Excerpted from G. Hamel, *Theoretische Mechanik*, Springer-Verlag, Berlin, 1949. Translated by D. H. Delphenich.

CHAPTER IX

NON-HOLONOMIC SYSTEMS WITH A FINITE NUMBER OF DEGREES OF FREEDOM

228. Introduction and method. – We restrict ourselves to systems with a finite number of degrees of freedom, so ones for which one has:

$$\mathbf{r} = \mathbf{r}\,(\mathbf{a};\, q_1,\, \ldots,\, q_n\, ;\, t)\,. \tag{1}$$

However, the q_ν shall not vary freely, but will be subject to constraints of the form:

$$f_\mu(q_\nu, \dot{q}_\nu, t) = 0, \qquad \mu = 1, 2, \ldots, m < n,$$

which cannot be reduced to finite equations between the q_ν and t alone.

Initially, we shall assume that the equations of constraint are linear in the $\dot{q}_\nu$; i.e., they take the form:

$$\sum_\nu b_{\mu;\nu}\, \dot{q}_\nu + c_\mu = 0\,. \tag{2}$$

§ 1. – The parametric method.

Corresponding to (2), one has:

$$\sum_\nu b_{\mu;\nu}\, \delta q_\nu = 0 \tag{3}$$

for the virtual displacements. **Lagrange's** principle:

$$\mathsf{S}\; dm\; \mathbf{w}\; \delta\mathbf{r} = \delta A$$

remains valid, and with:

$$\delta A = \sum K_\nu\, \delta q_\nu \tag{4}$$

and

$$\mathsf{S}\; dm\; \mathbf{w}\; \delta\mathbf{r} = \sum W_\nu\, \delta q_\nu\,, \tag{5}$$

in which:

$$W_\nu = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\mu} - \frac{\partial T}{\partial q_\mu}\,,$$

that principle will imply that:

$$\sum (W_\nu - K_\nu)\,\delta q_\nu = 0. \tag{6}$$

Together with (3), that will give:

$$W_\nu - K_\nu = \sum_{\mu=1}^{m} \lambda_\mu\, b_{\mu\nu}\,,$$

or

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q_\mu} - \frac{\partial T}{\partial q_\mu} = \sum_\nu \lambda_\mu\, b_{\mu\nu} + K_\nu\,. \tag{I}$$

The proof is precisely as it was in Chap. II, § **6**, if T is the kinetic energy:

$$\tfrac{1}{2}\,\mathbf{S}\,dm\,\mathbf{v}^2 = \tfrac{1}{2}\sum_{\nu,\mu} a_{\nu\mu}\,\dot q_\nu\,\dot q_\mu + \sum_\nu b_\nu\,\dot q_\nu + c\,.$$

One can regard $\lambda_\mu\, b_{\mu\nu}$ as the **Lagrangian** reaction force that is assigned to the μ^{th} constraint.

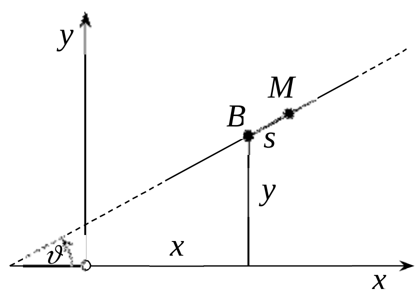

Figure 109.

229. The blade. – Example 1: The *blade* (cf., Chap. II, §§ **5** and **6**). Let its contact point with the *xy*-plane be *B*, whose coordinates we shall denote by *x*, *y*, in particular, and let it be regarded as a rigid body whose center of mass *M* lies at a distance of *s* from *B* in the direction of the blade. From Chap. III, § **3**, the kinetic energy of the motion, which is assumed to be planar, will be:

$$T = \tfrac{1}{2} m(\dot x^2 + \dot y^2) + m\,\dot\vartheta[\dot y\,(x^* - x) - \dot x\,(y^* - y)] + \tfrac{1}{2}\dot\vartheta^2 I_B\,,$$

or, since $x^* - x = s \cos\vartheta$, $y^* - y = s \sin\vartheta$:

$$T = \tfrac{1}{2} m(\dot x^2 + \dot y^2) + m\,s\,\dot\vartheta[\dot y\cos\vartheta - \dot x\sin\vartheta] + \tfrac{1}{2}\dot\vartheta^2 I_B\,.$$

However, the constraint equation reads:

$$\dot y\cos\vartheta - \dot x\sin\vartheta = 0.$$

The impulses are:

$$p_x = \frac{\partial T}{\partial \dot{x}} = m\dot{x} - ms\sin\vartheta\cdot\dot{\vartheta},$$
$$p_y = \frac{\partial T}{\partial \dot{y}} = m\dot{y} + ms\cos\vartheta\cdot\dot{\vartheta},$$
$$p_z = \frac{\partial T}{\partial \dot{z}} = ms(\dot{y}\cos\vartheta - \dot{x}\sin\vartheta) + I_B\,\dot{\vartheta},$$

which is why:

$$W_x = \frac{d}{dt}(m\dot{x} - ms\sin\vartheta\cdot\dot{\vartheta}),$$
$$W_y = \frac{d}{dt}(m\dot{y} + ms\cos\vartheta\cdot\dot{\vartheta}),$$
$$W_\vartheta = \frac{d}{dt}[ms(\dot{y}\cos\vartheta - \dot{x}\sin\vartheta)] + I_B\,\ddot{\vartheta} + ms\,\dot{\vartheta}(\dot{y}\sin\vartheta + \dot{x}\cos\vartheta).$$

In parametric form, the equations of motion will then read:

$$\frac{d}{dt}(m\dot{x} - ms\sin\vartheta\cdot\dot{\vartheta}) = -\lambda\sin\vartheta + X, \tag{a}$$
$$\frac{d}{dt}(m\dot{y} + ms\cos\vartheta\cdot\dot{\vartheta}) = \lambda\cos\vartheta + Y, \tag{b}$$
$$\frac{d}{dt}[ms(\dot{y}\cos\vartheta - \dot{x}\sin\vartheta)] + I_B\,\ddot{\vartheta} + ms\,\dot{\vartheta}(\dot{y}\sin\vartheta + \dot{x}\cos\vartheta) = M \tag{c}$$

when we set the virtual work done by the applied forces to:

$$\delta A_e = X\,\delta x + Y\,\delta y + M\,\delta\vartheta. \tag{d}$$

That must be combined with the constraint equation:

$$\dot{y}\cos\vartheta - \dot{x}\sin\vartheta = 0.$$

Naturally, after exhibiting the equations of motion, we can make use of that constraint equation, and thus simplify the third equation by dropping the first term. However, up to now, we have not been able to work with the expression for the kinetic energy that has been simplified by that constraint equation:

$$T = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \tfrac{1}{2}I_B\dot{\vartheta}^2.$$

It clearly gave false equations (cf., Chap. VI, § **3**).

We eliminate λ by multiplying (a) and (b) by cos ϑ and sin ϑ, resp., and adding them, which will give:

$$\cos\vartheta\frac{d}{dt}(m\dot{x}-m\,s\sin\vartheta\cdot\dot{\vartheta})+\sin\vartheta\frac{d}{dt}(m\dot{y}+m\,s\cos\vartheta\cdot\dot{\vartheta}) = X\cos\vartheta + Y\sin\vartheta,$$

or

$$m\cos\vartheta\, m\ddot{x}-m\,s\,\dot{\vartheta}^2\sin\vartheta+m\sin\vartheta\,\ddot{y} = Z,$$

in which Z means the traction in the direction of the blade, which also implies that:

$$I_B\ddot{\vartheta}+m\,s\,\vartheta(\dot{y}\sin\vartheta+\dot{x}\cos\vartheta) = M,$$

as well as:

$$\dot{y}\cos\vartheta-\dot{x}\sin\vartheta = 0.$$

If we want to get λ itself then we multiply (a) and (b) by – sin ϑ and + cos ϑ, resp., and get:

$$\lambda = X\sin\vartheta - Y\cos\vartheta - m\ddot{x}\sin\vartheta+m\ddot{y}\cos\vartheta+m\,s\,\ddot{\vartheta}.$$

In this, – X sin ϑ + Y cos ϑ means the applied force perpendicular to the blade, and $\ddot{y}\cos\vartheta-\ddot{x}\sin\vartheta+s\,\ddot{\vartheta}$ is the acceleration of center of mass in the same direction, since the first two terms mean the acceleration of the point B, to which one adds the relative acceleration $s\,\ddot{\vartheta}$ for M.

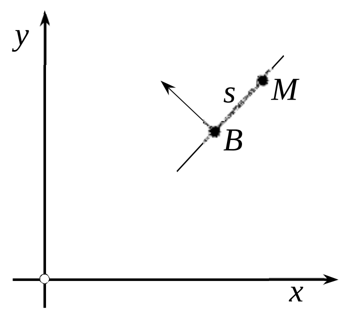

Figure 110.

We now introduce the velocity v in the direction of the blade by setting:

$$\dot{x} = v\cos\vartheta, \qquad \dot{y} = v\sin\vartheta,$$

and then:

$$\ddot{x} = \dot{v}\cos\vartheta - v\sin\vartheta\cdot\dot{\vartheta}, \qquad \ddot{y} = \dot{v}\sin\vartheta + v\cos\vartheta\cdot\dot{\vartheta}.$$

With that, the equations of motion will become:

$$m\dot{v}-m\,s\,\dot{\vartheta}^2 = Z,$$
$$I_B\,\ddot{\vartheta}+m\,s\,\dot{\vartheta}v = M.$$

The constraint equation is fulfilled by itself. v is not a total derivative of a coordinate constraint, because:

$$v\,dt = dx \cos \vartheta - dy \sin \vartheta$$

is not a total differential. We shall call such a quantity that is used to represent the velocity a *non-holonomic velocity parameter*.

We would now like to work through the case of *force-free motion.* Z and M are zero in that case, so the equations of motion will read:

$$m\dot{v} - m s \dot{\vartheta}^2 = 0,$$
$$I_B \ddot{\vartheta} + m s \dot{\vartheta} v = 0 .$$

They will then have the energy equation (cf., Chap. IV, § **3**) as a first integral:

$$T = \tfrac{1}{2} m v^2 + m \dot{\vartheta}(\dot{y}\, s \cos\vartheta - \dot{x}\, s \sin\vartheta) + \tfrac{1}{2} I_B \dot{\vartheta}^2 = h$$

or

$$m\,v^2 + I_B \dot{\vartheta}^2 = 2h .$$

This equation can be seen to be as a consequence of the equations of motion by differentiating it. We infer from it that:

$$\dot{\vartheta}^2 = \frac{2h}{I_B} - \frac{m}{I_B} v^2$$

and substitute that in the first equation of motion, which is the only one that we have to consider. We then get the first-order differential equation for v :

$$\dot{v} - \frac{2h\,s}{I_B} + \frac{m\,s}{I_B} v^2 = 0 .$$

If we set the positive-definite quantity $2h / m = v_0^2$ then we will get:

$$\dot{v} = \frac{m\,s}{I_B}(v_0^2 - v^2) = \frac{1}{a}(v_0^2 - v^2) ,$$

when we set $I_B / ms = a$. That will be a positive or negative length according to the sign of s. (When one does not have $v = v_0$) integration will give:

$$t = a \int \frac{dv}{v_0^2 - v^2} ,$$

or when $| v | < | v_0 |$:

$$v = v_0 \,\mathfrak{Tan}\, \frac{v_0\, t}{a}.$$

We will see that $|v| > |v_0|$ is impossible. The velocity will certainly become zero at some point in time; we have set $t = 0$ to be that moment. For $\dot{\vartheta}^2$, we get:

$$\dot{\vartheta}^2 = \frac{m(v_0^2 - v^2)}{I_B} = \frac{m}{I_B} v_0^2 \frac{1}{\mathfrak{Cos}^2 \dfrac{v_0\, t}{a}}.$$

Since we must have $\dot{\vartheta}^2 \geq 0$, $v^2 > v_0^2$ is excluded.

$$\dot{\vartheta} = v_0 \sqrt{\frac{m}{I_B}} \frac{1}{\mathfrak{Cos}\dfrac{v_0\, t}{a}}$$

gives:

$$\vartheta = v_0 \sqrt{\frac{m}{I_B}} \int \frac{dt}{\mathfrak{Cos}\dfrac{v_0\, t}{a}},$$

which is an elementary calculation.

For the sake of discussion, we can assume that s (and therefore a, as well) is positive. We can also take v_0 to be positive, since the sign of v_0 drops out of the formula for v. We will then have that $v > 0$ for $t > 0$ and $v < 0$ for $t < 0$; i.e., for $t > 0$, the center of mass is in front of B (in the direction of the motion) and for $t < 0$, it lies behind it. $v \to v_0$ for $t \to +\infty$, but $v \to -v_0$ for $t \to -\infty$.

The velocity then increases from $-v_0$ through zero to $+v_0$. ϑ will always be positive or always be negative according to the sign that we give to $\sqrt{m / I_B}$. From $t = 0$ to $t = \infty$, the blade rotates through the angle:

$$\Delta\vartheta = v_0 \sqrt{\frac{m}{I_B}} \int_0^\infty \frac{dt}{\mathfrak{Cos}\dfrac{v_0\, t}{s}} = a\sqrt{\frac{m}{I_B}} \int_0^\infty \frac{d\tau}{\mathfrak{Cos}\,\tau} = \frac{1}{s}\sqrt{\frac{I_B}{m}}\,\pi.$$

The same value will come about from the time from $-\infty$ to 0. Since the velocity v is zero for $t = 0$, but not $\dot{\vartheta}$, the curve must have a cusp there; by contrast, as $t \to \pm\infty$, it will have an asymptote, since ϑ tends to a finite value, like v. Naturally, the rectilinear motion with $\vartheta = 0$, $v = \sqrt{2h/m} = v_0$ is also a possible motion. However, one can also conclude that: If the center of mass lies in front of B, in the direction of motion, in this then the motion will be stable, since a perturbed motion will asymptotically approach the old motion. However, if M lies behind B then the motion will be unstable, because a perturbed motion must first pass through the cusp, at which point, an inversion of the sequence will take place, and from then on, the motion will be connected with an asymptotic line that is generally different. [See Carathéodory's discussion of the sled (= blade) ([1]).]

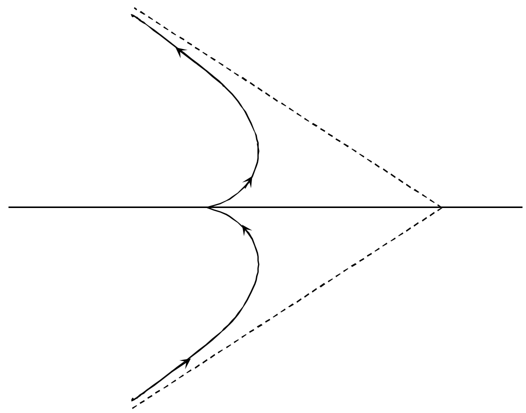

Figure 111.

Naturally, one can also represent the coordinates x and y by integrals:

$$x = \int v \cos \vartheta \, dt , \qquad y = \int v \sin \vartheta \, dt .$$

In the vicinity of $t = 0$, one has:

$$v \approx \frac{v_0^2 \, t}{a} .$$

If one makes the x-axis tangential to the cusp then one will have $\vartheta \approx 0$ or $\cos \vartheta \approx 1$ close to it, and:

$$x \approx \int_0^t \frac{v_0^2}{a} \, t \, dt = \frac{1}{2} \frac{v_0^2}{a} t^2 ,$$

whereas:

$$\vartheta \approx v_0 \sqrt{\frac{m}{I_B}} \int_0^t dt = v_0 \sqrt{\frac{m}{I_B}} \, t$$

and

$$y \approx \frac{v_0^2}{a} \sqrt{\frac{m}{I_B}} \int t^2 dt = \frac{1}{3} \frac{v_0^3 \, m \, s}{I_B} \sqrt{\frac{m}{I_B}} \, t^3 = \frac{1}{3} v_0^3 \, s \left(\sqrt{\frac{m}{I_B}} \right)^3 t^3 .$$

The curve is close to a **Neil** parabola, so the existence of a cusp is proved once more.

The blade is the simplest example of a non-holonomic scleronomic system, insofar as it has the lowest number of degrees of freedom. In fact, two degrees of freedom cannot give a non-holonomic system, since it is known that any differential expression:

([1]) **C. Carathéodory**, "Der Schlitten," Z. angew. Math. Mech. **13** (1933), 71-76.

$$P\,(x, y)\,dx + Q\,(x, y)\,dy$$

is associated with a multiplier M such that $M\,(P\,dx + Q\,dy)$ will be a total differential dz, so $P\,dx + Q\,dy = 0$ can be replaced with $dz = 0$; i.e., $z\,(x, y) = \text{const.}$

230. The tire. – Example 2: *The tire* on a planar floor. We think of it as a circular ring that rolls without slipping on a plane. In itself, the system has five degrees of freedom, namely, the coordinates x, y of the contact point, the direction angle ϑ of the contact tangent with respect to the x-axis (just like with the blade), then the angle of inclination ψ between the plane of the tire and the normal to the base plane, and fifth, the rolling angle φ in the plane of the tire, which is measured from a mark on the circle to the point of tangency. However, there exist the following two differential conditions, which express the rolling without slipping:

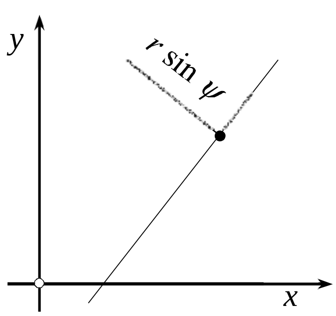

Figure 112.

$$dx = r\,d\varphi \cos\vartheta\,, \qquad dy = r\,d\varphi \sin\vartheta\,, \tag{1}$$

if r means the radius of the circle, because if the tire rolls without slipping then the contact point will displace through $r\,d\varphi$ further in the direction ϑ. The condition $dy = \tan\vartheta\,dx$ for the blade is included in (1). It is easy to prove that the conditions (1) cannot be replaced with finite equations:

$$f\,(x, y, \vartheta, \varphi) = 0,$$

which is already due to the fact that one can roll the tire from any initial position to any final position.

