the acceleration is determined for an instantaneously-given configuration and velocity in such a way that S is a minimum (**Gauss**'s principle), or what obviously amounts to the same thing, that E = const. and the curvature $1 / \rho$ is a minimum (*energy principle and Hertz's principle of the straightest path*). (That is because since S is the sum of two squares, it will obviously be a minimum when every term is as small as possible, so $1 / \rho$ will be a minimum and $\frac{d^2 s}{dt^2} = 0$, since ds / dt is given.

If we find that we are inside of the domain of admissible motions, such that the inverse of every motion is possible, then it will follow from the axiom that:

$$\mathsf{S}\, dm\, \mathfrak{w}\, \delta\mathfrak{w} = 0\,.$$

Now, if any admissible displacement is expressed by means of the free parameters $\mathfrak{q}$ as:

$$d\,\mathfrak{r} = \sum \mathfrak{a}_\lambda\, d\mathfrak{q}_\lambda + \mathfrak{b}\, dt\,,$$

so

$$\delta\mathfrak{r} = \sum \mathfrak{a}_\lambda \, \delta\mathfrak{q}_\lambda ,$$

and correspondingly:

$$\mathfrak{v} = \sum \mathfrak{a}_\lambda \, \dot{\mathfrak{q}}_\lambda + \dot{\mathfrak{b}} ,$$
$$\mathfrak{w} = \sum \mathfrak{a}_\lambda \, \ddot{\mathfrak{q}}_\lambda + \mathfrak{w}_0(\dot{\mathfrak{q}}, \ddot{\mathfrak{q}}, t) ,$$

then

$$\delta\mathfrak{w} = \sum \mathfrak{a}_\lambda \, \delta\ddot{\mathfrak{q}}_\lambda ,$$

and due to the freedom in $\delta\ddot{\mathfrak{q}}_\lambda$, it will follow that:

$$\mathsf{S} \, dm \; \mathfrak{w} \; \mathfrak{a}_\lambda = 0 ,$$

which is identical to:

$$\mathsf{S} \, dm \; \mathfrak{w} \; \delta\mathfrak{r} = 0 ,$$

which is d'Alembert's principle for force-free motion.

In the interior of the domain of admissible motion, Axiom II.5.a.α will be valid, as well as d'Alembert's principle for force-free motion [free of impressed forces]. It is further valid on the boundary (1):

Axiom II.5.a.β: General systems can be reduced to the ones that were characterized by *II.5.a.α* by adding ideal masses.

A conclusive examination of the extent to which that axiom of **Hertz** can serve as the foundation for a realistic theory of mechanics is still lacking, despite many isolated investigations (2).

24. Gauss's principle for general systems. – One can generalize the argument in the following way:

Axiom II.5.a. – Let $d\,\mathfrak{K}$ denote the impressed force that is applied to the volume element dV. The instantaneous acceleration for a given configuration and velocity will then be determined from Gauss's principle of least constraint:

$$\mathsf{S} \, dm \left(\mathfrak{w} - \frac{d\mathfrak{K}}{dm} \right)^2 = \text{minimum.}$$

(1) See **P. Stäckel**, Sitzungsber. Heidelberger Akad. (1919).

(2) For **Hertzian** mechanics, see, in particular: **A. Brill**, *Vorlesungen zur Einführung in die Mechanik raumerfüllender Massen*, Leipzig and Berlin, 1909.

The same method of proof as in the previous subsection will imply d'Alembert's principle for general forces in the interior of the domain of admissible motions (at least in regular cases). The principle is further true on the boundary, even in the irregular cases (cf., no. **25**. cf., also the work by **Stäckel** that was cited in the previous subsection).

35. The energy principle. – Axioms *II.5.a* and *b* will be replaced by other axioms:

Axiom II.5.c.α : Every closed, purely-mechanical system (conservative system) can be associated with a function U of the coordinates of all points of the system such that:

$$E + U = h = \text{const.} \qquad (\textit{energy principle}).$$

U is called the *potential energy* of the system.

It then follows from this that: If the system is instantaneously at rest ($E = 0$) and U is a minimum then the system will remain at rest. That is because in the other cases, E and U will increase, which would be incompatible with the previous axiom, or:

$$\delta U > 0 \qquad \text{or} \qquad \delta U = 0\,, \qquad \delta^2 U > 0\,, \qquad \text{etc.}$$

are *sufficient* conditions for equilibrium.

