"Zur Quantendynamik der Wellenfelder," Zeit. Phys. **56** (1929), 1-61.

On the quantum dynamics of wave fields

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(Received on 19 March 1929)

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Introduction. Up to now, in quantum theory, it has not been possible to connect mechanical and electrodynamical possibilities by, on the one hand, electrostatic and magnetostatic interactions and by radiation-mediated interactions, on the other, in a manner that is free of contradictions, and to consider both of them from a unified standpoint. In particular, no one has succeeded in considering the finite propagation speed of the electromagnetic force effects in the correct way. The purpose of the present paper is to fill that gap. In order to achieve that goal, it will be necessary to give a relativistically-invariant formalism that will allow one to treat the interaction between matter and the electromagnetic field, and thus also the one between matter and matter. This problem seems to be fundamentally linked with great difficulties that precluded **Dirac** from finding the relativistically-invariant formulation of the one-electron problem, up to now, and one will first arrive at a completely satisfactory solution to the problem that is posed here when one clarifies those fundamental difficulties. Nevertheless, it

gives the impression that the problem of retardation could be split into the aforementioned deeper-lying problems. Whereas it must be approached with no help on the part of classical theory, the retardation problem still seems soluble by corresponding considerations.

It is known in classical point mechanics that a relativistically-invariant formulation of the many-body problem with the help of **Hamilton**'s theory is not practicable. Therefore, one might also not hope that one could arrive at a relativistically-invariant treatment of the many-body problem by means of differential equations in configuration space (or the corresponding matrices) in quantum theory, especially since such a treatment would seem to be coupled inseparably with a quantization of electromagnetic waves that is equivalent to the introduction of light quanta. Thus, e.g., the equation ([*]) that **Eddington** gave for the two-electron problem, into which the four-dimensional distance between two world-points enters essentially, can hardly be brought into harmony with experiments if that equation yields interactions between the electrons that are qualitatively quite different from retarded potentials that one expects from **Maxwell**'s theory. That difference would also remain in the limiting case of high quantum numbers and many electrons, and would thus lead to contradictions. Moreover, the corresponding analogues to the theory that we strive for here will be, on the one hand, **Maxwell**'s theory, and on the other hand, the wave equation of the one-electron problem, when it is re-interpreted in the sense of a classical continuum theory. **Schrödinger** ([**]) has already achieved a formally-satisfactory combination of these two field theories. If one starts with the **Dirac** equation for the one-electron problem then that will exhibit the corresponding connection of **Tetrode** ([***]). The theory that we aim for here then relates to the aforementioned consequent field theories as quantum mechanics does to classical mechanics, in that it will, in fact, emerge from this field theory by quantization (i.e., introduction of non-commutative quantities or corresponding functionals), and in its formal content will define a consequent continuation of the investigations of **Dirac** ([****]), **Pauli** and **Jordan** ([†]) on radiation, and that of **Jordan, Klein**, and **Wigner** ([††]) on the many-body problem. A similar attempt was recently undertaken by **Mie** ([†††]). The corresponding analogue of that attempt is **Mie's** theory of the electron. For the time being, that theory generally remains a formal schema, as long as the classical field equation has not been found whose integration would yield electrons in a satisfactory way. Thus, **Mie**'s quantum theory of fields, which still exhibits many similarities with the theory that we seek here, is inapplicable in practice.

The theory that we seek here is also still afflicted with many defects. As was already mentioned, the fundamental difficulties in the relativistic formulation that were emphasized by **Dirac** remain unchanged ([††††]). Moreover, the formulas of the theory lead

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([*]) **A. S. Eddington**, Proc. Roy. Soc. **121** (1928), 524; **122** (1929), 358.
([**]) **E. Schrödinger**, Ann. d. Phys. **82** (1927), 265.
([***]) **H. Tetrode**, Zeit. Phys. **49** (1928), 858; cf., also **F. Möglich**, *ibidem*, **48** (1928), 852.
([****]) **P. A. M. Dirac**, Proc. Roy. Soc. (A) **114** (1927), 243 and 710.
([†]) **P. Jordan** and **W. Pauli, Jr.**, Zeit. Phys. **47** (1928), 151.
([††]) **P. Jordan** and **O. Klein**, *Ibidem*, **45** (1927), 751; **P. Jordan** and **E. Wigner**, *Ibidem*, **47** (1928), 631.
([†††]) **G. Mie**, Ann. d. Phys. (4) **85** (1928), 711.
([††††]) As **O. Klein** has shown [Zeit. Phys. **53** (1929), 157], these difficulties are especially striking due to the fact that according to **Dirac**'s theory, in some circumstances, the electron can pass through a

to an infinite zero-point energy for the radiation, and thus include the interaction of an electron with itself as an infinite additive constant. Naturally, the theory also yields no sort of information on the possibility of the radiation processes of the elementary electrical particles and on Nature's preference for antisymmetric wave function in configuration space over symmetric ones for many electrons or protons. However, these difficulties are of a sort that they do not interfere with the application of the theory to many physical problems. The methods that are developed here permit, e.g., the mathematical treatment of certain more detailed processes in the theory of the Auger effect and related problems, as well as the consideration of the retarded potential in the calculation of the energy values for the stationary states of atoms. The latter might be meaningful, in particular, for the theory of the fine structure of ortho-helium lines. Furthermore, the formalism that is developed here includes the previous methods (viz., quantum mechanics, **Dirac**'s theory of radiation) as special cases in the first approximation. In all, we may conclude from this that the later, ultimate theory will also have essential trains of through in common with the one that we seek here. Let it be mentioned that a quantization of the gravitational field, which seems to be necessary on physical grounds (*), is also practicable by means of a formalism that is completely analogous to the one employed here with no new difficulties.

I. General methods.

§ 1. Lagrangian and Hamiltonian form of the field equations, energy and impulse integrals. Let a **Lagrangian** function L be given that might depend upon certain continuous space-time functions Q_α (x_1, x_2, x_3, t), as well as upon their first derivatives with respect to the coordinates. The differential equations that the field quantities Q_α must satisfy might arise from the variational principle:

$$\delta \int L\left(Q_\alpha, \frac{\partial Q_\alpha}{\partial x_i}, \dot{Q}_\alpha \right) dV\, dt = 0 \qquad (1)$$

when the variation of the Q_α is assumed to vanish on the boundary of the domain of integration. In this, we have written $\dot{Q}_\alpha$ for the time derivative $\partial Q_\alpha / \partial t$ at a fixed spatial location, and the index α shall distinguish the various state quantities that are present in arbitrary, finite numbers, while the index i refers to the three spatial coordinates. In what follows, we shall always employ Greek symbols for indices of the former kind and Latin symbols for ones of the latter kind. As is known, the differential equations that follow from (1) read:

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potential jump whose order of magnitude is $V = mc^2 / e$, in contradiction to the classical energy theorem. For the time being, an analogous consequence of the theory also seems to frustrate a closer theoretical treatment of the structure of the nucleus.

(*) **A. Einstein**, Berl. Ber. (1916), 688; cf., esp., pp. 696, where the necessity of treating the emission of gravitational waves quantum-theoretically was emphasized. Furthermore, cf., **O. Klein**, Zeit Phys. **46** (1927), 188; cf., esp., the remark ** on pp. 188 of that paper.

$$\frac{\partial L}{\partial Q_\alpha} - \sum_i \frac{\partial}{\partial x_i} \frac{\partial L}{\partial \frac{\partial Q_\alpha}{\partial x_i}} - \frac{\partial}{\partial t} \frac{\partial L}{\partial \dot{Q}_\alpha} = 0. \tag{2}$$

In order to make the analogy with ordinary point mechanics emerge from this, we first introduce the Lagrange function that has been integrated over only the spatial volume:

$$\bar{L} = \int L \, dV \, . \tag{3}$$

For δQ_α that vanish on the boundaries, one will then get by partial integration:

$$\delta \bar{L} = \int \sum_\alpha \left(\frac{\partial L}{\partial Q_\alpha} - \sum_i \frac{\partial}{\partial x_i} \frac{\partial L}{\partial \frac{\partial Q_\alpha}{\partial x_i}} \right) \delta Q_\alpha \; dV.$$

On that basis:

$$\frac{\delta \bar{L}}{\delta Q_\alpha} = \frac{\partial L}{\partial Q_\alpha} - \sum_i \frac{\partial}{\partial x_i} \frac{\partial L}{\partial \frac{\partial Q_\alpha}{\partial x_i}} \tag{4}$$

is called the **Hamiltonian** or *functional* derivative of $\bar{L}$ with respect to Q_α at the spatial location P under scrutiny whose coordinates are x_1, x_2, x_3 . One can define it as the limit of the quotient:

$$\frac{\delta \bar{L}}{\delta Q_{\alpha;P}} = \lim \frac{\bar{L}(Q_\alpha + \delta Q_\alpha) - \bar{L}(Q_\alpha)}{\int \delta Q_\alpha dV},$$

such that the two values of $\bar{L}$ in the numerator differ only by the fact that one of the state quantities Q_α in one case is a different spatial function from the other case, while in the limit, not only should the integral in the denominator converge to zero, but also the interval in which δQ_α is assumed to be zero should collapse to a single spatial point P – viz., the one at which the functional derivative of $\bar{L}$ is to be ascertained. Since one trivially has:

$$\frac{\delta \bar{L}}{\delta Q_{\alpha;P}} = \left(\frac{\partial \bar{L}}{\partial \dot{Q}_\alpha} \right)_P,$$

the field equations will then read:

$$\frac{\partial}{\partial t} \frac{\delta \bar{L}}{\delta \dot{Q}_{\alpha;P}} = \frac{\delta \bar{L}}{\delta Q_{\alpha;P}}. \tag{2′}$$

Just as in point mechanics, equations (2) or (2′) determine the behavior of the state quantities at all subsequent time points when they, along with their first derivatives, are given at a certain time point. In place of the finitely-many state quantities q_i of point

mechanics, a continuum of state variables will appear here, or more precisely: finitely-many continua, namely, the state functions $Q_\alpha\ (x_1,\ x_2,\ x_3)$. By contrast, the spatial coordinates are not to be regarded as state quantities, but as parameters.

In fact, one can always arrive at the case of continuously-many degrees of freedom, where the state quantities are spatial functions, by passing to the limit from the case of finite-many degrees of freedom. For the sake of simplicity, let the volume domain in which the field quantities are defined be finite, and let it be subdivided into congruent parallelepiped cells with the edge lengths $\Delta x_1,\ \Delta x_2,\ \Delta x_3$. One then replaces the continuous spatial functions $Q_\alpha\ (x_1,\ x_2,\ x_3)$ by step functions that have constant values inside each cell. If one thinks of the cells as being characterized by three running numbers l, m, n, corresponding to the three spatial coordinates, then one will now have the finitely-many state quantities $Q_{\alpha,l,m,n}$. If one replace the integral in the expression for $\bar{L}$ with a sum and the spatial derivatives with differential quotients according to:

$$\frac{\partial Q_\alpha}{\partial x_1} = \frac{Q_{\alpha,l+1,m,n} - Q_{\alpha,l,m,n}}{\Delta x_1}$$

then with the Lagrange function:

$$\bar{L} = \Delta x_1\, \Delta x_2\, \Delta x_3 \sum_{l,m,n} L\left(Q_{\alpha,l,m,n}\, \frac{Q_{\alpha,l+1,m,n} - Q_{\alpha,l,m,n}}{\Delta x_1}, \cdots, \dot{Q}_{\alpha,l,m,n} \right) \tag{5}$$

the equations of motion of ordinary point mechanics will read:

$$\frac{d}{dt}\frac{\partial \bar{L}}{\partial \dot{Q}_{\alpha,l,m,n}} = \frac{\partial \bar{L}}{\partial Q_{\alpha,l,m,n}}. \tag{5′}$$

It shall now be shown that in the limit of a vanishing volume for the cells that are employed for the subdivision of space equations (2) or (2′) for a continuum of degrees of freedom will emerge precisely from equations (5′) of ordinary point mechanics (*). To that end, it will obviously suffice to show that:

$$\lim \frac{1}{\Delta x_1 \Delta x_2 \Delta x_3}\frac{\partial \bar{L}}{\partial Q_{\alpha,l,m,n}} \to \frac{\delta \bar{L}}{\delta Q_{\alpha;P}}.$$

Since the coordinates $Q_{\alpha,l,m,n}$ occur in terms that belong to the cell l, m, n, as well as to the cell $l-1$, m, n; l, $m-1$, n; l, m, $n-1$, in the sum over l, m, n, one will now have:

(*) Cf., on this, also **G. Mie**, *loc. cit.*, § 4 and § 5.

$$\frac{1}{\Delta x_1 \Delta x_2 \Delta x_3}\frac{\partial \bar{L}}{\partial Q_{\alpha,l,m,n}} = \left(\frac{\partial \bar{L}}{\partial Q_\alpha}\right)_{l,m,n} - \left[\left(\frac{\partial \bar{L}}{\partial \frac{\partial Q_\alpha}{\partial x_1}}\right)_{l,m,n} - \left(\frac{\partial \bar{L}}{\partial \frac{\partial Q_\alpha}{\partial x_1}}\right)_{l-1,m,n}\right]\frac{1}{\Delta x_1} - \cdots$$

and, in fact, for an arbitrary refinement of the cell subdivision, this will converge to:

$$\frac{\partial L}{\partial Q_\alpha} - \sum_i \frac{\partial}{\partial x_i}\frac{\partial L}{\partial \frac{\partial Q_\alpha}{\partial x_i}} = \frac{\delta \bar{L}}{\delta Q_{\alpha;P}},$$

as was asserted.

In analogy to point mechanics, we now come to the introduction of a **Hamiltonian** form for the field equations, instead of the **Lagrangian** one. First, one defines the "impulse" P_α that is canonically-conjugate to the field quantities Q_α:

$$P_\alpha = \frac{\partial L}{\partial \dot{Q}_\alpha}, \tag{6}$$

and then the **Hamiltonian** function *H*, according to:

$$H\left(P_\alpha, Q_\alpha, \frac{\partial Q_\alpha}{\partial x_i}\right) = \sum_\alpha P_\alpha \dot{Q}_\alpha - L. \tag{7}$$

By varying *H* with respect to the variables P_α, Q_α, it will follow from (6) that:

$$\begin{aligned}\delta H &= \sum_\alpha \left(\frac{\partial H}{\partial P_\alpha}\delta P_\alpha + \frac{\partial H}{\partial Q_\alpha}\delta Q_\alpha + \sum_i \frac{\partial H}{\partial \frac{\partial Q_\alpha}{\partial x_i}}\delta\frac{\partial Q_\alpha}{\partial x_i}\right) \\ &= \sum_\alpha \dot{Q}_\alpha\,\delta P_\alpha - \sum_\alpha \left(\frac{\partial L}{\partial Q_\alpha}\delta Q_\alpha + \sum_i \frac{\partial L}{\partial \frac{\partial Q_\alpha}{\partial x_i}}\delta\frac{\partial Q_\alpha}{\partial x_i}\right),\end{aligned}$$

so one will first have:

$$\frac{\partial H}{\partial P_\alpha} = \dot{Q}_\alpha, \tag{8}$$

and secondly:

$$-\left(\frac{\partial H}{\partial Q_\alpha}\right)_{P_\alpha} = \left(\frac{\partial L}{\partial Q_\alpha}\right)_{\dot{Q}_\alpha}, \qquad -\left(\frac{\partial H}{\partial \frac{\partial Q_\alpha}{\partial x_i}}\right)_{P_\alpha} = \left(\frac{\partial L}{\partial \frac{\partial Q_\alpha}{\partial x_i}}\right)_{\dot{Q}_\alpha} = P_{\alpha i}\,. \tag{9}$$

The variables inside the parentheses are to be kept constant for the differentiations in question, and we have introduced a new abbreviation $P_{\alpha i}$ for later purposes, moreover.

The canonical field equations follow from (8) and (9), when one recalls (2):

$$\dot{Q}_\alpha = \frac{\partial H}{\partial P_\alpha}, \qquad \dot{P}_\alpha = -\left[\frac{\partial H}{\partial Q_\alpha} - \sum_i \frac{\partial}{\partial x_i}\frac{\partial H}{\partial \frac{\partial Q_\alpha}{\partial x_i}}\right], \tag{10}$$

or when one introduces:

$$\bar{H} = \int H\, dV\,, \tag{11}$$

one will get the equations:

$$\dot{Q}_{\alpha;P} = \frac{\delta \bar{H}}{\delta P_{\alpha;P}}, \qquad \dot{P}_{\alpha;P} = -\frac{\delta \bar{H}}{\delta Q_{\alpha;P}}. \tag{I}$$

They arise from the variational principle:

$$\delta \int L\, dV\, dt = \delta \int \left[\sum_\alpha P_\alpha \dot{Q}_\alpha - H\left(P_\alpha, Q_\alpha, \frac{\partial Q_\alpha}{\partial x_i}\right)\right] dV\, dt = 0, \tag{12}$$

in which, P_α and Q_α are considered to be spatial functions that are varied independently and whose variations should vanish at the limits. The canonical field equations then determine the further temporal course of the spatial functions P_α and Q_α when they are given arbitrarily for a certain moment in time $t = t_0$.

Furthermore, only the form (12) for the variational principle will be used in the following calculations, and it is inessential whether the integrand of (12) can or cannot go to a function of Q_α, $\partial Q_\alpha / \partial x_i$, and $\dot{Q}_\alpha$ by just eliminating the P_α . One can also free oneself of the assumption that H does not include the spatial derivatives of the P_α, but that will not be necessary for the later applications.

We would now like to introduce the (hitherto unnecessary) assumption that the Hamiltonian function H does not include the time coordinate explicitly, and assert that the quantity $\bar{H}$ is not constant in time in that case. In fact, by partial integration, one will immediately find that:

$$\frac{d\bar{H}}{dt} = \int \sum_\alpha \left(\frac{\delta \bar{H}}{\delta P_{\alpha;P}} \dot{P}_{\alpha;P} + \frac{\delta \bar{H}}{\delta Q_{\alpha;P}} \dot{Q}_{\alpha;P}\right) dV_P\,,$$

in which the terms (*) that originate on the boundary of the domain of integration can generally be dropped (as in all of what follows when one verifies the temporal constancy of certain volume integrals). That means that the field quantities must vanish sufficiently rapidly when integrating over all space. If one assumes that, then the constancy of $\bar{H}$ in time will follow immediately from the given expression for $d\bar{H}/dt$ by using (I). *In all physical applications, the quantity* $\bar{H}$ *(just like the* ***Hamiltonian*** *function of point mechanics) can be interpreted as the total energy of the system for a suitable choice of the numerical factors.*

Other integrals exist besides the energy integral $\bar{H}$:

$$G_k = -\int \sum_\alpha P_\alpha \frac{\partial Q_\alpha}{\partial x_k} dV \qquad (k = 1, 2, 3) \qquad (13)$$

that can be interpreted as components of the *total impulse* of the system. Analogous to the energy integral, it must be assumed here that H does not contain the spatial coordinates explicitly either, but once again, one must allow the dropping of the outer surface integrals. In fact, it then follows from (13) by successive partial integration that:

$$\begin{aligned}\frac{dG_k}{dt} &= -\int \sum_\alpha \left(\dot{P}_\alpha \frac{\partial Q_\alpha}{\partial x_k} - \dot{P}_\alpha \frac{\partial P_\alpha}{\partial x_k} \dot{Q}_\alpha \right) dV \\ &= \int \sum_\alpha \left(\frac{\delta H}{\delta Q_\alpha} \frac{\partial Q_\alpha}{\partial x_k} + \frac{\delta H}{\delta P_\alpha} \frac{\partial P_\alpha}{\partial x_k} \right) dV,\end{aligned}$$

but by substituting the expressions:

$$\frac{\delta H}{\delta Q_\alpha} = \frac{\partial H}{\partial Q_\alpha} - \sum_i \frac{\partial}{\partial x_i} \frac{\partial H}{\partial \frac{\partial Q_\alpha}{\partial x_i}}, \qquad \frac{\delta H}{\delta P_\alpha} = \frac{\partial H}{\partial P_\alpha},$$

it will also follow that:

$$\frac{dG_k}{dt} = -\int \sum_\alpha \left[\sum_i \frac{\partial}{\partial x_i} \left(\frac{\partial H}{\partial \frac{\partial Q_\alpha}{\partial x_i}} \frac{\partial Q_\alpha}{\partial x_k} \right) - \frac{\partial H}{\partial x_k} \right] dV,$$

(*) That will give rise to the outer surface integral:

$$\int df \sum_i \left[\cos(n, x_i) \sum_\alpha \left(\frac{\partial H}{\partial \frac{\partial Q_\alpha}{\partial x_i}} \dot{Q}_\alpha + \frac{\partial H}{\partial \frac{\partial P_\alpha}{\partial x_i}} \dot{P}_\alpha \right) \right],$$

which can be interpreted as the energy flux through the boundary surface.

which can be converted completely into an outer surface integral, and will thus vanish, by assumption, such that G_k is, in fact, constant in time. If no spatial direction is distinguished from the outset – so L, and therefore also H, are invariant under spatial rotations of the coordinate axis cross – then the G_k will define the components of a vector, as they must.

§ 2. Canonical commutation relations (C. C. R.) for continuous space-time functions. Energy and impulse theorem in quantum dynamics. We are now sufficiently prepared to take the step from classical physics to quantum physics. For that, we first appeal to a method that corresponds to the employment of matrices or operators in quantum mechanics, while we will first briefly go into the methods that are analogous to the **Schrödinger** differential equation in coordinate space later on. The formal conversion of the latter methods to field physics encounters the mathematical complication of how to define a volume element on function space in a reasonable way. The former method has the advantage, moreover, that a greater freedom exists in the choice of the independent variables, in that canonical transformations can be performed more easily, and furthermore, that the form of the physical laws (which are the field equations and the expression for the **Hamiltonian** function, in our case) can be carried over from the classical theory directly. As is known, with that method, the difference between classical and quantum physics is expressed by the idea that the physical quantities will be generally replaced by non-commutative operators, moreover. In the case of quantum mechanics, these physical state quantities depend, firstly, upon time and secondly upon one (or more) discontinuous indices that distinguish the various degrees of freedom, so in the case of quantum dynamics of the field functions, the aforementioned indices (to some degree) go to continuously-varying spatial coordinates x_1, x_2, x_3, which are then to be regarded as ordinary numbers (i.e., *c*-numbers), just like the time t.