We would not like to treat this example with the parametric method, but we shall treat it by some other methods later. However, we shall calculate the kinetic energy. The center of the tire, which shall also be the center of mass, has the coordinates:

$$x^* = x - r \sin\psi \sin\vartheta,$$
$$y^* = y + r \sin\psi \cos\vartheta,$$
$$z^* = r \cos\psi\,.$$

As a result:

$$\dot{x}^* = \dot{x} - r\cos\psi \sin\vartheta\,\dot{\psi} - r\sin\psi\cos\vartheta\,\dot{\vartheta},$$
$$\dot{y}^* = \dot{y} + r\cos\psi\cos\vartheta\,\dot{\psi} - r\sin\psi\sin\vartheta\,\dot{\vartheta},$$
$$\dot{x}^* = - r\sin\psi\,\dot{\psi}\,.$$

Hence:

$$\begin{aligned}\dot{x}^{*2} + \dot{y}^{*2} + \dot{z}^{*2} = {} & \dot{x}^2 + \dot{y}^2 + r^2\dot{\psi}^2 + r^2\sin^2\psi\,\dot{\vartheta}^2 \\ & - 2\,\dot{x}\,\dot{\psi}\,r\cos\psi\sin\vartheta + 2\,\dot{y}\,\dot{\psi}\,r\cos\psi\cos\vartheta \\ & - 2\,\dot{x}\,\dot{\vartheta}\,r\sin\psi\cos\vartheta - 2\,\dot{y}\,\dot{\vartheta}\,r\sin\psi\sin\vartheta\,.\end{aligned}$$

The tire possesses the following rotational velocities:

1. $\dot{\varphi}$ around the axis perpendicular to it.

2. $\dot{\psi}$ around the tangent to the contact point, and for $\dot{\psi} > 0$, it appears to be directed to the left when one looks in the direction of rolling.

3. $\dot{\vartheta}$ around the vertical through the contact point.

The gives the components $-\dot{\psi}$ in the direction of the tangent, $\dot{\vartheta}\cos\psi$ in the plane of the tire (as seen from above), and $\dot{\varphi} - \dot{\vartheta}\sin\psi$ perpendicular to the tire. (In the figure, $\dot{\vartheta}$ is regarded as a vector that points up, while $\dot{\varphi}$ points down and left. One's line of sight points in the direction of rolling, so $\dot{\psi}$ points forward as a vector.)

As a result, if A, $B = A$, C are the principal moments of inertia, the kinetic energy will be (cf., Chap. VII, § **3**):

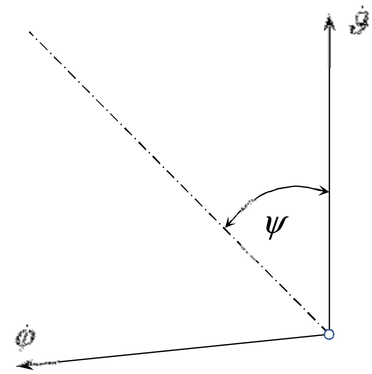

Figure 113.

$$T = \tfrac{1}{2} m(\dot{x}^2 + \dot{y}^2 + r^2\dot{\psi}^2 + r^2\sin^2\psi\,\dot{\vartheta}^2$$
$$- 2\dot{x}\dot{\psi}\, r\cos\psi\sin\vartheta + 2\dot{y}\dot{\psi}\, r\cos\psi\cos\vartheta - 2\dot{x}\dot{\vartheta} r\sin\psi\cos\vartheta - 2\dot{y}\dot{\vartheta} r\sin\psi\sin\vartheta)$$
$$+ \tfrac{1}{2} A(\dot{\psi}^2 + \cos^2\psi\,\dot{\vartheta}^2) + \tfrac{1}{2} C(\dot{\varphi} - \dot{\vartheta}\sin\varphi)^2 .$$

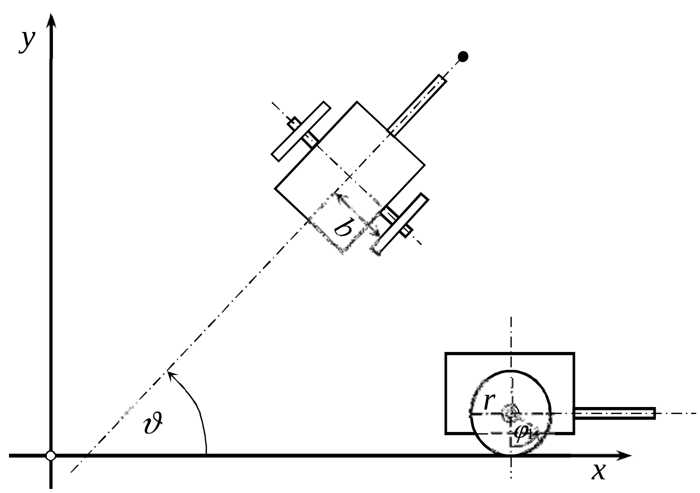

Figure 114.

231. The cart. Example 3: *The two-wheeled cart*, which shall also roll on a plane without slipping. It is also similar to the blade.

Let x, y be the coordinates of the intersection of the longitudinal axis with the transverse axis, along which the wheels are located. The angle ψ drops out, but we now have two rolling angles, namely, φ_1 for the right-hand wheel and φ_2 for the left-hand one. Thus, we again have five degrees of freedom. We will ignore the bobbing of the wagon around the transverse axis, which would then be a sixth degree of freedom. Let the distance from the center to the wheels be *b*, and let their radii be *r*. We will then have:

$$\dot{x}\cos\vartheta + \dot{y}\sin\vartheta + b\,\dot{\vartheta} + r\,\dot{\varphi}_1 = 0 \tag{1}$$

for the velocity of the contact point for the right wheel and:

$$\dot{x}\cos\vartheta + \dot{y}\sin\vartheta - b\,\dot{\vartheta} + r\,\dot{\varphi}_2 = 0 \tag{2}$$

for the left. They must be combined with the old condition the absence of transverse sliding:

$$\dot{x}\sin\vartheta - \dot{y}\cos\vartheta = 0. \tag{3}$$

There will then be three non-holonomic constraints for five degrees of freedom.

The kinetic energy of the wagon itself is the same as that of the sled, namely:

$$T = \tfrac{1}{2} m_w (\dot{x}^2 + \dot{y}^2) - m_w\, s\, \dot{x}\,\dot{\vartheta} \sin\vartheta + m_w\, s\, \dot{y}\,\dot{\vartheta} \cos\vartheta + \tfrac{1}{2} I_w \dot{\vartheta}^2 ,$$

if I_w is the moment of inertia of the wagon alone around the vertical through the point *x*, *y* . That must be combined with the kinetic energy of the wheels. If each wheel has a moment of inertia of C_R around the rotational axis and the moment of inertia A_R around a transverse axis to the wheel, then one will have:

$$\begin{aligned} T_R' &= \tfrac{1}{2} m_R [(\dot{x}\cos\vartheta + \dot{y}\sin\vartheta + b\,\dot{\vartheta})^2 + (\dot{x}\sin\vartheta - \dot{y}\cos\vartheta)^2] + \tfrac{1}{2} C_R\, \dot{\varphi}_1^2 + \tfrac{1}{2} A_R\, \dot{\vartheta}^2 \\ &= \tfrac{1}{2} m_R [(\dot{x}^2 + \dot{y}^2 + 2b\,\dot{\vartheta}(\dot{x}\cos\vartheta + \dot{y}\sin\vartheta)] + \tfrac{1}{2} C_R\, \dot{\varphi}_1^2 + \tfrac{1}{2} A_R\, \dot{\vartheta}^2 + \tfrac{1}{2} m_R\, \dot{\vartheta}^2 b^2 \end{aligned}$$

for the right-hand wheel and:

$$\begin{aligned} T_R'' &= \tfrac{1}{2} m_R [(\dot{x}\cos\vartheta + \dot{y}\sin\vartheta - b\,\dot{\vartheta})^2 + (\dot{x}\sin\vartheta - \dot{y}\cos\vartheta)^2] + \tfrac{1}{2} C_R\, \dot{\varphi}_2^2 + \tfrac{1}{2} A_R\, \dot{\vartheta}^2 \\ &= \tfrac{1}{2} m_R [\dot{x}^2 + \dot{y}^2 - 2b\,\dot{\vartheta}(\dot{x}\cos\vartheta + \dot{y}\sin\vartheta) + b^2 \dot{\vartheta}^2] + \tfrac{1}{2} C_R\, \dot{\varphi}_2^2 + \tfrac{1}{2} A_R\, \dot{\vartheta}^2 \end{aligned}$$

for the left-hand one. With $m_w + 2m_R = m$ (viz., the total mass of the wagon) and $I_w + 2m_R\, b^2 + 2A_R = I$ (viz., the total moment of inertia around the axis through *x*, *y*), that will give:

$$T = \tfrac{1}{2} m_R (\dot{x}^2 + \dot{y}^2) - m_w\, s\, \dot{x}\,\dot{\vartheta} \sin\vartheta + m_w\, s\, \dot{y}\,\dot{\vartheta} \cos\vartheta + \tfrac{1}{2} I\, \dot{\vartheta}^2 + \tfrac{1}{2} C_R (\dot{\varphi}_1^2 + \dot{\varphi}_2^2)$$

for the total kinetic energy. One can also replace $m_w\, s$ with $m\, a$, if *a* is the distance from the center of mass to the point *x*, *y* . Hence:

$$T = \tfrac{1}{2} m_R (\dot{x}^2 + \dot{y}^2) - m\,a\,\dot{x}\,\dot{\vartheta}\sin\vartheta + m\,a\,\dot{y}\,\dot{\vartheta}\cos\vartheta + \tfrac{1}{2} I\dot{\vartheta}^2 + \tfrac{1}{2} C_R (\dot{\varphi}_1^2 + \dot{\varphi}_2^2) .$$

The *bicycle* presents a more complicated problem, which was considered in a thorough study by **Carvallo** [J. Éc. poly. **215** (1901)]. There are references to older literature in the Enzyklopädie d. math. Wiss. IV, II. It would be desirable to treat the *automobile* similarly. One sees that non-holonomic systems are in no way rare, since almost all systems for which rolling without slipping takes place are non-holonomic.

§ 2. – The transition equations.

232. Deriving the equations. – We shall now develop a method that does not work with parameters, so it does not work with reaction forces, either, and it yields the desired replacement for the **Lagrange** equations, which might indeed be false (cf., Chap. VI, § **3**); we shall come back to that point. We might then prescribe m equations of the type:

$$\omega_\mu \equiv \sum_{\nu=1}^{n} b_{\mu,\nu}\,\dot{q}_\nu + c_\mu = 0, \qquad \mu = 1, 2, \ldots, m . \tag{1}$$

In addition, it can be preferable to further introduce:

$$\omega_\mu \equiv \sum_{\nu=1}^{n} b_{\mu,\nu}\,\dot{q}_\nu + c_\mu , \qquad \mu = 1, 2, \ldots, n, \tag{2}$$

which do not all have to be zero, so they imply no further conditions. For example, for the blade, we introduce:

$$v = \dot{x}\cos\vartheta + \dot{y}\sin\vartheta$$

as a *non-holonomic velocity parameter*. We extend that term to all ω_μ. Furthermore, we already know of such ω from the p, q, r of the top.

We shall also write:

$$d\vartheta_\mu \equiv \omega_\mu\,dt = \sum_{\nu=1}^{n} b_{\mu,\nu}\,dq_\nu + c_\mu\,dt . \tag{3}$$

However, the $d\vartheta_\mu$ are not generally total differentials. We correspondingly introduce the virtual displacements $\delta\vartheta_\mu$ according to:

$$\delta\vartheta_\mu \equiv \sum_{\nu=1}^{n} b_{\mu,\nu}\,\delta q_\nu . \tag{4}$$

The ω_μ shall be independent; i.e., let the determinant of the $b_{\mu,\nu}$ be non-zero:

$$\| b_{\mu,\nu} \| \neq 0,$$

such that we can solve equations (1) and (2) for the $\dot{q}_\nu$:

$$\dot{q}_\nu = \sum_{\mu=1}^{n} B_{\nu,\mu}\,\omega_\mu + C_\nu\,, \tag{5}$$

and correspondingly:

$$\delta q_\nu = \sum_{\mu=1}^{n} B_{\nu,\mu}\,\delta\vartheta_\mu + C_\nu\,. \tag{5}$$

Naturally, we can also take the ω_μ to be the $\dot{q}_\nu$. Now, if:

$$\mathbf{r} = \mathbf{r}\,(\mathbf{a}\,;\,q_1,\,q_2,\,\ldots\,q_n\,;\,t)$$

then

$$d\,\mathbf{r} = \sum \frac{\partial \mathbf{r}}{\partial q_\nu} dq_\nu + \frac{\partial \mathbf{r}}{\partial t} dt$$

$$\delta\,\mathbf{r} = \sum \frac{\partial \mathbf{r}}{\partial q_\nu} \delta q_\nu\,,$$

and we will then find the transition equations:

$$d\delta\,\mathbf{r} - \delta\,d\,\mathbf{r} = \sum \frac{\partial \mathbf{r}}{\partial q_\nu}(d\delta q_\nu - \delta d q_\nu)\,.$$

If we assume (as we can always do) that $d\,\delta q_\nu - \delta\,dq_\nu = 0$ then:

$$d\delta\,\mathbf{r} - \delta\,d\,\mathbf{r} = 0,$$

and conversely, the first equation will follow from the second one when the q_ν are not redundant. However, we would not like to make that assumption now with no further discussion, but instead, we shall calculate the relationship of the:

$$d\delta\,\mathbf{r} - \delta\,d\,\mathbf{r} \quad \text{and} \quad d\,\delta q_\nu - \delta\,dq_\nu \quad \text{to the} \quad d\,\delta\vartheta_\nu - \delta\,d\vartheta_\nu\,.$$

We can spare ourselves some work in writing if we introduce an $(n + 1)^{\text{th}}$ coordinate by way of:

$$q_{n+1} = t,$$

which belongs to the $(n + 1)^{\text{th}}$ constraint equation:

$$\dot{q}_{n+1} = 1$$

with

$$\delta q_{n+1} = 0 .$$

If we replace n + 1 with n that we can proceed as if the system were scleronomic. We would then have:

$$d\vartheta_\mu = \omega_\mu \, dt = \sum_{\nu=1}^{n} b_{\mu,\nu} \, dq_\nu \, , \tag{7}$$

$$\delta\vartheta_\mu = \sum_{\nu=1}^{n} b_{\mu,\nu} \, \delta q_\nu \, , \tag{8}$$

and the inverses:

$$dq_\nu = \sum_{\nu=1}^{n} B_{\nu\mu} \, d\vartheta_\mu \, , \tag{9}$$

$$\delta q_\nu = \sum_{\nu=1}^{n} B_{\nu\mu} \, \delta\vartheta_\mu \, . \tag{10}$$

The constraint equations read:

$$\delta\vartheta_\mu = 0, \qquad \mu = 1, 2, \ldots, m, \tag{11}$$

and

$$\omega_\mu = \text{const.}, \tag{12}$$

resp., in which one of those constants can be 1, while the others are zero. It follows from (9) and (10) that:

$$\begin{aligned} d\,\delta q_\nu - \delta\,dq_\nu &= \sum_\mu B_{\nu\mu}(d\,\delta\vartheta_\mu - \delta d\vartheta_\mu) + \sum_{\mu,\sigma} \frac{\partial B_{\nu\mu}}{\partial q_\sigma} dq_\sigma \, \delta\vartheta_\mu - \sum_{\mu,\sigma} \frac{\partial B_{\nu\mu}}{\partial q_\sigma} \delta q_\sigma \, d\vartheta_\mu \\ &= \sum_\mu B_{\nu\mu}(d\,\delta\vartheta_\mu - \delta d\vartheta_\mu) + \sum_{\mu,\sigma} \frac{\partial B_{\nu\mu}}{\partial q_\sigma} (dq_\sigma \, \delta\vartheta_\mu - \delta q_\sigma \, d\vartheta_\mu) \, , \end{aligned}$$

for which we can also write:

$$d\,\delta q_\nu - \delta\,dq_\nu = \sum_\mu B_{\nu\mu}(d\,\delta\vartheta_\mu - \delta d\vartheta_\mu) + \sum_{\mu,\sigma,\tau} \left(\frac{\partial B_{\nu\mu}}{\partial q_\sigma} B_{\sigma,\tau} - \frac{\partial B_{\nu\tau}}{\partial q_\sigma} B_{\sigma\mu} \right) dq_\sigma \, \delta\vartheta_\mu \, . \tag{13}$$

If one switches the summation indices τ and μ in the second term of the triple sum then one can also write:

$$d\,\delta q_\nu - \delta\,dq_\nu = \sum_\mu B_{\nu\mu}(d\,\delta\vartheta_\mu - \delta d\vartheta_\mu) + \sum_{\mu,\sigma,\tau} \frac{\partial B_{\nu\mu}}{\partial q_\sigma} B_{\sigma\tau} (dq_\tau \, \delta\vartheta_\mu - dq_\mu \, \delta\vartheta_\tau) . \tag{13b}$$

In a completely analogous way, we can also start from (7), (8) and get the solution of (13):

$$d\,\delta\vartheta_\mu - \delta\,d\vartheta_\mu = \sum_\nu b_{\mu\nu}(d\,\delta\vartheta_\nu - \delta d\vartheta_\nu) + \sum_{\nu,\sigma}\left(\frac{\partial b_{\mu\nu}}{\partial q_\sigma} - \frac{\partial b_{\nu\sigma}}{\partial q_\nu}\right) dq_\sigma\,\delta q_\nu\,, \tag{14}$$

or also:

$$d\,\delta\vartheta_\mu - \delta\,d\vartheta_\mu = \sum_\nu b_{\mu\nu}(d\,\delta q_\nu - \delta dq_\nu) + \sum_{\nu,\sigma,\tau,\rho}\left(\frac{\partial b_{\mu\nu}}{\partial q_\sigma} - \frac{\partial b_{\nu\sigma}}{\partial q_\nu}\right) B_{\sigma\tau}B_{\nu\rho}\,d\vartheta_\tau\,\delta\vartheta_\rho\,, \tag{15}$$

and with the abbreviation:

$$\sum_{\nu,\sigma}\left(\frac{\partial b_{\mu\nu}}{\partial q_\sigma} - \frac{\partial b_{\nu\sigma}}{\partial q_\nu}\right) B_{\sigma\tau}B_{\nu\rho} = \beta_\mu^{\tau,\rho}\,, \tag{16}$$

we can also write:

$$d\,\delta\vartheta_\mu - \delta\,d\vartheta_\mu = \sum_\nu b_{\mu\nu}(d\,\delta q_\nu - \delta dq_\nu) + \sum_{\tau,\rho}\beta_\mu^{\tau,\rho}\,d\vartheta_\tau\,\delta\vartheta_\rho\,. \tag{15a}$$

If we assume that $d\,\delta q_\nu - \delta\,dq_\nu = 0$ (which, as we said, we can always do) then we will get:

$$d\,\delta\vartheta_\mu - \delta\,d\vartheta_\mu = \sum_{\tau,\rho}\beta_\mu^{\tau,\rho}\,d\vartheta_\tau\,\delta\vartheta_\rho\,, \tag{15b}$$

or, since we clearly have:

$$\beta_\mu^{\tau,\rho} = -\beta_\mu^{\tau,\rho}$$

[by switching the summation indices in the summation in (16)], we can also write:

$$d\,\delta\vartheta_\mu - \delta\,d\vartheta_\mu = {}^{`}\!\!\sum \beta_\mu^{\tau,\rho}\,(d\vartheta_\tau\,\delta\vartheta_\rho - \delta\vartheta_\tau\,d\vartheta_\rho)\,, \tag{15c}$$

in which the sum is now extended over the combinations τ, ρ only once, which is suggested by the mark on the Σ .