Axiom II.5.c.β : $\delta U \geq 0$ for all virtual displacements is a *necessary* condition for equilibrium. It will then follow that in the interior of the admissible domain, $\delta U \leq 0$ is also necessary, so $\delta U = 0$ is a necessary and sufficient condition for equilibrium.

One can best recognize that this axiom is necessary (contrary to what is asserted in many textbooks) as follows: It will be false when one allows static friction. Equilibrium can also prevail when $\delta U < 0$ then. It is only by means of axiom *II.5.c.*β that the concept of a conservative system will be made precise, i.e., one that excludes friction.

Axiom II.5.c.γ : One has:

$$\delta U = \mathsf{S}\, dm\, \nabla u\, \delta \mathfrak{r},$$

in which u can be a finite scalar function that is associated with the coordinates of the individual mass-points.

$d\,\mathfrak{K} = -\,dm\,\nabla u$ is then called the *force* that acts upon the point dm and originates in the potential U.

Some famous examples of the application of the energy principle are:

1. The treatment of equilibrium in a cable on an inclined plane by **Simon Stevin**.

2. **Torricelli**'s law: A completely-flexible massive chain hangs in such a way that the center of mass lies as low as possible.

36. The principle of least action. – In the non-static case, one can regard the energy principle as saying that it gives the elapsed time along a known path. For one degree of freedom, it will suffice completely to determine the motion. For more than one degree of freedom, the path can be given independently of that principle by the following axiom:

Axiom II.5.c.δ: **Jacobi's principle of least action.** – The path is determined from:

$$\delta \int d\sigma = 0 ,$$

in which one sets:

$$d\,\sigma = \sqrt{(h-U)\,2E}\,dt .$$

The variation is performed with fixed limits and can then be understood to mean that that $\delta\,\mathfrak{r}$ represents a possible displacement, i. e, if every point *P* on the actual path is assigned a neighboring point *Q* then $\delta\,\mathfrak{r} = \overline{PQ}$ will be a possible displacement, but the set of all *Q* does not need to be a possible path.

It is only when Q defines a possible neighboring point – i.e., the system is holonomic – that the Jacobi axiom will be identical to the one that says that the integral $\int d\sigma$ *is a minimum in a sufficiently-small interval.*

Since $h - U = E$, the principle can also be formulated (**Euler**) as:

$$\delta \int_{t_1}^{t_2} 2E\,dt = 0 .$$

However, *t* must be varied in that *dt* in such a way that $E + U = h$ always remains constant. The principle can be once more converted into:

$$\int_{t_1}^{t_2} \left(2E\frac{\delta\,dt}{dt} + 2\,\delta E \right) dt = 0$$

or into:

$$\int_{t_1}^{t_2} \left(2E\frac{\delta\,dt}{dt} + \delta E - \delta U \right) dt = 0 .$$

Now, we assert that $\delta E + 2E\frac{\delta\, dt}{dt} = \delta' E$, where time *is not* varied in δ'.

Indeed, one will have:

$$\delta E = \delta \tfrac{1}{2}\mathsf{S}\, dm\, \mathfrak{v}^2 = \mathsf{S}\, dm\, \mathfrak{v}\, \delta\mathfrak{v} = \mathsf{S}\, dm\, \mathfrak{v}\frac{\delta\, d\mathfrak{r}}{dt} = \mathsf{S}\, dm\, \mathfrak{v}\left(\frac{\delta\, d\mathfrak{r}}{dt} - \frac{d\mathfrak{r}\,\delta\, d\mathfrak{r}}{dt^2}\right)$$

$$= \mathsf{S}\, dm\, \mathfrak{v}\,\delta'\mathfrak{v} - \mathsf{S}\, dm\, \mathfrak{v}^2\, \frac{\delta\, dt}{dt} = \delta' E - 2E\frac{\delta\, dt}{dt},$$

which was to be proved.