In order to arrive at C. C. R. (*) for the continuous field quantities, as in the previous paragraphs, we carry out the passage to the limit from the case of finitely-many degrees of freedom by starting with the Lagrange function (5), which will go to the Lagrange function (3) in the limit of an infinitely-fine cell decomposition of space. If we introduce the ordinary δ-symbol, which is defined by:

$$\delta_{ll'} = \left\{ \begin{array}{l} 0 \text{ for } l \neq l', \\ 1 \text{ for } l = l', \end{array} \right\} \tag{14}$$

and furthermore, the abbreviation:

$$\delta_{l,\,m,\,n;\,l',\,m',\,n'} = \delta_{ll'}\,\delta_{mm'}\,\delta_{nn'},$$

and the relation:

$$\Delta V = \Delta x_1\,\Delta x_2\,\Delta x_3$$

(*) Here, and in what follows, the abbreviation C. C. R. will always be employed for "canonical commutation relations."

for the volumes of the cells then, according to ordinary quantum mechanics, the C. C. R. for finitely-many degrees of freedom will read:

$$p_{\alpha,\,lmn}\,Q_{\beta,\,l'm'n'} - Q_{\beta,\,l'm'n'}\,p_{\alpha,\,lmn} = \delta_{l,\,m,\,n;\,l',\,m',\,n'}\,\delta_{\alpha\beta}\,, \tag{15}$$

to which, one can add the commutation of the various Q with each other, as well as the various p. In this, one has:

$$p_{\alpha,\,l\,m\,n} = \frac{\partial \bar{L}}{\partial \dot{Q}_{\alpha,lmn}} = \Delta V\,\frac{\partial L}{\partial \dot{Q}_{\alpha,lmn}}\,,$$

such that in the limit, one will have:

$$\lim_{\Delta V \to 0} \frac{1}{\Delta V}\; p_{\alpha,\,l,\,m,\,n} = P_\alpha\,(x_1,\,x_2,\,x_3).$$

If we were to pass to the limit $\Delta V \to 0$ in equation (14) after dividing by ΔV then we would get zero on the right-hand side. We would then obtain a reasonable result when we first multiplied (15) by an arbitrary step function f (i.e., a c-function) of the indices l' m' n' and then summed over all cells of a certain piece of space V', when we let the function f converge to a continuous spatial function $f\,(x_1,\,x_2,\,x_3)$ in the limit of $\Delta V \to 0$, in such a way that the sum:

$$\sum_{l'm'n'} f\,(l', m', n')\,\Delta V$$

would go to the integral:

$$\int_{V'} f\,(x_1', x_2', x_3')\,dV'$$

over the chosen spatial piece. We then obtain:

$$\sum_{l'm'n'} f\,(l', m', n')\,\Delta V \cdot \left[\frac{p_{\alpha,lmn}}{\Delta V} Q_{\beta,l'm'n'} - Q_{\beta,l'm'n'}\,\frac{p_{\alpha,lmn}}{\Delta V}\right]$$

$$= \frac{h}{2\pi i}\,\delta_{\alpha\beta}\begin{cases} f\,(l, m, n) & \text{when the cell } l, m, n \text{ is in } V', \\ 0 & \text{otherwise,} \end{cases}$$

and in the limit of an infinitely-fine cell decomposition:

$$\iiint_{V'} f\,(x_1', x_2', x_3')\,dV'\{P_\alpha(x_1, x_2, x_3)Q_\beta(x_1', x_2', x_3') - Q_\beta(x_1', x_2', x_3')P_\alpha(x_1, x_2, x_3)\}$$

$$= \frac{h}{2\pi i}\,\delta_{\alpha\beta}\begin{cases} f\,(x_1, x_2, x_3) & \text{when the cell } x_1, x_2, x_3 \text{ is in } V', \\ 0 & \text{otherwise.} \end{cases} \tag{16}$$

Moreover, the roles of x_1, x_2, x_3 and x_1', x_2', x_3' can also be switched in this. It is preferable to formulate this result by means of the singular function symbol $\delta(x)$ that **Dirac** introduced, which is defined by:

$$\int_a^b f(x)\,\delta(x)\,dx = \begin{cases} f(0), & \text{when } x = 0 \text{ in } (a,b), \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

It follows from this that one can always set $\delta(-x) = \delta(x)$. Furthermore, with the introduction of the vector $\mathfrak{r}$ whose components are x_1, x_2, x_3, and the abbreviations:

$$\left.\begin{aligned} &\delta(\mathfrak{r}) = \delta(x_1)\delta(x_2)\delta(x_3), \qquad \delta(\mathfrak{r},\mathfrak{r}') = \delta(\mathfrak{r}',\mathfrak{r}) = \delta(\mathfrak{r}-\mathfrak{r}'), \\ &\text{one will have :} \\ &\int_{V'} f(x_1',x_2',x_3')\,\delta(\mathfrak{r},\mathfrak{r}') = \begin{cases} f(x_1,x_2,x_3), & \text{when } x_1,x_2,x_3 \text{ in } V', \\ 0 & \text{otherwise.} \end{cases} \end{aligned}\right\} \tag{17'}$$

If we write P_α, Q_α for $P_\alpha(x_1, x_2, x_3)$, $Q_\alpha(x_1, x_2, x_3)$, for brevity, and P_α', Q_α' for $P_\alpha(x_1', x_2', x_3')$, $Q_\alpha(x_1', x_2', x_3')$, and if we introduce the bracket symbol:

$$[F, G] \equiv FG - GF$$

as a further abbreviation then the canonical commutation rules for continuous field quantities can be written as follows:

$$[Q_\alpha, Q_\beta'] = 0, \qquad [P_\alpha, P_\beta'] = 0, \qquad [P_\alpha, Q_\beta'] = [P_\alpha', Q_\beta] = \delta_{\alpha\beta}\,\delta(\mathfrak{r}, \mathfrak{r}'). \tag{II}$$

It should be remarked that these relations are true for two different spatial locations, but always at the same time point, and that nothing further will be said about the value of the bracket symbol in question of the field quantities at two different time points. By contrast, if we define the derivative of the δ-function in the usual way, namely:

$$\int_a^b f(x)\,\delta'(x)\,dx = \begin{cases} -f'(0) & \text{when } x = 0 \text{ in } (a,b), \\ 0 & \text{otherwise,} \end{cases} \tag{17''}$$

which arises from (17) formally by partial integration and dropping the terms that originate on the boundary, then the C. C. R. (II) can be differentiated by the spatial coordinates. One will then get, e.g.:

$$\left.\begin{aligned}
&\left[P'_\alpha, \frac{\partial Q_\alpha}{\partial x_i}\right] = \frac{h}{2\pi i}\frac{\partial}{\partial x_i}\delta(\mathfrak{r},\mathfrak{r}'), \qquad \left[P_\alpha, \frac{\partial Q'_\alpha}{\partial x'_i}\right] = \frac{h}{2\pi i}\frac{\partial}{\partial x'_i}\delta(\mathfrak{r},\mathfrak{r}') = -\frac{h}{2\pi i}\frac{\partial}{\partial x_i}\delta(\mathfrak{r},\mathfrak{r}'),\\
&\left[\frac{\partial P_\alpha}{\partial x_i}, Q'_\alpha\right] = \frac{h}{2\pi i}\frac{\partial}{\partial x_i}\delta(\mathfrak{r},\mathfrak{r}'), \qquad \left[\frac{\partial P'_\alpha}{\partial x'_i}, Q_\alpha\right] = \frac{h}{2\pi i}\frac{\partial}{\partial x'_i}\delta(\mathfrak{r},\mathfrak{r}') = -\frac{h}{2\pi i}\frac{\partial}{\partial x_i}\delta(\mathfrak{r},\mathfrak{r}'),
\end{aligned}\right\} \tag{18}$$

in which the last equation follows from $\delta(\mathfrak{r}, \mathfrak{r}') = \delta(\mathfrak{r} - \mathfrak{r}') = \delta(\mathfrak{r}' - \mathfrak{r})$.

In order to go further, we must define the differentiation of a function of non-commutating quantities with respect to one of the quantities, which happens, in a well-known manner, by way of:

$$\frac{\partial F(Q_1, Q_2, \ldots)}{\partial Q_1} = \lim_{\delta \to 0} \frac{F(Q_1 + \delta, Q_2, \ldots) - F(Q_1, Q_2, \ldots)}{\delta},$$

in which δ is a *c*-number (multiplied by the identity operator, which is not written down). With this definition, the usual rule for the differentiation of a product will be true:

$$\frac{\partial (F_1 F_2)}{\partial Q_1} = F_1 \frac{\partial F_2}{\partial Q_1} + \frac{\partial F_1}{\partial Q_1} F_2,$$

in which, one must be careful to preserve the sequence of the factors.

Now, let F be an arbitrary function of the P_α, $\partial P_\alpha / \partial x_i$, Q_α, $\partial Q_\alpha / \partial x_i$, which however might depend upon only the values of these functions at a single spatial location. In analogy with the corresponding development in ordinary quantum mechanics, one can easily prove that:

$$\left.\begin{aligned}
[F, Q'_\alpha] &= \frac{h}{2\pi i}\left[\frac{\partial F}{\partial P_\alpha}\delta(\mathfrak{r},\mathfrak{r}') + \sum_i \frac{\partial F}{\partial \frac{\partial P_\alpha}{\partial x_i}}\frac{\partial}{\partial x_i}\delta(\mathfrak{r},\mathfrak{r}')\right],\\
[P'_\alpha, F] &= \frac{h}{2\pi i}\left[\frac{\partial F}{\partial Q_\alpha}\delta(\mathfrak{r},\mathfrak{r}') + \sum_i \frac{\partial F}{\partial \frac{\partial Q_\alpha}{\partial x_i}}\frac{\partial}{\partial x_i}\delta(\mathfrak{r},\mathfrak{r}')\right].
\end{aligned}\right\} \tag{19}$$

According to (II) and (18), these relations will, in fact, obviously be correct when F is replaced with one of the field quantities P_α, Q_α, $\partial P_\alpha / \partial x_i$, $\partial Q_\alpha / \partial x_i$, and one will then further show that they will remain correct for $F_1 + F_2$ and $F_1 F_2$ when they are assumed to be correct for F_1 and F_2. One further obtains the corresponding C. R. for:

$$\bar{F} = \int F \, dV$$

immediately from (19) by partial integration, namely:

$$[\bar{F}, Q'_\alpha] = \frac{h}{2\pi i}\left(\frac{\partial F}{\partial P_\alpha} - \sum_i \frac{\partial}{\partial x_i}\frac{\partial F}{\partial \frac{\partial P_\alpha}{\partial x_i}}\right)_{\text{for } x_i = x'_i},$$

$$[P'_\alpha, \bar{F}] = \frac{h}{2\pi i}\left(\frac{\partial F}{\partial Q_\alpha} - \sum_i \frac{\partial}{\partial x_i}\frac{\partial F}{\partial \frac{\partial Q_\alpha}{\partial x_i}}\right)_{\text{for } x_i = x'_i}.$$

With the introduction of the symbols $\delta\bar{F}/\delta P_\alpha$ and $\delta\bar{F}/\delta Q_\alpha$, from (4), this can be easily written as:

$$[\bar{F}, Q_\alpha] = \frac{h}{2\pi i}\frac{\delta\bar{F}}{\delta P_\alpha}, \quad [\bar{F}, P_\alpha] = -\frac{h}{2\pi i}\frac{\delta\bar{F}}{\delta Q_\alpha}, \tag{20}$$

in which the variation is always performed at the same spatial location at which the field quantities that are found inside the bracket are found.

We are now sufficiently prepared to go on to a discussion of the field equations. We borrow them from classical theory in the canonical form (I):

$$\dot{Q}_\alpha = \frac{\delta\bar{H}}{\delta P_\alpha}, \qquad \dot{P}_\alpha = -\frac{\delta\bar{H}}{\delta Q_\alpha}, \tag{I}$$

with the special addition that the partial differentiations that enter into them are understood to have the sense that was defined above. In general, the special prescription about the sequence of factors in *H* that the classical paradigm cannot be determined uniquely will also be required. Thus, for later applications, *H* will be (essentially) a quadratic form in the field quantities, so the field equations will be (essentially) linear (*), such that the prescription (I) will say that the field equations read precisely the same as the corresponding classical ones.

The field equations can be written directly in form:

$$\dot{Q}_\alpha = \frac{2\pi i}{h}[\bar{H}, Q_\alpha], \qquad \dot{P}_\alpha = \frac{2\pi i}{h}[\bar{H}, P_\alpha]$$

by means of (20), from which, by an inductive argument that is similar to the one above, one can conclude that the quantity *F* that was considered there will satisfy the relation:

(*) The Hamiltonian function and the field equations for the matter waves contain products of the material field quantities ψ and ψ^* with the electromagnetic potentials Φ_ν. We will see that ψ and ψ^* commute with the Φ_ν in our theory, so that situation will not be affected.

$$\dot{F} = \frac{2\pi i}{h}[\bar{H}, F], \tag{21}$$

and thus, for:

$$\bar{F} = \int F\, dV,$$

one can conclude (*):

$$\dot{\bar{F}} = \frac{2\pi i}{h}[\bar{H}, \bar{F}]. \tag{21′}$$

Two conclusions can be drawn from this equation that are of fundamental importance for the consistent practicability of the theory. We first set $\bar{F} = \bar{H}$ in (21′), and since $[\bar{H}, \bar{H}] \equiv 0$, that will yield:

$$\dot{\bar{H}} = 0, \qquad \bar{H} = \text{const.} \tag{22}$$

The energy theorem is also true here (**) [in which, it is naturally assumed that H does not include time explicitly, since (21) applies to only quantities that fulfill that assumption]. Secondly, we replace F with one of the bracket symbols $[Q_\alpha , Q'_\beta]$, $[P_\alpha, P'_\beta]$, $[P_\alpha , Q'_\beta]$, $[P'_\alpha, Q_\beta]$ in (21). Since, from (II), these brackets are all c-numbers (more precisely: c-numbers multiplied by the identity operator), they will then commute with H, such that the temporal derivatives of the brackets (for fixed spatial positions) will vanish. *This means that if one assumes that the C. R.* (II) *is true for a certain time point* $t = t_0$ *then the C. R. will be reproduced for a neighboring time-point by means of the field equations* (I), *and thus, for all time.* The consistency of (I) and (II) is thus proved by that.

We now apply (20) to the impulse integral that is defined by (13):

$$G_k = -\int \sum_\alpha P_\alpha \frac{\partial Q_\alpha}{\partial x_k}\, dV. \tag{13}$$

If one identifies $\bar{F}$ with G_k in (20) then one will find that:

$$[G_k, Q_\alpha] = \frac{ih}{2\pi}\frac{\partial Q_\alpha}{\partial x_k}, \quad [G_k, P_\alpha] = \frac{ih}{2\pi}\frac{\partial P_\alpha}{\partial x_k}.$$

(*) The role that this relation plays is that it allows one to avoid the application of equations such as:

$$\dot{F} = \sum_\alpha \left[\frac{\partial F}{\partial Q_\alpha}\dot{Q}_\alpha + \frac{\partial F}{\partial P_\alpha}\dot{P}_\alpha + \sum_i \left(\frac{\partial F}{\partial \frac{\partial Q_\alpha}{\partial x_i}} \frac{\partial \dot{Q}_\alpha}{\partial x_i} + \frac{\partial F}{\partial \frac{\partial P_\alpha}{\partial x_i}} \frac{\partial \dot{P}_\alpha}{\partial x_i} \right) \right],$$

which would be inadmissible.

(**) One will remark that, in contrast to the older representation of quantum mechanics, we have not introduced the assumption that $\bar{H}$ has been brought into diagonal form, since that certainly represents an important case in physical applications of the equations, but not the only possible one.

It follows from this by induction for any quantity F (that does not include spatial coordinates explicitly) of the kind considered that:

$$\frac{\partial F}{\partial x_k} = -\frac{2\pi i}{h}[G_k, F], \tag{23}$$

which are relations that define one aspect of (21). Due to the fact that:

$$\int \frac{\partial F}{\partial x_k} dV = 0,$$

and by the existence of $\bar{F} = \int F\, dV$, it follows from that relation upon integrating over the spatial volume that:

$$[G_k, \bar{F}] = 0,$$

and, in particular, for $F = \bar{H}$, according to (21), one will have:

$$\dot{G}_k = 0, \qquad G_k = \text{const.}, \tag{24}$$

with which, the existence of the impulse integral in quantum dynamics will be proved. In regard to this, one must generally make a remark concerning the sequence of factors P_α and $\partial Q_\alpha / \partial x_k$ in (13). Indeed, the validity of (23) and (24) is independent of that sequence, but according to (18), $\left[P_\alpha, \frac{\partial Q_\alpha}{\partial x_k} \right]$ will be singular and indeterminate for $x = 0$, since those functions are to be taken at the same spatial location, and the same thing will be true for $\delta'(x)$. Which linear combination of expressions $P_\alpha \frac{\partial Q_\alpha}{\partial x_k}$ and $\frac{\partial Q_\alpha}{\partial x_k} P_\alpha$ should be employed in the integrands of G_k can therefore not be established from the outset.

By representing the operators that represent field quantities by matrices, and in the special case for which the energy and impulse G_k are matrices in *diagonal form*, the following differential equations for an arbitrary matrix element F_{nm} of F will result from (21) and (23):

$$\dot{F}_{nm} = \frac{2\pi i}{h}(\bar{H}_n - \bar{H}_m) F_{nm}, \qquad \frac{\partial F_{nm}}{\partial x_k} = -\frac{2\pi i}{h}(G_{k,n} - G_{k,m}) F_{nm},$$

such that the dependency of the element F_{nm} of space and time will necessarily have the form of a harmonic wave:

$$F_{nm} = a_{nm}\, e^{\frac{2\pi i}{h}[(\bar{H}_n - \bar{H}_m)t - (\mathfrak{G}_n - \mathfrak{G}_m)\mathfrak{r}]}, \tag{25}$$

if one understands $\mathfrak{G}$ to mean the impulse vector whose components are G_k. Independently of any special representation of the operators, it will follow from repeated application of (21) and (23) in a known way that for any quantity F:

$$= e^{\frac{2\pi i}{h}[\bar{H}(t'-t)-(\mathfrak{G},(\mathfrak{r}'-\mathfrak{r})]} \begin{matrix} F(x_1', x_2', x_3', t) \\ F(x_1, x_2, x_3, t) \end{matrix} e^{-\frac{2\pi i}{h}[\bar{H}(t'-t)-(\mathfrak{G},(\mathfrak{r}'-\mathfrak{r})]} . \qquad (26)$$

To conclude this paragraph, let us mention the development of the field quantities in eigen-oscillations as a method of integration; that is the only method that has proven to be practical up to now. One develops the field quantities in terms of their dependency upon the spatial coordinates in an orthogonal system:

$$P_\alpha = \sum_\rho a_{\alpha\rho}(t) u_\rho(x_1, x_2, x_3), \quad Q_\alpha = \sum_\rho b_{\alpha\rho}(t) u_\rho^*(x_1, x_2, x_3), \qquad (27)$$

in which:

$$\int u_\rho u_\sigma^* \, dV = \delta_{\rho\sigma}, \qquad (28)$$

and the inversion formulas read:

$$a_{\alpha\rho}(t) = \int P_\alpha u_\sigma^*(x_1, x_2, x_3) dV, \qquad b_{\alpha\rho}(t) = \int Q_\alpha u_\rho(x_1, x_2, x_3) dV. \qquad (27')$$

In this, the u_ρ are considered to be c-functions, while the a_α and b_α, like the P_α and Q_α, are considered to be q-numbers.

The fact that the orthogonal system is discrete can enforced by either consider the field in a cavity, on whose walls certain boundary conditions must be fulfilled (i.e., standing waves) or, as is customary in, e.g., the theory of crystal lattices, by the restriction of the field to spatially-periodic motions with sufficiently-large periods (i.e., travelling waves).

One gets the C. R. for the *a* and *b* from (II).

$$[a_{\alpha\rho}, b_{\beta\sigma}] = \int [P_\alpha, Q_\beta'] u_\rho^* u_\sigma' \, dV \, dV' = \frac{h}{2\pi i} \delta_{\alpha\beta} \int u_\rho^* u_\sigma' \, dV,$$

and the canonical form:

$$[a_{\alpha\rho}, b_{\beta\sigma}] = \frac{h}{2\pi i} \delta_{\alpha\beta} \delta_{\rho\sigma} \qquad (29)$$

will then follow from (28).

The Hamiltonian function $\bar{H}$ goes to a function of the *a* and *b* and gives rise to canonical equations in these variables. Naturally, one has complete freedom in the choice of the orthogonal system. If it so happens that one can choose $\bar{H}$ to be separable then all of the matrices will be easy to calculate. In other cases, one must turn to perturbation theory for a suitable initial system, whether by the introduction of **Schrödinger** functions $\varphi(b_1, b_2, \ldots)$ or by the original methods of matrix theory.

§ 3. Relativistic invariance of the C. C. R. for an invariant Lagrangian function. Up to now, we have spoken of only C. C. R. that couple the field quantities at two spatial locations at the same time point. However, the C. C. R. for two different time points are determined implicitly by the field equations (I). In order for the theory to be useful, it

must now be demanded that the C. C. R. must also preserve their forms when one goes from one coordinate system to another by a Lorentz transformation due to the relativistic invariance of the **Lagrangian** function. The problem of this paragraph is then to provide evidence that this condition is fulfilled.

If we go from one coordinate system to another by a Lorentz transformation then the values of the bracket symbol (II) will change on two grounds: First of all, the quantities P_α and Q_α are not generally scalars, so they will be transformed in a certain way at a well-defined world-point. Secondly, other world-points must be chosen in the C. C. R. in the primed coordinate system than the ones that were chosen in the unprimed system, where the latter exhibit a common t'-coordinate, while the former have an equal value of t. Meanwhile, the change in the bracket that is required by the latter situation would be difficult to ascertain, since we cannot exhibit general formulas for it in the case of finite differences in the time values at the two locations that come under consideration. However, one can circumvent that difficulty by restricting oneself to infinitesimal Lorentz transformations. In that case, in fact, any physical quantity $f\,(t')$ will be replaced by $f\,(t) + \partial f / \partial t\,(t' - t)$, and $\partial f / \partial t$, as well as the associated C. C. R., can be inferred from (I). The invariance of the system of equations under finite transformations will then follow from the group character of the invariance of the totality of these transformations. In what follows, we will proceed accordingly in such a way that we calculate the changes in the brackets under infinitesimal Lorentz transformations separately as a result of the two aforementioned facts, and then investigate the conditions under which they will be compensated.

