“Zur Quantentheorie der Wellenfelder, II” Zeit. Phys. **59** (1929), 168-190.

On the quantum theory of wave fields, II.

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(Received on 7 September 1929)

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The decomposition of total systems of terms into non-combining subsytems will be examined for the quantum theory of wave fields. The integrals of the equations of motion will be derived from the invariance properties of the Hamiltonian function. Furthermore, the consideration of gauge invariance will yield a satisfactory formulation of electrodynamics with no extra terms. The mathematical connection between wave theory and particle theory will be discussed.

Introduction. The relativistic formulation of the quantum theory of wave fields (*) has been plagued with difficult objections, up to now. In particular, the interaction of the electron with itself seems to make the application of the theory impossible in many cases, at the moment. We are thus still quite far from an ultimate formulation of the theory. Nevertheless, we would like to believe that it is precisely that construction of wave theory that will be vital to any further progress in quantum theory.

In the quantum theory of point mechanics, essential progress is achieved by the investigation of the invariance properties (**) of the **Hamiltonian** function. The distribution of systems of terms into non-combining groups of terms can be derived from these invariance properties, and likewise the simple integrals of the equations of motion are connected with such invariance properties of the **Hamiltonian** function. The invariance properties of the wave equations will be exploited in an entirely similar way in the following exposition.

§ 1. General method and impulse theorems. The basic idea of the method is generally this: If the **Hamiltonian** function $\bar{H}$ is invariant under certain operations then that means that a well-defined operator $\bar{H}$, which will be linear in all of the cases that are important to us, will remain unchanged; i.e., it will commute with $\bar{H}$. If one regards the operator as a quantum-theoretical variable (***) then it will follow that this variable is constant in time, so one thus will obtain an integral of the equations. If the aforementioned invariance remains under any changes or perturbations of the system then

(*) **W. Heisenberg** and **W. Pauli**, Zeit. Phys. **56** (1929), 1. This paper will be cited as I in what follows.

(**) Confer the summary presentation of **H. Weyl**, *Gruppentheorie und Quantenmechanik*, Leipzig, Hirzel, 1928.

(***) On this, cf., **P. A. M. Dirac**, Proc. Roy. Soc. London (A) **123** (1929), 714.

the changes in the values of the variables will be completely impossible, so every numerical value of the operator will represent a subsystem of terms that do not combine with the remaining terms.

As the simplest example, let the Hamiltonian function be invariant under the translation of the entire wave field in space. (The notations in the following formulas are taken from the paper I everywhere.) The translation $x_i \to x_i + \delta x_i$ corresponds to the change of the wave functions Q_α (cf., I, pp. 20):

$$Q_\alpha \to Q_\alpha - \frac{\partial Q_\alpha}{\partial x_i}\,\delta x_i\,. \tag{1}$$

The change in a functional F of the Q_α will then be:

$$F \to F - \int dV \sum_\alpha \frac{\delta F}{\delta Q_\alpha}\frac{\partial Q_\alpha}{\partial x_i}\delta x_i = \left(1 - \delta x_i \int dV \sum_\alpha \frac{\partial Q_\alpha}{\partial x_i}\frac{\delta}{\delta Q_\alpha}\right)F. \tag{2}$$

The translation by δx_i then corresponds to the operator:

$$1 - \delta x_i \int dV \sum_\alpha \frac{\partial Q_\alpha}{\partial x_i}\frac{\delta}{\delta Q_\alpha}\,. \tag{3}$$

Since the **Hamiltonian** function $\bar{H}$ is invariant under translations, the operator (3) must commute with $\bar{H}$. The quantum-theoretic variable that corresponds to it is then constant in time. Since the operator $\delta / \delta Q_\alpha$ corresponds to the variable $2\pi i / h\, P_\alpha$ [I, equation (20)], (3) will then yield the impulse theorem [I, equation (24)]:

$$\int dV \sum_\alpha \frac{\partial Q_\alpha}{\partial x_i} P_\alpha = \text{const.} \tag{4}$$

§ 2. Conservation of charge. The particular **Hamiltonian** function for the electrons (ψ_ρ) and radiation (Φ_ν) is invariant under the transformation:

$$\psi_\rho \to \psi_\rho\, e^{i\alpha}, \qquad \psi_\rho^* \to \psi_\rho^*\, e^{-i\alpha}, \tag{5}$$

in which α means a constant, or the corresponding infinitesimal transformation:

$$\psi_\rho \to \psi_\rho + i\,\delta\alpha\,\psi_\rho, \quad \psi_\rho^* \to \psi_\rho^* - i\,\delta\alpha\,\psi_\rho^*. \tag{6}$$

Since the ψ_ρ^* mean the impulses that are canonically-conjugate to the ψ_ρ, it will suffice to consider just the ψ_ρ. A functional of the ψ_ρ goes to:

$$F \to F + i\,\delta\alpha \int dV \frac{\delta F}{\delta\psi_\rho}\psi_\rho , \tag{7}$$

so the operator that belongs to the transformation (6) is then:

$$1 + i\,\delta\alpha \int dV\, \psi_\rho \frac{\delta}{\delta\psi_\rho} . \tag{8}$$