We then find that this is equivalent to **Hamilton**'s principle:

$$\int_{t_1}^{t_2} (\delta' E - \delta' U)\, dt = 0$$

or

$$\delta' \int_{t_1}^{t_2} (E - U)\, dt = 0 .$$

Time *is not varied* in that. Since:

$$\delta' U = \delta U = \mathsf{S}\, dm\, \nabla u\, \delta\mathfrak{r} = -\mathsf{S}\, d\,\mathfrak{K}\, \delta\mathfrak{r}$$

one can also write Hamilton's principle as:

$$\int_{t_1}^{t_2} \left(\delta' E + \mathsf{S}\, d\mathfrak{K}\, \delta\mathfrak{r}\right) dt = 0 .$$

In this form, it will be identical to the **Lagrangian** principle (d'Alembert's principle, plus the principle of virtual work) in the interior of the domain of admissible motions, because that principle reads (cf., no. **12**):

$$\frac{d}{dt}(\mathsf{S}\, dm\, \mathfrak{v}\delta\mathfrak{r}) = \mathsf{S}\, d\,\mathfrak{K}\, \delta\mathfrak{r} ,$$

from which the statement above will follow upon integration and fixing $\delta\mathfrak{r} = 0$ at the limits with no further assumptions. In that way, *one can construct a general theory of mechanics when one either* admits general forces $\delta\mathfrak{r}$ that do not originate in a potential *or expresses the axiom:*

Axiom II.5.c.η : One obtains non-conservative systems by assuming ideal motions (i.e., the motion of molecules when they are heated), and in such a way that U can also depend upon time.

In the latter case, Hamilton's principle is still valid, but the energy principle will be valid only if one adds further forms of energy that are more ideal for mechanics.

As is known, the energy principle follows from Hamilton's principle for conservative systems.

Remark: The identity of Hamilton's principle with d'Alembert's, in the Lagrangian form, was proved above. Conversely, one will get Jacobi's principle for conservative systems when one defines the still-free $\delta\, dt$ by $\delta(E + U) = 0$.

Concluding remark about *II.5* : We have always investigated the equivalence of new axioms with d'Alembert's principle in the Lagrangian formulation. In order to obtain a general theory of mechanics, we must always add Lagrange's liberation principle.

III. – NON-CLASSICAL MECHANICS.

37. The logical independence of the axioms. – The logical independence of the axioms must be proved by the fact that one can produce theories of mechanics with no contradictions that do not fulfill the individual axioms, but their agreement with experiments would then be unobservable. One part of the independence relationships was already discussed before here. Not all possible combinations of axioms shall be exhausted here, but only the essential independence relations shall be exhibited.

Above all, we shall treat the independence of **Newton**'s fundamental law *I.f*, and also that of the general axioms of natural philosophy. That shall be done in what follows, along with a decomposition of Newton's fundamental law into individual axioms.

a) The group-theoretic structure of Newton's fundamental law.

28. The three types of possible mechanical theories. – In addition to the general axioms *A*, *B*, *C*, *D* (no. **7**), we establish the following ones:

Axiom III.1.a.α : Relative to an absolute Euclidian space and an absolute time, there are three general laws of motion that take the form:

$$\left.\begin{aligned} f\,(\dot{\mathfrak{r}}, \ddot{\mathfrak{r}}, \ldots; \mu_1, \mu_2, \ldots) &= A, \\ g\,(\dot{\mathfrak{r}}, \ddot{\mathfrak{r}}, \ldots; \mu_1, \mu_2, \ldots) &= B, \\ h\,(\dot{\mathfrak{r}}, \ddot{\mathfrak{r}}, \ldots; \mu_1, \mu_2, \ldots) &= C, \end{aligned}\right\} \tag{1}$$

in which μ_1, μ_2, … are scalars or tensors that are assigned to the point in question, and *f*, *g*, *h* are functions of the quantities that they include that are fixed once and for all. *A*, *B*, *C*, the resultant force magnitudes, are functions of further force magnitudes A_1, B_1, C_1 ; A_2, B_2, C_2, …, that are determined by further causes in their own right, i.e., by processes and states at the point itself, as well as other points. $\mathfrak{r}$ and *t* do not occur explicitly, which carries with it the homogeneity of space and time (Axiom *A*).

Axiom III.1.a.β : The equations of motion are invariant under changes of coordinate system (viz., the isotropy of space, so this axiom is identical to *B*).

That includes the fact that there are equations of the form:

$$A' = \varphi\,(A, B, C\,;\,\alpha, \beta, \gamma)\,,$$

etc., that will allow one to calculate the force magnitudes A', B', C' in the new system in terms of the old *A*, *B*, *C*, and the Euler angles α, β, γ in the old system (cf., H.2).