If we would begin with the first-mentioned basis for the change in the bracket then we would first have to make some general statements about the type of transformations of the quantities P_α and Q_α under Lorentz transformations. It will be preferable to introduce the imaginary time coordinate $x_4 = ict$, and to further replace the quantities that were previously denoted by P_α with:

$$P_{\alpha 4} = \frac{\partial L}{\partial \dfrac{\partial Q_\alpha}{\partial x_4}} = ic\, P_\alpha, \tag{30}$$

such that:

$$\frac{\partial P_{\alpha 4}}{\partial x_4} = \dot{P}_\alpha = -\frac{\delta \bar{H}}{\delta Q_\alpha}, \qquad \frac{\partial Q_\alpha}{\partial x_4} = \frac{1}{ic}\dot{Q}_\alpha = \frac{1}{ic}\frac{\delta \bar{H}}{\delta P_\alpha} = \frac{\delta \bar{H}}{\delta P_{\alpha 4}}, \tag{30'}$$

$$[P_{\alpha 4},\ \bar{Q}_\beta\,] = \frac{hc}{2\pi}\delta_{\alpha\beta}\delta(\mathfrak{r}, \bar{\mathfrak{r}}), \tag{31}$$

in which we now apply the overbar in order to characterize the space-time location of Q_β, in order to preserve the prime symbol for the transition to another coordinate system. In what follows, the quantities that were already introduced in (9), namely:

$$P_{\alpha i} = \frac{\partial L}{\partial \dfrac{\partial Q_\alpha}{\partial x_i}} = -\frac{\partial H}{\partial \dfrac{\partial Q_\alpha}{\partial x_i}}, \tag{9}$$

will enter in with the same status as the $P_{\alpha 4}$, although no C. C. R. exist between Q_β and $P_{\beta 4}$ that are as simple as equation (31). However, in regard to that, one must especially emphasize that the agreement of the two expressions for $P_{\alpha i}$ that were used in (9) is not generally guaranteed, due to the non-commutation of certain factors in H. It is only when L is a quadratic form in the $\dot{Q}_\alpha$ and $\partial Q_\alpha / \partial x_i$ with constant coefficients (with the possible addition of a function of only the Q_α) that the argument that was used in the derivation of (9) can be adopted immediately. One then generally has:

$$P_{\alpha\mu} = \frac{\partial L}{\partial \dfrac{\partial Q_\alpha}{\partial x_\mu}}, \tag{32}$$

if here, and in what follows, the indices μ, ν, … always run 1 to 4, in contrast to the Latin indices, which only refer to spatial coordinates and run from 1 to 3, and in contrast to the indices α, β, …, which distinguish the various quantities $P_{\alpha\mu}$ and Q^μ. We shall not make any special assumption about the transformation law of the latter quantities. However, it can be easily concluded from (32) that (*): If one transforms the Q quantities by the orthogonal coordinate transformation (one sums over indices that appear twice):

$$x'_\mu = a_{\mu\nu} x_\nu, \qquad a_{\mu\rho} a_{\nu\rho} = \delta_{\nu\mu}, \tag{33}$$

according to:

$$Q'_\alpha = A_{\alpha\beta} Q_\beta, \tag{34}$$

then the $P_{\alpha\mu}$ will transform according to:

$$P'_{\alpha\mu} = a_{\mu\nu} B_{\alpha\beta} P_{\beta\nu}, \tag{35}$$

in which the B coefficients depend upon the A coefficients according to:

$$A_{\alpha\gamma} B_{\beta\gamma} = \delta_{\alpha\beta}. \tag{36}$$

That is, the matrix of B is the reciprocal of the transpose of the matrix A. It then follows from this that the P and Q will always be contracted over equal indices in such a way that, e.g., $\sum_\alpha P_{\alpha\mu} Q_\alpha$ is a vector and $\sum_\alpha P_{\alpha\mu} \dfrac{\partial Q_\alpha}{\partial x_\nu}$ is a tensor.

If we go from the finite transformations to the infinitesimal ones, where $a_{\mu\nu} = \delta_{\mu\nu} + \varepsilon s_{\mu\nu}$, $A_{\alpha\beta} = \delta_{\alpha\beta} + \varepsilon t_{\alpha\beta}$, and according to (33) and (36), $s_{\mu\nu} = -s_{\nu\mu}$, $s_{\nu\nu} = 0$, $B_{\alpha\beta} = \delta_{\alpha\beta} - \varepsilon t_{\alpha\beta}$, and if one neglects quantities of order ε^2, then (33), (34), and (35) will become:

$$x'_\mu = x_\mu + \varepsilon s_{\mu\nu} x_\nu, \qquad s_{\mu\nu} = -s_{\nu\mu}, \tag{33′}$$

(*) This will be true independently of whether L, along with $\dot{Q}_\alpha$, does or does not include the $P_{\alpha 4}$ explicitly.

$$Q'_\alpha = Q_\alpha + \varepsilon\, t_{\alpha\beta}\, Q_\beta, \tag{34'}$$

$$P'_{\alpha\mu} = P_{\alpha\mu} - \varepsilon\, t_{\beta\alpha}\, P_{\beta\mu} + \varepsilon\, s_{\mu\nu}\, P_{\alpha\nu}\ . \tag{35'}$$

Moreover, we calculate the brackets of the primed field quantities, while we nonetheless first let the world-points (x_μ) and ($\bar{x}_\mu$) at which the field quantities are taken be fixed.

One will then have:

$$\begin{aligned}
[Q'_\alpha, \bar{Q}'_\beta] &= [Q_\alpha,\ \bar{Q}_\beta] + \varepsilon\, t_{\alpha\gamma}[Q_\gamma,\ \bar{Q}_\beta] + \varepsilon\, t_{\beta\delta}[Q_\alpha,\ \bar{Q}_\delta],\\
[P'_{\alpha 4}, \bar{Q}'_\beta] &= [P_{\alpha 4}\ ,\ \bar{Q}_\beta] + \varepsilon\, s_{4\nu}[P_{\alpha\nu},\ \bar{Q}_\beta] - \varepsilon\, t_{\gamma\alpha}[P_{\gamma 4}\ ,\ \bar{Q}_\delta] + \varepsilon\, t_{\beta\delta}[P_{\alpha 4}\ ,\ \bar{Q}_\delta],\\
[P'_{\alpha 4}, \bar{P}'_{\beta 4}] &= [P_{\alpha 4}\ ,\ \bar{P}_{\beta 4}] + \varepsilon\, s_{4\nu}[P_{\alpha\nu},\ \bar{P}_{\beta 4}] + \varepsilon\, s_{4\nu}[P_{\alpha 4}\ ,\ \bar{P}_{\beta\nu}]\\
&\quad - \varepsilon\, t_{\gamma\alpha}[P_{\gamma 4},\ \bar{P}_{\beta 4}] - \varepsilon\, t_{\gamma\beta}[P_{\alpha 4}\ ,\ \bar{P}_{\gamma 4}]\ .
\end{aligned}$$

The expressions will simplify substantially when we substitute the values (II) [(31), resp.] of the brackets for the unprimed coordinate system. In fact, all terms that contain the $t_{\alpha\beta}$ as a factor will then vanish. That is trivial in the first and last equation, but in the second equation that will give a contribution (up to a common constant factor):

$$-\, t_{\gamma\alpha}\, \delta_{\gamma\beta} + t_{\beta\gamma}\, \delta_{\alpha\gamma} = -\, t_{\beta\alpha} + t_{\beta\alpha} = 0.$$

All that will remain are terms with the factor $s_{4\nu}$, in which it follows, moreover, that $s_{44} = 0$, so ν can be replaced with the index k that runs from 1 to 3, such that one will have:

$$\left.\begin{aligned}
&[Q'_\alpha, \bar{Q}'_\beta] = 0,\\
&[\bar{P}'_{\alpha 4}, Q'_\beta] = \frac{hc}{2\pi}\, \delta(\mathfrak{x}, \bar{\mathfrak{x}})\, \delta_{\alpha\beta} + \varepsilon\, s_{4k}[P_{\alpha k}, \bar{Q}_\beta],\\
&[P'_{\alpha 4}, \bar{P}'_{\beta 4}] = \varepsilon s_{4k}[P_{\alpha k}, \bar{P}_{\beta 4}] + \varepsilon\, s_{4k}[P_{\alpha 4}, \bar{P}_{\beta k}].
\end{aligned}\right\} \tag{37}$$

We can now go on to the calculation of the second part of the change in the brackets, namely, the one that originates in the change of the world points. Now, it is always permissible for us to move the origin of the coordinate system to either of the two world-points that then remain fixed. Which of these world-points is irrelevant, since is has already been shown that the transition from one slice t = const, to a *parallel* neighboring slice in the four-dimensional world will not change the C. C. R. If we choose the first point P to be the fixed one then the second one $\bar{P}$ will have the values $x_i = \bar{x}_i$, $x_4 = 0$ in the unprimed coordinate system, while the point $\bar{P}'$ will possess coordinates with the same values $x'_i = \bar{x}_i$, $x'_4 = 0$ in the primed system. The point $\bar{P}$ will then have the coordinates ($\bar{x}_i$, 0) in the unprimed system, but the point $\bar{P}'$ will have the coordinates, $(\bar{x}_i - \varepsilon\, s_{ik}\bar{x}_k, -\varepsilon\, s_{4k}\bar{x}_k)$, since $x_4 = 0$. Thus, for any two quantities F_1, F_2, one will have:

$$[F_1(P),\, F_2(\bar{P}')\,] = [F_1(P),\, F_2(\bar{P})\,] - \varepsilon\, s_{\nu k}\ \bar{x}_k \left[F_1(P), \frac{\partial F_2(\bar{P})}{\partial \bar{x}_\nu} \right].$$

If we once more leave the origin of the coordinate system arbitrary then the total change in the brackets will become:

$$\left.\begin{aligned}
[Q'_\alpha, Q'_\beta(\bar{P}') - [Q_\alpha, Q_\beta(\bar{P})] &= -\varepsilon\, s_{\nu k}(\bar{x}_k - x_k) \left[Q_\alpha, \frac{\partial \bar{Q}_\beta}{\partial \bar{x}_\nu} \right], \\
[Q'_\alpha, Q'_\beta(\bar{P}') - [Q_\alpha, Q_\beta(\bar{P})] &= -\varepsilon\, s_{\nu k}(\bar{x}_k - x_k) \left[Q_\alpha, \frac{\partial \bar{Q}_\beta}{\partial \bar{x}_\nu} \right] + \varepsilon\, s_{4k} [P_{\alpha k}, \bar{Q}_\beta], \\
[P'_{\alpha 4}, P'_{\beta 4}(\bar{P}') - [P_{\alpha 4}, P_{\beta 4}(\bar{P})] &= -\varepsilon\, s_{\nu k}(\bar{x}_k - x_k) \left[P_{\alpha 4}, \frac{\partial \bar{P}_{\beta 4}}{\partial \bar{x}_\nu} \right] \\
&\quad + \varepsilon\, s_{4k} \{ [P_{\alpha k}, \bar{P}_{\beta 4}] + [P_{\alpha 4}, \bar{P}_{\beta k}] \}.
\end{aligned}\right\} \qquad (38)$$

In regard to this, we next remark that the summation index ν can be restricted to 4 in the terms with $s_{\nu k}$. That is trivial for the first and last equation, since the brackets in question vanish, but for the middle equation, it will follow from the fact that the terms that are endowed with the factor $(\bar{x}_k - x_k)$ need to be preserved only when that factor is once more cancelled by the addition of a derivative of the δ-function with respect to x_k, which will also be important in what follows. For $\nu = 1, 2, 3$, $\left[P_{\alpha 4}, \frac{\partial Q_\beta}{\partial \bar{x}_\nu} \right]$ is now proportional to $\frac{\partial}{\partial x_\nu} \delta(\mathfrak{r}, \bar{\mathfrak{r}})$, while the term will be made to vanish for $\nu = k$ due to the fact that $s_{kk} = 0$, such that only $\nu = 4$ will remain here, as well.

Now, should the C. C. R. (II) also remain valid for non-parallel neighboring slices $t' =$ const. then for all skew-symmetric $s_{\mu\nu}$ – and thus for all s_{4k} – the terms in ε in the formulas that were written down must compensate for them. That is, one must have:

$$\left.\begin{aligned}
(\bar{x}_k - x_k) \left[Q_\alpha, \frac{\partial \bar{Q}_\beta}{\partial x_4} \right] &= 0, \\
(\bar{x}_k - x_k) \left[P_{\alpha 4}, \frac{\partial \bar{Q}_\beta}{\partial x_4} \right] &= [P_{\alpha k}, \bar{Q}_\beta], \\
(\bar{x}_k - x_k) \left[P_{\alpha 4}, \frac{\partial \bar{P}_{\beta 4}}{\partial x_4} \right] &= [P_{\alpha k}, \bar{P}_{\beta 4}] + [P_{\alpha 4}, \bar{P}_{\beta k}].
\end{aligned}\right\} \qquad (39)$$

We now go on to the verification of equations (39), for which we assume that H does not include the spatial derivatives of P. The first equation is then fulfilled in its own

right, since the bracket $\left[Q_\alpha, \frac{\partial \bar{Q}_\beta}{\partial x_4} \right]$ will then include only the function δ itself, but not its spatial derivatives; the second equation is also easy to confirm. Next, from (II) and (19), the right-hand side will become:

$$\frac{2\pi}{hc}[P_{\alpha k}, \bar{Q}_\beta] = \frac{\partial P_{\alpha k}}{\partial P_{\beta 4}} \delta(\mathfrak{r}, \bar{\mathfrak{r}}) = - \frac{\partial^2 H}{\partial P_{\beta 4} \partial \frac{\partial Q_\alpha}{\partial x_k}} \delta(\mathfrak{r}, \bar{\mathfrak{r}}),$$

while the left-hand side will yield:

$$\frac{2\pi}{hc}(\bar{x}_k - x_k)\left[P_{\alpha 4}, \frac{\partial \bar{Q}_\beta}{\partial x_4} \right] = \frac{2\pi}{hc}(\bar{x}_k - x_k)\left[P_{\alpha 4}, \frac{\partial \bar{H}}{\partial P_\beta} \right]$$

$$= (\bar{x}_k - x_k) \frac{\partial}{\partial Q_\alpha} \frac{\partial H}{\partial P_\beta} \delta + (\bar{x}_k - x_k) \sum_i \frac{\partial}{\partial \frac{\partial Q_\alpha}{\partial x_i}} \frac{\partial H}{\partial P_\beta} \frac{\partial}{\partial \bar{x}_i} \delta .$$

Since the factor of $(\bar{x}_k - x_k)$ must be compensated by a derivative of the δ-function, all that will remain as a result of partial differentiation with respect to $(\bar{x}_k - x_k)$ is:

$$- \frac{\partial}{\partial \frac{\partial Q_\alpha}{\partial x_k}} \frac{\partial H}{\partial P_\beta},$$

which agrees with the value on the right-hand side (*). The last of equations (39) requires somewhat more calculation. It then follows from (19) that the value of the right-hand side of this equation is:

$$\frac{2\pi}{hc}\{[P_{\alpha k}, \bar{P}_{\beta 4}] + [P_{\alpha 4}, \bar{P}_{\beta k}]\} = \left(- \frac{\partial P_{\alpha k}}{\partial Q_\beta} + \frac{\partial P_{\beta k}}{\partial Q_\alpha} \right) \delta + \sum_i \left(\frac{\partial P_{\alpha k}}{\partial \frac{\partial Q_\beta}{\partial x_i}} + \frac{\partial P_{\beta k}}{\partial \frac{\partial Q_\alpha}{\partial x_i}} \right) \frac{\partial}{\partial \bar{x}_i} \delta$$

$$= \left(\frac{\partial^2 H}{\partial Q_\beta \partial \frac{\partial Q_\alpha}{\partial x_k}} - \frac{\partial^2 H}{\partial Q_\alpha \partial \frac{\partial Q_\beta}{\partial x_k}} \right) \delta - \sum_i \left(\frac{\partial^2 H}{\partial \frac{\partial Q_\beta}{\partial x_i} \partial \frac{\partial Q_\alpha}{\partial x_k}} + \frac{\partial^2 H}{\partial \frac{\partial Q_\alpha}{\partial x_i} \partial \frac{\partial Q_\beta}{\partial x_i}} \right) \frac{\partial}{\partial \bar{x}_i} \delta . \qquad (*)$$

The left-hand side of the last equation (39) becomes:

(*) The commutability of the differentiations with respect to the various variables is also rigorously true for differentiations with respect to matrices, as becomes clear from the definition of that operation that was given in the previous paragrapgh.

$$\frac{2\pi}{hc}(\overline{x}_k - x_k)\left[P_{\alpha 4}, \frac{\partial \overline{P}_{\beta 4}}{\partial x_4}\right]$$

$$= \frac{2\pi}{hc}(\overline{x}_k - x_k)\left[P_{\alpha 4}, \frac{\partial H}{\partial Q_\beta}\right] + \frac{2\pi}{hc}(\overline{x}_k - x_k)\sum_i \frac{\partial}{\partial \overline{x}_i}\left[P_{\alpha 4}, \frac{\partial \overline{H}}{\partial \dfrac{\partial Q_\beta}{\partial x_i}}\right]$$

$$= \frac{\partial^2 H}{\partial Q_\beta \partial \dfrac{\partial Q_\alpha}{\partial x_k}} + (\overline{x}_k - x_k)\sum_i \frac{\partial}{\partial \overline{x}_i}\left[\frac{\partial^2 H}{\partial Q_\beta \partial \dfrac{\partial Q_\beta}{\partial x_i}}\delta + \sum_j \frac{\partial^2 H}{\partial \dfrac{\partial Q_\alpha}{\partial x_j}\partial \dfrac{\partial Q_\beta}{\partial x_i}}\frac{\partial \delta}{\partial \overline{x}_j}\right]$$

$$= \left(\frac{\partial^2 H}{\partial Q_\beta \partial \dfrac{\partial Q_\alpha}{\partial x_k}} - \frac{\partial^2 H}{\partial Q_\alpha \partial \dfrac{\partial Q_\beta}{\partial x_k}}\right)\delta + \sum_i \left(\frac{\partial}{\partial \overline{x}_i}\frac{\partial^2 H}{\partial \dfrac{\partial Q_\alpha}{\partial x_k}\partial \dfrac{\partial Q_\beta}{\partial x_i}}\right)\cdot \delta$$
$$\times \sum_i \sum_j (\overline{x}_k - x_k)\frac{\partial^2 \delta}{\partial \overline{x}_i \partial \overline{x}_j}\frac{\partial^2 H}{\partial \dfrac{\partial Q_\alpha}{\partial x_j}\partial \dfrac{\partial Q_\beta}{\partial x_i}}.$$

The first term already agrees with the corresponding one in (*), while the last one will give something non-vanishing only for $i = k$ or for $j = k$. In the first case, one will get the contribution:

$$-\sum_i \frac{\partial^2 H}{\partial \dfrac{\partial Q_\alpha}{\partial x_i}\partial \dfrac{\partial Q_\beta}{\partial x_k}}\frac{\partial \delta}{\partial \overline{x}_i},$$

while in the second case, one will get:

$$-\sum_i \frac{\partial^2 H}{\partial \dfrac{\partial Q_\alpha}{\partial x_k}\partial \dfrac{\partial Q_\beta}{\partial x_i}}\frac{\partial \delta}{\partial \overline{x}_i},$$

which agrees precisely with the terms in (*) that are multiplied by $\partial\delta/\partial\overline{x}_i$. The only term that will remain is:

$$\sum_i \left(\frac{\partial}{\partial \overline{x}_i}\frac{\partial^2 H}{\partial \dfrac{\partial Q_\alpha}{\partial x_k}\partial \dfrac{\partial Q_\beta}{\partial x_i}}\right)\cdot \delta(\mathfrak{r}, \overline{\mathfrak{r}}).$$

Its vanishing seems to be a special auxiliary condition that is required by the relativistic invariance of the C. C. R.:

$$\sum_i \left(\frac{\partial}{\partial x_i} \frac{\partial^2 H}{\partial \frac{\partial Q_\alpha}{\partial x_k} \partial \frac{\partial Q_\beta}{\partial x_i}} \right) = 0. \tag{40}$$

Indeed, it is not fulfilled for an *arbitrary* relativistically-invariant Lagrangian function L and the associated Hamiltonian function H, but only for all of them for which equation (9) is guaranteed, and which we will encounter in physical applications. The quadratic terms in the spatial derivatives of the Q_α (higher ones will not appear at all) will then always have constant coefficients. With that, we have then proved the invariance of the C. C. R. to the extent that will be necessary in the following applications.

It then follows from the form of the C. C. R. that the brackets vanish (infinitesimal character of the C. C. R.) for all world-points with space-like connecting directions (i.e., $\sum_i \Delta x_i^2 - c^2 \Delta t^2 > 0$) that are at *finite* distances from each other. It follows from closer considerations of a different kind that this state of affairs does *not* generally remain true for points on a light cone or with time-like connecting directions. In that case, the values of the brackets can also be given explicitly for points with finite, non-zero, separation distances, and only in special cases. In quantum mechanics, that situation corresponds to the fact that perhaps the coordinate $q(t)$ at time t does not commute with the coordinate at time t'; the brackets in question cannot generally be given explicitly.

It then follows further from (21) and (23) that:

$$\left.\begin{aligned} J_k &= -ic\,G_k = \int \sum_\alpha P_{\alpha 4} \frac{\partial Q_\alpha}{\partial x_k} dV, \\ J_4 &= \int \left(\sum_\alpha P_{\alpha 4} \frac{\partial Q_\alpha}{\partial x_k} - L \right) dV = \bar{H} = E \end{aligned}\right\} \tag{41}$$

define the components of a four-vector that combines the total energy and the total impulse. Those relations will then take the form:

$$\frac{\partial F}{\partial x_\nu} = \frac{2\pi}{hc} [J_\nu, F]. \tag{42}$$

Later on, we will confirm the vector character of J_k by direct calculation.

II. Presentation of the fundamental equations for the theory of electromagnetic fields and matter waves.

§ 4. Difficulties in electrodynamics, the quantization of Maxwell's equations, necessity of extra terms. We shall next seek to apply the schema for the C. C. R. of the previous chapter to the equations of vacuum electrodynamics. The physical state quantities here are the components Φ_α of the four-potential [$\Phi_i = \mathfrak{A}_i$, $\Phi_4 = i\,\Phi_0$], from which the field strengths follow by differentiation:

$$\left.\begin{aligned} F_{\alpha\beta} &= \frac{\partial \Phi_\beta}{\partial x_\alpha} - \frac{\partial \Phi_\alpha}{\partial x_\beta}, \\ F_{4k} = i\,\mathfrak{E}_k,\ (F_{23}, F_{31}, F_{12}) &= (\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{H}_3),\ F_{\alpha\beta} = -F_{\beta\alpha}. \end{aligned}\right\} \tag{43}$$

It is known that the usual **Maxwell** equations of vacuum electrodynamics:

$$\frac{\partial F_{\alpha\beta}}{\partial x_\beta} = 0 \tag{44}$$

will follow by variation from the action principle:

$$\delta \int L\, dV\, dt = 0$$

when one substitutes the expression (*):

$$L = -\tfrac{1}{4} F_{\alpha\beta} F_{\alpha\beta} = \tfrac{1}{2}(\mathfrak{E}^2 - \mathfrak{H}^2) \tag{45}$$

for the Lagrangian function L.