Now, one should note: *If ϑ_μ is a true coordinate – i.e., $d\vartheta_\mu$ is a total differential – then all of the $\beta_\mu^{\tau,\rho}$ will be zero, and one will have:*

$$d\,\delta\vartheta_\mu - \delta\,d\vartheta_\mu = 0,$$

when one assumes that $d\,\delta q_\nu - \delta\,dq_\nu = 0$, because one will then have:

$$b_{\mu\nu} = \frac{\partial\vartheta_\mu}{\partial q_\nu} \qquad \text{and} \qquad \frac{\partial b_{\mu\nu}}{\partial q_\sigma} - \frac{\partial b_{\mu\sigma}}{\partial q_\nu} = \frac{\partial^2\vartheta_\mu}{\partial q_\sigma\partial q_\nu} - \frac{\partial^2\vartheta_\mu}{\partial q_\nu\partial q_\sigma} = 0.$$

One then recognizes that the non-holonomity of a "quasi-coordinate" (which is what we say when ω_μ is a non-holonomic velocity parameter) means that the $\beta_\mu^{\tau,\rho}$ are not all zero in the transition equation (15a).

233. Critique. – Furthermore, the following should be observed:

If $d\vartheta_\mu / dt$ = const. is a (generally non-holonomic) condition equation then $\delta\vartheta_\mu = 0$, and naturally, $\delta\, d\vartheta_\mu = 0$, as well. By contrast, it would be false to conclude that $d\delta\vartheta_\mu = 0$ with no further assumptions.

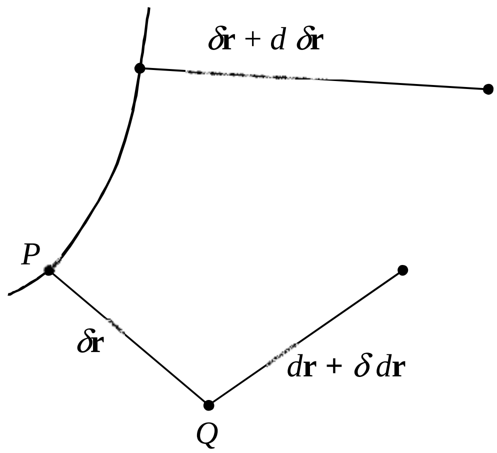

Figure 115.

Naturally, when the δq_ν are functions of the q_μ, the $d\delta q_\nu$ are also defined. By contrast, the δdq_ν are by no means given from the outset by way of the dq_ν. The same thing is true for the $\delta\, d\mathbf{r}$ and the $\delta\, d\vartheta$; i.e., it is not necessary to combine the neighboring points $Q_1, Q_2, \ldots$ into a path and denote the change $d\mathbf{r} + \delta\, d\mathbf{r}$ by $Q_1 Q_2$; as Fig. 115 suggests, that can be assumed in some way. Under the stated assumption, (15a) will imply that:

$$-\delta\, d\vartheta_\mu = \sum_\nu b_{\mu\nu}(d\delta q_\nu - \delta dq_\nu) + \sum_{\tau,\rho} \beta_\mu^{\tau,\rho}\, d\vartheta_\tau\, \delta\vartheta_\rho \,.$$

If one assumes that $d\delta q_\nu - \delta dq_\nu = 0$ then it will follow that:

$$-\delta\, d\vartheta_\mu = \sum_{\tau,\rho} \beta_\mu^{\tau,\rho}\, d\vartheta_\tau\, \delta\vartheta_\rho \,,$$

and that shows that one *cannot* generally regard $\delta\, d\vartheta_\mu$ as zero. In the literature, one often finds the remark that one cannot set $d\,\delta q_\nu - \delta\, dq_\nu = 0$ for non-holonomic systems. That false assertion comes about because the authors tacitly assume that one also has $\delta\, d\vartheta_\mu = 0$. Naturally, one must assume that the condition:

$$0 = \sum_\nu b_{\mu\nu}(d\delta q_\nu - \delta dq_\nu) + \sum_{\tau,\rho} \beta_\mu^{\tau,\rho}\, d\vartheta_\tau\, \delta\vartheta_\rho$$

is true for $d\,\delta q_\nu - \delta\, dq_\nu$, which generally excludes the possibility that $d\,\delta q_\nu - \delta\, dq_\nu = 0$. The tacit assumption that $\delta\, d\vartheta_\mu = 0$ mostly arises from the fact that the authors do not introduce the ω_μ at all, but solve the constraint equations for some suitable dq_ν, so, e.g., for the blade, one has:

$$dy = dx \cdot \tan\vartheta\,, \qquad \text{(a)}$$

$$\delta y = \delta x \cdot \tan\vartheta\,. \qquad \text{(b)}$$

It would then seem to follow from this that:

$$d\,\delta y - \delta\, dy = \tan\vartheta \cdot (d\,\delta x - \delta\, dx) + \frac{1}{\cos^2\vartheta}(d\vartheta\,\delta x - \delta\vartheta\, dx)\,.$$

If the author then makes the assumption that $d\,\delta x - \delta\,dx = 0$ then the foregoing equation will naturally exclude the assumption that $d\,\delta y - \delta\,dy = 0$, which obviously contradicts our assertion. We will say about this that: (a) and (b) are correct. It also follows from (b) that:

$$d\,\delta y = d\,\delta x \tan\vartheta + \frac{1}{\cos^2\vartheta}\,\delta x\, d\vartheta,$$

but (a) cannot be varied with no further assumptions, since $\delta\, d\vartheta_1$ is not zero. (Here, we have assumed that $d\vartheta_1 = dy - dx \tan\vartheta$.) For that reason already, the entirely generally idea that $d\,\delta\mathbf{r} - \delta\,d\mathbf{r}$ or $d\,\delta q_\nu - \delta\,dq_\nu$ are not to be set equal to zero is to be preferred. One will then be completely free. That is especially necessary when one does not wish to corrupt the transition to **Lie**'s theory of infinitesimal transformations.

With **Lie**, we would also like to regard:

$$\delta\mathbf{r} = \sum_\nu \frac{\partial\mathbf{r}}{\partial q_\nu}\delta q_\nu = \sum_{\nu,\mu}\frac{\partial\mathbf{r}}{\partial q_\nu}B_{\nu\mu}\,\delta\vartheta_\mu$$

as an *infinitesimal transformation.* However, in the theory of those transformations, the $\delta\vartheta_\mu$ are regarded as constants, as well as the $d\vartheta_\mu$. But then, the $\delta\,d\vartheta_\mu$, as well as the $d\,\delta\vartheta_\mu$, will both be zero. However, in general, the $d\,\delta q_\nu - \delta\,dq_\nu$, and therefore also the $d\,\delta\mathbf{r} - \delta\,d\mathbf{r}$, cannot be assumed to be zero now.

We then see that equations (15a):

$$d\,\delta\vartheta_\mu - \delta\,d\vartheta_\mu = \sum_\nu b_{\mu\nu}(d\,\delta q_\nu - \delta\,dq_\nu) + \sum_{\tau,\rho}\beta_\mu^{\tau,\rho}\,d\vartheta_\tau\,\delta\vartheta_\rho,$$

along with:

$$d\,\delta\mathbf{r} - \delta\,d\mathbf{r} = \sum\frac{\partial\mathbf{r}}{\partial q_\nu}(d\,\delta q_\nu - \delta\,dq_\nu),$$

are the definitive equations of transition.

234. Example. – *Example 1: The blade*. If we set:

$$d\vartheta_1 = dy\cos\vartheta - dx\sin\vartheta,$$
$$d\vartheta_2 = dx\cos\vartheta + dx\sin\vartheta = v\,dt,$$
$$d\vartheta_3 = d\vartheta,$$

and correspondingly:

$$\delta\vartheta_1 = \delta y\cos\vartheta - \delta x\sin\vartheta,$$
$$\delta\vartheta_2 = \delta x\cos\vartheta + \delta y\sin\vartheta,$$
$$\delta\vartheta_3 = \delta\vartheta$$

then:

$$d\,\delta\vartheta_1 - \delta\,d\vartheta_1 = (d\,\delta y - \delta\,dy)\cos\vartheta - (d\,\delta x - \delta\,dx)\sin\vartheta - \delta y\,d\vartheta\sin\vartheta - \delta x\,d\vartheta\cos\vartheta$$
$$+ dy\,\delta\vartheta\sin\vartheta + dx\,\delta\vartheta\cos\vartheta,$$

$$d\,\delta\vartheta_2 - \delta\,d\vartheta_2 = (d\,\delta x - \delta\,dx)\cos\vartheta + (d\,\delta y - \delta\,dy)\sin\vartheta - \delta x\,d\vartheta\sin\vartheta + \delta y\,d\vartheta\cos\vartheta$$
$$+ dx\,\delta\vartheta\sin\vartheta + dy\,\delta\vartheta\cos\vartheta,$$

$$d\,\delta\vartheta_3 - \delta\,d\vartheta_3 = d\,\delta\vartheta - \delta\,d\vartheta,$$

or

$$d\,\delta\vartheta_1 - \delta\,d\vartheta_1 = -(d\,\delta x - \delta\,dx)\sin\vartheta + (d\,\delta y - \delta\,dy)\cos\vartheta - d\vartheta_3\,\delta\vartheta_2 + \delta\vartheta_3\,d\vartheta_2,$$

$$d\,\delta\vartheta_2 - \delta\,d\vartheta_2 = (d\,\delta x - \delta\,dx)\cos\vartheta + (d\,\delta y - \delta\,dy)\sin\vartheta + d\vartheta_3\,\delta\vartheta_1 - \delta\vartheta_3\,d\vartheta_1,$$

$$d\,\delta\vartheta_3 - \delta\,d\vartheta_3 = d\,\delta\vartheta - \delta\,d\vartheta.$$

One sees from this that all β are constants that are either ± 1 or zero. All $\beta_3^{\tau,\sigma}$ are zero, since $\vartheta_3 = \vartheta$ is a true coordinate. $\beta_1^{2,3} = -\beta_1^{3,2} = 1$, $\beta_2^{3,1} = -\beta_2^{1,3} = 1$; all other β are zero. According to **Lie**, the constancy of β means that the associated infinitesimal transformations define a group; viz., the group of motions of the blade. We shall not go into that, though.

Example 2: The tire. One can perhaps set:

$$\begin{aligned} dx - r\,d\varphi\cos\vartheta &= d\vartheta_1, \\ dy - r\,d\varphi\sin\vartheta &= d\vartheta_2, \\ d\vartheta &= d\vartheta_3, \\ d\varphi &= d\vartheta_4, \\ dx\cos\vartheta + dy\sin\vartheta &= d\vartheta_5. \end{aligned}$$

Since ϑ_3 and ϑ_4 are true coordinates, all β with the index 3 or 4 will be zero. However, it is simpler, here as well (as with the blade), to set:

$$-dx\sin\vartheta + dy\cos\vartheta = d\vartheta_1 = 0$$

and

$$r\,d\varphi - dx\cos\vartheta - dy\sin\vartheta = d\vartheta_2 = 0.$$

Those constraint equations are equivalent to the old ones. Now, ϑ_1, ϑ_3 are the same as they were for the blade, and ϑ_5 is the same as ϑ_2 in that example. Hence, as before, one will have:

$$d\,\delta\vartheta_1 - \delta\,d\vartheta_1 = -(d\,\delta x - \delta\,dx)\sin\vartheta + (d\,\delta y - \delta\,dy)\cos\vartheta - d\vartheta_3\,\delta\vartheta_5 + \delta\vartheta_3\,d\vartheta_5,$$

$$d\,\delta\vartheta_5 - \delta\,d\vartheta_5 = (d\,\delta x - \delta\,dx)\cos\vartheta + (d\,\delta y - \delta\,dy)\sin\vartheta + d\vartheta_3\,\delta\vartheta_1 - \delta\vartheta_3\,d\vartheta_1.$$

Since $d\vartheta_2 = r\,d\varphi - d\vartheta_5$, we find that for the new ϑ_2, we have:

$$\begin{aligned} d\,\delta\vartheta_2 - \delta\,d\vartheta_2 &= r\,(d\,\delta\varphi - \delta\,d\varphi) - (d\,\delta\vartheta_5 - \delta\,d\vartheta_5) \\ &= r\,(d\,\delta\varphi - \delta\,d\varphi) - (d\,\delta\vartheta_5 - \delta\,d\vartheta_5) - (d\,\delta y - \delta\,dy) - d\vartheta_3\,\delta\vartheta_1 + \delta\vartheta_3\,d\vartheta_1 . \end{aligned}$$

All of the β are constant again, namely, ± 1 or zero.

Example 3: The two-wheeled wagon. Here, we also set:

$$\begin{aligned} -dx\sin\vartheta + dy\cos\vartheta &= d\vartheta_1 = 0, \\ dx\cos\vartheta + dy\sin\vartheta &= d\vartheta_2 = 0, \\ d\vartheta &= d\vartheta_3 . \end{aligned}$$

We further set:

$$r\,(d\varphi_1 + d\varphi_2) + 2\,(dx\cos\vartheta + dy\sin\vartheta) = d\vartheta_4 = 0$$

or

$$r\,(d\varphi_1 + d\varphi_2) + 2\,d\vartheta_2 = d\vartheta_4 = 0$$

or

$$2b\,d\vartheta + r\,(d\varphi_1 + d\varphi_2) = d\vartheta_5 = 0 .$$

Those two constraint equations are equivalent to the older ones (1) and (2). Naturally, we get the same transition equations for ϑ_1, ϑ_2, ϑ_3 that we did for the blade, namely:

$$d\,\delta\vartheta_4 - \delta\,d\vartheta_4 = r\,(d\,\delta\varphi_1 - \delta\,d\varphi_1) + r\,(d\,\delta\varphi_2 - \delta\,d\varphi_2) + 2\,(d\,\delta\vartheta_2 - \delta\,d\vartheta_2)$$

$$\begin{aligned} &= r\,(d\,\delta\varphi_1 - \delta\,d\varphi_1) + r\,(d\,\delta\varphi_2 - \delta\,d\varphi_2) \\ &\quad + 2\,(d\,\delta x - \delta\,dx)\cos\vartheta + 2\,(d\,\delta y - \delta\,dy)\sin\vartheta + 2\,(d\,\delta\vartheta - \delta\,d\vartheta), \end{aligned}$$

$$d\,\delta\vartheta_4 - \delta\,d\vartheta_4 = 2b\,(d\,\delta\vartheta - \delta\,d\vartheta) + r\,(d\,\delta\varphi_1 - \delta\,d\varphi_1) .$$

One sees that $d\vartheta_5 = 0$ is an integrable combination, so $\vartheta_5 = 2b\,\vartheta + r\,(\varphi_1 - \varphi_2)$ is a true coordinate. All β are also constant here.

§ 3. – Deriving the equations of motion.

235. The Lagrange-Euler equations. – We have **Lagrange**'s principle as our foundation, and together with the generalized central equation that will yield:

$$\frac{d}{dt}\mathsf{S}\,dm\,\mathbf{v}\,dt - \delta T - \mathsf{S}\,dm\,\mathbf{v}\frac{d\,\delta\mathbf{r} - \delta\,d\mathbf{r}}{dt} = \delta A = \sum K_\nu\,\delta q_\nu .$$

Now, $T = T(q_\nu,\ \dot{q}_\nu,\ t)$, and $\mathsf{S}\ dm\ \mathbf{v}\ \delta\mathbf{r} = \sum p_\nu\ \delta q_\nu$, where $p_\nu = \partial T / \partial \dot{q}_\nu$ are the components of the impulse.