It then follows that there are three possible types of mechanical theories:

1. Transitive ones ([1]), for which A, B, C are equivalent to a rotation, so they will be the defining data for a unit quaternion.

2. A scalar one, for which A, B, C are scalars, and the equations above will read $A = A'$, etc.

3. A vectorial one, for which A, B, C are equivalent to the three components of a vector (equivalent means: there is a one-to-one correspondence between them).

An example of a transitive theory of mechanics is:

$$\mu_1 \mathfrak{w} = \lambda_1 \mathfrak{k}, \quad \mu_2 \mathfrak{w} = \lambda_2 \mathfrak{k} \qquad (\mathfrak{v} = \dot{\mathfrak{r}}, \mathfrak{w} = \ddot{\mathfrak{r}}),$$

where

$$|\mathfrak{k}| = 1, \qquad |\mathfrak{s}| = 1, \qquad \mathfrak{k}\,\mathfrak{s} = 0.$$

The remaining pieces of $\mathfrak{s}$ and $\mathfrak{k}$ must be addressed as force magnitudes. They are equivalent to a unit quaternion, namely, the rotation that takes the zweibein $\mathfrak{s}$, $\mathfrak{k}$ from a basic configuration to its instantaneous one. λ_1 and λ_2 are unknown parameters. One has to imagine that eliminating them will produce the required three independent equations from the five that exist.

An example of a scalar theory of mechanics is:

$$\mu \frac{d}{dt}(\mathfrak{v}^2) = A, \qquad \mu \frac{d}{dt}(\mathfrak{w}^2) = B, \qquad \mu \frac{d}{dt}(\dot{\mathfrak{w}}^2) = C.$$

As one easily sees, in this theory of mechanics, the inertial paths are ordinary helices that are traversed with constant velocity. Thus, if a point were released suddenly, i.e., in such a way that it would be subject to no further forces, then there would be conserved quantities in this theory of mechanics, namely, velocity, radius of curvature, and radius of torsion, which would be precisely the same as they were at the moment of release.

Such a theory of mechanics would contradict no general law of thought (nor the causality principle, either), nor in particular the basic laws of transcendental philosophy. The attempts of **Schopenhauer** and others to prove the *a priori* nature of the Galilean law of inertia from such principles are then false, as **Mach** had pointed out before.

39. The parallelogram of forces.

Axiom III.1.b : The functions that represent A, B, C as the resultants of A_1, B_1, C_1 ; A_2, B_2, C_2 ; etc., satisfy the following conditions:

([1]) Transitive means: Any triple A, B, C can be obtained from any other one by a rotation, so there is no invariant coupling of the A, B, C (**Lie**).

1. The composition is commutative and associative.
2. Any triple is uniquely determined by the others.
3. The composition formulas are invariant under changes of coordinate system.
4. They are continuous.

It will then follow that:

α) A transitive theory of mechanics is impossible.

β) In the other two cases, one can introduce three new force magnitudes in place of A, B, C such that, first of all, either A, B, C are the components of a vector or a scalar, and that, secondly, the composition formulas read (H. 2):

$$A = A_1 + A_2 + \ldots, \qquad B = B_1 + B_2 + \ldots, \qquad C = C_1 + C_2 + \ldots$$

In what follows, we would always like to think that the force magnitudes have been normalized in that way.

As opposed to what was true before in no. **6**, one still has the possibility of replacing the A, B, C with functions of them such that Axiom $I.d'.\delta$ will become a definition. Namely, as one can easily show, if one drops that axiom then the single freedom that one has consists of introducing functions of A, B, C in place of the latter such that $I.d'.\delta$ will still be fulfilled.

40. The axiom of continuity.

Axiom $III.1.c.\beta$: For every volume dV, there are, first of all, spatially-distributed force:

$$A = \xi\, dV, \qquad B = \eta\, dV, \qquad C = \zeta\, dV,$$

and secondly, surface forces:

$$A_n = \alpha_n\, dF, \quad B_n = \beta_n\, dF, \quad C_n = \gamma_n\, dF,$$

(cf., Axiom *II.b*), such that our basic equation will read:

$$\mu\, f\,(\dot{\mathfrak{r}}, \ddot{\mathfrak{r}}, \ldots; \mu_1, \mu_2, \ldots) = \sum \xi + \lim_{dV \to 0} \frac{1}{dV} \mathsf{S}\, \alpha_n\, dF, \qquad \text{etc.}$$

That implies the *fundamental law:* The forces are determined by the two types of axioms up to now up to a homogeneous linear transformation with constant coefficients, and in vectorial mechanics, they are even determined up to the choice of coordinate system and units. The concept of force then means a *form* into which an empirical mechanical law can be brought in essentially only one way (H.2).