We now define the impulses that are canonically conjugate to the Φ_α:

$$P_{\alpha 4} = \frac{\partial L}{\partial \dfrac{\partial \Phi_\alpha}{\partial x_4}}$$

according to the prescription of the first chapter and find that:

$$P_{k4} = -F_{4k} \quad (k = 1, 2, 3), \quad P_{44} = 0. \tag{46}$$

The identical vanishing of the impulse that is conjugate to Φ_4 represents a remarkable degeneracy of the Lagrangian function of electrodynamics, and brings certain

(*) One always sums over indices that appear twice, and indeed over each index independently of the other ones. Greek indices run from 1 to 4 and Latin ones from 1 to 3. Furthermore, it must be remarked that we shall employ the **Heaviside** units for the field strengths throughout this chapter.

complications with it. Above all, the P_{k4} can no longer be specified as arbitrary spatial functions on a world-slice t = const., due to the coupling:

$$\sum_{k=1}^{3} \frac{\partial P_{k4}}{\partial x_4} = -\operatorname{div} \mathfrak{E} = 0 \tag{44′}$$

[which follows from (44) for $\alpha = 4$]. The C. C. R. of Chapter I [cf., also (30)], which would yield:

$$[\Phi_\alpha\,,\ \Phi'_\beta] = 0, \qquad [F_{4i}\,,\ F'_{4k}] = 0, \qquad [F_{4k},\ \Phi'_\alpha] = \delta_{k\alpha}\,\frac{-hc}{2\pi}\,\delta(\mathfrak{r},\,\mathfrak{r}') \tag{47}$$

in our case, are therefore not applicable with no further assumptions, and the extent to which they are compatible with the auxiliary condition (44′) must be established. We immediately find that this is not the case for the equations that were just written down, since it would follow from them that:

$$\left[\frac{\partial F_{4k}}{\partial x_k}, \Phi'_i\right] = \frac{-hc}{2\pi}\frac{\partial}{\partial x_i}\ \delta(\mathfrak{r},\,\mathfrak{r}'),$$

while, from (44′), that expression must vanish. Generally, the C. R. are useful that emerge from the given ones by elimination of the potentials by differentiation of the field strengths:

$$[F_{ik}\,,\ F'_{lm}] = 0, \qquad [F_{4i}\,,\ F'_{4k}] = 0, \qquad [F_{4k},\ F'_{lm}] = \frac{-hc}{2\pi}\left(\delta_{kl}\frac{\partial\delta}{\partial x_m} - \delta_{km}\frac{\partial\delta}{\partial x_l}\right), \tag{47}$$

or, when written three-dimensionally:

$$[\mathfrak{H}_i,\ \mathfrak{H}'_k] = 0, \quad [\mathfrak{E}_i,\ \mathfrak{E}'_k] = 0, \quad [\mathfrak{E}_1\,,\ \mathfrak{H}'_2] = -[\mathfrak{E}_2\,,\ \mathfrak{H}'_1] = \frac{hc}{2\pi i}\frac{\partial\delta}{\partial x_3}. \tag{47″}$$

It will then follow from the last of the equations that were written down that:

$$\left[\sum_k \frac{\partial F_{4k}}{\partial x_k}, F'_{lm}\right] = \frac{-hc}{2\pi}\left(\frac{\partial^2\delta}{\partial x_l\,\partial x_m} - \frac{\partial^2\delta}{\partial x_m\,\partial x_l}\right) = 0$$

as required.

The C. R. (47′) are, in fact, equivalent to the quantization of electromagnetic waves by the notion of light quanta, as one might perhaps recognize by the introduction of eigen-oscillations according to the method that was given at the end of § 2. However, the fact remains that the general schema of the C. C. R. that was developed in Chapter I can be employed in electrodynamics with no further assumptions.

It seems natural from our Ansatz regarding the relativistic treatment of the many-body problem that we should first account for the presence of particles by the

introduction of the associated calculations for matter waves. The transition from the classical theory to the quantum theory will come about in two steps: First, there is the transition from classical point mechanics to the wave equations of the quantum-mechanical one-body problem (a particle in a prescribed electromagnetic field) and the interpretation of the differential equations thus-obtained in the sense of a classical continuum theory. Secondly, there is the transition to the many-body problem, in which the four-current that the matter waves produce according to **Maxwell**'s equations can be regarded as having been generated by an electromagnetic field, and likewise since matter is also an electromagnetic field (and both fields move through the space-time manifold) will be subjected to quantization. However, that process will have the result that the fundamental difficulties that are attached to each of the relativistic theories of the quantum-mechanical one-body problem that have been posed up to now, and which originate in the possibility of having two different signs for the energy for a given impulse according to the relativistic formulation of the energy-impulse theory for a particle, will also persist in our theory and remain completely unsolved.

Here, we will use the **Dirac** theory of one particle, which accounts for spin, as a basis, and therefore, before we go into a further discussion of the complication in electrodynamics that we just spoke of, we shall next summarize the equations of that theory, to the extent that they are important to us. Four functions ψ_ρ ($\rho = 1, \ldots, 4$) will be introduced, along with four four-rowed matrices γ_μ whose elements are $\gamma^\mu_{\rho\sigma}$, and which satisfy the relations:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\, \delta_{\mu\nu}\,. \tag{48}$$

The ψ_ρ then satisfy the field equations:

$$\sum_\mu \sum_\sigma \gamma^\mu_{\rho\sigma} \left(\frac{h}{2\pi i} \frac{\partial}{\partial x_\mu} + \frac{e}{c} \Phi_\mu \right) \psi_\rho \; - i\, mc\; \psi_\rho = 0. \tag{49}$$

Likewise, the $\psi^\dagger_\rho$ satisfy the adjoint equations:

$$\sum_\mu \sum_\sigma \gamma^\mu_{\rho\sigma} \left(\frac{h}{2\pi i} \frac{\partial}{\partial x_\mu} - \frac{e}{c} \Phi_\mu \right) \psi^\dagger_\sigma + i\, mc\, \psi^\dagger_\rho \; = 0. \tag{50}$$

The electron charge has been set to $-e$ (with e positive) in this. We now assert that both equations will follow from the variational principle:

$$\delta \int L\, dV\, dt = 0$$

when one sets:

$$L = - \sum_\mu \sum_\sigma \sum_\rho \left[\gamma^\mu_{\rho\sigma} \psi^\dagger_\sigma \left(\frac{h}{2\pi i} \frac{\partial}{\partial x_\mu} + \frac{e}{c} \Phi_\mu \right) \psi_\sigma - i\, mc^2\, \psi^\dagger_\rho \psi_\rho \right], \tag{51}$$

and $\psi^\dagger$ and ψ are varied independently of each other. This is trivial for equation (49), but for equation (50), the statement will follow from this that *L* differs from:

$$L' = + \sum_\mu \sum_\sigma \sum_\rho \left[\gamma^\mu_{\rho\sigma} \left(\frac{h}{2\pi i} \frac{\partial \psi^\dagger_\sigma}{\partial x_\mu} - \frac{e}{c} \Phi_\mu \psi^\dagger_\sigma \right) \psi_\sigma + i\, mc^2\, \psi^\dagger_\rho \psi_\rho \right] \qquad (51')$$

only by terms that can be written as divergences, and which then make no contribution to the variation of $\int L\, dV\, dt$:

$$L - L' = - \sum_\mu \sum_\sigma \sum_\rho \frac{hc}{2\pi i} \gamma^\mu_{\rho\sigma} \frac{\partial}{\partial x_\mu} (\psi^\dagger_\rho \psi_\sigma) \,. \qquad (52)$$

It is important to remark that we do not need to take $\psi^\dagger$ and ψ as commuting in these calculations, and $\psi^\dagger$ can always be placed to the left of ψ. Since *L*, as well as *L'*, vanish for the non-varied field motion according to (49) and (50), the same thing will be true for the difference $L - L'$. It is then possible to identify:

$$s_\mu = (-e) \sum_{\rho,\sigma} \gamma^\mu_{\rho\sigma} \psi^\dagger_\rho \psi_\sigma \qquad (53)$$

with the current vector $\left(s_k = \frac{1}{c} i_k, s_4 = i\rho \right)$, where the factor $-e$ will naturally enter in as a result of the negative electron charge. As a result of (49) and (50), one will then have:

$$\frac{\partial s_\mu}{\partial x_\mu} = 0, \qquad (54)$$

and it will follow further from (51) and (53) that:

$$\frac{\partial L}{\partial \Phi_\mu} = s_\mu \,. \qquad (55)$$

In order to obtain this relation, each of the expressions (49) and (50) will also be multiplied by *c* in the definition of *L*.

The variational principle from which we have derived the **Dirac** field equations immediately takes on the **Hamilton** form (12), which is characterized by the independent variation of the P_α and Q_α and the linearity of the Lagrangian function in $\dot{Q}_\alpha$ $\left(\text{in } \frac{\partial \psi_\sigma}{\partial x_4}, \text{in our case} \right)$. We then have:

$$P_{\sigma 4} = \frac{\partial L}{\partial \frac{\partial \psi_\sigma}{\partial x_4}} = -\frac{hc}{2\pi i}\sum_\rho \gamma^4_{\rho\sigma}\psi^\dagger_\rho = -\frac{hc}{2\pi}\psi^*_\sigma .$$

The latter notation is justified by the fact that the expression:

$$\psi^*_\sigma = \frac{1}{i}\sum_\rho \gamma^4_{\rho\sigma}\psi^\dagger_\rho \tag{56}$$

can be chosen by means of the differential equations (49) and (50) for the function ψ^*_σ that is the complex-conjugate of ψ_σ, when the γ_α are **Hermitian** matrices, such that the expression for the particle density will be given by:

$$\frac{1}{(-e)}\frac{1}{i}\, s_4 = \sum_\rho \psi^*_\rho \psi_\rho , \tag{53′}$$

moreover. The C. R. (II) [(31), resp.] then assume the simple form:

$$\left.\begin{aligned} &[\psi_\rho, \psi'_\sigma] = 0, \\ &[\psi_\rho, \psi^{*\prime}_\sigma] = \frac{1}{i}\sum_\tau \gamma^4_{\tau\sigma}[\psi_\sigma, \psi^{\dagger\prime}_\tau] = \delta_{\rho\sigma}\delta(\mathfrak{r}, \mathfrak{r}'), \\ &[\psi^*_\rho, \psi^{*\prime}_\sigma] = [\psi^\dagger_\tau, \psi^{\dagger\prime}_\tau] = 0 \end{aligned}\right\} \tag{57}$$

here. The transformation laws for the quantities ψ_ρ and $\psi^\dagger_\rho$ under Lorentz transformations does not need to be discussed in detail here, since it will suffice to remark that they are in harmony with the general rules of § 3, and therefore the relativistic invariant of the C. R. (57) can also be considered to have been proved.

At this point, we might discuss the well-known peculiarity that the C. R. (57) represent only one of two possibilities that are both completely justified, formally speaking, and indeed one of them corresponds to the symmetric solutions of the usual quantum-mechanical equations in configuration space (i.e., Einstein-Bose statistics), while the other case corresponds to the antisymmetric solutions (i.e., Fermi-Dirac statistics) of the *q*-number relations that arise from (57) when one replaces the – signs in the brackets with + signs everywhere. If we then introduce the abbreviation:

$$[F, G]_+ = FG + GF$$

then we will get:

$$\left.\begin{aligned} &[\psi_\rho, \psi^{*\prime}_\sigma] = \frac{1}{i}\sum_\tau \gamma^4_{\tau\sigma}[\psi_\rho, \psi^{\dagger\prime}_\sigma]_+ = \delta_{\rho\sigma}\delta(\mathfrak{r}, \mathfrak{r}'), \\ &[\psi_\rho, \psi'_\sigma]_+ = [\psi^*_\rho, \psi^{*\prime}_\sigma]_+ = [\psi^\dagger_\rho, \psi^{\dagger\prime}_\sigma] = 0 \end{aligned}\right\} \tag{57a}$$

here.

It is necessary for us to go into the changes in the general equations (19) and (20) of § 2 that arise from this. It is clear that in these equations the bracket must be given the + sign *in the event that F is linear in* ψ *or* $\psi^\dagger$ *(or their derivatives).* One will see that this is the case for all brackets that appear in the invariance proof of § 3, and that this will carry over directly to the case that is presently before us. By contrast, the proof of (19) gives some insight into the combination of F_1 and F_2 into $F_1 F_2$. One then has:

$$[F_1 F_2, Q_\alpha]_- = F_1 (F_2 Q_\alpha + Q_\alpha F_2) - (F_1 Q_\alpha + Q_\alpha F_1) F_2,$$
$$[P_\alpha, F_1 F_2]_- = (P_\alpha F_1 + F_1 P_\alpha) F_2 - F_1 (P_\alpha F_2 + F_2 P_\alpha),$$

only for the usual bracket with the – sign, while $[F_1 F_2, Q_\alpha]_+$ and $[P_\alpha, F_1 F_2]_+$ cannot be reduced to the corresponding symbols for F_1 and F_2 individually. Thus, if F is a *bilinear form* in $\psi^\dagger$, $\partial\psi^\dagger / \partial x_i$, ψ, $\partial\psi / \partial x_i$, in which the $\psi^\dagger$ always stands to the left of the ψ, then the *usual brackets* will be taken with the – signs. Therefore, the C. R. (21) and (23) [(42), resp.] for energy and impulse are also valid with the usual brackets, which is of decisive significance for the feasibility of the theory.

One sees that the two types of solutions – namely, Einstein-Bose statistics, on the one hand, and the exclusion principle (forbidden equivalence), on the other – seem to still be completely justified formally from the standpoint of the quantization of the waves and the relativistically-invariant treatment of the many-body problem, as well, and a satisfactory explanation for the preference of the second possibility by nature can therefore not be given (†). Special ψ-functions are introduced for protons, as well as for electrons, which commute with the latter, moreover. However, since the equations for these two read completely the same – apart from the fact that – e gets replaced with + e and m with M – we do not need to go into that further.

We can now consider the interaction of matter waves with the electromagnetic field, which is produced by the variational principle:

$$\delta \int L \, dV \, dt = 0,$$

when one replaces L with the sum of the radiation part $L^{(s)}$ [equation (45)] and the matter part $L^{(m)}$ [equation (51)]. As a result of (55), it follows from this that:

$$\frac{\partial F_{\alpha\beta}}{\partial x_\beta} = s_\alpha,$$

when the expression (53) is substituted for s_α. Physically, this means that this current vector is definitive for not only the effect of an external field on matter, but also conversely serves to generate a field. However, this once more raises the complication of exhibiting the C. R. that is in harmony with the condition:

(†) The information that **P. Jordan** gave in regard to this in Ergebisse der exakten Naturwissenschaftern **7** (1929), 206 was incorrect. In neither of the two cases would a zero-point energy appear for matter waves, moreover.

$$\text{div } \mathfrak{E} = \frac{1}{i} s_4 = \rho = (-e) \sum_\sigma \psi_\sigma^* \psi_\sigma \, .$$

Namely, if we define:

$$[\text{div } \mathfrak{E}, \, \psi_\sigma^{*\prime}] = (-e) \, \delta(\mathfrak{r}, \mathfrak{r}') \, \psi_\sigma^{*\prime},$$

which applies to (57), as well as (57a), then it will follow by integrating (x_i) over a finite volume that contains the point (x_i') that:

$$\left[\int \mathfrak{E}_n df, \psi_\sigma^{*\prime} \right] = (-e) \, \psi_\sigma^{*\prime} .$$

However, that means that the electric field strength $\mathfrak{E}$ cannot commute with the matter field ψ for *finite* distances between the spatial points (x_i) and (x_i'), as well. Devising a theory with such non-infinitesimal C. R. seems practically hopeless, especially since the proof of the relativistic invariance of such C. R. might be linked with great complications.

However, it is possible to avoid that complication by a formal trick that consists in adding small extra terms to the Lagrangian function $L^{(s)}$ of electrodynamics, that likewise contain only first derivatives of the potential Φ_α and do not affect the linearity of the field equations, but which imply that P_{44} no longer vanishes identically. One then counts these altered equations with the canonical C. R. and then first lets the coefficients of the extra terms converge to zero in the physical applications in the final results. The simplest possibility for such an extra terms is expressed by the Ansatz:

$$-L^{(s)} = \frac{1}{4} F_{\alpha\beta} F_{\alpha\beta} - \frac{\varepsilon}{2} (\text{Div } \Phi)^2, \qquad \text{Div } \Phi = \sum_\alpha \frac{\partial \Phi_\alpha}{\partial x_\alpha} . \tag{58}$$

Yet another possibility is:

$$-L'^{(s)} = \frac{1}{4} (1 + \varepsilon) F_{\alpha\beta} F_{\alpha\beta} - \frac{\varepsilon}{2} \frac{\partial \Phi_\alpha}{\partial x_\beta} \frac{\partial \Phi_\alpha}{\partial x_\beta} .$$

It can then be easily shown that the difference between the variations of the integrals of L' and L vanish identically. The C. R. would generally be different in the two cases, but it can be assumed that all physical final results will be the same in the limit as $\varepsilon \to 0$. The Ansatz (58) for L shall then be retained in what follows. The modified **Maxwell** equations then read:

$$\frac{\partial F_{\alpha\beta}}{\partial x_\beta} + \varepsilon \frac{\partial}{\partial x_\alpha} (\text{Div } \Phi) = (1 + \varepsilon) \frac{\partial}{\partial x_\alpha} (\text{Div } \Phi) - \Box \Phi_\alpha = s_\alpha . \tag{59}$$

Moreover, in place of (46), the impulses that are conjugate to the Φ_α are now:

$$P_{k4} = -F_{4k}, \qquad P_{44} = \varepsilon \,\mathrm{Div}\,\Phi, \tag{60}$$

and the C. C. R. will become:

$$\left.\begin{aligned} &[\Phi_\alpha, \Phi'_\beta] = 0, && [F_{4i}, F'_{4k}] = 0, && [F_{4i}, \mathrm{Div}\,\Phi] = 0, \\ &[F_{4i}, \Phi'_\alpha] = \delta_{i\alpha} \frac{-hc}{2\pi} \delta(\mathfrak{r}, \mathfrak{r}'), && && [\mathrm{Div}\,\Phi, \Phi'_4] = \frac{1}{\varepsilon} \frac{hc}{2\pi} \delta(\mathfrak{r}, \mathfrak{r}'). \end{aligned}\right\} \tag{61}$$

One further sees that the previously-written C. R. (47) remain correct, but are extended by ones that contain the $\partial\Phi_4 / \partial x_4$; the equations (47′) or (47″) then remain true, as well. Furthermore, equation (59) is now no longer affected for $\alpha = 4$, since the second temporal derivative of Φ_4 enters into it, such that now the Φ_α and the conjugate $P_{\alpha 4}$ can, in fact, be given for a certain time points as arbitrary spatial functions. Let it be remarked that the complete invariance of the theory under variations of the potentials that leave the field strengths unchanged, namely, ones for which:

$$\Phi'_\alpha = \Phi_\alpha + \frac{\partial \lambda}{\partial x_\alpha},$$

no longer exists now, but this invariance will probably remain when one subjects the function λ to the auxiliary condition:

$$\Box \lambda = \text{const.}$$

The relativistic invariance of C. R. (61) is also proved rigorously for $\varepsilon \neq 0$ by the considerations of § 2.

The essential basic assumptions of our theory are contained in the expression (58) and (51) for the Lagrangian function of radiation and matter, and the associated C. C. R. (61) and (57) or (57a). We extend them by writing down the expressions for the **Hamiltonian** functions. According to (60) and (58), one will have:

$$\begin{aligned} H^{(s)} = P_{\nu 4} \frac{\partial \Phi_\nu}{\partial x_4} - L^{(s)} &= -F_{4k} \frac{\partial \Phi_k}{\partial x_4} + \varepsilon \,\mathrm{Div}\,\Phi \frac{\partial \Phi_\nu}{\partial x_4} + \tfrac{1}{4} F_{ik} F_{ik} - (\mathrm{Div}\,\Phi)^2 \\ &= -\tfrac{1}{4} F_{4k} F_{4k} + \tfrac{1}{4} F_{ik} F_{ik} - F_{4k} \frac{\partial \Phi_k}{\partial x_4} + \varepsilon (\mathrm{Div}\,\Phi)^2 - \varepsilon \,\mathrm{Div}\,\Phi \frac{\partial \Phi_k}{\partial x_k} \end{aligned} \tag{58′}$$

for the radiation part, and according to (51) and (56), one will have:

$$\begin{aligned} H^{(m)} = -\frac{hc}{2\pi i} \gamma^4_{\rho\sigma} \psi^\dagger_\rho \frac{\partial \psi_\sigma}{\partial x_4} - L^{(m)} &= +\frac{hc}{2\pi i} \gamma^4_{\rho\sigma} \psi^\dagger_\rho \frac{\partial \psi_\sigma}{\partial x_k} - imc^2 \psi^\dagger_\sigma \psi_\sigma + e \gamma^\mu_{\rho\sigma} \psi^\dagger_\rho \psi_\sigma \Phi_\mu \\ &= +\frac{hc}{2\pi i} \alpha^k_{\rho\sigma} \psi^*_\rho \frac{\partial \psi_\sigma}{\partial x_k} + mc^2 \alpha^\nu_{\rho\sigma} \psi^\dagger_\sigma \psi_\sigma + e \alpha^k_{\rho\sigma} \psi^*_\rho \psi_\sigma \Phi_\mu + ei \psi^*_\rho \psi_\sigma \Phi_4 \end{aligned} \tag{51′}$$

for the matter part, if one sets:

$$\alpha^k = i\,\gamma^4\gamma^k, \quad \alpha^4 = \gamma^4, \qquad \text{with} \quad \alpha^\mu\,\alpha^\nu + \alpha^\nu\,\alpha^\mu = 2\,\delta_{\nu\mu}\,. \tag{48′}$$

Furthermore, one must add the statement that is contained in the C. C. R. for the total field that all electromagnetic field quantities (potential, field strengths, and Div Φ) *commute* with all field quantities of the matter waves (ψ_ρ, $\psi_\rho^\dagger$) (for the same time-point). This fact, which includes an essential difference between our theory and the theory of **Jordan** and **Klein**, which is valid in the limit $c \to \infty$, implies a great simplification in the calculations. On the other hand, the splitting of the Lagrangian function into two summands that are completely independent logically and correspond to the matter and light waves (if one also considers protons then there will be three independent summands) corresponds to the provisional character of our theory and can probably be modified later in favor of a unified conception of all genera of wave fields.