However, since:

$$d\ \delta\mathbf{r} - \delta\ d\mathbf{r} = \sum \frac{\partial \mathbf{r}}{\partial q_\nu}(d\ \delta q_\nu - \delta\ dq_\nu)$$

from the transition equations, and:

$$p_\nu = \mathsf{S}\ dm\ \mathbf{v}\ \frac{\partial \mathbf{r}}{\partial q_\nu},$$

we will have:

$$\mathsf{S}\,dm\,\mathbf{v}\frac{d\ \delta\mathbf{r} - \delta\ d\mathbf{r}}{dt} = \sum p_\nu \frac{d\ \delta q_\nu - \delta\ dq_\nu}{dt}.$$

We can then write the central equation as:

$$\frac{d}{dt}\sum_\nu p_\nu\ \delta q_\nu - \delta T - \sum_\nu p_\nu \frac{d\ \delta q_\nu - \delta\ dq_\nu}{dt} = \sum K_\nu\ \delta q_\nu\ .$$

If we now set:

$$\delta q_\nu = \sum B_{\nu\mu}\ \delta\vartheta_\mu$$

from § **2**, formula (10), and convert T into $\mathbf{T}(q_\nu,\ \omega_\mu)$, with:

$$\dot{q}_\nu = \sum B_{\nu\mu}\ \omega_\mu$$

(we have set t = one of the q !) then we might set:

$$\sum_\nu p_\nu\ \delta q_\nu = \sum_{\nu,\mu} p_\nu\ B_{\nu\mu}\ \delta\vartheta_\mu = \sum_\nu P_\nu\ \delta\vartheta_\nu\ ,$$

$$\sum_\nu K_\nu\ \delta q_\nu = \sum_{\nu,\mu} K_\nu\ B_{\nu\mu}\ \delta\vartheta_\mu = \sum \mathsf{K}_\mu\ \delta\vartheta_\mu\ .$$

Theorem:

One has:

$$P_\mu = \frac{\partial \mathsf{T}}{\partial \omega_\mu}.$$

Proof:

$$P_\mu = \sum_\nu p_\nu\ B_{\nu\mu}\ ,$$

but

$$\frac{\partial \mathsf{T}}{\partial \omega_\mu} = \sum_\nu \frac{\partial T}{\partial \dot{q}_\nu}\frac{\partial \dot{q}_\nu}{\partial \omega_\mu} = \sum_\nu p_\nu\ B_{\nu\mu} = P_\mu\ .$$

In order to find the equations of motion, we can (and we would like to) set $d\,\delta q_\nu - \delta\,dq_\nu = 0$. The central equation will then give:

$$\frac{d}{dt}\sum P_\mu\,\delta\vartheta_\mu - \sum\frac{\partial \mathbf{T}}{\partial\omega_\mu}\delta\omega_\mu - \sum\frac{\partial \mathbf{T}}{\partial q_\nu}\delta q_\nu = \sum \mathsf{K}_\mu\,\delta\vartheta_\mu\,,$$

or since:

$$P_\mu = \frac{\partial \mathbf{T}}{\partial\omega_\mu}\,,$$

we will have:

$$\sum\frac{dP_\mu}{dt}\delta\vartheta_\mu + \sum P_\mu\left(\frac{d\,\delta\vartheta_\mu}{dt} - \frac{\delta\,d\vartheta_\mu}{dt}\right) - \sum\frac{\partial \mathbf{T}}{\partial q_\nu}B_{\nu\mu}\,\delta\vartheta_\nu = \sum \mathsf{K}_\mu\,\delta\vartheta_\mu\,. \tag{I}$$

We now introduce the transition equations:

$$d\,\delta\vartheta_\mu - \delta\,d\vartheta_\mu = \sum_{\tau,\rho}\beta_\mu^{\tau,\rho}\,d\vartheta_\tau\,\delta\vartheta_\rho$$

and get:

$$\sum\frac{dP_\mu}{dt}\delta\vartheta_\mu + \sum_{\mu,\rho,\tau}\delta\vartheta_\mu(P_\rho\,\beta_\rho^{\tau,\mu}\,\omega_\tau) - \sum\frac{\partial \mathbf{T}}{\partial q_\nu}B_{\nu\mu}\,\delta\vartheta_\nu = \sum \mathsf{K}_\mu\,\delta\vartheta_\mu\,.$$

Now, we have $\delta\vartheta_\mu = 0$ for $\mu = 1, 2, 3, \ldots, m$. The corresponding terms then drop out. By contrast, the $\delta\vartheta_\mu$ are free for $\mu = m + 1, \ldots, n$. Hence, one has the equations:

$$\frac{dP_\mu}{dt} + \sum_{\rho,\tau}\beta_\rho^{\tau,\mu}\,P_\rho\,\omega_\tau - \left(\frac{\partial \mathbf{T}}{\partial\vartheta_\mu}\right) = \mathsf{K}_\mu \qquad (\mu = m + 1, \ldots, n). \tag{II}$$

We have allowed ourselves to write $\sum_\nu\frac{\partial \mathbf{T}}{\partial q_\nu}B_{\nu\mu} = \left(\frac{\partial \mathbf{T}}{\partial\vartheta_\mu}\right)$ in this. Namely, if ϑ_μ were a true coordinate then from the chain rule, one would have:

$$\sum_\nu\frac{\partial \mathbf{T}}{\partial q_\nu}B_{\nu\mu} = \sum\frac{\partial \mathbf{T}}{\partial q_\nu}\frac{\partial q_\nu}{\partial\vartheta_\mu} = \frac{\partial \mathbf{T}}{\partial\vartheta_\mu}\,.$$

Analogously, we can write:

$$\mathsf{K}_\mu = -\left(\frac{\partial U}{\partial\vartheta_\mu}\right)$$

in the case of a potential U. Just to be careful, we put the expression in parentheses, since it certainly does not need to be a true derivative.

In v. 50 of the Zeit. f. Math. u. Physik, 1904, we referred to equations (II) as the **Lagrange-Euler** equations, because they include the **Lagrange** equations as a special case, namely, for the case in which all β are zero, and therefore all ϑ are true coordinates, as well as the **Euler** equations for the rigid body, which we still have to prove (in § **5**).

Equations (II) were exhibited independently by the Italian **Volterra**, the Russian **Voronetz**, and by **Poincaré** for the case of constant β. Boltzmann also come close to exhibiting them. The methods of those authors differ in places.

236. The case $m = 0$. – Naturally, it can happen that $m = 0$, so there are no non-holonomic constraints present, but one must still introduce non-holonomic velocity parameters. That is the case for, e.g., the rigid body.

In complete generality, one can use a linear transformation of the $\dot{q}_\nu$:

$$\dot{q}_\nu = \sum_\mu B_{\nu\mu}\,\omega_\mu$$

to transform the positive-definite form:

$$T = \tfrac{1}{2}\sum a_{\rho\tau}\,\dot{q}_\sigma\,\dot{q}_\tau$$

into the form:

$$\mathsf{T} = \tfrac{1}{2}\sum \omega_\mu^2 \,.$$

One will then have:

$$P_\mu = \omega_\mu \,,$$

and the equations of motion (II) will read:

$$\frac{d\omega_\mu}{dt} + \sum_{\rho,\tau} \beta_\rho^{\tau,\mu}\,\omega_\rho\,\omega_\tau = \mathsf{K}_\mu \,. \tag{IIa}$$

However, the β will still depend upon the q, in general. Equations (IIa), together with the equations:

$$\dot{q}_\nu = \sum_\mu B_{\nu\mu}\,\omega_\mu \,,$$

define a simultaneous system of $2n$ first-order differential equations for the ω_μ and q_μ . Their validity is completely general.

237. Warning and remark. – If condition equations $\omega_\mu = 0$, $\mu = 1, 2, \ldots, m$ are present then the sum over t in (II) will extend from only $m + 1$ to n . However, one must be careful to set the ω_1, ω_2, … equal to zero in $\mathsf{T} = T$ from the outset and to use the risky T^+ that arises in that way.

Since one needs the derivatives with respect to ω_ρ in order to calculate the P_ρ, one must let them be variable. One can probably set quadratic terms with vanishing ω equal to zero from the outset.

If one sets $q_n = t$ (for rheonomic systems), so $\omega_n = 1$, then $\delta\vartheta_n = 0$, and the n^{th} equation must be dropped in order for one to set $\omega_n = 1$. However, one must also only do that afterwards, since one must differentiate with respect to ω_n in order to construct P_n.

If one would like to employ the generalized central equation then one would have to set:

$$d\,\delta\vartheta_\mu - \delta\,d\vartheta_\mu = \sum_\nu b_{\mu\nu}(d\,\delta q_\nu - \delta\,dq_\nu) + \sum_{\tau,\rho}\beta_\mu^{\tau,\rho}\,d\vartheta_\tau\,\delta\vartheta_\rho$$

and then get:

$$\sum_{\mu,\nu} P_\mu\, b_{\mu\nu}(d\,\delta q_\nu - \delta\,dq_\nu) - \sum_\nu p_\nu\,\frac{d\,\delta q_\nu - \delta\,dq_\nu}{dt},$$

in addition. However, that is zero, since one has $\sum P_\mu\, b_{\mu\nu} = p_\nu$, since that is the solution to:

$$P_\mu = \sum_\nu B_{\nu\mu}\, p_\nu .$$

§ 4. – Examples.

238. The blade. – If one would like to exhibit the equations of motion (II) then it is not always perhaps practical to exhibit the table of β's, but much simpler to revert to the form:

$$\sum_\mu \frac{dP_\mu}{dt}\delta\theta_\mu + \sum_\rho P_\rho\left(\frac{d\,\delta\vartheta_\rho}{dt} - \frac{\delta\,d\vartheta_\rho}{dt}\right) - \sum\frac{\partial \mathsf{T}}{\partial q_\nu}B_{\nu\mu}\,\delta\vartheta_\mu = \sum \mathsf{K}_\mu\,\delta\vartheta_\mu\,, \tag{I}$$

after calculating T and the impulse $P_\mu = \partial\mathsf{T} / \partial\omega_\mu$, and to look for the terms with $\delta\vartheta_\mu$ in the expression for $d\,\delta\vartheta_\mu - \delta\,d\vartheta_\mu$. That is how we shall proceed.

Example 1: The blade. – From § **1**, one has:

$$T = \tfrac{1}{2}m v^2 + m s\,\dot\vartheta(\dot y\cos\vartheta - \dot x\sin\vartheta) + \tfrac{1}{2}I_B\,\dot\vartheta^2,$$
$$\mathsf{T} = \tfrac{1}{2}m\,\omega_2^2 + m s\,\omega_3\,\omega_1 + \tfrac{1}{2}I_B\,\omega_3^2.$$

Hence:

$$P_1 = \frac{\partial\mathsf{T}}{\partial\omega_1} = m\,s\,\omega_3\,,$$
$$P_2 = \frac{\partial\mathsf{T}}{\partial\omega_2} = m\,\omega_2\,,$$

$$P_3 = \frac{\partial \mathsf{T}}{\partial \omega_3} = m\,s\,\omega_1 + I_B\,\omega_3 = I_B\,\omega_3\,.$$

Since one has $\delta\vartheta_1 = 0$, we only have to exhibit the two equations that belong to $\delta\vartheta_2$ and $\delta\vartheta_3$. Since:

$$\delta x = \delta\vartheta_2 \cos\vartheta - \delta\vartheta_1 \sin\vartheta,$$
$$\delta y = \delta\vartheta_2 \sin\vartheta + \delta\vartheta_1 \cos\vartheta,$$

one will have $\mathsf{K}_2 = X\cos\vartheta + Y\sin\vartheta = Z$, and:

$$\mathsf{K}_3 = M\,.$$

From § **2**, $\delta\vartheta_2$ belongs to $-\omega_3\,P_1$, and $\delta\vartheta_3$ belongs to $\omega_2\,P_1 - \omega_2\,P_1 = \omega_2\,P_1$. Since T is independent of the coordinates, the equations of motion will then read:

$$\frac{dP_1}{dt} - \omega_3\,P_1 = Z \qquad \text{or} \qquad m\frac{d\omega_2}{dt} - \omega_3^2\,m\,s = Z,$$

$$\frac{dP_3}{dt} + \omega_2\,P_1 = M \qquad\qquad I_B\frac{d\omega_3}{dt} + \omega_2\,m\,s\,\omega_3 = M\,.$$

However, since $v = \omega_2$, $\dot{\vartheta} = \omega_3$, those are the same equations as in § **1**.

Remark: We must actually write $\frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m(v^2 + \omega_1^2)$ in T instead of $\frac{1}{2}mv^2$. However, we can drop the purely-quadratic term $\frac{1}{2}m\,\omega_1^2$, since it contributes the term $m\,\omega_1$ to P_1, which vanishes.

We will devote a special paragraph to Example 2, namely, the tire; for now, we shall turn to:

239. Example 3: the two-wheeled wagon. – With:

$$\omega_1 = -\,\dot{x}\sin\vartheta + \dot{y}\cos\vartheta = 0,$$
$$\omega_2 = \dot{x}\cos\vartheta + \dot{y}\sin\vartheta = 0,$$
$$\omega_3 = \dot{\vartheta},$$
$$\omega_4 = r(\dot{\varphi}_1 + \dot{\varphi}_2) + 2\omega_2 = 0,$$
$$\omega_5 = 2b\,\dot{\vartheta} + r(\dot{\varphi}_1 - \dot{\varphi}_2) = 0,$$

one has:

$$\mathsf{T} = \tfrac{1}{2}m\,\omega_2^2 + m\,a\,\omega_3\,\omega_1 + \tfrac{1}{2}I\,\omega_3^2 + \frac{1}{2}\frac{C_B}{r^2}(-2\omega_2\,\omega_4 + 2\omega_2^2 - 2b\,\omega_3\,\omega_5 + 2b^2\omega_3^2)\,.$$

One then has:

$$2r\,\dot{\varphi}_1 = \omega_4 + \omega_5 - 2\omega_2 - 2b\,\omega_3\,,$$
$$2r\,\dot{\varphi}_2 = \omega_4 - \omega_5 - 2\omega_2 + 2b\,\omega_3\,,$$

and as a result:

$$\dot{\varphi}_1^2 + \dot{\varphi}_2^2 = [(\omega_4 - 2\omega_2)^2 + (\omega_5 - 2\omega_3)^2]\,.$$

We have dropped the terms in ω_1^2, ω_2^2, ω_3^2 from our calculations. As a result:

$$P_1 = \frac{\partial \mathsf{T}}{\partial \omega_1} = m\,a\,\omega_3\,,$$
$$P_2 = \frac{\partial \mathsf{T}}{\partial \omega_2} = m\,\omega_2 - \frac{C_B}{r^2}\omega_4 + 2\frac{C_B}{r^2}\omega_2 = m\,\omega_2 + 2\frac{C_B}{r^2}\omega_2 = \omega_2\left(m + 2\frac{C_B}{r^2}\right),$$
$$P_3 = \frac{\partial \mathsf{T}}{\partial \omega_3} = m\,a\,\omega_1 + I\,\omega_3 - \frac{C_B}{r^2}b\,\omega_5 + 2\frac{C_B}{r^2}b^2\omega_3 = \omega_3\left(I + 2\frac{C_B}{r^2}b^2\right),$$
$$P_4 = \frac{\partial \mathsf{T}}{\partial \omega_4} = -\frac{C_B}{r^2}\omega_2\,,$$
$$P_5 = \frac{\partial \mathsf{T}}{\partial \omega_5} = -\frac{C_B}{r^2}\omega_3\,b\,.$$

From the transition equations, one can associate:

$\delta\vartheta_2$	with	$-P_1\,\omega_3\,,$
$\delta\vartheta_3$	with	$P_1\,\omega_2 - \omega_2 P_1 - 2\omega_2 P_1 = P_1\,\omega_2\,.$

Since T is once more independent of the coordinates, and:

$$\begin{cases} X\,\delta x + Y\,\delta y + M\,\delta\vartheta = X(\cos\vartheta\,\delta\vartheta_2 - \sin\vartheta\,\delta\vartheta_1) + Y(\cos\vartheta\,\delta\vartheta_1 + \sin\vartheta\,\delta\vartheta_2) + M\,\delta\vartheta_3 \\ \qquad = Z\,\delta\vartheta_2 + M\,\delta\vartheta_3, \end{cases}$$

the equations of motion (II) will then read:

$$\left.\begin{aligned} \frac{d}{dt}P_2 - P_1\,\omega_1 &= Z, \\ \frac{d}{dt}P_3 + P_1\,\omega_2 &= M \end{aligned}\right\} \qquad \text{or} \qquad \left\{\begin{aligned} \frac{d\omega_2}{dt}m' - m\,a\,\omega_3^2 &= Z, \\ \frac{d\omega_3}{dt}I' + m\,a\,\omega_3\,\omega_2 &= M. \end{aligned}\right.$$

We have set $m + 2\dfrac{C_R}{r^2} = m'$, $I + 2\dfrac{C_R}{r^2}b^2 = I'$.

The equations are essentially identical to those of the blade.