41. The mechanics of special relativity and Newtonian mechanics. – The left-hand sides of equations (1) in no. **38** are still general, except that the *f*, *g*, *h* must also be the components of a vector or a scalar. Despite the axioms up to now, one can still construct a great number of non-Newtonian theories of mechanics.

The mechanics of special relativity belongs to that number. It is vectorial and can be written ([1]):

$$\mu \, dV \frac{d}{dt}\left(\frac{\mathfrak{v}}{\sqrt{1-v^2}}\right) = d\,\mathfrak{k} \, .$$

Instead of the Galilean principle of relativity (no. **3**), one now has the Einsteinian principle of special relativity (i.e., the Lorentz transformation).

Newtonian mechanics can then be obtained from the following further axioms:

Axiom III.1.d.α: The theory of mechanics is vectorial.

Axiom III.1.d.β: The energy principle:

$$\frac{d}{dt}\left(\tfrac{1}{2}\, dm\,\mathfrak{v}^2\right) = \mathfrak{v}\; d\,\mathfrak{k}$$

shall be a consequence of the laws of motion.

Axiom III.1.d.γ: The Galilean principle of relativity is valid.

It follows from the last axiom that the left-hand side of our equations (1) of no. **38**, which we can now write:

$$dm\,\mathfrak{f}\,(\mathfrak{w}, \dot{\mathfrak{w}}, \ldots; \mu_1, \mu_2, \ldots) = d\,\mathfrak{k} \, ,$$

from the first axiom, will not include $\mathfrak{v}$ explicitly. However, it follows from the second axiom that:

$$dm\; \mathfrak{f}\; \mathfrak{v} = dm\; \mathfrak{v}\; \mathfrak{w} \, ,$$

so

$$(\mathfrak{f} - \mathfrak{w})\; \mathfrak{v} = 0 \, .$$

([1]) See Chap. 10 in this volume of the *Handbuch*.

However, since $\mathfrak{f}$ does not include $\mathfrak{v}$, so $\mathfrak{f} - \mathfrak{w}$ is independent of $\mathfrak{v}$, the equation can be fulfilled only by $\mathfrak{f} \equiv \mathfrak{w}$, which was to be proved.

Schütz ([1]) proved that Axioms *III.1.d.*β and γ already suffice to serve as the basis for Newton's laws.

Phillip Frank ([2]) also arrived at the special relativistic theory of mechanics in a similar group-theoretic way by requiring Einstein's principle of special relativity instead of the Galilean principle of relativity. A later work of **Phillip Frank** and **H. Rothe** ([3]) includes an extension of those considerations that is based upon a more general group of transformations. In all cases, kinetic energy is introduced as the most basic invariant.

A. Kneser and **Gertrud Weyl** ([4]) combined the Galilean principle of relativity with a principle of least action in the form of $\delta \int \mathfrak{L}(q, \dot{q})\, dt = 0$ that showed the far-reaching meaning of both assumptions.

In connection with that, one must also mention the critical examination of **Painlevé** ([5]), which did not, however, represent a rigorous axiomatic system.

b) Non-Boltzmann system mechanics.

42. The generalized law of moments. – If one drops the Boltzmann Axiom *II.1.c* about the symmetry of the stress dyadic then one will obtain a somewhat-more-generalized law in place of the law of moments.

In the dyadic of internal stresses:

$$\begin{matrix} X_x & X_y & X_z \\ Y_x & Y_y & Y_z \\ Z_x & Z_y & Z_z \end{matrix}$$

is *asymmetric* then, as is known, the three quantities:

$$\tfrac{1}{2}(Z_y - Y_z)\,, \qquad \tfrac{1}{2}(X_z - Z_x)\,, \qquad \tfrac{1}{2}(Y_x - X_y)$$

will define the components of a vector that we would like to denote by $\mathfrak{T}$.