§ 5. On the relationship between the equations that were presented here and the previous Ansätze for the quantum electrodynamics of charge-free fields. In a previous paper by **Jordan** and **Pauli** (*), the C. R. of electrodynamics were formulated in the special case of the absence of charged particles from a somewhat different standpoint, for which the four-dimensional integrals (over space and time) were considered with brackets in integrands, which might then be referred to as the four-dimensional standpoint, for that reason. There, a Δ-function was defined by the relation:

$$\left.\begin{aligned} &\int_{V_4} f(x,\ldots,t)\,\Delta(x,\ldots,t)\,dV\,dt \\ &= \int_{V_3^+} f(x_1,x_2,x_3,ct=-r)\,\frac{1}{r}\,dx_1dx_2dx_3 - \int_{V_3^-} f(x_1,x_2,x_3,ct=r)\,\frac{1}{r}\,dx_1dx_2dx_3, \end{aligned}\right\} \tag{62}$$

which was assumed to be valid for any function f (x, …, t), such that this function presents a singularity on the light cone $ct = -r$ and $ct = +r$, and indeed with the opposite signs for the past and future. In that sense, one can also set:

$$\Delta\,(x,\ldots,t) = \frac{1}{r}\,[\delta(r+ct) - \delta(r-ct)], \tag{62′}$$

if one understands δ to once more mean the ordinary δ-function.

One now asks what follows from the relation (62′) for the Δ-function when we always introduce only three-dimensional integrals over space $t = 0$, instead of four-dimensional integrals over space-time, consistent with the standpoint that we have assumed here.

We then first obtain:

$$\Delta = 0 \quad \text{for integrals over } t = 0 \tag{63}$$

(*) **P. Jordan** and **W. Pauli**, Zeit. Phys. **47** (1928), 151. This section is not required for an understanding of what follows.

from (62). The same thing is then true for the spatial derivatives of Δ and the second time derivatives, since they can be expressed in terms of the spatial ones by:

$$\sum_{\alpha=1}^{4} \frac{\partial^2 \Delta}{\partial x_\alpha^2} = 0.$$

However, something interesting happens when we specialize $\frac{1}{c}\frac{\partial \Delta}{\partial t}$ to three-dimensional integrals using (62′). We then get:

$$\frac{1}{c}\frac{\partial \Delta}{\partial t} = \delta'(r),$$

which can, however, be converted further. Namely, let $f(x_1, x_2, x_3)$ be an arbitrary function of the three spatial coordinates, and evaluate:

$$\int f \frac{2}{r}\delta'(r)\, dx_1\, dx_2\, dx_3 .$$

We introduce polar coordinates and set the function f that is integrated over the angle equal to $\Phi(r)$:

$$\Phi(r) = \int f\, d\Omega, \qquad \text{so} \qquad \Phi(0) = 4\pi f(0),$$

so:

$$\int f \frac{2}{r}\delta'(r)\, dV = \int_0^\infty \Phi(r)\, 2\delta'(r)\, r\, dr .$$

If we think of $\Phi(r)$ as being an even function for negative r (such that $\Phi(r)$ remains continuous for $r = 0$ and $r\,\Phi(r)$ still has a continuous derivative for $r = 0$) then, since $\delta'(r)$ is an odd function, we can also write:

$$\int f \frac{2}{r}\delta'(r)\, dV = \int_{-\infty}^{+\infty} \Phi\, \delta'(r)\, r\, dr = -\frac{d}{dr}(r\Phi)\bigg|_{r=0} = -\Phi(0) = -4\pi f(0).$$

Since the integral that is being calculated has the value $-4\pi f(0)$ for all $f(x_1, x_2, x_3)$, we can say that we have:

$$\frac{1}{c}\frac{\partial \Delta}{\partial t} = -4\pi\, \delta(x_1, x_2, x_3) \qquad \text{for integrals over } t = 0. \tag{64}$$

We can directly adapt the C. R. for the field strengths in the paper of **Jordan** and **Pauli**, namely:

$$[\mathfrak{E}_i, \mathfrak{E}'_k] = [\mathfrak{H}_i, \mathfrak{H}'_k] = \frac{ihc}{8\pi^2}\left(\frac{\partial^2}{\partial x_i\, \partial x_k} - \delta_{ik}\frac{\partial^2}{c^2 \partial t^2}\right) \Delta(P - P),$$

$$[\mathfrak{E}_1,\ \mathfrak{E}'_2] = -[\mathfrak{H}_1,\ \mathfrak{E}'_2] = \frac{ihc}{8\pi^2}\frac{\partial}{\partial x_{3\,k}}\frac{1}{c}\frac{\partial}{\partial t}\ \Delta(P-P),$$

into the ones (47″) that are employed here, which actually emerge from the ones that were written down by means of (63) and (64).

The four-dimensional viewpoint has the advantage over the three-dimensional one that it makes the relativistic invariance of the C. R. immediately obvious, while that must be verified by the somewhat circumstantial methods of § 2 for the three-dimensional viewpoint that is assumed here. Nevertheless, we have many grounds for believing that the three-dimensional viewpoint should be preferred in the formulation of the C. R. First of all, from the four-dimensional viewpoint, the fact that the generalization to other waves than light waves is not entirely clear is a result of the fact that not just neighboring points will contribute to the integral over the bracket here. Even for force-free matter waves, it would be a result of the dependency of their phase velocity on the wave length that a four-dimensional integral over the interior of the light cone would appear in the definition of the associated Δ-function, along with the three-dimensional integral over that light cone. For matter waves in an external electromagnetic field, the analogue of the Δ-function can, in fact, be defined on the basis of its properties, but it can no longer be calculated explicitly, in general. Finally, in all physical applications, it is always just the three-dimensional integral over t = const., that is in question, such that the three-dimensional viewpoint also has a closer connection to the physical content of the theory than the four-dimensional one.

§ 6. Differential and integral form of the conservation law for the energy and impulse of the total wave field. In Chapter I, it was shown how one could always give temporally-constant volume integrals for the total energy and total impulse [see equations (7), (13), and (41)], from the canonical form of the field equations, namely:

$$J_\mu = \int \left(\sum_\alpha P_{\alpha 4} \frac{\partial Q_\alpha}{\partial x_\alpha} - \delta_{\mu 4} L \right) dV, \tag{41′}$$

in which the components of the four-vector J_ν for ν = 1, 2, 3, which represent the components of the impulse, multiplied by – ic, while $J_4 = \bar{H}$ determines the total energy. However, it was not shown there whether energy and impulse can also be conserved in the differential form:

$$\frac{\partial T_{\mu\nu}}{\partial x_\nu} = 0, \tag{65}$$

in which $T_{\mu\nu}$ in represents the tensor of stress and energy and impulse density in a known way, from which the constancy of:

$$J_\mu = \int T_{\mu 4}\, dV \tag{66}$$

then follows. Here, we would like to show that this is, in fact, the case, and that indeed the integrands in (41′) and (66) do not agree, but the values of the integrals can probably be always assumed to vanish when they are resting (i.e., two-dimensional) outer surface integrals. However, it should be emphasized that one should expect conservation laws for only the total field, which is composed of the electromagnetic and matter waves.

We begin with the discussion of the contribution of the matter waves to the energy tensor. It is well-known, and was calculated most completely by **Tetrode** (*). The calculations in question shall again be sketched out here, but with consideration given to the non-commutation of the factors that are present. Starting with the expression (51) for the matter part of the Lagrangian function and the relation (55), we would like to see how the expression $F_{\mu\nu}\, s_\nu$ for the Lorentz force can be converted into a four-dimensional divergence. One then has:

$$F_{\mu\nu}\, s_\nu = \left(\frac{\partial \Phi_\nu}{\partial x_\mu} - \frac{\partial \Phi_\mu}{\partial x_\nu} \right) s_\nu = \frac{\partial \Phi_\nu}{\partial x_\mu} s_\nu - \frac{\partial}{\partial x_\nu} (\Phi_\mu s_\nu),$$

in which, from what was said already, use was made of the following important relation (54), viz.:

$$\frac{\partial s_\nu}{\partial x_\nu} = 0, \tag{54}$$

which follows from the field equations for matter waves. If we further make note of (55) then it will follow that:

$$F_{\mu\nu}\, s_\nu = \frac{\partial L}{\partial \Phi_\nu} \frac{\partial \Phi_\nu}{\partial x_\mu} - \frac{\partial}{\partial x_\nu} (\Phi_\mu s_\nu). \tag{67}$$

From this, it is important that the current components must commute with all electromagnetic field quantities, since they are expressible in terms of only ψ and $\psi^\dagger$, such that the sequence of factors does not enter into (67). On the same grounds (i.e., the commutation of $\partial \Phi_\nu / \partial x_\mu$ with ψ and $\psi^\dagger$), we can set:

$$\frac{\partial L}{\partial x_\mu} = \frac{\partial L}{\partial \Phi_\nu} \frac{\partial \Phi_\nu}{\partial x_\mu} + \frac{\partial \psi_\rho^\dagger}{\partial x_\mu} \frac{\partial L}{\partial \psi_\rho^\dagger} + \frac{\partial^2 \psi_\rho^\dagger}{\partial x_\mu \partial x_\nu} \frac{\partial L}{\partial \dfrac{\partial \psi_\rho^\dagger}{\partial x_\nu}} + \frac{\partial L}{\partial \psi_\rho} \frac{\partial \psi_\rho}{\partial x_\mu} + \frac{\partial L}{\partial \dfrac{\partial \psi_\rho}{\partial x_\nu}} \frac{\partial^2 \psi_\rho}{\partial x_\nu \partial x_\mu},$$

in which one must be careful that the factors that contain $\psi^\dagger$ must always appear to the left of the factors that contain ψ. One finally has the field equations:

$$\frac{\partial L}{\partial \psi_\rho^\dagger} = \frac{\partial}{\partial x_\nu} \frac{\partial L}{\partial \dfrac{\partial \psi_\rho^\dagger}{\partial x_\nu}}, \qquad \frac{\partial L}{\partial \psi_\rho} = \frac{\partial}{\partial x_\nu} \frac{\partial L}{\partial \dfrac{\partial \psi_\rho}{\partial x_\nu}},$$

(*) **H. Tetrode**, Zeit. Phys., *loc. cit.*

which emerge from the vanishing of the variation of L, such that we finally get from (67):

$$F_{\mu\nu}\, s_\nu = -\frac{\partial \psi_\rho^\dagger}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\frac{\partial L}{\partial \frac{\partial \psi_\rho^\dagger}{\partial x_\nu}} - \frac{\partial^2 \psi_\rho^\dagger}{\partial x_\mu\, \partial x_\nu}\frac{\partial L}{\partial \frac{\partial \psi_\rho^\dagger}{\partial x_\nu}} - \left(\frac{\partial}{\partial x_\nu}\frac{\partial L}{\partial \frac{\partial \psi_\rho}{\partial x_\nu}}\right)\frac{\partial \psi_\rho}{\partial x_\mu}$$
$$- \frac{\partial L}{\partial \frac{\partial \psi_\rho}{\partial x_\nu}}\frac{\partial^2 \psi_\rho}{\partial x_\nu\, \partial x_\mu} + \frac{\partial L}{\partial x_\mu} + \frac{\partial}{\partial x_\nu}(\Phi_\mu s_\nu)\,;$$

that is, the energy-impulse tensor $T_{\mu\nu}^{(m)}$ of matter, which is defined by:

$$T_{\mu\nu}^{(m)} = \frac{\partial \psi_\rho^\dagger}{\partial x_\mu}\frac{\partial}{\partial x_\nu}\frac{\partial L}{\partial \frac{\partial \psi_\rho^\dagger}{\partial x_\nu}} + \frac{\partial L}{\partial \frac{\partial \psi_\rho}{\partial x_\nu}}\frac{\partial \psi_\rho}{\partial x_\mu} - \delta_{\mu\nu} L + \Phi_\mu s_\nu\,, \tag{68}$$

satisfies the relations:

$$F_{\mu\nu}\, s_\nu = -\frac{\partial T_{\mu\nu}^{(m)}}{\partial x_\nu}\,. \tag{69}$$

The first term in (68) drops out, in turn, by the choice (51) of L, so L itself can also be set equal to zero then, and it is interesting to point out that its derivation by the relation (69) will also remain valid when one replaces L with L' [see equation (51′)] in (68), or with $(L + L')/2$. With the expression (51) for L, we get from (68):

$$-T_{\mu\nu}^{(m)} = \frac{hc}{2\pi i}\sum_{\rho,\sigma}\gamma_{\sigma\rho}^\nu \psi_\sigma^\dagger \frac{\partial \psi_\rho}{\partial x_\mu} - \Phi_\mu s_\nu = \sum_{\rho,\sigma}\gamma_{\sigma\rho}^\nu \psi_\sigma^\dagger\left(\frac{hc}{2\pi i}\frac{\partial}{\partial x_\mu} + e\Phi_\mu\right)\psi_\rho\,. \tag{70}$$

This expression for the tensor $T_{\mu\nu}^{(m)}$ is not symmetric in μ and ν. As **Tetrode** has shown, an expression for the energy tensor that is symmetric in μ and ν can also be used. However, since it will lead to the same integral value for energy and impulse as the asymmetric expressions (68) or (70), we do not need to go into that point any further here (*).

We now go on to the part of the energy-impulse tensor that the electromagnetic field contributes. Moreover, we will demand that this part $T_{\mu\nu}^{(s)}$, along with the expression (54):

$$s_\nu = \frac{\partial F_{\nu\rho}}{\partial x_\rho} + \varepsilon\frac{\partial}{\partial x_\nu}(\mathrm{Div}\,\Phi) \tag{59}$$

(*) By contrast, it is necessary to employ the expression for $T_{\mu\nu}^{(m)}$ that is symmetric in μ and ν in the calculation of the total angular impulse.

for the current, must fulfill the relation:

$$\frac{\partial T_{\mu\nu}^{(s)}}{\partial x_\nu} = F_{\mu\nu} s_\nu . \tag{71}$$

If the term that is multiplied by ε were not present then $- T_{\mu\nu}^{(s)}$ could be taken to be the **Maxwell** tensor:

$$S_{\mu\nu} = F_{\mu\rho} F_{\nu\rho} - \tfrac{1}{4} F_{\rho\sigma} F_{\rho\sigma} \delta_{\mu\nu} \tag{72}$$

in a known way, since, with consideration given to the identity (43), it will fulfill the identity:

$$\frac{\partial S_{\mu\nu}}{\partial x_\nu} = - F_{\mu\nu} \frac{\partial F_{\nu\rho}}{\partial x_\rho} . \tag{73}$$

We must then look for extra terms that are proportional to ε that are counted in the second term in the expression for the current. However, in this we will keep in mind the fact that according to the equation:

$$\frac{\partial s_\nu}{\partial x_\nu} = 0,$$

which follows from the field equations for matter waves as a result of (59), one will have:

$$\Box \ \mathrm{Div}\, \Phi = \sum_{\mu,\nu} \frac{\partial^2}{\partial x_\mu^2} \frac{\partial \Phi_\nu}{\partial x_\nu} = 0. \tag{74}$$

We then assert that the extra term $\varepsilon \Sigma_{\mu\nu}$, with:

$$\Sigma_{\mu\nu} = \overline{\Phi_\nu \frac{\partial}{\partial x_\mu} (\mathrm{Div}\, \Phi)} + \overline{\Phi_\mu \frac{\partial}{\partial x_\nu} (\mathrm{Div}\, \Phi)} - \overline{\frac{\partial}{\partial x_\rho} (\Phi_\rho \mathrm{Div}\, \Phi)}\, \delta_{\mu\nu} + \frac{1}{2} (\mathrm{Div}\, \Phi)^2 \delta_{\mu\nu} , \tag{75}$$

that gets added to (72) will give what is desired. (The overbars mean symmetrization, due to the non-commutation of factors.) We then obtain:

$$\begin{aligned} \frac{\partial \Sigma_{\mu\nu}}{\partial x_\nu} = {} & \overline{(\mathrm{Div}\, \Phi) \frac{\partial}{\partial x_\mu} \mathrm{Div}\, \Phi} + \overline{\Phi_\nu \frac{\partial^2}{\partial x_\nu \partial x_\mu} (\mathrm{Div}\, \Phi)} + \overline{\frac{\partial \Phi_\mu}{\partial x_\nu} \frac{\partial}{\partial x_\rho} (\mathrm{Div}\, \Phi)} - \overline{\Phi_\mu \Box\, \mathrm{Div}\, \Phi} \\ & - \overline{\frac{\partial \Phi_\nu}{\partial x_\mu} \frac{\partial}{\partial x_\nu} (\mathrm{Div}\, \Phi)} - \overline{\mathrm{Div}\, \Phi \frac{\partial}{\partial x_\mu} \mathrm{Div}\, \Phi} - \overline{\Phi_\nu \frac{\partial^2}{\partial x_\nu \partial x_\mu} (\mathrm{Div}\, \Phi)} + \overline{\frac{\partial \Phi_\mu}{\partial x_\nu} \frac{\partial}{\partial x_\mu} (\mathrm{Div}\, \Phi)} . \end{aligned}$$

As a result of (74), all that remains in this is:

$$\frac{\partial \Sigma_{\mu\nu}}{\partial x_\nu} = \left(\frac{\partial \Phi_\mu}{\partial x_\nu} - \frac{\partial \Phi_\nu}{\partial x_\mu} \right) \frac{\partial}{\partial x_\nu} (\mathrm{Div}\, \Phi) = - F_{\mu\nu} \frac{\partial}{\partial x_\nu} (\mathrm{Div}\, \Phi), \tag{76}$$

so when one employs the expression (58) for $L^{(s)}$, one will get:

$$- T_{\mu\nu}^{(s)} = S_{\mu\nu} + \varepsilon \Sigma_{\mu\nu}$$

$$= F_{\mu\rho} F_{\nu\rho} + \varepsilon \overline{\Phi_\nu \frac{\partial}{\partial x_\mu} (\mathrm{Div}\, \Phi)} + \varepsilon \overline{\Phi_\mu \frac{\partial}{\partial x_\nu} (\mathrm{Div}\, \Phi)} - \varepsilon \overline{\frac{\partial}{\partial x_\rho} (\Phi_\rho \mathrm{Div}\, \Phi)}\ \delta_{\mu\nu} + L^{(s)}\, \delta_{\mu\nu}, \tag{77}$$

which is, in fact, relation (71). Using (59), one can convert this into:

$$- T_{\mu\nu}^{(s)} = F_{\nu\rho} \frac{\partial \Phi_\rho}{\partial x_\mu} - \frac{\partial}{\partial x_\rho} (\Phi_\mu F_{\nu\rho}) - \varepsilon \frac{\partial \Phi_\nu}{\partial x_\mu} \mathrm{Div}\, \Phi + \Phi_\mu s_\nu$$

$$+ \varepsilon \overline{\frac{\partial}{\partial x_\mu} (\Phi_\nu \mathrm{Div}\, \Phi)} - \varepsilon \overline{\frac{\partial}{\partial x_\rho} (\Phi_\rho \mathrm{Div}\, \Phi)}\ \delta_{\mu\nu} + L^{(s)}\, \delta_{\mu\nu}. \tag{77′}$$

It is now essential that the terms in $\Phi_\mu s_\nu$ *must cancel precisely when one adds* (77′) *and* (68). One obtains the difference as the sum of the integrands of (41′):

$$T_{\mu\nu} = T_{\mu\nu}^{(s)} + T_{\mu\nu}^{(s)}, \tag{78}$$

which, in fact, fulfills the conservation law (65), from (69) and (71), and with consideration to the value (60) of the impulse of the electromagnetic field, one gets, for $\nu = 4$:

$$T_{\mu 4}^{(s)} + T_{\mu 4}^{(s)} - \left(\sum_\alpha P_{\alpha 4} \frac{\partial Q_\alpha}{\partial x_\mu} - L\, \delta_{\mu 4} \right)$$

$$= + \frac{\partial}{\partial x_\rho} (\Phi_\mu F_{4\rho}) - \varepsilon \frac{\partial}{\partial x_\mu} (\Phi_4 \mathrm{Div}\, \Phi) + \varepsilon \frac{\partial}{\partial x_\rho} (\Phi_\rho \mathrm{Div}\, \Phi)\ \delta_{\mu 4}.$$

However, these expressions contain only *spatial* derivatives, so they will vanish upon integrating over spatial volumes. Since $F_{44} = 0$, this is trivial for the first term, and for the other ones, it will be immediately obvious for $\mu = 1, 2, 3$, while for $\mu = 4$, what will remain is:

$$- \varepsilon \frac{\partial}{\partial x_\mu} (\Phi_4 \mathrm{Div}\, \Phi) + \varepsilon \frac{\partial}{\partial x_\rho} (\Phi_\rho \mathrm{Div}\, \Phi) = \varepsilon \sum_{k=1}^{3} \frac{\partial}{\partial x_k} (\Phi_k \mathrm{Div}\, \Phi).$$

With that, the desired proof is completed, and the connection between the differential form and the canonical integral form of the conservation laws is exhibited in our case. At

the same time, the vector character of J_k is also verified once more. However, it must be emphasized that the given expression for the electromagnetic part of the energy and impulse contains a zero-point energy for the radiation, as well as a self-energy for the electrons and protons, which does not correspond to reality (*). The extent to which this fundamental gap in the theory that is being developed here does not, nevertheless, affect the calculations in special physical problems will be discussed in the following chapter.

III. Approximation methods for the integration of the equations and physical applications.

§ 7. Presentation of the difference equations for the probability amplitudes. The calculations of this chapter will be based upon the **Hamiltonian** function H whose radiation and matter wave parts are given by (58′) and (51′). For this, it becomes convenient to introduce a real t according to $x_4 = ict$, and to correspondingly also set $\Phi_4 = i\,\Phi_0$. Furthermore, with regard to the applications, we shall again go from the **Heaviside** units to the ordinary ones, such that the Φ_μ will be replaced with $\frac{1}{\sqrt{4\pi}}\Phi_\mu$, while s_μ is replaced with $\sqrt{4\pi}\,s_\mu$. Finally, it is convenient to introduce potentials Φ^0_μ (c-numbers) of external, "applied" forces whose sources are not counted with the system. For example, due to the large mass of the atomic nuclei, it will often be convenient to consider the forces that originate with them in the Φ^0_μ, and thus, to neglect the reaction.