240. A rheonomic example. – As a *fourth example*, we shall treat a *rheonomic system.* Place wheels of radius r_1 and r_2 be placed along the axis of a sliding shaft of variable cross-section that rotates with an angular velocity of ω_0, such that they inevitably rotate with it (perhaps by means of gears), as well as raising and lowering. The displacement along the longitudinal axis happens in a given way, but by contrast let ω_0 not be given. If φ_1, φ_2 are the angles of rotation and x_1, x_2 are the distances from the centers to the rotational axis then:

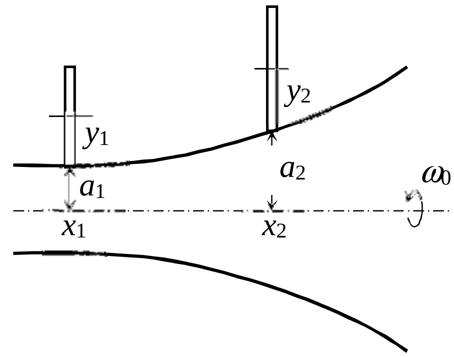

Figure 116.

$$r_1\,\dot{\varphi}_1 = a_1\,\omega_0\,, \qquad r_2\,\dot{\varphi}_2 = a_2\,\omega_0\,,$$

so $\dot{\varphi}_1 = f(t)\,\dot{\varphi}_2$, if we set $\frac{a_1}{a_2}\frac{r_2}{r_1} = f(t)$. Although r_1, r_2 are fixed, a_1 and a_2 are, however, dependent upon t by way of the given displacement of the shaft. In this, we have a system with four degrees of freedom that is nonetheless rheonomic. If we set q_3 then we will have the equations of motion:

$$x_1 = a_1(t) + r_1\,, \qquad x_1 = a_1(t) + r_1\,,$$

so

$$\dot{x}_1 = \dot{a}_1\,\omega_3\,, \qquad \dot{x}_2 = \dot{a}_2\,\omega_3\,,$$

$$d\varphi_1 - f(q_3)\,d\varphi_2 = d\vartheta_1 = 0\,.$$

We let:

$$\varphi_2 = q_2 = \vartheta_2$$

in this. The kinetic energy is:

$$T = \tfrac{1}{2}m_1\,\dot{x}_1^2 + \tfrac{1}{2}m_2\,\dot{x}_2^2 + \tfrac{1}{2}I_1\,\dot{\varphi}_1^2 + \tfrac{1}{2}I_2\,\dot{\varphi}_2^2\,,$$

$$\mathsf{T} = \tfrac{1}{2}F(q_3)\,\omega_3^2 + \tfrac{1}{2}I_1[\omega_1 + f(q_3)\omega_2]^2 + \tfrac{1}{2}I_2\,\omega_2^2\,,$$

with

$$F(q_3) = m_1\,\dot{a}_1^2 + m_2\,\dot{a}_2^2\,.$$

Thus:

$$P_1 = \frac{\partial \mathsf{T}}{\partial \omega_1} = I_1\,[\omega_1 + f(q_3)\,\omega_2] = I_1\,f(q_3)\,\omega_2\,,$$

$$P_2 = \frac{\partial \mathsf{T}}{\partial \omega_2} = f(q_3)\,I_1\,[\omega_1 + f(q_3)\,\omega_2] + I_2\,\omega_2 = I_1\,f(q_3)\,\omega_2 + I_2\,\omega_2\,,$$

$$P_3 = \frac{\partial \mathsf{T}}{\partial \omega_3} = F\,(q_3)\;\omega_2 = F\,(q_3)\;.$$

The transition equations read:

$$\begin{aligned} d\,\delta\vartheta_1 - \delta\,d\vartheta_1 &= d\,\delta\varphi_1 - \delta\,d\varphi_1 - f\,(q_3)\,(d\,\delta\varphi_2 - \delta\,d\varphi_2) - \dot f\;(\,dq_3\;\delta\varphi_2 - \delta\,q_3\;d\varphi_2) \\ &= d\,\delta\varphi_1 - \delta\,d\varphi_1 - f\,(q_3)\,(d\,\delta\varphi_2 - \delta\,d\varphi_2) - \dot f\;(\,d\vartheta_3\;\delta\vartheta_2 - \delta\,\vartheta_3\;d\vartheta_2), \\ d\,\delta\vartheta_2 - \delta\,d\vartheta_2 &= d\,\delta\varphi_2 - \delta\,d\varphi_2\,, \\ d\,\delta\vartheta_3 - \delta\,d\vartheta_3 &= d\,\delta\varphi_3 - \delta\,d\varphi_3\,. \end{aligned}$$

Therefore, $\delta\vartheta_2$ is associated with:

$$-P_1\,\dot f\,\omega_2 = -P_1\,\dot f\;.$$

Moreover, one has:

$$\frac{\partial \mathsf{T}}{\partial \varphi_1} = 0, \qquad \frac{\partial \mathsf{T}}{\partial \varphi_2} = 0, \qquad \frac{\partial \mathsf{T}}{\partial \varphi_3} = \tfrac{1}{2}\dot F(q_3) + I_1[\omega_1 + f\,(q_3)\,\omega_2]\,\dot f(q_3)\,\omega_2$$

$$= \tfrac{1}{2}\dot F(t) + I_1\,f\,(t)\,\omega_2^2\;\dot f\;.$$

However, the relations:

$$\dot q_\nu = \sum B_{\nu\mu}\,\omega_\mu$$

read

$$\begin{aligned} \dot q_1 &= \dot\varphi_1 = \omega_1 + f\,\omega_2\,, \\ \dot q_2 &= \dot\varphi_2 = \omega_2\,, \\ \dot q_3 &= \omega_3 \end{aligned}$$

here. As a result:

$$\left(\frac{\partial \mathsf{T}}{\partial \vartheta_1}\right) = \frac{\partial \mathsf{T}}{\partial q_1} = 0, \qquad \left(\frac{\partial \mathsf{T}}{\partial \vartheta_2}\right) = \frac{\partial \mathsf{T}}{\partial q_1}\,f + \frac{\partial \mathsf{T}}{\partial q_2} = 0, \qquad \left(\frac{\partial \mathsf{T}}{\partial \vartheta_3}\right) = \frac{\partial \mathsf{T}}{\partial q_3} = \tfrac{1}{2}\dot F + I_1 f\;\dot f\;\omega_2^2\,.$$

If forces X_1 and X_2 act upon the system, along with moments M_1 and M_2, then the virtual work done will be:

$$\delta A = X_1\;\delta x_1 + X_2\;\delta x_2 + M_1\;\delta\varphi_1 + M_2\;\delta\varphi_2 = M_1\,(\delta\vartheta_1 + f\;\delta\vartheta_2) + M_2\;\delta\vartheta_2\;.$$

Hence:

$$\mathsf{K}_2 = M_1\,f + M_2\;.$$

The equation of motion then reads:

$$\frac{d}{dt}P_2 - P_1\dot f = \mathsf{K}_2 \qquad \text{or} \qquad \frac{d}{dt}(I_1 f^2\dot\varphi_2 + I_2\,\dot\varphi_2) - I_1 f\;\dot f\dot\varphi_2 = M_1\,f + M_2\;.$$

However, it would wrong to use the illegitimate form of the kinetic energy:

$$T^{+} = \tfrac{1}{2}F(t) + \tfrac{1}{2}I_1 f^2 \dot{\varphi}_2^2 + \tfrac{1}{2}I_2\,\dot{\varphi}_2^2$$

and construct the **Lagrange** equation from that:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\varphi}_2} - \frac{\partial T}{\partial \varphi_2} = K_2\,,$$

such that:

$$\frac{d}{dt}(I_1 f^2 + I_2)\,\dot{\varphi}_2 = K_2 = M_1 f + M_2\,.$$

This equation is clearly different from the one above. The difference lies in the term $-I_1 f\;\dot{f}\,\dot{\varphi}_2$, which drops out because of the illegitimate – i.e., premature – use of the equations of constraint.

Remark: We have treated the shaft as merely a massless control element.

241. A holonomic example. – As a *fifth* example, we shall take one that is intrinsically holonomic, but into which we would like to introduce a non-holonomic velocity parameter in such a way that $\mathsf{T} = \tfrac{1}{2}\sum \omega_\mu^2$.

With two degrees of freedom, let:

$$T = \tfrac{1}{2}[\dot{q}_1^2 + 2q_1\,\dot{q}_1\,\dot{q}_2 + (q_1^2 + 1)\dot{q}_2^2].$$

If $\omega_1 = \dot{q}_1 + q_1\,\dot{q}_2$, $\omega_2 = \dot{q}_2$, whose inverses are $\dot{q}_2 = \omega_2$, $\dot{q}_1 = \omega_1 - q_1\,\omega_2$, then:

$$\mathsf{T} = \tfrac{1}{2}(\omega_1^2 + \omega_2^2)\,.$$

The transition equations read:

$$\begin{aligned} d\,\delta\vartheta_1 - \delta\,d\vartheta_1 &= d\,\delta q_1 - \delta\,dq_1 + q_1\,(d\,\delta q_2 - \delta\,dq_2) + dq_1\,\delta q_2 - \delta q_1\,dq_2 \\ &= d\,\delta q_1 - \delta\,dq_1 + q_1\,(d\,\delta q_2 - \delta\,dq_2) + (d\vartheta_1 - q_1\,d\vartheta_2)\,\delta\vartheta_1 - (\delta\vartheta_1 - q_1\,\delta\vartheta_2)\,d\vartheta_2\,, \\ &= d\,\delta q_1 - \delta\,dq_1 + q_1\,(d\,\delta q_2 - \delta\,dq_2) + d\vartheta_1\,\delta\vartheta_2 - \delta\vartheta_1\,d\vartheta_2\,, \end{aligned}$$

and

$$d\,\delta\vartheta_2 - \delta\,d\vartheta_2 = d\,\delta q_2 - \delta\,dq_1\,.$$

$\delta\vartheta_1$ is then associated with $-\,\omega_2\,P_2 = -\,\omega_2\,\omega_1$, $\delta\vartheta_2$ is then associated with $\omega_1\,P_1 = \omega_1^2$. Now, if the forces K_1 and K_2 act then:

$$\begin{aligned} \delta A &= K_1\,\delta q_1 + K_2\,\delta q_2 = K_1\,(\delta\vartheta_1 - q_1\,\delta\vartheta_1) + K_2\,\delta\vartheta_2 \\ &= \mathsf{K}_1\,\delta\vartheta_1 + \mathsf{K}_2\,\delta\vartheta_2 \qquad \text{with} \quad \mathsf{K}_1 = K_1 \quad \text{and} \quad \mathsf{K}_2 = -\,q_1\,K_1 + K_2\,, \end{aligned}$$

and the equations of motion will read:

and

$$\left.\begin{aligned} \frac{d}{dt}\omega_1 - \omega_1\,\omega_2 &= K_1, \\ \frac{d}{dt}\omega_2 + \omega_1^2 \quad &= K_2. \end{aligned}\right\} \tag{II}$$

The **Lagrange** equations, which are entirely legitimate here, read:

$$\left.\begin{aligned} \frac{d}{dt}(\dot{q}_1 - q_1\,\dot{q}_2) - \dot{q}_1\,\dot{q}_2 - q_1\,\dot{q}_2^2 &= K_1, \\ \frac{d}{dt}[q_1\,\dot{q}_2 + (q_2^2+1)\,\dot{q}_2] &= K_2. \end{aligned}\right\} \tag{III}$$

The first equations are identical. If one adds the first of equations (II) to the second one, multiplied by q_1, then one will get:

$$\ddot{q}_2 + (\dot{q}_1 + q_1\,\dot{q}_2)^2 + q_1(\ddot{q}_1 + q_1\,\ddot{q}_2 + \dot{q}_1\,\dot{q}_2) - (\dot{q}_1 + q_1\,\dot{q}_2)\,\dot{q}_2\,q_1 = K_2$$

or

$$q_1\ddot{q}_2 + \dot{q}_1^2 + (1+q_1^2)\,\ddot{q}_2 + 2\,\dot{q}_1\,\dot{q}_2\,q_1 = K_2\ ;$$

i.e., the second equation in (III). Naturally, both (II) and (III) are identical. However, since the β are constant, the form (II) is certainly more convenient to integrate for force-free motion, which is then the same as for the blade. The **Euler** equations of the rigid body also belong to this case, which we would like to treat in a special section as a *sixth example.*

§ 5. – The rigid body.

242. New derivation of Euler's equations. – The rigid body shall rotate about a fixed point, such that its kinetic energy will be:

$$T = \tfrac{1}{2}(A\,p^2 + B\,q^2 + C\,r^2)\ .$$

In order to have precisely the form $\frac{1}{2}\sum \omega_\mu^2$, we need only to set $\sqrt{A}p = \omega_1$, $\sqrt{B}p = \omega_2$, $\sqrt{C}p = \omega_3$, but that is not inessential, so we shall not do that.

We now need the transition equations for the p, q, r. We can derive them from the given relations (Chap. II, § **8**):

$$p = \dot{\chi}\cos\psi + \dot{\varphi}\sin\chi\sin\varphi\,, \qquad q = -\dot{\chi}\sin\psi + \dot{\varphi}\sin\chi\cos\varphi\,, \qquad r = \dot{\varphi}\cos\chi + \dot{\psi}\ ;$$

however, it is more convenient to recall the **Euler** formulas:

$$d\mathbf{r} = d\boldsymbol{\vartheta} \times \mathbf{r}, \quad \text{and} \quad \delta\mathbf{r} = \delta\boldsymbol{\vartheta} \times \mathbf{r},$$

It follows directly from this that:

$$\begin{aligned} d\,\delta\mathbf{r} - \delta d\,\mathbf{r} &= (d\,\delta\boldsymbol{\vartheta} - \delta d\boldsymbol{\vartheta}) \times \mathbf{r} + \delta\boldsymbol{\vartheta} \times d\,\mathbf{r} - d\boldsymbol{\vartheta} \times \delta\mathbf{r} \\ &= (d\,\delta\boldsymbol{\vartheta} - \delta d\boldsymbol{\vartheta}) \times \mathbf{r} + \delta\boldsymbol{\vartheta} \times (d\boldsymbol{\vartheta} \times d\,\mathbf{r}) - d\,\boldsymbol{\vartheta} \times (\delta\boldsymbol{\vartheta} \times \mathbf{r}) . \end{aligned}$$

However, from the known formula $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) \equiv 0$, that is also:

$$(d\,\delta\boldsymbol{\vartheta} - \delta d\boldsymbol{\vartheta}) \times \mathbf{r} - \mathbf{r} \times (\delta\boldsymbol{\vartheta} \times d\boldsymbol{\vartheta}) .$$

If one assumes that $d\,\delta\mathbf{r} - \delta d\mathbf{r} = 0$ then (since $\mathbf{r}$ is arbitrary) the transition formula:

$$d\,\delta\boldsymbol{\vartheta} - \delta d\boldsymbol{\vartheta} = -\,\delta\boldsymbol{\vartheta} \times d\boldsymbol{\vartheta}$$

will follow. However, that refers to the rest coordinate system. If we denote our derivatives relative to the moving coordinate system, which is fixed in the body, by putting primes on d (δ, resp.) then from Chap. VIII, § **1**:

$$d\,\delta\boldsymbol{\vartheta} = d'\delta\boldsymbol{\vartheta} + d\boldsymbol{\vartheta} \times \delta\boldsymbol{\vartheta}$$

and

$$\delta d\boldsymbol{\vartheta} = \delta' d\boldsymbol{\vartheta} + \delta\boldsymbol{\vartheta} \times d\boldsymbol{\vartheta} .$$

Hence:

$$d\,\delta\boldsymbol{\vartheta} - \delta d\boldsymbol{\vartheta} = d'\delta\boldsymbol{\vartheta} - \delta' d\boldsymbol{\vartheta} + 2\, d\boldsymbol{\vartheta} \times \delta\boldsymbol{\vartheta} .$$

Therefore:

$$\begin{aligned} d'\delta\boldsymbol{\vartheta} - \delta' d\boldsymbol{\vartheta} &= d\,\delta\boldsymbol{\vartheta} - \delta d\boldsymbol{\vartheta} - 2\, d\boldsymbol{\vartheta} \times \delta\boldsymbol{\vartheta} \\ &= -\,\delta\boldsymbol{\vartheta} \times d\boldsymbol{\vartheta} - 2\, d\boldsymbol{\vartheta} \times \delta\boldsymbol{\vartheta} , \\ &= -\,d\boldsymbol{\vartheta} \times \delta\boldsymbol{\vartheta} . \end{aligned}$$

Lagrange already found this.

Therefore, when the equations of motion are referred to a system that is fixed in the body:

$$\left(\frac{\partial}{\partial t}\frac{\partial T}{\partial \boldsymbol{\omega}}\right)\delta\boldsymbol{\vartheta} - (\boldsymbol{\omega} \times \delta\boldsymbol{\vartheta})\frac{\partial T}{\partial \boldsymbol{\omega}} = \mathbf{M}\;\delta\boldsymbol{\vartheta},$$

or, with $\mathbf{D} = \partial T / \partial \boldsymbol{\omega}$:

$$\frac{\partial}{\partial t}\mathbf{D} + \boldsymbol{\omega} \times \mathbf{D} = \mathbf{M},$$

and that is the **Euler** equation, which also represents a special case of the **Lagrange-Euler** equations.

We shall now take up the *second example.*

§ 6. – The tire.

243. Exhibiting the equations of motion. – From § **1**, with:

$$
\begin{aligned}
d\vartheta_1 &= -\,dx \sin\vartheta + dy \cos\vartheta = 0,\\
d\vartheta_2 &= \;\; r\,d\varphi - dx\cos\vartheta - dy\cos\vartheta = r\,d\psi - d\vartheta_5 = 0,\\
d\vartheta_3 &= d\vartheta,\\
d\vartheta_4 &= d\psi,\\
d\vartheta_5 &= dx\cos\vartheta + dy\sin\vartheta,
\end{aligned}
$$

one will have:

$$
\begin{aligned}
\mathsf{T} = {}& \tfrac{1}{2} m\,(\omega_1^2 + \omega_5^2 + r^2\omega_4^2 + r^2\sin^2\psi\,\omega_3^2 + 2r\cos\psi\,\omega_1\,\omega_4 - 2r\sin\psi\,\omega_5\,\omega_3)\\
& + \tfrac{1}{2} A(\omega_4^2 + \cos^2\psi\,\omega_3^2) + \tfrac{1}{2} C\left[\frac{1}{r}(\omega_2 + \omega_5) - \sin\psi\,\omega_3\right]^2 ,
\end{aligned}
$$

or, when we drop the terms with ω_1^2, ω_2^2:

$$
\begin{aligned}
\mathsf{T} = {}& \tfrac{1}{2} m\,(\omega_5^2 + r^2\omega_4^2 + r^2\sin^2\psi\,\omega_3^2 + 2r\cos\psi\,\omega_1\,\omega_4 - 2r\sin\psi\,\omega_5\,\omega_3)\\
& + \tfrac{1}{2} A(\omega_4^2 + \cos^2\psi\,\omega_3^2) + \tfrac{1}{2} C\left[\frac{1}{r^2}(2\,\omega_2\,\omega_5 + \omega_3^2) + \sin^2\psi\,\omega_3^2 - \frac{2}{r}(\omega_2 + \omega_3)\sin\psi\,\omega_3\right].
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
P_1 &= \frac{\partial \mathsf{T}}{\partial \omega_1} = m\,r\cos\psi\,\omega_4 ,\\
P_2 &= \frac{\partial \mathsf{T}}{\partial \omega_2} = \frac{C}{r^2}\omega_5 - \frac{C}{r}\sin\psi\,\omega_3 ,\\
P_3 &= \frac{\partial \mathsf{T}}{\partial \omega_3} = m\,r^2\sin^2\psi\,\omega_3 - m\,r\sin\psi\,\omega_5 + A\cos^2\psi\,\omega_3 - C\sin^2\psi\,\omega_3 - \frac{C}{r}\sin\psi\,\omega_5 ,\\
P_4 &= \frac{\partial \mathsf{T}}{\partial \omega_4} = m\,r^2\,\omega_4 + A\,\omega_4 ,\\
P_5 &= \frac{\partial \mathsf{T}}{\partial \omega_5} = m\,\omega_5 - m\,r\sin\psi\,\omega_3 + A\,\omega_4 + \frac{C}{r^2}\omega_5 - \frac{C}{r}\sin\psi\,\omega_3 .
\end{aligned}
$$

We have now dropped the vanishing terms. From the transition equations in § **2**, $\delta\vartheta_3$ is associated with:

$$P_1\,\omega_5 + P_2\,\omega_1 - P_5\,\omega_1 = P_1\,\omega_5 ,$$

nothing is associated with $\delta\vartheta_4$, and $\delta\vartheta_5$ is associated with $-P_1\,\omega_3$.