One easily obtains from the equations:

$$\mu\,\mathfrak{w} = \mathfrak{k} + \frac{\partial \mathfrak{s}_x}{\partial x} + \frac{\partial \mathfrak{s}_y}{\partial y} + \frac{\partial \mathfrak{s}_z}{\partial z}\,,$$

([1]) **Schütz**, "Das Prinzip der absoluten Erhaltung der Energie," Göttinger Nachr., math. phys. Klasse (1897).

([2]) **Phillip Frank**, Wiener Ber., math.-nat. Klasse **118** (1909), pp. 373.

([3]) **Phillip Frank** and **H. Rothe**, Wiener Ber., math.-nat. Klasse **119** (1910), pp. 615.

([4]) **A. Kneser**, Math. Zeit. **2** (1918), pp. 326. **Gertrud Weyl**, *ibidem*, **11** (1921), pp. 97.

([5]) **Painlevé**, "Les axiomes de la mécanique," and "Note sur la propagation de la lumière," Paris, 1922, in the collection: *Les maîtres de la pensées scientifiques*.

upon outer multiplication by $\mathfrak{r}\,dV$ and integration, as well as an application of Gauss's law, that:

$$\mathsf{S}_V dm[\mathfrak{r}\,\mathfrak{w}] = \mathsf{S}_V [\mathfrak{r}\,\mathfrak{w}]dV + \mathsf{S}_O [\mathfrak{r}\,\mathfrak{s}_n]dF - 2\mathsf{S}_V \mathfrak{T}\,dV .$$

The departure consists of the last term. One can also regard the equation now as the law of moments, but as it would be given for any volume element dV that is endowed with an impressed moment $-2\,\mathfrak{T}\,dV$, which plays no role in the center of mass law but must be added to the law of moments.

Yet another conception of things is possible: One can decompose the stress dyadic, when assumed to be asymmetric, additively into two parts, a symmetric one with:

$$X'_y = Y'_x = \tfrac{1}{2}(X_y + Y_x)\,, \qquad \text{etc.,}$$

and an anti-symmetric one. If we denote all quantities that refer to the symmetric part by primes then the equations of motion can be written:

$$\mu\,\mathfrak{w} = \mathfrak{k} + \frac{\partial \mathfrak{s}'_x}{\partial x} + \frac{\partial \mathfrak{s}'_y}{\partial y} + \frac{\partial \mathfrak{s}'_z}{\partial z} - \text{rot}\,\mathfrak{T}\,,$$

and the center of mass law will assume the form:

$$\mathsf{S}_V dm\,\mathfrak{w} = \mathsf{S}_V \mathfrak{k}\,dV - \mathsf{S}_V \text{rot}\,\mathfrak{T}\,dV + \mathsf{S}_O \mathfrak{s}'_n\,dF\,,$$

while the law of moments reads:

$$\mathsf{S}_V dm[\mathfrak{r}\,\mathfrak{w}] = \mathsf{S}_V [\mathfrak{r}\,\mathfrak{w}]dV + \mathsf{S}_O [\mathfrak{r}\,\mathfrak{s}'_n]dF - \mathsf{S}_V [\mathfrak{r}\,\text{rot}\,\mathfrak{T}]\,dV .$$

One can also reduce non-Boltzmann mechanics to the Boltzmann case then by adding an impressed force $-\text{rot}\,\mathfrak{T}\,dV$ everywhere.

That explains the independence of the Boltzmann axiom by reducing it to classical mechanics.

Examples of non-Boltzmann mechanics:

a) The example of **W. Thomson** ([1]): $\mathfrak{T} = [\mathfrak{c}\ \text{rot}\ \mathfrak{v}]$, in which $\mathfrak{c}$ means a vector that is assigned to position. This is realized by the introduction of invisible tops. Statics is the same as in the classical case ($\mathfrak{v} = 0$). *The independence of d'Alembert's principle follows from that.* However, the bodies are not isotropic, due to $\mathfrak{c}$. That can be achieved when one takes $C\,\mathfrak{v}$ in place of $\mathfrak{c}$. Since

([1]) For the literature and an explanation, see, e.g., **Brill**, *Mechanik raumerfüllendender Massen*, pp. 135, 170.