With the introduction of the impulses that are conjugate to the potentials Φ_μ (when measured in ordinary units) according to:

$$\left.\begin{aligned} \Pi_k &= -\frac{1}{4\pi c}\mathfrak{E}_k = +\frac{1}{4\pi c}\left(\frac{1}{c}\frac{\partial \Phi_k}{\partial t} + \frac{\partial \Phi_0}{\partial x_k}\right), \\ \Pi_0 &= \frac{1}{4\pi c}\mathfrak{E}_k = \frac{\varepsilon}{4\pi c}\left(\frac{\partial \Phi_k}{\partial x_k} + \frac{1}{c}\frac{\partial \Phi_0}{\partial t}\right), \end{aligned}\right\} \qquad (60')$$

such that the C. R. read:

$$[\Pi_\rho\,,\ \Phi'_\sigma\,] = \frac{h}{2\pi i}\,\delta_{\rho\sigma}\,\delta(\mathfrak{r},\,\mathfrak{r}'), \qquad (61')$$

the radiation part of the Hamiltonian function will assume the form:

(*) It is known that **Klein** and **Jordan** were able to eliminate certain factors in the expression for energy by reordering in their theory of the self-energy of the electron. That reordering is equivalent to the addition of certain terms that include the bracket $[\Phi_k\,,\,\psi]$ to the energy density. In our theory, in which Φ_k commutes with ψ at the same time point, it seems that no analogue to the **Klein-Jordan** trick exists that would be quite so simple.

$$\bar{H}^{(s)} = \int dV \left\{ \frac{1}{16\pi} \left(\frac{\partial \Phi_i}{\partial x_k} - \frac{\partial \Phi_k}{\partial x_i} \right)^2 + 2\pi c^2 \Pi_k^2 - e \frac{\partial \Phi_0}{\partial x_k} \Pi_k + \frac{2\pi c^2}{\varepsilon} \Pi_0^2 - e \frac{\partial \Phi_k}{\partial x_k} \Pi_0 \right\}. \quad (79)$$

Correspondingly, one then gets the matter part of the Hamiltonian in the form of:

$\bar{H}^{(m)} =$

$$\int dV \left[\frac{hc}{2\pi i} \alpha^k_{\rho\sigma} \psi^*_\rho \frac{\partial \psi_\sigma}{\partial x_x} + mc^2 \alpha^k_{\rho\sigma} \psi^*_\rho \psi_\sigma + e(\Phi^0_k + \Phi_k) \alpha^k_{\rho\sigma} \psi^*_\rho \psi_\sigma - e(\Phi^0_0 + \Phi_0) \psi^*_\rho \psi_\sigma \right]. \quad (79a)$$

The associated C. R. are determined from (57) and (57a), as before:

a) Einstein-Bose statistics:

$$\psi_\rho(\mathfrak{r})\ \psi^*_\sigma(\mathfrak{r}') - \psi^*_\sigma(\mathfrak{r}')\ \psi_\rho(\mathfrak{r}) = \delta_{\rho\sigma}\, \delta(\mathfrak{r}, \mathfrak{r}'). \quad (57)$$

b) Exclusion principle (forbidden equivalence):

$$\psi_\rho(\mathfrak{r})\ \psi^*_\sigma(\mathfrak{r}') + \psi^*_\sigma(\mathfrak{r}')\ \psi_\rho(\mathfrak{r}) = \delta_{\rho\sigma}\, \delta(\mathfrak{r}, \mathfrak{r}'). \quad (57a)$$

In order to solve the quantum-theoretical problem that is defined by equation (79a), one conveniently develops the ψ (Φ, resp.) in a suitable orthogonal system. The classical solutions of the field equations that one obtains when one deletes the interaction terms (and thus, the terms of the form $\psi^* \alpha \psi \Phi$) from (79a) lend themselves to that development in a natural way.

We therefore first assume that the **Dirac** equations for the matter waves are integrated for the potentials Φ^0_μ, which we assume are constant in time. [When the Φ^0_μ contain a component that is variable in time, one can split it off and, if convenient, treat it together with the interaction terms of (79).] Each eigenvalue E^s of the "unperturbed" problem that was solved belongs to a system of eigenfunctions (ρ = 1, 2, 3, 4), which is normalized according to the equation:

$$\int dV\, u^{*r}_\rho u^s_\rho = \delta_{rs}\,. \quad (80)$$

One further has the "inverse" orthogonality relations:

$$\sum_s u^{*s}_\rho(\mathfrak{r}) u^s_\sigma(\mathfrak{r}') = \delta_{\rho\sigma}\, \delta(\mathfrak{r}, \mathfrak{r}'). \quad (80')$$

We then set:

$$\psi_\rho = \sum_s a_s\, u^s_\rho\,, \qquad \psi^*_\rho = \sum_s a^*_s\, u^{*s}_\rho\,. \quad (81)$$

The *a* quantities satisfy the C. R.:

$$\left.\begin{aligned} &\text{Bose - Einstein statistics:} \quad a_s a_t^* - a_t^* a_s = \delta_{st}, \\ &\text{Exclusion principle:} \quad a_s a_t^* + a_t^* a_s = \delta_{st}. \end{aligned}\right\} \tag{82}$$

The same process shall further be applied to the cavity radiation with no interaction with matter. However, on grounds that will be explained later, we will not start with the **Hamiltonian** function (79) then, but with a somewhat modified function (doubly-appearing indices will always be summed over):

$$H_0^{(s)} = \int dV \left\{ \frac{1}{16\pi}\left(\frac{\partial \Phi_i}{\partial x_k} - \frac{\partial \Phi_k}{\partial x_i}\right)^2 + 2\pi c^2 \Pi_k^2 - e\frac{\partial \Phi_0}{\partial x_k}\Pi_k + \frac{2\pi c^2}{\varepsilon+\delta}\Pi_0^2 - \frac{\varepsilon c}{\varepsilon+\delta}\frac{\partial \Phi_k}{\partial x_k}\Pi_0 \right.$$
$$\left. - \frac{\varepsilon\delta}{8\pi(\varepsilon+\delta)}\left(\frac{\partial \Phi_k}{\partial x_k}\right)^2 + \frac{\delta}{8\pi}\left(\frac{\partial \Phi_0}{\partial x_k}\right)^2 \right\}; \tag{83}$$

δ is a small parameter. We now look for the solutions to the classical wave problem that are associated with (82). To that end, we set, in a known way:

$$\left.\begin{aligned} \Phi_1 &= \sqrt{\frac{8}{L^3}}\, q_1^r \cos\frac{\pi}{L}\kappa_r x \cdot \sin\frac{\pi}{L}\lambda_r y \cdot \sin\frac{\pi}{L}\mu_r z, \\ \Phi_2 &= \sqrt{\frac{8}{L^3}}\, q_2^r \sin\frac{\pi}{L}\kappa_r x \cdot \cos\frac{\pi}{L}\lambda_r y \cdot \sin\frac{\pi}{L}\mu_r z, \\ \Phi_3 &= \sqrt{\frac{8}{L^3}}\, q_3^r \sin\frac{\pi}{L}\kappa_r x \cdot \sin\frac{\pi}{L}\lambda_r y \cdot \cos\frac{\pi}{L}\mu_r z, \\ \Phi_0 &= \sqrt{\frac{8}{L^3}}\, q_0^r \sin\frac{\pi}{L}\kappa_r x \cdot \sin\frac{\pi}{L}\lambda_r y \cdot \sin\frac{\pi}{L}\mu_r z. \end{aligned}\right\} \tag{84}$$

In this, L means the edge length of the (cubic) cavity, and κ_r, λ_r, μ_r are whole numbers that belong to the oscillation with the index r. We likewise set:

$$\left.\begin{aligned} \Pi_1 &= \sqrt{\frac{8}{L^3}}\, p_1^r \cos\frac{\pi}{L}\kappa_r x \cdot \sin\frac{\pi}{L}\lambda_r y \cdot \sin\frac{\pi}{L}\mu_r z, \\ \Pi_2 &= \sqrt{\frac{8}{L^3}}\, p_2^r \sin\frac{\pi}{L}\kappa_r x \cdot \cos\frac{\pi}{L}\lambda_r y \cdot \sin\frac{\pi}{L}\mu_r z, \\ \Pi_3 &= \sqrt{\frac{8}{L^3}}\, p_3^r \sin\frac{\pi}{L}\kappa_r x \cdot \sin\frac{\pi}{L}\lambda_r y \cdot \cos\frac{\pi}{L}\mu_r z, \\ \Pi_0 &= \sqrt{\frac{8}{L^3}}\, p_0^r \sin\frac{\pi}{L}\kappa_r x \cdot \sin\frac{\pi}{L}\lambda_r y \cdot \sin\frac{\pi}{L}\mu_r z. \end{aligned}\right\} \tag{85}$$

The **Hamiltonian** function thus goes to:

$$\left.\begin{aligned}\bar{H}^{(s)} = 2\pi c^2 &\left(p_1^{r\,2} + p_2^{r\,2} + p_3^{r\,2} + \frac{1}{\varepsilon+\delta}\, p_0^{r\,2} \right) \\ &- \frac{c\pi}{L} q_0^r (\kappa_r p_1^r + \lambda_r p_2^r + \mu_r p_3^r) + \frac{\varepsilon\, c\, \pi}{L(\varepsilon+\delta)}\, p_0^r (q_1^r \kappa_r + q_2^r \lambda_r + q_3^r \mu_r) \\ &- \frac{\pi}{8L^2} (q_1^r \kappa_r + q_2^r \lambda_r + q_3^r \mu_r)^2 \frac{\varepsilon\,\delta}{\varepsilon+\delta} + \frac{\delta\pi}{8L^2}\, q_0^{r\,2} (\kappa_r p_1^r + \lambda_r p_2^r + \mu_r p_3^r) \\ &+ \frac{\pi}{8L^2} [(q_1^r \lambda_r - q_2^r \kappa_r)^2 + (q_1^r \mu_r - q_3^r \kappa_r)^2 + (q_2^r \mu_r - q_3^r \lambda_r)^2].\end{aligned}\right\} \tag{86}$$

This function belongs to the corresponding canonical equations, which read, after eliminating p:

$$\left.\begin{aligned}\left(\frac{L}{c\pi}\right)^2 \ddot{q}_1^r + (\kappa_r^2 + \lambda_r^2 + \mu_r^2) q_1^r &= (1+\varepsilon)\left(-\frac{L}{c\pi} \dot{q}_0^r + \kappa_r q_1^r + \lambda_r q_2^r + \mu_r q_3^r \right) \kappa_r , \\ \left(\frac{L}{c\pi}\right)^2 \ddot{q}_2^r + (\kappa_r^2 + \lambda_r^2 + \mu_r^2) q_2^r &= (1+\varepsilon)\left(-\frac{L}{c\pi} \dot{q}_0^r + \kappa_r q_1^r + \lambda_r q_2^r + \mu_r q_3^r \right) \lambda_r , \\ \left(\frac{L}{c\pi}\right)^2 \ddot{q}_3^r + (\kappa_r^2 + \lambda_r^2 + \mu_r^2) q_3^r &= (1+\varepsilon)\left(-\frac{L}{c\pi} \dot{q}_0^r + \kappa_r q_1^r + \lambda_r q_2^r + \mu_r q_3^r \right) \mu_r , \\ (1-\delta)\left[-\left(\frac{L}{c\pi}\right)^2 \ddot{q}_0^r + (\kappa_r^2 + \lambda_r^2 + \mu_r^2) q_0^r \right] &= -(1+\varepsilon)\left(-\frac{L}{c\pi} \dot{q}_0^r + \kappa_r q_1^r + \lambda_r q_2^r + \mu_r q_3^r \right).\end{aligned}\right\} \tag{87}$$

For every value of r (i.e., for every system of values of κ_r, λ_r, μ_r), equations (87) describe the motions of four coupled oscillators. The classical solution of such a problem will be found by the Ansatz $q_0^r = b_0 \cos 2\pi \nu_r t$, $q_1^r = b_1 \cos 2\pi \nu_r t$, $q_2^r = b_2 \cos 2\pi \nu_r t$, $q_3^r = b_3 \cos 2\pi \nu_r t$. Equation (87) then goes to a system of linear equations with the determinant $\left(\frac{2L}{c} \nu_r = \nu_r', X_r = \kappa_r^2 + \lambda_r^2 + \mu_r^2 - \nu_r'^2 \right)$:

$$\begin{vmatrix} \frac{X_r}{1+\varepsilon} - \kappa_r^2 & -\kappa_r \lambda_r & -\kappa_r \mu_r & -\kappa_r \nu_r' \\ -\lambda_r \kappa_r & \frac{X_r}{1+\varepsilon} - \lambda_r^2 & -\lambda_r \mu_r & -\lambda_r \nu_r' \\ -\mu_r \kappa_r & -\mu_r \lambda_r & \frac{X_r}{1+\varepsilon} - \mu_r^2 & -\mu_r \nu_r' \\ +\nu_r' \kappa_r & \nu_r' \lambda_r & \nu_r' \mu_r & \frac{X_r (1+\delta)}{1+\varepsilon} + \nu_r'^2 \end{vmatrix} . \tag{88}$$

By setting the determinant equal to zero, one obtains a triple root $\nu_r'^2 = \kappa_r^2 + \lambda_r^2 + \mu_r^2$, and a single root:

$$\nu_r'^2 = \frac{\varepsilon - \varepsilon\delta}{\varepsilon + \delta} \cdot (\kappa_r^2 + \lambda_r^2 + \mu_r^2).$$

We denote the four roots by $\nu_{r,1}$; $\nu_{r,2}$; $\nu_{r,3}$; $\nu_{r,0}$. The three roots $\nu_{r,1} = \nu_{r,2} = \nu_{r,3}$ belong to three linearly-independent solutions that satisfy the condition:

$$b_1\ \kappa_r + b_2\ \lambda_r + b_3\ \mu_r + b_0\ \nu'_{r,1} = 0. \tag{89}$$

$\nu_{r,0}$ belongs to the (un-normalized) solution:

$$b_1 = \kappa_r, \qquad b_2 = \lambda_r, \qquad b_3 = \mu_r, \qquad b_0 = -\frac{\nu_{r,0}}{1-\delta}. \tag{90}$$

In the limiting case $\delta = 0$, one will always have $\nu_{r,0} = \nu_{r,1}$, and the fourth oscillation will no longer be linearly-independent of the first three. There will then exist only three proper, periodic, linearly-independent solutions of (87). The fourth linearly-independent solution of (87) will then be aperiodic and can be obtained by passing to the limit $\delta \to 0$ in the following way: For $\delta \neq 0$, we combine the two solutions:

$$\begin{aligned}
q_1^r &= \kappa_r \sin 2\pi \nu_{r,1}\, t, & q_1^r &= \kappa_r \sin 2\pi \nu_{r,0}\, t,\\
q_2^r &= \lambda_r \sin 2\pi \nu_{r,1}\, t, & q_2^r &= \lambda_r \sin 2\pi \nu_{r,0}\, t,\\
q_3^r &= \mu_r \sin 2\pi \nu_{r,1}\, t, & q_3^r &= \mu_r \sin 2\pi \nu_{r,0}\, t,\\
q_0^r &= -\nu'_{r,1} \cos 2\pi \nu_{r,1}\, t, & q_0^r &= -\frac{\nu'_{r,0}}{1-\delta} \cos 2\pi \nu_{r,0}\, t
\end{aligned}$$

by subtraction to form a beat:

$$\left.\begin{aligned}
q_1^r &= 2\kappa_r \cos 2\pi \frac{\nu_{r,1}+\nu_{r,0}}{2} t \sin 2\pi \frac{\nu_{r,1}-\nu_{r,0}}{2} t,\\
q_2^r &= 2\lambda_r \cos 2\pi \frac{\nu_{r,1}+\nu_{r,0}}{2} t \sin 2\pi \frac{\nu_{r,1}-\nu_{r,0}}{2} t,\\
q_3^r &= 2\mu_r \cos 2\pi \frac{\nu_{r,1}+\nu_{r,0}}{2} t \sin 2\pi \frac{\nu_{r,1}-\nu_{r,0}}{2} t,\\
q_0^r &= 2\nu'_{r,1} \sin 2\pi \frac{\nu_{r,1}+\nu_{r,0}}{2} t \sin 2\pi \frac{\nu_{r,1}-\nu_{r,0}}{2} t - \left(\nu'_{r,1} - \frac{\nu'_{r,0}}{1-\delta}\right) \cos 2\pi \nu_{r,0} t.
\end{aligned}\right\} \tag{91}$$

In the limit $\delta \to 0$, one will have $\nu_{r,0} = \nu_{r,1}\left(1 - \frac{\delta}{2} - \frac{\delta}{2\varepsilon}\right)$; if one multiplies the values of q

by $\varepsilon / 2$ and goes to $\delta = 0$ then one will obtain:

$$\left.\begin{aligned} q_1^r &= 2\pi(1+\varepsilon)\nu_r t \cdot \kappa_r \cos 2\pi\nu_r t, \\ q_2^r &= 2\pi(1+\varepsilon)\nu_r t \cdot \lambda_r \cos 2\pi\nu_r t, \\ q_3^r &= 2\pi(1+\varepsilon)\nu_r t \cdot \mu_r \cos 2\pi\nu_r t \\ q_0^r &= 2\pi(1+\varepsilon)\nu_r t \cdot \nu_r' \sin 2\pi\nu_r t - (1-\varepsilon)\nu_r' \cos 2\pi\nu_r t. \end{aligned}\right\} \qquad (92)$$

Aperiodic solutions of (87) then exist for $\delta = 0$. If one defines the associated partial oscillation of the field strengths then that will yield:

$$q_1^r \lambda_r - q_2^r \kappa_r = 0, \ldots, \qquad q_0^r \kappa_r + \frac{L}{c\pi}\dot{q}_1^r = 2\varepsilon\, \nu_r' \kappa_r \cdot \cos 2\pi \nu_r t, \ldots \qquad (93)$$

Thus, the aperiodic changes in the potential belong to periodic oscillations of the field strengths, which vanish when $\varepsilon \to 0$, moreover. The aperiodic solutions that are considered here have the simple form of the C. R. (61′) to thank, since they guarantee the commutation of Φ_0 and Φ_k . However, the transition $\varepsilon \to 0$ can be completed with no difficulties in all physical questions, since no aperiodic solutions of the kind (92) exist for the field strengths.

Nevertheless, it would be inconvenient to calculate with these aperiodic initial solutions; for that reason, we have added the δ-term into the function $\bar{H}_0^2$. The introduction of the δ-term then has a reason that is similar to the introduction of the cavity: It implies a discrete eigenvalue spectrum. Cavities and δ-terms generally disturb the invariance of the equations under spatial and temporal transformations. In the final result, however, if we go to the limit of an infinitely large cavity and the limit of $\delta = 0$ then the invariance will once more be restored.

The transition to the quantum-theoretical solution of equation (86) comes about in such a way that one introduces the impulse and coordinates of the principal oscillations P^r, Q^r (there are four of them for each r), in place of the p^r, q^r, and indeed an elementary calculation will yield the possible schema:

$$
\left.
\begin{aligned}
\frac{1}{\sqrt{4cL}}\,q_1^r &= \frac{\lambda_r}{\sqrt{\nu'_{r,1}(\lambda_r^2+\kappa_r^2)}}\,Q_1^r + \frac{\mu_r\kappa_r}{\nu'_{r,1}\sqrt{\nu'_{r,1}(\lambda_r^2+\kappa_r^2)}}\,Q_2^r + \frac{\kappa_r}{\sqrt{\delta\nu'^3_{r,1}}}\,Q_3^r + \frac{\kappa_r\sqrt{1-\delta}}{\sqrt{\delta\nu'^2_{r,1}\nu'_{r,0}}}\,P_0^r,\\
\frac{1}{\sqrt{4cL}}\,q_2^r &= -\frac{\kappa_r}{\sqrt{\nu'_{r,1}(\lambda_r^2+\kappa_r^2)}}\,Q_1^r + \frac{\mu_r\kappa_r}{\nu'_{r,1}\sqrt{\nu'_{r,1}(\lambda_r^2+\kappa_r^2)}}\,Q_2^r + \frac{\lambda_r}{\sqrt{\delta\nu'^3_{r,1}}}\,Q_3^r + \frac{\lambda_r\sqrt{1-\delta}}{\sqrt{\delta\nu'^2_{r,1}\nu'_{r,0}}}\,P_0^r,\\
\frac{1}{\sqrt{4cL}}\,q_3^r &= -\frac{\sqrt{\lambda_r^2+\kappa_r^2}}{\nu'_{r,1}\sqrt{\nu'_{r,1}}}\,Q_2^r + \frac{\mu_r}{\sqrt{\delta\nu'^3_{r,1}}}\,Q_3^r + \frac{\mu_r\sqrt{1-\delta}}{\sqrt{\delta\nu'^2_{r,1}\nu'_{r,0}}}\,P_0^r,\\
\frac{1}{\sqrt{4cL}}\,q_0^r &= -\frac{\nu'_{r,1}}{\sqrt{\delta\nu'_{r,1}}}\,P_3^r - \frac{\nu'_{r,0}}{\sqrt{(1-\delta)\delta\nu'_{r,0}\nu'^3_{r,1}}}\,Q_0^r,\\
\sqrt{4cL}\;p_1^r &= \lambda_r\sqrt{\frac{\nu'_{r,1}}{\lambda_r^2+\kappa_r^2}}\,P_1^r + \frac{\mu_r\kappa_r}{\sqrt{\nu'_{r,1}(\lambda_r^2+\kappa_r^2)}}\,P_2^r - \kappa_r\sqrt{\frac{\delta\nu'_{r,0}}{(1-\delta)\nu'^2_{r,1}}}\,Q_0^r,\\
\sqrt{4cL}\;p_2^r &= -\kappa_r\sqrt{\frac{\nu'_{r,1}}{\lambda_r^2+\kappa_r^2}}\,P_1^r + \frac{\mu_r\lambda_r}{\sqrt{\nu'_{r,1}(\lambda_r^2+\kappa_r^2)}}\,P_2^r - \lambda_r\sqrt{\frac{\delta\nu'_{r,0}}{(1-\delta)\nu'^2_{r,1}}}\,Q_0^r,\\
\sqrt{4cL}\;p_3^r &= -\frac{\sqrt{\lambda_r^2+\kappa_r^2}}{\sqrt{\nu'_{r,1}}}\,P_2^r - \mu_r\sqrt{\frac{\delta\nu'_{r,0}}{(1-\delta)\nu'^2_{r,1}}}\,Q_0^r,\\
\sqrt{4cL}\;p_0^r &= \sqrt{\delta\nu'_{r,1}}\,Q_0^r.
\end{aligned}
\right\} \qquad (94)
$$

One then has:

$$P_i^r = \frac{1}{2\pi\nu_{r,i}}\dot{Q}_i^r, \qquad P_0^r = -\frac{1}{2\pi\nu_{r,0}}\dot{Q}_0^r,$$

and furthermore, the C. R.:

$$[P_i^r, Q_k^s]_- = \delta_{ik}\,\delta_{rs}\,\frac{h}{2\pi i}, \quad [P_i^r, P_k^s]_- = 0, \quad [Q_i^r, Q_k^s]_- = 0 \qquad (i = 1, 2, 3, 0;\ k = 1, 2, 3, 0).$$

The equation $P_0^r = -\dfrac{1}{2\pi\nu_{r,0}}\dot{Q}_0^r$ shows that the **Hamiltonian** function includes $(P_0^r)^2$ and $(Q_0^r)^2$ with negative signs:

$$
\begin{aligned}
H_0 &= 2\pi\nu_{r,1}\,\tfrac{1}{2}[(P_1^r)^2 + (Q_1^r)^2] + 2\pi\nu_{r,2}\,\tfrac{1}{2}[(P_2^r)^2 + (Q_2^r)^2]\\
&\quad + 2\pi\nu_{r,3}\,\tfrac{1}{2}[(P_3^r)^2 + (Q_3^r)^2] - 2\pi\nu_{r,0}\,\tfrac{1}{2}[(P_0^r)^2 + (Q_0^r)^2].
\end{aligned} \qquad (95)
$$

In order to not be forced to always write the principal oscillation separately with the index 0 in what follows, we introduce:

$$\nu_{r,4} = -\nu_{r,0},$$

and furthermore:

$$P^{r,4} = -\,Q_0^r\,, \qquad\qquad Q^{r,4} = \,P_0^r\,. \tag{96}$$

From now on, we enumerate the principal oscillations with an index λ that runs from 1 to 4:

$$Q^{r,\lambda} \equiv (Q_1^r\,,\; Q_2^r\,,\; Q_3^r\,,\; P_0^r\,),$$
$$P^{r,\lambda} \equiv (P_1^r\,,\; P_2^r\,,\; P_3^r\,,\; -\,Q_0^r\,).$$

With the help of (94) and (84), the potentials can now be written in a form that is analogous to (81):

$$\left.\begin{aligned} \Phi_i &= Q^{r\lambda}\upsilon_i^{r\lambda}, & \Phi_0 &= P^{r\lambda}\upsilon_0^{r\lambda}, \\ \Pi_i &= \frac{1}{4cL}P^{r\lambda}w_i^{r\lambda}, & \Pi_0 &= \frac{1}{4cL}Q^{r\lambda}w_0^{r\lambda}. \end{aligned}\right\} \tag{97}$$

The $\upsilon_i^{r\lambda}$ and $w_i^{r\lambda}$ mean the orthogonal system of eigenfunctions of the cavity.