Therefore, since only:

$$\frac{\partial \mathrm{T}}{\partial \psi} = m r^2 \sin\psi\cos\psi\,\omega_3^2 - m r\cos\psi\,\omega_5\,\omega_3 - A\cos\psi\sin\psi\,\omega_3^2 + C\sin\psi\cos\psi\,\omega_3^2 - \frac{C}{r}\cos\psi\,\omega_5\,\omega_3$$

is non-zero, the three equations of motion will read:

$$\frac{d}{dt}P_3 + P_1\,\omega_5 - \frac{\partial \mathrm{T}}{\partial \psi}\left(\frac{\partial \psi}{\partial \vartheta_3}\right) = \mathrm{K}_3\,,$$

$$\frac{d}{dt}P_4 - \frac{\partial \mathrm{T}}{\partial \psi}\left(\frac{\partial \psi}{\partial \vartheta_4}\right) = \mathrm{K}_4\,,$$

$$\frac{d}{dt}P_5 - P_1\,\omega_3 - \frac{\partial \mathrm{T}}{\partial \psi}\left(\frac{\partial \psi}{\partial \vartheta_5}\right) = \mathrm{K}_5\,.$$

However, one has: $d\psi = d\vartheta_4$, so one will have $\left(\frac{\partial \psi}{\partial \vartheta_3}\right) = 0$, $\left(\frac{\partial \psi}{\partial \vartheta_4}\right) = 1$, $\left(\frac{\partial \psi}{\partial \vartheta_5}\right) = 0$. Let gravity be the only applied force, so it will have the potential:

$$U = m\,g\,r\cos\psi\,.$$

Hence, $\mathrm{K}_3 = 0$, $\mathrm{K}_4 = -\partial U / \partial\psi = m\,g\,r\sin\psi$, $\mathrm{K}_5 = 0$, and we will get the *equations of motion:*

$$\frac{d}{dt}(mr^2\sin^2\psi\,\omega_3 - mr\sin\psi\,\omega_5 + A\cos^2\psi\,\omega_3 + C\sin^2\psi\,\omega_3 - \frac{C}{r}\omega_5\sin\psi) + mr\cos\psi\,\omega_4\,\omega_5 = 0,$$

$$\frac{d}{dt}(mr^2\psi\,\omega_3 + A\omega_4) - mr^2\sin\psi\cos\psi\,\omega_3^2 + mr\cos\psi\,\omega_5\,\omega_3 + A\cos\psi\sin\psi\,\omega_5^2 - C\sin\psi\cos\psi\,\omega_3^2$$
$$+\frac{C}{r}\cos\psi\,\omega_5\,\omega_3 = mgr\sin\psi,$$

$$\frac{d}{dt}(m\,\omega_5 - mr\sin\psi\,\omega_3 + \frac{C}{r^2}\omega_5 - \frac{C}{r}\sin\psi\,\omega_3) - mr\cos\psi\,\omega_3\,\omega_4 = 0.$$

Hence, $\omega_3 = \dot{\vartheta}$, $\omega_3 = \dot{\psi}$, $\omega_3 = r\,\dot{\varphi}$.

244. Question of stability. – Naturally, those equations have the solution $\psi = 0$, $\omega_3 = 0$, $\omega_5 = r\,\omega_0 = \text{const}$: The tire rolls upright with constant speed. *Is that motion stable?*

In order to clarify this question using **Lagrange**'s method of small oscillations, we now regard ψ, $\dot{\psi}$, $\dot{\vartheta}$ as small first-order quantities and neglect higher-order terms. When will then remain is:

$$\frac{d}{dt}(-mr\,\psi\,\omega_5 + A\,\dot{\vartheta} - \frac{C}{r}\,\omega_5\,\psi) + mr\,\dot{\psi}\,\omega_5 = 0,$$

$$(A+mr^2)\frac{d^2\psi}{dt^2} + mr\,\omega_5\,\dot{\vartheta} + \frac{C}{r}\,\omega_5\,\dot{\vartheta} = m\,g\,r\,\psi,$$

$$\frac{d}{dt}(m\,\omega_5 + \frac{C}{r^2}\,\omega_5) = 0\,.$$

The last equation gives constant ω_5. With the abbreviations:

$$\frac{C}{r}\omega_5 = B, \qquad mr\ \omega_5 + \frac{C}{r}\omega_5 = D, \qquad A + mr^2 = E, \quad mgr = F,$$

we can write:

$$A\ddot{\vartheta} - B\dot{\psi} = 0,$$
$$D\,\dot{\vartheta} + E\,\ddot{\psi} - F\,\dot{\psi} = 0.$$

The Ansatz $\vartheta = \Theta\,e^{i\alpha t}$, $\psi = \Psi\,e^{i\alpha t}$ gives:

$$\Theta\,(-A\,\alpha^2) - \Psi\,B\,i\alpha = 0,$$
$$\Theta\,(+D\,i\alpha) + \Psi\,(B\alpha^2 - F) = 0.$$

Hence, one must have that the determinant:

$$\begin{vmatrix} -A\alpha^2 & -Bi\alpha \\ +Di\alpha & -E\alpha^2 - F \end{vmatrix} = 0,$$

or

$$A\ \alpha^2\,(E\ \alpha^2 + F) - D\,B\ \alpha^2 = 0.$$

In addition to the double root $\alpha = 0$, this equation also has the root:

$$\alpha^2 = \frac{BD - AF}{AE}\,.$$

α will be real when $BD > AF$ – i.e., $\frac{C}{r}\left(mr + \frac{C}{r}\right)\omega_5^2 > A\ m\ g\ r$ – so when the tire moves fast enough. There is a small oscillation in that case.

The double root corresponds to a possible solution with ψ = const., ϑ = const. One can find it as follows: When the first equation is integrated, that will give:

$$A\dot{\vartheta} - B\,\psi = \kappa A \qquad \text{(integration constant)}.$$

When that is substituted in the second one, that will give:

$$D\,\kappa + \frac{B}{A}D\psi + E\ddot{\psi} - F\,\psi = 0\,.$$

When $BD > AF$, this equation will have not only the small oscillation, but also the solution $\psi = \dfrac{AD\kappa}{AF - BD}$ = const., which also corresponds to constant $\dot{\vartheta}$. For a small, constant inclination, the tire can also rotate with a corresponding constant $\dot{\vartheta}$, which does not affect the stability. For an exact solution of the differential equations, see Problems 180 and 181.

§ 7. – The principle of least action.

245. First proof. – We already said that the principle of least action is always true in the form:

$$\delta\int_{t_1}^{t_2}(T - U)\,dt = 0$$

(in case a potential is present), when we treat the virtual displacements as possible, but that might make the neighboring motions impossible for non-holonomic constraints. We now see how that is quite obvious. If the constraints $\omega_\mu = 0$ exist ($\mu = 0, 1, \ldots, m$), so we also have $\delta\vartheta_\mu = 0$, then the possible displacements will be characterized by $\delta\vartheta_\mu = 0$, from which, it will follow that $d\delta\vartheta_\mu = 0$. However, should the neighboring path be possible, one would need to have $\omega_\mu = 0$, as well as $\delta\omega_\mu = 0$, which would be impossible for a non-holonomic system under the assumption that:

$$d\,\delta\mathbf{r} - \delta\,d\mathbf{r} = 0.$$

We would now like to ask whether it is possible to formulate the principle by comparing with possible neighboring paths.

To that end, we start from the generalized central equation:

$$\frac{d}{dt}\sum p_\nu\,\delta q_\nu - \sum p_\nu\,\frac{d\,\delta q_\nu - \delta\,dq_\nu}{dt} - \delta T = -\,\delta U,$$

and when we integrate from t_1 to t_2, and set $\delta q_\nu = 0$ at the ends of the interval, that will give:

$$\int_{t_1}^{t_2}\left[\delta(T-U)+\sum p_\nu \frac{d\,\delta q_\nu - \delta dq_\nu}{dt}\right]dt = 0.$$

We then convert this with the transition equation:

$$d\ \delta\vartheta - \delta\, d\vartheta = \sum_{\mu,\nu} b_{\mu,\nu}(d\ \delta q_\nu - \delta\, dq_\nu) + \sum_{\tau,\rho}\beta_\mu^{\tau,\rho}\,d\vartheta_\tau\ \delta\vartheta_\rho\ .$$

We solve this for $d\ \delta q_\nu - \delta\, dq_\nu$, which we do by multiplying by $B_{\nu\mu}$ and summing over μ, since $\omega_\mu = \sum_\mu b_{\nu\mu}\ \dot q_\mu$ has the solution $\dot q_\nu = \sum_\mu B_{\nu\mu}\ \omega_\mu$; hence:

$$\dot q_\nu = \sum_\mu B_{\nu\mu}\ b_{\mu\sigma}\ \omega_\mu\ .$$

Hence, $\sum_\mu B_{\nu\mu}\ b_{\mu\sigma} = \delta_{\nu\sigma}$ is equal to the **Kronecker** symbol. One will then get:

$$\sum_\mu B_{\nu\mu}\ (d\ \delta\vartheta_\mu - \delta\ d\vartheta_\mu) = d\ \delta q_\nu - \delta\, dq_\nu + \sum_{\mu,\nu,\tau,\sigma} B_{\nu\mu}\ p_\nu\ \beta_\mu^{\tau,\rho}\ \frac{d\vartheta_\tau}{dt}\,\delta\vartheta_\rho\ .$$

However, one has $\sum_\mu B_{\nu\mu}\ p_\nu = P_\mu$ (see § **3**). We then get:

$$\sum_\nu p_\nu\,\frac{d\ \delta q_\nu - \delta\, dq_\nu}{dt} = \sum_\mu P_\mu\,\frac{d\ \delta q_\mu - \delta\, dq_\mu}{dt} - \sum_{\mu,\tau,\rho} P_\mu\ \beta_\mu^{\tau,\rho}\omega_\tau\ \delta\vartheta_\rho\,.$$

However, with that, the principle will assume the form:

$$\int_{t_1}^{t_2}\left[\delta(T-U)+\sum_\mu P_\mu\,\frac{d\ \delta q_\mu - \delta\, dq_\mu}{dt} - \sum_{\substack{\mu=1,2,\ldots,n,\\ \tau,\rho=m+1,\ldots,n}} P_\mu\ \beta_\mu^{\tau,\rho}\omega_\tau\ \delta\vartheta_\rho\right]dt = 0.$$

We can now arrange the virtual displacements to be such that not only the displacements are possible for which:

$$\delta\vartheta_\mu = 0, \qquad \mu = 1, 2, \ldots, m,$$

but also the ones for the neighboring paths:

$$\delta\omega_\mu = 0, \qquad \mu = 1, 2, \ldots, m\ .$$

We must now only arrange that, from (1), the $d\ \delta q_\nu - \delta\, dq_\nu$ must satisfy:

$$d\,\delta q_\nu - \delta\, dq_\nu = \sum_{\mu=m+1,\dots,n} B_{\nu\mu}(d\,\delta\vartheta_\mu - \delta\, d\vartheta_\mu) - \sum_{\mu,\tau,\rho=m+1,\dots,n} B_{\nu\mu}\,\beta_\mu^{\tau,\rho} d\vartheta_\tau\,\delta\vartheta_\rho\,. \tag{2}$$

Indeed, for $\mu > m$, we can even establish the commutation relation:

$$d\,\delta\vartheta_\mu - \delta\, d\vartheta_\mu = 0$$

[which will make the first sum on the right in (2) drop out], which will imply no restriction on the $\delta\vartheta_\rho$ for $\rho > m$, and is otherwise a proper definition for the $\delta\omega_\mu$. We will then get the variational principle:

$$\int_{t_1}^{t_2}\left[\delta(T-U) - \sum_{\substack{\mu=1,2,\dots n,\\ \tau,\rho=m+1,\dots,n}} P_\mu\,\beta_\mu^{\tau,\rho}\omega_\tau\,\delta\vartheta_\rho\right]dt = 0.$$

In this form, the least-action principle is now possible for not only the displacements, but also the neighboring paths; We can replace T with T^+.

246. Second proof. – There is a *second proof* that is probably simpler. In:

$$\delta\int_{t_1}^{t_2}(\mathsf{T} - U)\,dt = 0,$$

one sets:

$$\delta\mathsf{T} = \sum_{\nu=1}^{m}\frac{\partial\mathsf{T}}{\partial\omega_\nu}\delta\omega_\nu + \sum_{\nu=m+1}^{m}\frac{\partial\mathsf{T}}{\partial\omega_\nu}\delta\omega_\nu + \sum_{\nu=1}^{m}\frac{\partial\mathsf{T}}{\partial q_\nu}\delta q_\nu\,.$$

Now, one can make use of the non-holonomic constraints in the second and third terms from the outset; i.e., one can replace T with T^+. The second and third terms together then give δT^+, and one will get:

$$\int_{t_1}^{t_2}\left(\delta T^+ - \delta U + \sum_{\nu=1}^{m} P_\nu\,\delta\omega_\nu\right)dt = 0.$$

That is then the desired form, since the constraints were indeed used in T^+, so the neighboring paths were possible. Using the transition equations:

$$\delta\omega_\mu - \frac{d}{dt}\,\delta\vartheta_\mu = -\sum_{\tau,\rho=m+1,\dots,n}\beta_\mu^{\tau,\rho}\omega_\tau\,\delta\vartheta_\rho\,,$$

we can convert the correction term $\sum_{\nu}^{m} P_\nu\ \delta\omega_\nu$ into:

$$-\sum_{\substack{\mu=1,2,\ldots n,\\ \tau,\rho=m+1,\ldots,n}} P_\mu\ \beta_\mu^{\tau,\rho}\omega_\tau\ \delta\vartheta_\rho\ ,$$

and thus get the principle:

$$\int_{t_1}^{t_2}\left[\delta(T^+ - U) - \sum_{\substack{\mu=1,2,\ldots m,\\ \tau,\rho=m+1,\ldots,n}} P_\mu\ \beta_\mu^{\tau,\rho}\omega_\tau\ \delta\vartheta_\rho\right]dt\ = 0.$$

There is an obvious difference between the two forms. In the first formulation, the sum extends over all μ from 1 to n, while in the second, it only goes from 1 to m . However, the difference is zero, because as we know we indeed have:

$$-\sum_{\tau,\rho}\beta_\mu^{\tau,\rho}\ \omega_\tau\ \delta\vartheta_\rho\ = \left(\delta\omega_\mu - \frac{d\ \delta\vartheta_\mu}{dt}\right),$$

and for $\mu > m$, that is set equal to zero. That theorem goes back to **Voronetz**, and the method of proof is partly found in Math. Ann. **92** (1924). For more details, see Math. Ann. **111** (1935).

§ 8. – Nonlinear constraint equations.

247. The first form. – Now, one can prescribe nonlinear constraint equations and also introduce nonlinear velocity parameters. Hence:

$$f_\nu(\dot{q}_\mu, q) = \omega_\nu\ , \qquad \nu = 1, 2, \ldots, n\ . \tag{1}$$

Let it be established that:

$$\omega_\nu = 0, \qquad \nu = 1, 2, \ldots, m < n \tag{2}$$

in that.

Let those equations be mutually independent and soluble for the $\dot{q}$:

$$\dot{q}_\mu = F_\mu(\omega, q)\ . \tag{3}$$

Correspondingly, one has:

$$\delta\omega_\nu = \sum\frac{\partial f_\nu}{\partial\dot{q}_\sigma}\delta\dot{q}_\sigma = \sum f_{\nu,\sigma}\ \delta\dot{q}_\sigma\ , \tag{4}$$

$$\delta\dot{q}_\sigma\ = \sum\frac{\partial F_\sigma}{\partial\omega_\mu}\delta\omega_\mu = \sum F_{\sigma,\mu}\ \delta\omega_\mu\ . \tag{5}$$

It then follows that:

$$\sum_{\nu} f_{\nu,\sigma} F_{\sigma,\mu} = \delta_{\nu,\mu} , \tag{6}$$

$$\sum_{\mu} F_{\sigma,\mu} \; f_{\nu,\sigma} = \delta_{\sigma,\mu} , \tag{7}$$

in which δ is the **Kronecker** symbol. Solubility assumes the non-vanishing of the determinant $f_{\nu,\mu}$.

We also write:

$$\omega_\nu = \frac{d\vartheta_\nu}{dt} .$$

However, $\delta\vartheta_\nu$ must now be redefined, because if one would like to write, say, $\delta q_\mu = F_\mu\,(\delta\vartheta, q)$ then that would make no sense in the nonlinear case. **Lagrange**'s principle will then break down, but ***Gauss****'s principle of least constraint* will help us further. That demands that:

$$\mathsf{S}\;(dm\,\mathbf{w} - \delta\mathbf{K}_e)\;\delta\mathbf{w} = 0,$$

and since:

$$\mathbf{w} = \frac{d^2\mathbf{r}}{dt^2} = \sum \frac{\partial \mathbf{r}}{\partial q_\nu}\ddot{q}_\nu + \text{terms with no } \ddot{q}_\nu ,$$

$$\mathsf{S}\;(dm\,\mathbf{w} - \delta\mathbf{K}_e)\;\frac{\partial \mathbf{r}}{\partial q_\nu}\delta\ddot{q}_\nu = 0.$$

By contrast, **Lagrange**:

$$\mathsf{S}\;(dm\,\mathbf{w} - \delta\mathbf{K}_e)\;\frac{\partial \mathbf{r}}{\partial q_\nu}\delta q_\nu = 0.$$

Now, if (3) is true then one will also have:

$$\ddot{q}_\mu = \sum \frac{\partial F_\mu}{\partial \omega_\sigma}\dot{\omega}_\sigma + \text{terms with no } \dot{\omega} ,$$

so

$$\delta\ddot{q}_\mu = \sum_{\sigma} F_{\nu,\sigma}\;\delta\vartheta_\sigma .$$

Gauss*'s principle then implies that:*

$$\mathsf{S}\;(dm\,\mathbf{w} - \delta\mathbf{K}_e)\;\frac{\partial \mathbf{r}}{\partial q_\nu}F_{\nu,\sigma}\;\delta\dot{\omega}_\sigma = 0.$$

If one of the ω_ν is to be zero then that must mean that it must not enter into F_μ, so it will not enter into $\delta\ddot{q}_\mu$, either. However, that means that $\delta\dot{\omega}_\sigma$ must be set to zero.