$\mathfrak{T}\,\text{rot}\,\mathfrak{v} = 0$, the law of energy in its classical form will be valid rigid bodies. D'Alembert's principle is also not a consequence of the law of energy then.

b) $\mathfrak{T} = c\,\text{rot}\,\mathfrak{v}$. The law of energy is not true in its classical form. The body is isotropic.

c) $\mathfrak{T} = c\,\text{div}\,\mathfrak{v}\,\text{rot}\,\mathfrak{v}$. Things are the same as before. In addition, classical mechanics is true, so d'Alembert's principle for rigid bodies ($\text{div}\,\mathfrak{v} = 0$), with which *the independence* of Boltzmann's principle from d'Alembert's *is verified for general systems* (cf., no. 24).

c) Glimpse into the mechanics of Einstein's theory of relativity.

43. The gravitational field. – The mechanics of special relativity were already touched upon briefly in no. **41**, and they can be easily annexed into Newtonian mechanics.

In the general theory of relativity, the mechanics of free points in a gravitational field (which is the only case that we shall confine ourselves to) can be based upon the following axioms:

Axiom III.2.a : Physical processes play out in a four-dimensional space-time continuum that has a Euclidian metric at the infinitesimal level:

$$ds^2 = \sum_{i,k=1}^{4} g^{i,k} dx_i\, dx_k\,.$$

That quadratic form can be transformed into the form:

$$-dx_1^2 - dx_2^2 - dx_3^2 + dx_4^2$$

at any real location.

Axiom III.2.b : The motion of a point in the field of gravity results along the "straightest line":

$$\delta \int ds = 0\,.$$

(Generalization of the Jacobi principle)

Axiom III.2.c : The $g^{i,k}$ satisfy ten second-order partial differential equations (six of which are independent) that are linear in the second derivatives and invariant under an arbitrary transformation of x (i.e., the relativity principle), i.e., if one introduces new x' in place of x as twice-continuously-differentiable functions of the x and recalculates ds^2 in terms of $\sum g'\,dx'_i\,dx'_k$ then the g' will satisfy differential equations that arise from the old ones by recalculation.

Axiom III.2.d : $\int ds$ means the time that a light clock that moves with the point would measure (i.e., proper time).

Axiom III.2.e : For light rays, one has $ds = 0$, so under the transformation of ds^2 into $-dx_1^2 - dx_2^2 - dx_3^2 + dx_4^2$, one will have:

$$\frac{dx_4}{\sqrt{dx_1^2 + dx_2^2 + dx_3^2}} = 1 .$$

(viz., the principle of the constancy of the speed of light)

The question of an absolute space and absolute time is obviously entirely meaningless here, since the "straight lines" of world geometry are defined to be the paths of free material points in a gravity field, and it will therefore no longer matter how one visualizes those "straight lines" (on this, cf., no. **3**).

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IV. – THE CONSISTENCY OF THE AXIOMS.

44. General overview. – Since all of physics reverts back to mechanics in terms of its causes, consistency can be verified only piece-wise, namely, only to the extent that something more definite is given about the forces that are implied by the causes.

Painlevé ([1]) showed that contradictions can appear for arbitrary assumptions on the forces in the example of friction.

The proof of consistency is achieved under certain restricting conditions:

1. *for stereomechanics*, **A. Mayer** ([2]), **E. Zermelo** ([3]), and **G. Hamel** (H.1).

2. *for elastomechanics*, **Korn**, *et al* ([4]).

3. *for hydromechanics*, **L. Lichtenstein** ([5]).

We would not like to go into the details of cases 2 and 3 here. Every individual problem that is worked out to its conclusion implies a small sub-domain of the proof of consistency.

45. Working out the details for stereomechanics. – In the case of stereomechanics, this comes down to showing that the accelerations can be determined in such a way that either the constraints cease to be in effect (e.g., the bodies detach from each other) or that the normal pressure can be counted as positive. Mathematically, the statement of the consistency will be equivalent to the following mathematical *theorem:* One has $m = n + p$ linear equations of the form:

$$\sum_{k=1}^{n} a_{i,k}\, x_k + \sum_{l=1}^{p} a_{i,n+l}\, z_l = b_i + y_i \qquad (i = 1, 2, \ldots, n),$$

$$\sum_{k=1}^{n} a_{i,k}\, x_k + \sum_{l=1}^{p} a_{i,n+l}\, z_l = b_i \qquad (i = n + 1, \ldots, m),$$

in which the quadratic form:

$$\sum_{i,k=1}^{n} a_{i,k}\, x_i\, x_k$$

([1]) **Painlevé**, C. R. Acad. Sci. Paris **120** (1895), pp. 596, and *Leçons sur le frottement*, 1895. Cf., also the discussion in the Zeit. Math. Phys. **58** (1910).