In place of the coefficients a, a^* in equation (81) and the P, Q in equation (97), one now introduces the number of corpuscles in corresponding quantum states as a variable, as **Dirac** did for the first time in his theory of radiation. Let the number of electrons in the state s be N_s , and let the number of light quanta in the state r be M_r . Call the canonically-conjugate angles Θ_s (χ_r, resp.).

One shall then have (†):

$$\left.\begin{aligned} &\text{Bose - Einstein statistics :} \\ &\qquad a_s = e^{-\frac{2\pi i}{h}\Theta_s}N_s^{1/2}, \qquad a_s^* = N_s^{1/2}e^{\frac{2\pi i}{h}\Theta_s}, \\ &\text{Exclusion principle :} \\ &\qquad a_s = \mathcal{V}_s\,e^{-\frac{2\pi i}{h}\Theta_s}N_s^{1/2}, \qquad a_s^* = N_s^{1/2}e^{\frac{2\pi i}{h}\Theta_s}\mathcal{V}_s, \\ &\text{Radiation :} \\ &\qquad P^{r\lambda} = \sqrt{\frac{h}{4\pi}}\left(M_{r,\lambda}^{1/2}\,e^{\frac{2\pi i}{h}\chi_{r,\lambda}} + e^{-\frac{2\pi i}{h}\chi_{r,\lambda}}M_{r,\lambda}^{1/2}\right), \\ &\qquad Q^{r\lambda} = \frac{1}{i}\sqrt{\frac{h}{4\pi}}\left(M_{r,\lambda}^{1/2}\,e^{\frac{2\pi i}{h}\chi_{r,\lambda}} - e^{-\frac{2\pi i}{h}\chi_{r,\lambda}}M_{r,\lambda}^{1/2}\right). \end{aligned}\right\} \tag{98}$$

The quantities $\mathcal{V}_s$ are the sign functions that were introduced by **Jordan** and **Wigner**:

$$\mathcal{V}_s = \prod_{t\le s}\;(1 - 2\,N_t). \tag{99}$$

(†) Cf., the repeatedly-cited paper of **Jordan** and **Wigner** on this.

The exponential functions of the phase angles can be regarded as operators, and have the following properties:

$$\left.\begin{aligned} &\text{Bose - Einstein statistics:}\\ &\qquad e^{\frac{2\pi i}{h}\Theta_s} \text{ converts } N_s \text{ into } N_s - 1,\\ &\text{Exclusion principle:}\\ &\qquad e^{\frac{2\pi i}{h}\Theta_s} \quad " \quad N_s \quad " \; 1 - N_s,\\ &\text{Radiation:}\\ &\qquad e^{\frac{2\pi i}{h}\chi_r} \quad " \quad M_r \quad " \; M_r - 1. \end{aligned}\right\} \tag{100}$$

We now go on to the presentation of the **Schrödinger** equation that belongs to the **Hamiltonian** function (79) plus (79a). The probability function φ shall depend upon the variables N_s and M_r, so $|\varphi|^2$ shall then give the probability that N_s electrons will be found in the state s, and M_r light quanta shall be found in the state r. One obtains the differential equation that belongs to φ when one expresses the H in (79) and (79a) in terms of N, Θ, and M, χ with the help of (81), (97), (98), and then regards the angles as operators and sets $(H - E)\,\varphi = 0$ (E is the total energy of the system). One will then use the the fact that the u_s and $v_{r\lambda}$ are solutions of **Hamilton**'s equation (79) [(79a), resp.] with no interaction terms to good advantage. The "unperturbed" energy will then have the simple form:

$$E = \sum_s E_s N_s + \sum_{r,\lambda} (M_{r,\lambda} + \tfrac{1}{2}) h\nu_{r,\lambda}\,.$$

The term $\frac{1}{2}\sum h\nu$ means an infinite additive zero-point energy of the radiation cavity. Since that term enters into the total energy only as an additive constant, it has no physical meaning, and can then be dropped (cf., pp. 52). If one expresses the interaction term in H in terms of u and v then the following integral will appear:

$$\left.\begin{aligned} c_{st}^{r\lambda} &= \int u_\rho^{*s}\, \alpha_{\rho\sigma}^{i}\, u_\sigma^{t}\, v_i^{r\lambda} dV,\\ d_{st}^{r\lambda} &= \int u_\rho^{*s}\, u_\sigma^{t}\, v_0^{r\lambda} dV. \end{aligned}\right\} \tag{101}$$

The difference equation for the probability amplitude $\varphi\,(N_1, N_2, \ldots; M_1, M_2, \ldots)$ then ultimately reads (in what follows, the summation sign will again be written out):

a) In the case of **Bose-Einstein** statistics for matter:

$$(-E + \sum_s N_s E_s + \sum_r M_{r\lambda} h\nu_{r\lambda})\;\; \varphi\,(N_1, N_2, \ldots; M_1, M_2, \ldots)$$

$$= e\sqrt{\frac{h}{4\pi}} \sum'_{s,t,r,\lambda} N_s^{1/2}(N_t+1)^{1/2}$$
$$\times [M_{r\lambda}^{1/2}(d_{st}^{r\lambda} - ic_{st}^{r\lambda})\varphi(N_1,\ldots,N_s-1,\ldots,N_t+1,\ldots;\ M_1,\ldots,M_{r\lambda}-1,\ldots)$$
$$+ (M_{r\lambda}+1)^{1/2}\ (d_{st}^{r\lambda} + ic_{st}^{r\lambda})\ \ \varphi(N_1,\ \ldots,\ N_s-1,\ \ldots,\ N_t+1,\ \ldots;\ M_1,\ \ldots,\ M_{r\lambda}+1,\ \ldots)]$$

$$+\ e\sqrt{\frac{h}{4\pi}} \sum_s N_s$$
$$\times [M_{r\lambda}^{1/2}(d_{ss}^{r\lambda} - ic_{ss}^{r\lambda})\varphi(N_1,\ldots;\ M_1,\ldots,M_{r\lambda}-1,\ldots)$$
$$+ (M_{r\lambda}+1)^{1/2}\ (d_{ss}^{r\lambda} + ic_{ss}^{r\lambda})\ \ \varphi(N_1,\ \ldots;\ M_1,\ \ldots,\ M_{r\lambda}+1,\ \ldots)]\ . \qquad (102)$$

b) In the case of the exclusion principle for matter:

$$(-E + \sum_s N_s E_s + \sum_r M_{r\lambda} h\nu_{r\lambda})\ \ \varphi(N_1,\ N_2,\ \ldots;\ M_1,\ M_2,\ \ldots)$$
$$= e\sqrt{\frac{h}{4\pi}} \sum'_{s,t,r,\lambda} \mathcal{V}_s(N_1,\ldots,1-N_s,\ldots)\mathcal{V}_t(N_1,\ldots,1-N_t,\ldots)$$
$$\times [M_{r\lambda}^{1/2}(d_{st}^{r\lambda} - ic_{st}^{r\lambda})\varphi(N_1,\ldots,1-N_s,1-N_t,\ldots;\ M_1,\ldots,M_{r\lambda}-1,\ldots)$$
$$+ (M_{r\lambda}+1)^{1/2}\ (d_{st}^{r\lambda} + ic_{st}^{r\lambda})\ \ \varphi(N_1,\ \ldots,\ 1-N_s,\ \ldots,\ 1-N_t,\ \ldots;\ M_1,\ \ldots,\ M_{r\lambda}+1,\ \ldots)]$$

$$+\ e\sqrt{\frac{h}{4\pi}} \sum_s N_s$$
$$\times [M_{r\lambda}^{1/2}(d_{ss}^{r\lambda} - ic_{ss}^{r\lambda})\varphi(N_1,\ldots;\ M_1,\ldots,M_{r\lambda}-1,\ldots)$$
$$+ (M_{r\lambda}+1)^{1/2}\ (d_{ss}^{r\lambda} + ic_{ss}^{r\lambda})\ \ \varphi(N_1,\ \ldots;\ M_1,\ \ldots,\ M_{r\lambda}+1,\ \ldots)]\ . \qquad (103)$$

The summand $r = s$ is excluded from the summation Σ'.

§ 8. Calculation of the perturbed eigenvalues up to second order in the interaction terms. If one regards the interaction terms in equations (102), (103) as small perturbations then one can attempt to integrate (102), (103) by successive approximations. In the unperturbed system, say, N_s^0 electrons are found in the state s, while no light quanta at all are present. We exclude dispersion and absorption processes from this initial solution, which are of no interest to us, at first. The unperturbed probability amplitude will read:

$$\varphi_0(N_1,\ldots;M_1,\ldots) = \delta_{N_1,N_1^0}\delta_{N_2,N_2^0}\cdots\delta_{M_1,0}\delta_{M_2,0}\cdots, \qquad \delta_{N_1,N_1^0} = \begin{cases} 1 & \text{for } N_1 = N_1^0, \\ 0 & \text{for } N_1 \neq N_1^0. \end{cases} \qquad (104)$$

We substitute this value of φ_0 in the interaction term for equation (102) [(103), resp.] and thus find the perturbation to the first approximation φ_1 of the probability amplitude $\varphi = \varphi_0 + \varphi_1 + \ldots$ That yield:

a) **Bose-Einstein** statistics:

$$\left.\begin{aligned}
&\varphi_1(N_1^0,\ldots,N_s^0+1,\ldots,N_t^0-1,\ldots;0,0,\ldots,\overset{r\lambda}{1},0,0)\\
&\quad=\frac{e\sqrt{\frac{h}{4\pi}}}{E_s-E_t+h\nu_{r\lambda}}(N_s^0+1)^{1/2}(N_t^0)^{1/2}(d_{st}^{r\lambda}-ic_{st}^{r\lambda}),\\
&\varphi_1(N_1^0,\ldots;0,0,\ldots,\overset{r\lambda}{1},0,0)\\
&\quad=\frac{e\sqrt{\frac{h}{4\pi}}}{h\nu_{r\lambda}}\sum_s N_s^0(d_{st}^{r\lambda}-ic_{st}^{r\lambda}).
\end{aligned}\right\}\qquad(105)$$

b) Exclusion principle:

$$\left.\begin{aligned}
&\varphi_1(N_1^0,\ldots,1-N_s^0,\ldots,1-N_t^0,\ldots;0,0,\ldots,\overset{r\lambda}{1},0,0)\\
&\quad=\frac{e\sqrt{\frac{h}{4\pi}}}{E_s-E_t+h\nu_{r\lambda}}\mathcal{V}_s(N_1^0,\ldots,1-N_t^0,\ldots)\mathcal{V}_t(N_1^0,\ldots,1-N_t^0,\ldots)(d_{st}^{r\lambda}-ic_{st}^{r\lambda})\\
&+\frac{e\sqrt{\frac{h}{4\pi}}}{E_t-E_s+h\nu_{r\lambda}}\mathcal{V}_t(N_1^0,\ldots,1-N_s^0,\ldots)\mathcal{V}_s(N_1^0,\ldots,1-N_s^0,\ldots)N_s^0(1-N_t^0)(d_{st}^{r\lambda}-ic_{st}^{r\lambda}),\\
&\varphi_1(N_1^0,\ldots;0,\ldots,\overset{r\lambda}{1},0,\ldots)=\frac{e\sqrt{\frac{h}{4\pi}}}{h\nu_{r\lambda}}\sum_s N_s^0(d_{ss}^{r\lambda}-ic_{ss}^{r\lambda}).
\end{aligned}\right\}\qquad(106)$$

At all other locations in N_{1^-}, N_{2^-}, … space, one will have $\varphi_1 = 0$. By substituting φ_1 from (105) and (106) in (102) and (103), one will obtain the eigen-perturbation $E^{(2)}$ ($E = E^0 + E^{(1)} + E^{(2)} + \ldots$) if the equation at the location N_1^0, N_2^0, …; 0, … has been written out. The temporal mean of the perturbation terms, and thus, the perturbing energy $E^{(1)}$, will vanish.

Calculation yields:

a) **Bose-Einstein** statistics:

$$-E^{(2)} = \sum_{s,t,r,\lambda}{}' \frac{e^2 \frac{h}{4\pi}}{E_s - E_t + h\nu_{r\lambda}} (N_s^0 + 1) N_t^0 (d_{st}^{r\lambda} - i c_{st}^{r\lambda})(d_{st}^{r\lambda} + i c_{st}^{r\lambda})$$
$$+ \sum_{s,t,r,\lambda} \frac{e^2}{4\pi \nu_{r\lambda}} N_s^0 (d_{ss}^{r\lambda} - i c_{ss}^{r\lambda}) N_t^0 (d_{tt}^{r\lambda} + i c_{tt}^{r\lambda}) . \quad (107)$$

b) Exclusion principle:

$$-E^{(2)} = \sum_{s,t,r,\lambda}{}' \frac{e^2 \frac{h}{4\pi}}{E_s - E_t + h\nu_{r\lambda}} N_t^0 (1 - N_s^0)(d_{st}^{r\lambda} - i c_{st}^{r\lambda})(d_{st}^{r\lambda} + i c_{st}^{r\lambda})$$
$$+ \sum_{s,t,r,\lambda} \frac{e^2}{4\pi \nu_{r\lambda}} N_s^0 (d_{ss}^{r\lambda} - i c_{ss}^{r\lambda}) N_t^0 (d_{tt}^{r\lambda} + i c_{tt}^{r\lambda}) . \quad (108)$$

Small denominators of the form $E_s - E_t + h\nu_{r\lambda}$ can possibly appear on the right-hand sides of formulas (105) to (108) that will affect the convergence of the process. Their physical meaning is the following one: In order for $E_s - E_t + h\nu_{r\lambda}$ to be small, one must have $E_s - E_t \sim h\nu_{r\lambda}$; i.e., the unperturbed system is capable making a jump from the state t to the state s with the emission of a light quantum $h\nu_{r\lambda}$. The further discussion of the small denominators would then proceed in precisely the same way as **Dirac**'s theory of radiation. Since we are more interested in the eigen-perturbations here, we would like to assume that the terms in question do not affect the result appreciably; that is case for, e.g., the normal state of an atom from which no emission is possible. However, even in the excited states, it might very well be meaningful to consider the interaction of the electrons, while neglecting the radiation force. Since we are aiming for the calculation of the interaction, we will not comment further upon the appearance of small denominators.

In what follows, it shall be proved that the eigen-perturbation that is calculated from (107), (108) is, to a certain approximation, identical with the second-order eigen-perturbation that one obtains when one imposes electrostatic interactions between the electrons in the usual way and solved the **Schrödinger** equation in configuration space. To that end, we remark that the quantities $c_{st}^{r\lambda}$ have to do with the currents, and therefore with the magnetic interactions of electrons, while the quantities $d_{st}^{r\lambda}$ relate to the electric interaction. Since the magnetic interactions are equal in order of magnitude to the relativistic effects that can still not be treated in configuration space, we will neglect them for our proof of the $c_{st}^{r\lambda}$ terms. What then remains is the calculation of sums of the type:

$$\sum_{r,\lambda} \frac{1}{E_s - E_t + h\nu_{r\lambda}} d_{st}^{r\lambda} d_{ts}^{r\lambda} . \quad (109)$$

The integrals $d_{st}^{r\lambda}$ ($r \neq t$) will first become noticeably large for $\nu_{r\lambda}$ for which the wave length of light is comparable to the atomic dimensions, and thus, for very large $\nu_{r\lambda}$.

Furthermore, since the number of eigen-oscillations increases very rapidly with increasing ν, we will suspect that the main contribution to the sum comes from very large values of ν. It therefore seems justified to consider the sum:

$$\sum_{r,\lambda} \frac{1}{h\nu_{r\lambda}} d_{st}^{r\lambda} d_{ts}^{r\lambda} \tag{110}$$

in the first approximation, instead of (111). As an estimate will show, the error is not greater than the error that is introduced by neglecting the $c_{st}^{r\lambda}$. The sum (110) is easy to evaluate; one finds, somewhat more generally, that:

$$\sum_{r,\lambda} \frac{1}{h\nu_{r\lambda}} d_{st}^{r\lambda} d_{nm}^{r\lambda} = \sum_{r,\lambda} \int dV\, dV' \left(u_\rho^{*s} u_\rho^t v_0^{r\lambda}\right)_P \cdot \left(u_\sigma^{*n} u_\sigma^m v_0^{r\lambda}\right)_{P'} \cdot \frac{1}{h\nu_{r\lambda}}. \tag{111}$$

(Here, P and P' shall index points in the volumes considered.)

The sum:

$$\sum_{r,\lambda} \frac{v_0^{r\lambda}(P) v_0^{r\lambda}(P')}{\nu_{r\lambda}} = G(P, P')$$

appears in this. In order to calculate it, one defines $\Delta_P G(P, P')$. From (94) and (84), that will yield:

$$\begin{aligned}
\Delta_P G(P, P') &= \sum_{r,\lambda} \frac{\Delta v_0^{r\lambda}(P) \cdot v_0^{r\lambda}(P')}{\nu_{r\lambda}} = \left(\frac{\pi}{L}\right)^2 \sum_{r,\lambda} \frac{-(v'_{r,1}) v_0^{r\lambda}(P) \cdot v_0^{r\lambda}(P')}{\nu_{r\lambda}} \\
&= \left(\frac{\pi}{L}\right)^2 \cdot \frac{8}{L^3} 4cL \left[-\frac{1}{\delta} + \frac{1}{(1-\delta)\delta}\right] \frac{2L}{c} \\
&\times \sum_r \sin\frac{\pi}{L}\kappa_r x_P \sin\frac{\pi}{L}\lambda_r y_P \sin\frac{\pi}{L}\mu_r z_P \sin\frac{\pi}{L}\kappa_r x_{P'} \sin\frac{\pi}{L}\lambda_r y_{P'} \sin\frac{\pi}{L}\mu_r z_{P'} \\
&= \frac{8\pi^2}{1-\delta} \cdot \delta(P - P').
\end{aligned} \tag{112}$$

$\delta(P - P')$ means the **Dirac** δ-function of the points P and P'. If the cavity is sufficiently large then the solution to the differential equation (112) for $G(P - P')$ will read:

$$G(P - P') = -\frac{2\pi}{1-\delta} \frac{1}{r_{PP'}}. \tag{113}$$

It follows from this that in the limit of $\delta = 0$:

$$-\sum_{r,\lambda} \frac{1}{2\pi\nu_{r\lambda}} d_{st}^{r\lambda} d_{nm}^{r\lambda} = A_{st,\,nm} = \int dV\, dV' \frac{u_\rho^{*s}(P) u_\rho^t(P) u_\sigma^{*n}(P') u_\sigma^m(P')}{r_{PP'}}. \tag{114}$$

The integral $A_{st,\,nm}$ is then the known exchange integral that appears in perturbation theory when one treats the many-body problem with quantum mechanics in the usual way. Ultimately, one gets the energy perturbation $E^{(2)}$, up to terms of order δ, as:

a) **Bose-Einstein** statistics:

$$E^{(2)} = \frac{e^2}{2}\left[\sum_{s,t}{}'(N_s^0+1)N_t^0 A_{st,ts} + \sum_{s,t} N_s^0 N_t^0 A_{ss,tt}\right].$$

b) Exclusion principle:

$$E^{(2)} = \frac{e^2}{2}\left[\sum_{s,t}{}' N_t^0(1-N_s^0)A_{st,ts} + \sum_{s,t} N_s^0 N_t^0 A_{ss,tt}\right].$$

The terms that appear here still include infinite sums of the form:

$$\sum_s A_{st,ts} = S_t\,.$$

It follows from (114) that:

$$\begin{aligned} S_t &= \sum_s A_{st,ts} = \int \sum_s \frac{u_\rho^{*s}(P)u_\rho^t(P)u_\sigma^{*n}(P')u_\sigma^m(P')}{r_{PP'}}\,dV\,dV' \\ &= \int \frac{u_\rho^{*t}(P)u_\rho^t(P)\delta(P-P')\delta_{\rho\sigma}}{r_{PP'}}\,dV\,dV' = \int \frac{(u_\rho^{*t}u_\rho^t)}{r_{PP'}}\,dV \\ &= \frac{1}{r_{PP}'}\int u_\rho^{*t}u_\rho^t dV = \frac{1}{r_{PP}'}\,. \end{aligned} \tag{115}$$

The quantities S_t then correspond to the self-interaction of a particle that was discussed by **Jordan** and **Klein**, and become infinitely large. S_t does not depend upon the distance t; i.e., the self-interaction of an electron is the same at any distance. Thus, the terms S_t, like the zero point of the radiation, generally imply an infinite additive constant for the total energy. In the theory that is being developed here, there are no processes in which the electron number changes. Therefore, the additive extra terms will have no effect, since *one is only interested in the energy differences* (†); we then drop the self-interaction of the electrons from $E^{(2)}$ and obtain:

a) **Bose-Einstein** statistics:

(†) Translator's note: italics mine. To me, this suggests that one is dealing with non-conservative forces that do not admit non-singular potential energy functions; the energy difference is the path-dependent work integral between the initial and final states.

$$E^{(2)} = e^2 \sum_{s>t} N_s^0 N_t^0 (A_{st,ts} + A_{ss,tt}) + e^2 \sum_s \frac{N_s^0 (N_s^0 - 1)}{2} A_{ss,ss} + \text{const.} \tag{116}$$

b) Exclusion principle:

$$E^{(2)} = e^2 \sum_{s>t} N_s^0 N_t^0 (-A_{st,ts} + A_{ss,tt}) + e^2 \sum_s \frac{N_s^0 (N_s^0 - 1)}{2} A_{ss,ss} + \text{const.} \tag{117}$$

These are precisely the formulas that conventional quantum mechanics gives when one considers the electrostatic interaction of the electrons in the first approximation. The theory that we are pursuing here leads naturally to these formulas only under certain omissions that shall be discussed briefly.