The foregoing equation can be brought into the **Lagrangian** form when we define the $\delta\vartheta$ by way of:

$$\delta q_\nu = \sum_\sigma F_{\nu,\sigma}\,\delta\vartheta_\sigma\,, \tag{8}$$

and conversely:

$$\delta\vartheta_\mu = \sum_\sigma f_{\sigma,\tau}\,\delta q_\tau\,, \tag{9}$$

in which $\delta\vartheta_\nu = 0$ for $\nu = 1, 2, \ldots, m$.

In order to remain consistent with ***Gauss****'s principle, one can then define virtual displacements to be the differential variations of the velocities. One will then get* ***Lagrange****'s principle. This new definition of virtual displacements includes the old one for linear constraints.*

We can easily exhibit the equations of motion now. As far as *force* is concerned, we get:

$$\delta A = \sum_\nu K_\nu\,\delta q_\nu = \sum_{\nu,\mu} K_\nu\,F_{\nu,\mu}\,\delta\vartheta_\mu = \sum \mathsf{K}_\mu\,\delta\vartheta_\mu\,,$$

with

$$\mathsf{K}_\mu = \sum_\nu K_\nu\,F_{\nu,\mu} = \sum K_\nu\,\frac{\partial F_\nu}{\partial\omega_\mu}. \tag{10}$$

We then recalculate the *kinetic energy:*

$$T(q,\ \dot q) = T(q, F) = \mathsf{T}(q,\ \omega)$$

and the work done by momentum:

$$\sum_\nu p_\nu\,\delta q_\nu = \sum_\nu \frac{\partial T}{\partial\dot q_\nu}\frac{\partial\dot q_\nu}{\partial\omega_\mu}\delta\vartheta_\mu = \sum \frac{\partial\mathsf{T}}{\partial\omega_\mu}\delta\vartheta_\mu\,. \tag{11}$$

Hence, we introduce the *impulse components:*

$$P_\mu = \frac{\partial\mathsf{T}}{\partial\omega_\mu}\,. \tag{12}$$

We now need *transition equations.* It follows from:

$$\frac{d\vartheta_\nu}{dt} \equiv \omega_\nu = f_\nu(\dot q\,,\, q), \qquad \delta\vartheta_\nu = \sum\frac{\partial f_\nu}{\partial\dot q_\sigma}\delta\dot q_\sigma$$

that:

$$\frac{d\,\delta\vartheta_\nu}{dt} - \frac{\delta\,d\vartheta_\nu}{dt} = \sum\frac{\partial f_\nu}{\partial\dot q_\sigma}\frac{d\,\delta\dot q_\sigma}{dt} + \sum\frac{d}{dt}\frac{\partial f_\nu}{\partial\dot q_\sigma}\delta q_\sigma - \sum\frac{\partial f_\nu}{\partial\dot q_\sigma}\delta\dot q_\sigma - \sum\frac{\partial f_\nu}{\partial q_\sigma}\delta q_\sigma\,,$$

or, if we now demand that:

$$d\ \delta q_\sigma - \delta\,dq_\sigma = 0$$

that

$$\frac{d\,\delta\vartheta_\nu}{dt} - \frac{\delta\,d\vartheta_\nu}{dt} = \sum \left(\frac{d}{dt}\frac{\partial f_\nu}{\partial \dot{q}_\sigma} - \frac{\partial f_\nu}{\partial q_\sigma} \right) \delta q_\sigma . \tag{13}$$

The ***Lagrangian*** *central equation:*

$$\frac{d}{dt}\sum p_\nu\,\delta q_\nu - \delta T = \delta A$$

now yields:

$$\sum \frac{dP_\mu}{dt}\delta\vartheta_\mu + \sum P_\mu \frac{d\,\delta\vartheta_\mu - \delta\,d\vartheta_\mu}{dt} - \sum \frac{\partial \mathrm{T}}{\partial q_\nu}\frac{\partial F_\nu}{\partial \omega_\mu}\delta\vartheta_\mu = \sum \mathrm{K}_\mu\,\delta\vartheta_\mu ,$$

or, with the use of (13):

$$\sum \frac{dP_\mu}{dt}\delta\vartheta_\mu + \sum P_\nu \left(\frac{d}{dt}\frac{\partial f_\nu}{\partial \dot{q}_\sigma} - \frac{\partial f_\nu}{\partial \dot{q}_\sigma} \right) \frac{\partial F_\sigma}{\partial \omega_\mu}\delta\vartheta_\mu - \sum \frac{\partial \mathrm{T}}{\partial q_\nu}\frac{\partial F_\nu}{\partial \omega_\mu}\delta\vartheta_\mu = \sum \mathrm{K}_\mu\,\delta\vartheta_\mu .$$

Now, since the first $\delta\vartheta_\mu$ are equal to zero, while the others are arbitrary, for $\mu = m + 1, \ldots, n$, that will yield the *equations of motion:*

$$\frac{dP_\mu}{dt} + \sum_{\nu,\sigma} P_\nu \left(\frac{d}{dt}\frac{\partial f_\nu}{\partial \dot{q}_\sigma} - \frac{\partial f_\nu}{\partial \dot{q}_\sigma} \right) \frac{\partial F_\sigma}{\partial \omega_\mu} - \sum_\nu \frac{\partial \mathrm{T}}{\partial q_\nu}\frac{\partial F_\nu}{\partial \omega_\mu} = \mathrm{K}_\mu . \tag{I}$$

However, the *first form* has the disadvantage that the $\dot{q}$ and $\ddot{q}$ still enter into the calculation of the second term. For that reason, we would like to give a conversion that includes only the q_ν and the ω.

We start from (3) and (8) and then obtain:

$$\frac{d\,\delta q_\nu - \delta\,dq_\nu}{dt} = \sum_\sigma F_{\nu,\sigma}\frac{d\,\delta\vartheta_\sigma - \delta\,d\vartheta_\sigma}{dt} + \sum_\sigma \frac{dF_{\nu,\sigma}}{dt}\delta\vartheta_\sigma - \sum_\mu \frac{\partial F_\nu}{\partial q_\mu}\delta q_\mu$$

$$= \sum_\sigma F_{\nu,\sigma}\frac{d\,\delta\vartheta_\sigma - \delta\,d\vartheta_\sigma}{dt} + \sum_\sigma \left(\frac{dF_{\nu,\sigma}}{dt} - \sum_\mu \frac{\partial F_\nu}{\partial q_\mu}F_{\nu,\sigma} \right)\delta\vartheta_\sigma ;$$

hence, with $d\,\delta q_\nu - \delta\,dq_\nu = 0$, we will have:

$$\sum_\sigma F_{\nu,\sigma}\frac{d\,\delta\vartheta_\sigma - \delta\,d\vartheta_\sigma}{dt} = -\sum_\sigma \left(\frac{dF_{\nu,\sigma}}{dt} - \sum_\mu \frac{\partial F_\nu}{\partial q_\mu}F_{\nu,\sigma} \right)\delta\vartheta_\sigma .$$

When one uses (8) to symbolically write:

$$F_{\nu,\sigma} = \left(\frac{\partial F_\nu}{\partial \vartheta_\sigma}\right),$$

one can also write the foregoing transition equation as:

$$\sum_\sigma F_{\nu,\sigma} \frac{d\,\delta\vartheta_\sigma - \delta\,d\,\vartheta_\sigma}{dt} = -\sum_\sigma \left(\frac{d}{dt}\frac{\partial F_\nu}{\partial \omega_\sigma} - \left(\frac{\partial F_\nu}{\partial \vartheta_\sigma}\right)\right)\delta\vartheta_\sigma .$$

We can likewise write:

$$\sum_\nu \frac{\partial \mathrm{T}}{\partial q_\nu}\frac{\partial F_\nu}{\partial \omega_\mu} = \left(\frac{\partial \mathrm{T}}{\partial \vartheta_\mu}\right)$$

in (I).

248. The second form. – A *second form of the transition equations* (13) follows by multiplying (6) by $f_{\mu\nu}$ and summing over ν:

$$\frac{d\,\delta\vartheta_\mu - \delta\,d\,\vartheta_\mu}{dt} = -\sum_{\nu,\sigma} f_{\mu,\nu}\left[\frac{d}{dt}\frac{\partial F_\nu}{\partial \omega_\sigma} - \left(\frac{\partial F_\nu}{\partial \vartheta_\sigma}\right)\right]\delta\vartheta_\mu . \tag{13a}$$

However, (14) will then imply that:

$$\sum \frac{dP_\mu}{dt}\delta\vartheta_\mu - \sum_{\mu,\nu,\sigma} P_\mu\, f_{\mu,\nu}\left[\frac{d}{dt}\frac{\partial F_\nu}{\partial \omega_\sigma} - \left(\frac{\partial F_\nu}{\partial \vartheta_\sigma}\right)\right]\delta\vartheta_\mu - \sum\left(\frac{\partial \mathrm{T}}{\partial \vartheta_\mu}\right)\delta\vartheta_\mu = \sum \mathrm{K}_\mu\,\delta\vartheta_\mu ,$$

and we will then have the *second form of the equations of motion:*

$$\frac{dP_\mu}{dt} - \sum_{\sigma,\nu} P_\mu\, f_{\mu,\nu}\left[\frac{d}{dt}\frac{\partial F_\nu}{\partial \omega_\sigma} - \left(\frac{\partial F_\nu}{\partial \vartheta_\sigma}\right)\right] - \sum\left(\frac{\partial \mathrm{T}}{\partial \vartheta_\mu}\right) = \mathrm{K}_\mu = -\left(\frac{\partial U}{\partial \vartheta_\mu}\right), \quad (\mu = m+1, \ldots, n), \tag{II}$$

in case a potential exists.

These equations are found in the works of **Leif Johnson**, but their derivation is flawed ([2]). In regard to that, see a paper by the author ([3]).

Indeed, these equations still include $f_{\mu,\nu}$, which is a function of $\dot{q}$ and q, but it follows from:

$$\delta\omega_\sigma = \sum f_{\sigma,\nu}\,\delta\dot{q}_\nu$$

([2]) **JOHNSON**, Leif, "Dynamique générales des systèmes non holonomes," Kon. Norske Vid. Selskab. Skrifter.
([3]) **HAMEL**, Georg, "Nichtholonomer Systeme höherer Art," Sitz. Math. Ges. Berlin, v. XXXVII.

and

$$\delta \dot{q}_\nu = \sum F_{\nu,\mu}\, \delta\omega_\nu$$

that the $f_{\sigma,\nu}$ are the sub-determinants of $\| F_{\nu,\mu} \|$, divided by the total determinant, so they are obtained by means of linear algebra and can be converted into functions of ω_μ and q_ν. One does not need to revert to equations (1) and (3).

Special case: If one takes T itself to be ω_n then $\mathsf{T} = \omega_n$, $P_n = 1$, all other P are zero, and $\left(\dfrac{\partial \mathsf{T}}{\partial \vartheta_\mu}\right)$ = 0. Hence, the equations of motion will read:

$$-\sum_\nu f_{n,\nu}\left[\frac{d}{dt}\frac{\partial F_\nu}{\partial \omega_\mu} - \left(\frac{\partial F_\nu}{\partial \vartheta_\mu}\right)\right] = \mathsf{K}_\mu, \qquad \mu = m + 1, \ldots, n$$

in this case.

249. Example. – There do not seem to be any examples from daily life, as in the linear case. The value of such nonlinear constraints lies not so much in their presentation as in the possibility of introducing some sort of combination $f(\dot{q}, q)$ that might be useful as a variable, if perhaps an integral = const. exists. We shall satisfy ourselves with an artificial example.

Let the object be a point in three-dimensional space. Take ω_3 to be:

$$T = \tfrac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2)\,.$$

Let:

$$\omega_1 = \tfrac{1}{2}(\dot{x}_3^2 - \dot{x}_1^2 - \dot{x}_2^2) = 0$$

be prescribed, i.e.:

$$\dot{x}_3 = \pm\sqrt{\dot{x}_1^2 + \dot{x}_2^2}\ ;$$

i.e., the velocity in the vertical direction is equal to the one in the horizontal direction. In other words: The angle of inclination is 45°, which can indeed be achieved by means of wheels (*Steuern*). It follows from:

$$\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 = 2\omega_3\,,$$
$$\dot{x}_1^2 + \dot{x}_2^2 - \dot{x}_3^2 = -2\omega_3$$

that

$$\dot{x}_3^2 = \omega_2 + \omega_1\,, \qquad \dot{x}_3 = \sqrt{\omega_3 + \omega_1}\,.$$

We then set:

$$\dot{x}_1 = \sqrt{\omega_3 - \omega_1}\,\cos\omega_2\,,$$

$$\dot{x}_2 = \sqrt{\omega_3 - \omega_1}\ \sin\ \omega_2\ .$$

Therefore:

$$f_1 = \omega_1 = \tfrac{1}{2}(\dot{x}_3^2 - \dot{x}_1^2 - \dot{x}_2^2)\,,$$

$$f_2 = \omega_2 = \arctan\frac{\dot{x}_2}{\dot{x}_1}\,,$$

$$f_3 = \omega_3 = \tfrac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2)\,.$$

The inverses are:

$$F_1 = \dot{x}_1 = \sqrt{\omega_3 - \omega_1}\ \cos\ \omega_2\ ,$$

$$F_2 = \dot{x}_2 = \sqrt{\omega_3 - \omega_1}\ \sin\ \omega_2\ ,$$

$$F_3 = \dot{x}_3 = \sqrt{\omega_3 + \omega_1}\ .$$

It follows from this that:

$$F_{1,1} = -\frac{\cos\omega_2}{2\sqrt{\omega_3 - \omega_1}}\,,\qquad F_{1,2} = -\sqrt{\omega_3 - \omega_1}\ \sin\ \omega_2\,,\qquad F_{1,3} = -\frac{\cos\omega_2}{2\sqrt{\omega_3 - \omega_1}}\,,$$

$$F_{2,1} = -\frac{\sin\omega_2}{2\sqrt{\omega_3 - \omega_1}}\,,\qquad F_{2,2} = \sqrt{\omega_3 - \omega_1}\ \cos\ \omega_2\,,\qquad F_{2,3} = \frac{\sin\omega_2}{2\sqrt{\omega_3 - \omega_1}}\,,$$

$$F_{3,1} = \frac{1}{2\sqrt{\omega_3 + \omega_1}}\,,\qquad F_{3,2} = 0\,,\qquad F_{3,3} = \frac{1}{2\sqrt{\omega_3 + \omega_1}}\,.$$

Since one does not further partially differentiate with respect to ω_1, one can already set $\omega_1 = 0$. The determinant is then:

$$\Delta = \begin{vmatrix} -\dfrac{\cos\omega_2}{2\sqrt{\omega_3}} & -\sqrt{\omega_3}\sin\omega_2 & \dfrac{\cos\omega_2}{2\sqrt{\omega_3}} \\ -\dfrac{\sin\omega_2}{2\sqrt{\omega_3}} & \sqrt{\omega_3}\sin\omega_2 & \dfrac{\sin\omega_2}{2\sqrt{\omega_3}} \\ \dfrac{1}{2\sqrt{\omega_3}} & 0 & \dfrac{1}{2\sqrt{\omega_3}} \end{vmatrix} = -\frac{1}{2\sqrt{\omega_3}}\,.$$

One only needs $f_{3,\nu}$. One finds it the fastest here from the fact that:

$$f_3 = \tfrac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2)\,,$$

which leads to:

$$f_{3,1} = \dot{x}_1 = \sqrt{\omega_3}\ \cos\ \omega_3\ ,$$

$$f_{3,2} = \dot{x}_2 = \sqrt{\omega_3}\ \sin\ \omega_3\ ,$$
$$f_{3,3} = \dot{x}_3 = \sqrt{\omega_3}\ .$$

One can now write down the two equations of motion. They read:

$$-\sqrt{\omega_3}\left[\cos\omega_2\frac{d}{dt}(-\sqrt{\omega_3}\sin\omega_2)+\sin\omega_2\frac{d}{dt}(\sqrt{\omega_3}\cos\omega_2)\right] = \mathsf{K}_2\ ,$$
$$-\sqrt{\omega_3}\left[\cos\omega_2\frac{d}{dt}\frac{\cos\omega_2}{2\sqrt{\omega_3}}+\sin\omega_2\frac{d}{dt}\frac{\cos\omega_2}{2\sqrt{\omega_3}}+\frac{d}{dt}\frac{2}{2\sqrt{\omega_3}}\right] = \mathsf{K}_3\ ,$$

because the F_ν are independent of the coordinates. One can get the forces from:

$$\mathsf{K}_2\ \delta\vartheta_2 + \mathsf{K}_3\ \delta\vartheta_3 = X_1\ \delta\vartheta_1 + X_2\ \delta\vartheta_2 + X_3\ \delta\vartheta_3$$

$$= \left(X_1\frac{\partial F_1}{\partial\omega_2}+X_2\frac{\partial F_2}{\partial\omega_2}+X_3\frac{\partial F_3}{\partial\omega_2}\right)\delta\vartheta_2+\left(X_1\frac{\partial F_1}{\partial\omega_3}+X_2\frac{\partial F_2}{\partial\omega_3}+X_3\frac{\partial F_3}{\partial\omega_3}\right)\delta\vartheta_3\ ,$$

namely:

$$\mathsf{K}_2 = -\ X_1\sqrt{\omega_3}\sin\omega_2 - X_2\sqrt{\omega_3}\cos\omega_2\ ,$$
$$\mathsf{K}_3 = \quad X_1\frac{\cos\omega_2}{2\sqrt{\omega_3}}\sin\omega_2 + X_2\frac{\sin\omega_2}{2\sqrt{\omega_3}} + X_3\frac{1}{2\sqrt{\omega_3}}\ .$$

Differentiating these will give the simple equations:

$$\omega_3\frac{d\omega_2}{dt} = \mathsf{K}_2 \quad \text{and} \quad \frac{1}{2\omega_3}\frac{d\omega_3}{dt} = \mathsf{K}_3\ .$$

Naturally, in the force-free case, that will give the energy integral:

$$\omega_3 = \text{const.}$$

If perhaps $X_1 = 0$, $X_2 = 0$, $X_3 = K$ = const. then it will follow that:

$$\mathsf{K}_2 = 0, \qquad \mathsf{K}_3 = \frac{K}{2\sqrt{\omega_3}}, \qquad \omega_3\frac{d\omega_2}{dt} = 0, \qquad \frac{1}{2\omega_3}\frac{d\omega_3}{dt} = \frac{K}{2\sqrt{\omega_3}}\ ,$$

so

$$\omega_2 = \text{const.}, \qquad \sqrt{\omega_3} = \tfrac{1}{2}K\,t+\sqrt{\omega_0}\ , \qquad \mathsf{T} = \omega_3 = (\tfrac{1}{2}K\,t+\sqrt{\omega_0})^2,$$

$$\dot{x}_1 = \omega_3 \cos \omega_2 = (\tfrac{1}{2} K t + \sqrt{\omega_0}) \cos \omega_2 ,$$
$$\dot{x}_2 = \omega_3 \sin \omega_2 = (\tfrac{1}{2} K t + \sqrt{\omega_0}) \cos \omega_2 ,$$
$$\dot{x}_3 = \omega_3 \qquad = \tfrac{1}{2} K t + \sqrt{\omega_0} ,$$

which can be integrated by elementary methods, since ω_2 is constant.