([2]) **Adolf Mayer**, Leipziger Ber., math.-phys. Klasse **51** (1899), pp. 224.

([3]) **Zermelo**, Göttinger Nachr., math.-phys. Klasse (1899), 306. A further bibliography is in *Enzykl. d. math. Wiss.*, Bd. IV, 1, art. 6 (**Stäckel**), pp. 460, and also in **Stäckel**'s previously-cited treatise in the Sitzungsber. Heidelberger Akad. Ber. (1919).

([4]) See *Enzykl. d. math. Wiss.*, Bd. IV 4, art. 24 (**Tedone**), pp. 55.

([5]) **L. Lichtenstein**, Math. Zeit. **23** (1925), pp. 89, as well as *ibid.* **26** (1927).

is positive-definite. Those equations in the unknowns x, y, z will then admit one and only one solution under the auxiliary conditions:

$$x_i\, y_i = 0 \qquad (i = 1, 2, \ldots, n)\ ; \qquad x_i \geq 0\ , \qquad y_i \geq 0\ .$$

Therefore, when one x (an acceleration magnitude) is positive, the corresponding y (a normal pressure) will be zero, and conversely. The z are subject to no further restrictions.

Ostrogradsky ([1]), who was deeply concerned with inequalities in mechanics, along with **Fourier** and **Gauss**, believed that the theorem could be expressed thus: Those x are zero that would become negative when the y are set to zero.

That theorem is false, as **Adolf Mayer** had proved, although not quite satisfactorily. One therefore *cannot* say: If dropping some normal pressures is going to imply that some of the inequality constraints are violated then those inequality constraints must remain valid. If there is no violation then the constraint can be dropped. Since **Adolf Mayer** gave no examples, here are two simple ones:

1. Let a point x, y be subject to the constraints:

$$x \geq 0\ , \qquad y - z \geq 0\ .$$

If the force $\mathfrak{k}$ lies such that:

$$X < 0\ , \qquad Y < 0\ , \qquad |\,Y\,| < |\,X\,|$$

then motion that does not observe the inequalities will initially ($x = 0$, $y = 0$; $\dot{x} = 0$, $\dot{y} = 0$) give:

$$\text{indeed} \qquad \ddot{x} < 0\ , \quad \text{but} \quad \frac{d^2}{dt^2}(y - x) > 0\ .$$

Nonetheless, both of them will obviously remain true when one observes the inequalities, i.e., the point remains at the corner of the admissible region.

2. The opposite hypothesis: viz., if *all* constraints are violated in the absence of normal pressures then *all* of them will remain valid under the actual motion, is also false, as the following example might show: The constraints are:

$$x \geq 0\ ,\ y + x \geq 0\ , \qquad X < 0\ ,\ Y < 0\ , \qquad |\,Y\,| > |\,X\,|\ .$$

([1]) **Ostrogradsky**, Petersburger Ber. (1834) and (1838).

Obviously, for this force, when $x = 0$, $y = 0$, $\dot{x} = 0$, $\dot{y} = 0$, and in the absence of normal pressures, it would follow that $\ddot{x} < 0$, $\frac{d^2}{dt^2}(y + x) < 0$. When one observes the inequalities, it will indeed follow that $\frac{d^2}{dt^2}(y + x) = 0$, but $\frac{d^2 x}{dt^2} > 0$.

Adolf Mayer and **E. Zermelo**, who first proved the uniqueness, employed Gauss's principle of least constraint, and restricted themselves to point-systems in order to prove it. One can also extend the theorem to arbitrary systems of rigid bodies and arbitrary constraints (even friction) by employing some elementary tools (cf., no. **25**, H.1).

A discussion of possible collisions will be passed over here. It generally consists of the unproved conjecture that collisions will not occur when the continuity hypothesis is implemented consistently without idealizations such as rigid bodies, and the like.