The magnetic terms $c_{st}\, d_{ts}$ and $c_{st}\, c_{ts}$ will be dropped. Since the sum:

$$\sum_{r,\lambda} \frac{1}{h\nu_{r\lambda}} c_{st} d_{nm} ,$$

which is analogous to (110), vanishes, as the calculations will show, the terms $c_{st}\, c_{ts}$ will chiefly play a role in the magnetic interactions that gives rise to exchange terms of the form:

$$\int \frac{u_\rho^{*s}(P)\, \alpha_{\rho\sigma}^i\, u_\sigma^t(P)\, u_\mu^{*n}(P')\, \alpha_{\mu\nu}^i\, u_\nu^m(P')}{r_{PP'}} dV\, dV' .$$

Its magnitude is small of order $(v / c)^2$ relative to the values $E^{(2)}$. In addition, the exact formula (107) [(108), resp.] contains extra terms of the kind $\sum \frac{E_s - E_t}{(E_s - E_t + h\nu_r)\, h\nu_r} d_{st} d_{ts}$, which can be neglected in the transition from (109) to (110), and which originate in the retarding of the potential. Ultimately, $E^{(2)}$ still does not give the exact eigenvalue, but $E^{(3)}$, $E^{(4)}$, etc., must be considered in E, as well. In many cases, $E^{(3)}$ will greater than the terms that have been neglected up to now. The calculation of $E^{(3)}$ and a comparison of the values with the corresponding perturbing terms of the treatment in configuration space will, however, lead to extremely tedious calculations. If would be very desirable to have another method of integrating the fundamental equations of the theory, in which the interaction of the electrons is not assumed to be small and is developed in powers of $1 / e$. It would also be necessary to investigate the role of the self-energy of the electrons more precisely in terms of order $(v / c)^2$.

§ 9. On the light emission that one might expect from the theory when an electron passes through a potential jump. The calculations of the previous section will show that the theory that is sought here includes the results of the previous theories as special cases. The proof of that can also be easily carried out for radiation phenomena for which equations (102), (103) lead to essentially the same results as **Dirac**'s theory of

radiation. It also yields nothing new in the problem of the sharpness of the energy definition in the stationary states.

By contrast, some experiments shall be discussed here that still have not been treated from the standpoint of the theories up to now (*). To cite a particular example: Let a helium atom in the normal state be under the influence of a strong electric field. That field can ionize the helium atom with a certain probability. In a similar way, it is know that for an α-particle in the **Gamov-Gurney-Condon** theory of the **Geiger-Nutall** law, a certain probability exists that it can pass through the nuclear potential jump. Such a transition is found in quantum mechanics in such a way that the electron can leave its atom with a well-defined energy that is given by the difference between the energies that were originally present in the normal states of helium and the remaining energy of the positive helium ion. However, if one considers the interaction of matter and radiation in the manner that has been set down here then a certain probability will also exist for the emission of electrons of significantly lower energy values, such that the energy law will be justified by the emission of a corresponding light quantum. Here, the blurring of the energy of the emitted electrons has nothing to do with the lifetimes of the states in question, so the effect comes about entirely independently of the ionization probability. Moreover, the theory leads one to expect phenomena that are completely similar (*mutatis mutandis*) to the **Auger** jumps.

One can summarize the mathematical treatment of the aforementioned effects under the title of "transitions between states of equal energy." We then assume that there is a discrete, radiation-less state of the atom in the unperturbed system (viz., the normal state or the metastable state), and there is a continuum of translational states in the vicinity of that energy value that will resolve into a sequence of discrete, very closely spaced terms under the quantization in a cavity. If one first treats that problem with conventional quantum mechanics then a certain eigenfunction φ^a of the electron coordinates will belong to the discrete initial state of the atom. Eigenfunctions φ^t will belong to the individual translation states that will be represented to a sufficient approximation by a product of the eigenfunctions of the ions and the translational eigenfunctions (viz., plane waves) of a single electron. If one denotes the kinetic energy of the electron by E_t then the mean distance ΔE_t between two neighboring translational states of energy E_t will have the value:

$$\Delta E_t = \frac{h^3}{16\pi (2m)^{3/2} L^3 E_t^{1/2}} \tag{118}$$

(L = edge length of the cavity).

According to **Dirac** (**), the transition probability for the process considered – viz., the transition of an electron from the atomic context to the energetically-corresponding translational state – is then given by:

(*) If one restricts oneself to the first approximation, as will be done in what follows, then one will get results that can be derived from **Dirac**'s theory of radiation. However, that will no longer be correct in the higher approximations, since a unified treatment of the interaction forces and radiation forces will be required then, which is still not contained in **Dirac**'s theory of radiation.

(**) Proc. Roy. Soc. **114** (1927), 243; see, esp., pp. 264, equation (32).

$$\left|\Phi^0_{at}\right|^2 \frac{4\pi^2}{h\,\Delta E_t} = \left|\Phi^0_{at}\right|^2 \frac{64\pi^3}{h^4}(2m)^{3/2} L^3 E_t^{1/2}\,. \tag{119}$$

In this, Φ^0_{at} means the matrix element of the perturbing potential that belongs to the transition considered. One then has:

$$\Phi^0_{at} = -\,e \int \varphi^{*a} \Phi^0 \varphi^t \, d\Omega\,. \tag{120}$$

$d\Omega$ means the volume element in configuration space. Here, in the case of the **Gamov** transition, the potential jump itself enters as the perturbing potential, in essence, so the smallness of Φ^0_{at} originates in the fact that the product $\varphi^{*a}\ \varphi^t$ is small everywhere. (The eigenfunction of the translation and that of the atom decrease exponentially inside of the jump.) In the case of the photoelectric effect, Φ^0 means the potential of an external perturbing light wave (*), while in the **Auger** process, it will be the potential of the **Coulomb** interactions of the electrons. Since the eigenfunction φ^t is normalized over the entire cavity in the coordinates of an electron, one easily sees that Φ^0_{at} behaves like $L^{3/2}$ as a function of L. The transition probability (119) is then independent of L, as it must be.

Equation (120) becomes especially simple when the interaction of the electrons is generally regarded as small. In the case of the exclusion principle, one will then have, in the highest approximation:

$$\Phi^0_{at} = -\,e \int dV u^{a*} \Phi^0 u^t\,,$$

in which the u^a means the eigenfunction of the state from which the electron is removed by the transition; u^t is the translation eigenfunction, so the integral is only extended over a three-dimensional space, namely, the coordinates of *one* electron.

If one treats the same problem with the method that was described in this paper then a perturbing term of the form $-\,e\ \Phi^0\ \psi^*\ \psi$ will first enter in place of the perturbing potential V in the **Hamiltonian** function. If one again expresses ψ^*_ρ and ψ_ρ in terms of N_s and Θ_s then it will follow from (98) in the case of the exclusion principle that:

$$\begin{aligned} H_1 &= -\,e\,\Phi^0 \psi^*_\rho \psi_\rho \\ &= N_s\,(1 - N_t)\ \mathcal{V}_s\,(N_1, \ldots, 1 - N_s, \ldots)\ \mathcal{V}_s\,(N_1, \ldots, 1 - N_s, \ldots)\ a_{st}\ \ldots, \end{aligned} \tag{121}$$

in which:

$$a_{st} = -\,e \int \Phi^0\, u^{*s}_\rho\, u^t_\rho\, dV\,. \tag{121a}$$

We can again adapt the **Dirac** equation for the transition probabilities (119) directly, if we now have:

(*) Cf., on this, **G. Wentzel**, Phys. Zeit. **29** (1928), 321.

$$\Phi^0_{at} = \sum_{N_1,N_2,\ldots,M_1,M_2} \varphi^{*a}(N_1,\ldots,M_1,\ldots)\, H_1 \varphi^t \tag{122}$$

for Φ^0_{at} ; here, φ^t means the probability amplitudes for the initial and final states in N_1, …, M_1, …space. If one next asks about the transitions *without* the emission of light quanta then one will have:

$$\left.\begin{aligned} \varphi^a &= \delta_{N_1 N_1^0} \cdot \delta_{N_2 N_2^0} \cdots \delta_{N_{a,1}} \cdots \delta_{N_{t,0}} \cdots \delta_{M_{1,0}} \cdots + \text{higher-order terms}, \\ \varphi^t &= \delta_{N_1 N_1^0} \cdots \delta_{N_{a,0}} \cdots \delta_{N_{t,1}} \cdots \delta_{M_{1,0}} \cdots + \text{higher-order terms}. \end{aligned}\right\} \tag{123}$$

H_1 is to be regarded as an operator [cf., (100)] in (122), and one will get, in the zeroth approximation:

$$\Phi^0_{at} = a_{at} = -e \int dV \Phi^0 u^{*a}_\rho u^t_\rho , \tag{124}$$

in agreement with the previous result.

Transitions with the emission of light quanta ($h\nu_{r\lambda}$) are then also present. The eigenfunction for the state *a* remains as before, except that φ^a must be calculated up to the terms of first order that are given in equation (106). By contrast, the eigenfunction of the final state *t* in the zeroth approximation is now:

$$\varphi^t_0 = \delta_{N_1,N_1^0} \ldots \delta_{N_{a,0}} \ldots \delta_{N_{t,1}} \ldots \delta_{M_{1,0}} \ldots \delta_{M_{r\lambda,1}} \ldots$$

Analogous to (106), the perturbing terms of first order in φ^t are given by:

$$\left.\begin{aligned} &\varphi_1(N_t^0,\ldots,1-N_a^0,\ldots,1-N_s^0,\ldots,N_t^0,\ldots,0,\ldots,1^2,\ldots) \\ &\qquad (N_t^0=0,\quad N_a^0=1), \\ &= \frac{e\sqrt{\dfrac{h}{4\pi}}}{E_s - E_t - h\nu_{r\lambda}} \mathcal{V}_s(N_1^0,\ldots,N_t^0,\ldots)\mathcal{V}_t(N_1^0,\ldots,N_t^0,\ldots)\cdot(1-N_s^0+\delta_{as}-\delta_{st})(d^{r\lambda}_{st}+ic^{r\lambda}_{st}), \\ &\varphi_1(N_1^0,\ldots,N_t^0,\ldots,0,0,\ldots) = \frac{e\sqrt{\dfrac{h}{4\pi}}}{-h\nu_{r\lambda}} \sum (N_s^0 - \delta_{as} + \delta_{st})(d^{r\lambda}_{st}+ic^{r\lambda}_{st}). \end{aligned}\right\} \tag{125}$$

Other values of φ_1 can enter in place of the N_1, …, M_1 space that are of no interest to us, since the enter into the sum (122).

We thus obtain, in the first approximation (the terms of order zero drop out):

$$\Phi^0_{at,h\nu_{r\lambda}} = \sum_{N,\ldots,M,\ldots} \varphi^{*a}(N_1,\ldots,M_1,\ldots)\, H_1\, \varphi^{t,h\nu_{r\lambda}}$$

$$= \sum_s \frac{e\sqrt{\frac{h}{4\pi}}}{E_s - E_a + h\nu_{r\lambda}} \mathcal{V}_s(N_1^0,\ldots,1-N_a^0,\ldots)\,\mathcal{V}_a(N_1^0,\ldots,1-N_a^0,\ldots)$$
$$(1-N_s^0)(d_{as}^{r\lambda} + ic_{as}^{r\lambda})\cdot \mathcal{V}_s(N_1^0,\ldots,1-N_a^0,\ldots)\mathcal{V}_t(N_1^0,\ldots,1-N_a^0,\ldots)\,a_{st}$$

$$+ \sum \frac{e\sqrt{\frac{h}{4\pi}}}{h\nu_{r\lambda}} N_s^0 (d_{ss}^{r\lambda} + ic_{ss}^{r\lambda})\,a_{at}\cdot \mathcal{V}_a(N_1^0,\ldots,1-N_a^0,\ldots)\,\mathcal{V}_t(N_1^0,\ldots,1-N_a^0,\ldots)$$

$$+ \sum a_{as}\mathcal{V}_a(N_1^0,\ldots,1-N_a^0,\ldots)\mathcal{V}_s(N_1^0,\ldots,1-N_a^0,\ldots)$$
$$\frac{e\sqrt{\frac{h}{4\pi}}}{E_s - E_a + h\nu_{r\lambda}} \mathcal{V}_s(N_1,\ldots,1-N_a^0,\ldots)\,\mathcal{V}_t(N_1,\ldots,1-N_a^0,\ldots)$$
$$\cdot\,(1-N_s^0+\delta_{as}-\delta_{st})(d_{st}^{r\lambda} + ic_{st}^{r\lambda})$$

$$+ a_{as}\mathcal{V}_a(N_1^0,\ldots,1-N_a^0,\ldots)\mathcal{V}_s(N_1^0,\ldots,1-N_a^0,\ldots)$$
$$\cdot \frac{e\sqrt{\frac{h}{4\pi}}}{-h\nu_{r\lambda}} \sum (N_s^0 - \delta_{as} + \delta_{st})(d_{ss}^{r\lambda} + ic_{ss}^{r\lambda}). \qquad (126)$$

Use of the notation $d_{st}^* = d_{ts}$ is made in this. Combining the various terms yields:

$$\Phi_{at,h\nu_{r\lambda}}^0 = \mathcal{V}_a(N_1^0,\cdots,1-N_a^0,\ldots)\mathcal{V}_t(N_1^0,\cdots,1-N_a^0,\ldots)\cdot\left[a_{at}\frac{e\sqrt{\frac{h}{4\pi}}}{h\nu_{r\lambda}}(d_{aa}+ic_{aa}-d_{tt}-ic_{tt})\right.$$
$$+ \sum_s \frac{e\sqrt{\frac{h}{4\pi}}}{E_s - E_a + h\nu_{r\lambda}}(1-N_s^0)(d_{as}^{r\lambda}+ic_{as}^{r\lambda})\,a_{st}$$
$$\left. + \sum_s \frac{e\sqrt{\frac{h}{4\pi}}}{E_s - E_a + h\nu_{r\lambda}}(1-N_s^0+\delta_{as}-\delta_{st})(d_{st}^{r\lambda}+ic_{st}^{r\lambda})\,a_{as}\right]$$

$$= \mathcal{V}_a(N_1^0,\cdots,1-N_a^0,\ldots)\mathcal{V}_t(N_1^0,\cdots,1-N_a^0,\ldots)$$

$$\times \left[\sum_s \frac{e\sqrt{\frac{h}{4\pi}}}{E_s - E_a + h\nu_{r\lambda}} (1 - N_s^0 + \delta_{as}) \cdot (d_{as}^{r\lambda} + ic_{as}^{r\lambda})\, a_{st} \right.$$

$$\left. + \sum_s \frac{e\sqrt{\frac{h}{4\pi}}}{E_s - E_a + h\nu_{r\lambda}} (1 - N_s^0 + \delta_{as}) \cdot (d_{st}^{r\lambda} + ic_{st}^{r\lambda})\, a_{as} \right]. \qquad (127)$$

By substituting $\Phi^0_{at,h\nu_{r\lambda}}$ into (119), one will obtain the probability of the transition from the state *a* to a state of *equal energy*, during which, a light quantum $h\nu_{r\lambda}$ is excited and an electron is emitted in the state E_t . For the values λ = 1, 2, the energy of the aforementioned final state will differ only slightly from the sum $h\nu_{r\lambda}$ + energy of the atomic system in the state *t* in which no light quanta are present (which is equal to E_{ion} + E_t + $h\nu_{r\lambda}$). For λ = 3 or 4, however, the energy of the final state will differ from the corresponding sum by quantities of order 1 / δ, as one infers from a consideration that is analogous to the one that led from (104) to (108). Should the energy of the final state be equal to that of the initial state then for λ = 3, 4, and small δ, either $h\nu_{r\lambda}$ must be very large or E_t must be very large; the corresponding probability amplitude will then be very small, so the contributions of λ = 3 and λ = 4 should be dropped in the limit δ = 0.

From (119), the total probability for the emission of a light quantum with a frequency between ν and $\nu + \Delta\nu$ and the simultaneous emission of an electron with the "corresponding" energy"

$$E_t = E_a - E_{\text{ion}} - h\nu \qquad (128)$$

then amounts to:

$$\frac{4\pi^2}{h\,\Delta E_t} \sum_{\lambda=1,2} \sum_{\nu}^{\nu+\Delta\nu} \left| \sum_s \frac{e\sqrt{\frac{h}{4\pi}} (d_{as}^{r\lambda} + ic_{as}^{r\lambda}) a_{st}}{E_s - E_a + h\nu_{r\lambda}} (1 - N_s^0 + \delta_{as}) \right.$$

$$\left. + \sum_s \frac{e\sqrt{\frac{h}{4\pi}} a_{as} (d_{st}^{r\lambda} + ic_{st}^{r\lambda})}{E_s - E_a + h\nu_{r\lambda}} (1 - N_s^0 + \delta_{as}) \right|^2. \qquad (129)$$

Since λ can assume only the values 1, 2, ν_0, and therefore d_{as}, will vanish from equation (94). The sums between the lines are then converted into:

$$\left| \sum_s e\sqrt{\frac{h}{4\pi}} (1 - N_s^0 + \delta_{as}) \left[\frac{c_{as}^{r\lambda} a_{st}}{E_s - E_a + h\nu_{r\lambda}} + \frac{a_{as} c_{st}^{r\lambda}}{E_s - E_t + h\nu_{r\lambda}} \right] \right| \qquad (130)$$

If the wave length of light that belongs to $h\nu_{r\lambda}$ is large compared to the atomic dimensions then one can set:

$$c_{mn}^{r\lambda} = \sum_l \int u_\rho^{*m} \alpha_{\rho\sigma}^l u_\sigma^n u_l^{r\lambda} = \sum_l (\nu_l^{r\lambda})_A \frac{1}{c} \dot{x}_l^{nm} \,. \tag{131}$$

In this, the index A means that the value of the function of position in question is to be taken at the location of the atom. If one performs the summation over the values of r, λ between ν and $\nu + \Delta\nu$ then, from (84) and (94), one will get:

$$\frac{1}{\Delta E_t} \frac{16\pi^2 e^2}{3c^3} \nu \Delta\nu \sum_l \left| \sum_s (1 - N_s^0 + \delta_{as}) \left[\frac{\dot{x}_l^{as} a_{st}}{E_s - E_a + h\nu} + \frac{a_{as} \dot{x}_l^{st}}{E_s - E_t + h\nu} \right] \right|^2 \tag{132}$$

for the transition probability (129). The factor $1 / \Delta E_t$ will drop out when the matrices a_{st} and $\dot{x}_l^{st}$ are calculated with the eigenfunctions, which are normalized according to the scale of dE_t .

If ν and $\Delta\nu$ have an order of magnitude E / h then the relative frequency of processes with the emission of light quanta in comparison to the frequency of ordinary transitions will have the order of magnitude:

$$\sim \frac{e^2}{hc} \left(\frac{\dot{x}}{c} \right)^2 . \tag{133}$$

The probability for the transition that is considered here will then be small of the same order as the radiation effects relative to the probability of the ordinary processes.

If one applies this result to the **Gamov-Gurney-Condon** theory of the radioactive decay of the nucleus then one will conclude that primary β-ray spectra cannot be sharp, since all radiation effects of the electrons on the nucleus have the relative order of magnitude 1. Admittedly, from the theory that is presented here, the associated continuous γ-ray spectra must also always appear, since in this theory, the validity of the energy law will always remain true. This theory then gives no insight into the complications that are linked with the apparent non-existence of those γ-ray spectra.

Our argument for the *continuous* primary β-spectra has a certain similarity to a consideration of **Rosseland** (*), which said that the electrons would be forced to emit radiation as a result of the acceleration that they acquire upon leaving the nucleus. However, upon closer comparison, there are still some differences between the theories. The **Rosseland** transitions correspond to only the terms in the sum (129) for which $E_s - E_t - h\nu_{r\lambda}$ can be considered to be very small. Due to the smallness of the associated coefficients d_{st}, c_{st}, they will produce only a minor contribution to the total result.

Equation (129) is also applicable to the photo-electric effect. However, that will produce nothing new, but will give the known probability formula for the Compton

(*) **S. Rosseland**, Zeit. Phys. **14** (1923), 173.

effect. If one were to choose the state t to be in a discrete spectrum then one would again obtain a derivative of the **Ladenburg-Kramer** dispersion formula from (129).

From the final formula (132), one can recognize that it also contains the jump $+ mc^2 \rightarrow - mc^2$ that **Dirac** discussed, which will naturally influence the result. Since that jump undoubtedly does not happen in reality, we have not considered it in the discussion of (132). That fact is inconsequential to the theory that is discussed here, which one must consider as long as the **Dirac** difficulty is still unexplained.