250. Linearization. – One can often treat nonlinear problems like:

$$dx^2 + dy^2 + dz^2$$

in a different way by linearizing it using the introduction of auxiliary variables, which are initially only apparent. Indeed, one can introduce the auxiliary variable ϑ by that fact:

$$dx = dz \cos \vartheta ,$$
$$dy = dz \sin \vartheta .$$

One will then have:

$$\ddot{x} = \ddot{z} \cos \vartheta - \dot{z} \sin \vartheta \, \dot{\vartheta} ,$$
$$\ddot{y} = \ddot{z} \sin \vartheta + \dot{z} \cos \vartheta \, \dot{\vartheta} .$$

However, in constructing $\delta \ddot{x}$, $\delta \ddot{y}$, one must now vary $\dot{\vartheta}$, since the position has nothing to do with ϑ at all, which first makes its appearance in the representation of the velocity. $\dot{\vartheta}$ is a quantity that is meaningful for the acceleration, and must then varied according to **Gauss**'s principle. Hence:

$$\delta \ddot{x} = \delta \ddot{z} \cos \vartheta - \dot{z} \sin \vartheta \, \delta \dot{\vartheta} ,$$
$$\delta \ddot{y} = \delta \ddot{z} \sin \vartheta + \dot{z} \cos \vartheta \, \delta \dot{\vartheta} .$$

For $m = 1$, that now gives:

$$\ddot{x}\,\delta\ddot{x} + \ddot{y}\,\delta\ddot{y} + \ddot{z}\,\delta\ddot{z} = X \cos \vartheta + Y \sin \vartheta - Z - \ddot{x}\,\dot{z} \sin \vartheta + \ddot{y}\,\dot{z} \cos \vartheta$$
$$= - X\,\dot{z} \sin \vartheta + Y\,\dot{z} \cos \vartheta .$$

Since $\dot{z} = 0$ does not come into question (except for the case of rest), two equations for z and ϑ will follow by subsequently introducing ϑ and dropping $\dot{z}$ that:

$$2\,\ddot{z} = X \cos \vartheta + Y \sin \vartheta + Z$$

and

$$\dot{z}\,\dot{\vartheta} = - X \sin \vartheta + Y \cos \vartheta .$$

For $X = 0$, $Y = 0$, that will yield:

$$\dot{\vartheta} = 0 \qquad \text{and} \qquad 2\ddot{z} = Z .$$

However, if one eliminates ϑ instead then one will get:

$$\dot{z}^2 = \dot{x}^2 + \dot{y}^2 ,$$

$$2\ddot{z} = X\frac{\dot{x}}{\dot{z}} + Y\frac{\dot{y}}{\dot{z}} + Z \quad \text{and} \quad -\ddot{x}\dot{y} + \dot{x}\ddot{y} = -X\dot{y} + Y\dot{x} .$$

One easily proves the agreement between this and the old results.

Up to now, ϑ was an auxiliary mathematical quantity. However, one can also give it a real interpretation upon completing the system with additional masses that one can allow to go to zero. [For this, see various notes by **Appell**, *et al.*, in the Rendiconti di Palermo 1911 and 1912, as well as **Delassus**, especially "Sur les liaisons et les mouvements…" in the Ann. École normale (3) **29** (1912). **Appell**'s 1913 book on the dynamics of material systems also includes the essentials.] A true linearization will then be realized in that way.

In our example, we can think of the horizontal plane as playing a role; let ξ, η be the coordinates of the contact point B, and let ϑ be the angle of inclination of the plane of rolling with respect to the x-axis. At a distance of ρ from the point B, the rolling object (which is supported by the plane without friction) carries a vertical along which a mass-point m can move. Let its coordinates be x, y, z. For constant z, that will be the blade, in essence. Now however, the mass-point m shall be coupled to the rolling body by a cord such that it rises in proportion to the angle of rotation φ, so one will have:

$$dz = b\, d\varphi . \tag{1}$$

The non-holonomic constraints are then:

$$x = \xi + \rho \cos \vartheta , \qquad y = \eta + \rho \sin \vartheta, \tag{2}$$

$$d\xi = a \cos \vartheta\, d\varphi, \qquad d\eta = a \sin \vartheta\, d\varphi ; \tag{3}$$

a is the radius of the rolling body. With:

$$\dot{x} = \dot{\xi} - \rho \sin \vartheta \dot{\vartheta}, \qquad \dot{y} = \dot{\eta} + \rho \cos \vartheta \dot{\vartheta}, \tag{4}$$

one will have:

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{\mu}{2}(\dot{\xi}^2 + \dot{\eta}^2) + \frac{1}{2}A\dot{\vartheta}^2 + \frac{1}{2}B\dot{\varphi}^2 ,$$

when the rolling body, which is thought to be centered, has a mass of μ and moments of inertia A, B.

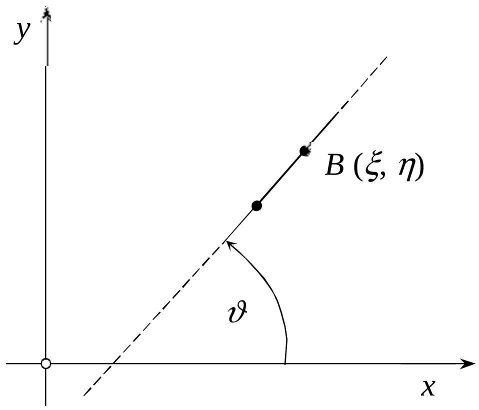

Figure 117,

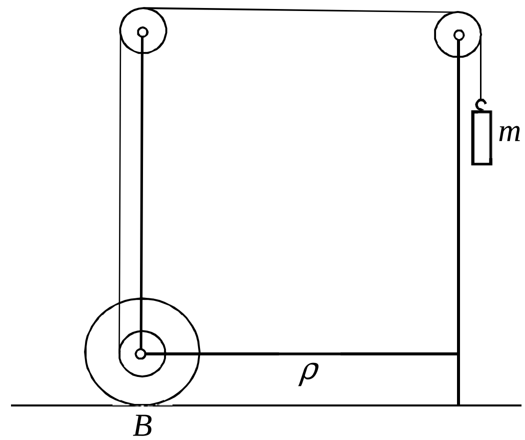

Figure 118

When one substitutes the values of $\dot{x}$, $\dot{y}$ in (4), as will as the value of $\dot{z}$ in (1), that will give:

$$T = \tfrac{1}{2}(m+\mu)(\dot{\xi}^2+\dot{\eta}^2) - m\dot{\xi}\,\dot{\vartheta}\,\rho\sin\vartheta + m\dot{\eta}\,\dot{\vartheta}\,\rho\cos\vartheta + \tfrac{1}{2}(A+m\rho^2) + \tfrac{1}{2}(B+mb^2)\,\dot{\varphi}^2 .$$

The further examination proceeds similarly to the case of the blade. We set:

$$\begin{aligned} d\vartheta_1 &= -\,d\xi\sin\vartheta + d\eta\cos\vartheta = 0,\\ d\vartheta_2 &= \quad d\xi\cos\vartheta + d\eta\sin\vartheta - a\,d\varphi = 0,\\ d\vartheta_3 &= d\vartheta,\\ d\vartheta_4 &= d\varphi . \end{aligned}$$

We then get the transition equations:

$$\begin{aligned} d\,\delta\vartheta_1 - \delta\,d\vartheta_1 &= (d\vartheta_2\,\delta\vartheta_2 - \delta\vartheta_2\,d\vartheta_2) + a\,(d\vartheta_4\,\delta\vartheta_3 - \delta\vartheta_4\,d\vartheta_3),\\ d\,\delta\vartheta_2 - \delta\,d\vartheta_2 &= d\vartheta_3\,\delta\vartheta_1 - d\vartheta_1\,d\vartheta_3 ,\\ d\,\delta\vartheta_3 - \delta\,d\vartheta_3 &= 0,\\ d\,\delta\vartheta_4 - \delta\,d\vartheta_4 &= 0 . \end{aligned}$$

When we drop the terms ω_1^2, ω_2^2, the kinetic energy will become:

$$\mathsf{T} = \tfrac{1}{2}(m+\mu)\,2\omega_2\,\omega_4 + m\,\rho\,\omega_2\,\omega_4 + \tfrac{1}{2}\omega_3^2(A+m\,\rho^2) + [(m+\mu)\,a^2 + B + m\,b^2] .$$

We then have:

$$\begin{aligned} P_1 &= m\,\rho\,\omega_2 ,\\ P_2 &= (m+\mu)\,\omega_4 ,\\ P_3 &= (A+m\rho^2)\,\omega_3 ,\\ P_4 &= [(m+\mu) + B + mb^2]\,\omega_4 . \end{aligned}$$

Now, if gravity $Z = - m\,g$ acts upon m, along with X, Y, and forces Ξ, Η, and the moment M act upon the rolling body then:

$$\delta A = - mg\ \delta z + \Xi\ \delta\xi + \mathrm{H}\ \delta\eta + M\ \delta\vartheta + X\ \delta x + Y\ \delta y$$
$$= [(X + \Xi)\ a \cos\vartheta + (Y + \mathrm{H})\ a \sin\vartheta - mg\ b]\ \delta\varphi + [M + \rho\,(- X \sin\vartheta + Y \cos\vartheta)]\ \delta\vartheta\,.$$

We will then get:

$$(A + m\,\rho^2)\ \ddot{\vartheta} + m\,\rho\,\dot{\vartheta}\,a\,\dot{\varphi} = M + \rho\,(- X \sin\vartheta + Y \cos\vartheta), \tag{5}$$

which comes from $\delta\vartheta_3$, and:

$$[(m + \mu)\ a^2 + B + mb^2]\ \ddot{\varphi} - m\,\rho\,a\,\dot{\vartheta}^2 = (X + \Xi)\ a \cos\vartheta + (Y + \mathrm{H})\ a \sin\vartheta - mg\ b, \tag{6}$$

which comes from $\delta\vartheta_4$. *Those are the two equations of motion.*

251. Passing to the limit. – If we now neglect the mass of the rolling body, i.e., set:

$$\mu = 0, \qquad A = 0, \qquad B = 0,$$

then we will get:

$$m\,\rho^2\ \ddot{\vartheta} + m\rho\,a\,\dot{\vartheta}\,\varphi = M + \rho\,(- X \sin\vartheta + Y \cos\vartheta),$$
$$m\ (a^2 + b^2)\ \ddot{\varphi} - ma\rho\,\dot{\vartheta}^2 = (X + \Xi)\ a \cos\vartheta + (Y + \mathrm{H})\ a \sin\vartheta - mg\ b.$$

This is all still quite normal. However, if we further set $\rho = 0$ then we must have $M = 0$ in order for no contradiction to arise, and the first equation will drop out, so only the second one will remain:

$$m\ (a^2 + b^2)\ \ddot{\varphi} = (X + \Xi)\ a \cos\vartheta + (Y + \mathrm{H})\ a \sin\vartheta - mg\ b.$$

The problem is now indeterminate.

If $M = 0$ is given, but $\rho \neq 0$, then it will follow from (5) that:

$$m\,\rho\,\ddot{\vartheta} + m\,a\,\dot{\vartheta}\,\dot{\varphi} = - X \sin\vartheta + Y \cos\vartheta\,.$$

If one now passes to the limit:

$$\rho \to 0$$

then that will give:

$$m\,a\,\dot{\vartheta}\,\dot{\varphi} = - X \sin\vartheta + Y \cos\vartheta\,.$$

However, that equation, together with the second equation for $\dot{\varphi}$ with Ξ = 0, H = 0, is the same system of equations that we will get by eliminating ϑ from the constraint equation that arises from (3) in that way, namely:

$$d\xi^2 + d\eta^2 = a^2\,d\varphi^2 = \frac{a^2}{b^2}dz^2 \ ,$$

which will go to:

$$dx^2 + dy^2 = \frac{a^2}{b^2}dz^2$$

for $\rho = 0$.

We first set $m = 0$, $A = 0$, $B = 0$, so we have neglected the mass of the rolling body, and then let $\rho \to 0$. However, if one lets those four quantities go to zero simultaneously and one has:

$$M = \rho\, M'$$

then one will arrive at:

$$\lim \frac{A}{\rho} = \alpha\,.$$

One will then get:

$$\alpha\,\ddot{\vartheta} + ma\,\dot{\vartheta}\dot{\varphi} = M' - X\sin\vartheta + Y\cos\vartheta,$$

instead of eq. (5). The indeterminacy has vanished, but the parameter α has appeared, which depends upon the passage to the limit that one carries out with the mass distribution μ, A, B of the rolling body and the geometric quantity ρ. **Appell** set $\alpha = 0$, which implies a generalization of our way of doing things. However, that is not intrinsically necessary. One must then be careful with any assumptions that involve setting masses and lengths of control elements equal to zero when the control element is important for the motion of the system. (Cf., Problem 151, *et seq.* on this.)

§ 9. – Second-class non-holonomic systems.

252. A questionable state of affairs. – Let us point out the systems in which the constraint equations also include the accelerations $\ddot{q}$:

$$f_\nu(\ddot{q}, \dot{q}, q) = 0, \qquad \nu = 1, 2, \ldots, m\,. \tag{1}$$

At best, one will again work with **Gauss**'s principle, which will yield:

$$\sum\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_\mu} - \frac{\partial T}{\partial q_\mu}\right)\delta\ddot{q}_\mu = \sum K_\nu\,\delta\ddot{q}_\nu$$

or also

$$\delta S \equiv \sum \frac{\partial S}{\partial \ddot{q}_\nu} \delta \ddot{q}_\nu = \sum K_\nu \, \delta \ddot{q}_\nu \, .$$

(S is **Appell**'s acceleration function; cf., Chap. VII, § **7.**) One now eliminates some of the $\ddot{q}$ with the help of (1), such that only the free ones will still remain.

Example: Let a point in space that is initially free be subject to the condition that $\ddot{x}_3 = \ddot{x}_1 \ddot{x}_2$. It will then follow from $S = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2)$ that:

$$S = \tfrac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_1^2 \, \dot{x}_2^2)$$

and $\sum_1^3 X_\nu \, \delta \ddot{x}_\nu$ will become:

$$X_1 \, \delta \ddot{x}_1 + X_2 \, \delta \ddot{x}_2 + X_3 (\ddot{x}_2 \, \delta \ddot{x}_2 + \ddot{x}_3 \, \delta \ddot{x}_3) \, .$$

As a result, the equations of motion will read:

$$\ddot{x}_1 (1 + \dot{x}_2^2) = X_1 + X_3 \, \ddot{x}_2 \, ,$$
$$\ddot{x}_2 (1 + \dot{x}_1^2) = X_2 + X_3 \, \ddot{x}_1 \, .$$

However, that system has two solutions:

1) $\ddot{x}_1 = 0, \ \ddot{x}_2 = 0, \ \ddot{x}_3 = 0$; despite the fact that $\mathsf{K} \neq 0$.

2) $$\ddot{x}_1 = \mathsf{K} \frac{\ddot{x}_2}{1 + \ddot{x}_2^2} \, ,$$

or when substituted in the second equation:

$$\ddot{x}_2 \left(1 + \mathsf{K}^2 \frac{\ddot{x}_2^2}{(1 + \ddot{x}_2^2)^2} \right) = \mathsf{K} \frac{\ddot{x}_2}{1 + \ddot{x}_2^2} \, ,$$

or when $\ddot{x}_2 \neq 0$:

$$(1 + \ddot{x}_2^2)^2 + \mathsf{K}^2 \, \ddot{x}_2^2 = \mathsf{K}^2 (1 + \ddot{x}_2^2) \qquad \text{or} \qquad (1 + \ddot{x}_2^2)^2 = \mathsf{K}^2,$$

$$\ddot{x}_2^2 = |\mathsf{K}| - 1, \qquad \ddot{x}_2 = \sqrt{|\mathsf{K}| - 1} \, ,$$

$$\ddot{x}_1 = \mathsf{K} \frac{\sqrt{|\mathsf{K}| - 1}}{|\mathsf{K}|} \, ,$$

$$\ddot{x}_3 = \frac{\mathsf{K}}{|\mathsf{K}|} (|\mathsf{K}| - 1) = \mathsf{K} - \frac{\mathsf{K}}{|\mathsf{K}|} \, .$$

(This assumes that $|\mathsf{K}| > 1$.)

What is the correct solution? If one calculates:

$$S\, dm \left(\mathbf{w} - \frac{d\mathbf{K}_e}{dm}\right)^2 = \ddot{x}_1^2 + \ddot{x}_2^2 + (\ddot{x}_3 - K)^2$$

then the first solution will be K^2, while the second one will be:

$$|K| - 1 + |K| - 1 + 1 = 2|K| - 1.$$

Now, one has:

$$K^2 - 2|K| + 1 = (|K| - 1)^2 > 0\ ;$$

hence, the second solution gives the true minimum. However, whether or not **Gauss**'s principle can be extended in that way is still unproven physically. We then meet up with the fact that this entire situation is questionable. Just as we already would not actually like to think of the forces as depending upon the accelerations (at most improperly by a process of elimination), constraints in which the accelerations factor will also seem to be debatable, and above all, ones in which even higher derivatives are involved.

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