It then follows that:

$$\int dV\, \psi_\rho \psi_\rho^* = \text{const.} \tag{9}$$

One can rearrange the factors on the left-hand side of (9) with affecting the temporal constancy of the integral. One then obtains the theorem of the conservation of charge. If the **Hamiltonian** function contains matter waves for electrons and protons ($\psi_\rho^{(e)}$ and $\psi_\rho^{(p)}$) then the theorem of the conservation of charge will have the form:

$$\int dV \cdot (-\psi_\rho^{*(e)}\psi_\rho^{(e)} + \psi_\rho^{*(p)}\psi_\rho^{(p)}) = \text{const.} \tag{10}$$

In order for this equation to be justified, the Hamiltonian function must be invariant under the following transformation:

$$\left.\begin{array}{ll} \psi_\rho^{(e)} \to \psi_\rho^{(e)} e^{i\alpha}, & \psi_\rho^{*(e)} \to \psi_\rho^{*(e)} e^{-i\alpha}, \\ \psi_\rho^{(p)} \to \psi_\rho^{(p)} e^{-i\alpha}, & \psi_\rho^{*(p)} \to \psi_\rho^{*(p)} e^{i\alpha}. \end{array}\right\} \tag{11}$$

The usual form of the Hamiltonian function up to now contains two independent summands that each depend upon only $\psi_\rho^{(e)}$ or $\psi_\rho^{(p)}$ alone. That function is invariant under (11), so one even has the conservation of charge for the protons and electrons separately. However, one sees from (11) that one can possibly introduce terms of the form:

$$\psi_\rho^{(e)}\psi_\rho^{(p)} F_{ik} \mathfrak{s}_{\rho\sigma}^{ik} + \psi_\rho^{*(e)}\psi_\rho^{*(p)} F_{ik} s_{\rho\sigma}^{*ik} \tag{12}$$

into the Hamiltonian function with altering (10). ($\mathfrak{s}^{ik}$ means the components of the **Dirac** spin tensor.) Such extra terms make it possible for there to exist "annihilation processes," in which an electron and a proton combine into a light quantum (†). The annihilation processes can then be introduced into the mathematical framework of the quantum theory of waves with no difficulty, although it is known that they have no place in particle theory.

§ 3. The transformation $\Phi_\nu \to \Phi_\nu + \partial\chi / \partial x_\nu$, $\psi_\rho \to e^{-\frac{2\pi i}{h}\frac{e}{c}\chi} \cdot \psi_\rho$. For the sake of simplicity, let there be just one kind of matter (ψ_ρ) in the following calculations. As long

(†) Translator's note: Of course, this paper was published before the positron was discovered.

as one ignores extra terms with ε and δ (I, pp. 31), the Hamiltonian function will be invariant under the following transformation (†):

$$\Phi_\nu \to \Phi_\nu + \frac{\partial \chi}{\partial x_\nu}, \qquad \psi_\rho \to e^{-\frac{2\pi i}{h}\frac{e}{c}\chi} \cdot \psi_\rho, \qquad \psi^*_\rho \to \psi^*_\rho \, e^{\frac{2\pi i}{h}\frac{e}{c}\chi}, \tag{13}$$

in which χ represents an arbitrary function of space and time. (In this, χ shall commute with all variables ψ_ρ, Φ_α, and in addition, the values of χ and $\partial\chi / \partial t$ at different spatial locations must commute.) As is known, this invariance will be perturbed by the extra terms in ε and δ. This is a blemish in the theory that seems unavoidable when one carries over **Maxwell**'s equations to quantum theory in the usual way. However, if one examines the integrals that belong to (13) then that will suggest the possibility of avoiding the extra terms altogether (*).

For the following calculations, we start with the Lagrangian function with no ε and δ terms; its radiation part is then called (when one employs **Heaviside** units) simply:

(†) **Weyl** used the term "gauge invariance" for this in *loc. cit.*

(*) In a paper by **E. Fermi** that appeared in the meantime [Rendiconti d. R. Acc. dei Lincei (6) **9** 1st half (1929), pp. 881], another interesting method of quantization was given in which the gauge invariance was perturbed by auxiliary conditions, instead of extra terms. The **Fermi** method can be characterized as follows from the viewpoint that is assumed in this paper: One introduces:

$$\bar{L}^{(s)} = \int \frac{1}{2}\sum_{\mu,\nu}\left(\frac{\partial \Phi_\mu}{\partial x_\nu}\right)^2 dV$$

as the radiation part of the Lagrangian function, such that the field equations that arise by varying Φ_μ will read:

$$-\sum_\nu \frac{\partial^2 \Phi_\mu}{\partial x_\nu^2} = s_\mu .$$

For the quantities:

$$K = \sum_\mu \frac{\partial \Phi_\mu}{\partial x_\mu},$$

the relation:

$$\sum_\nu \frac{\partial^2 K}{\partial x_\nu^2} = 0$$

will follow from these equations by means of $\sum_\mu \frac{\partial s_\mu}{\partial x_\mu} = 0$. In order to make this result agree with **Maxwell**'s equations, **Fermi** added the auxiliary conditions:

$$K = 0 \text{ and } \dot{K} = 0$$

on a slice t = const. in a known way, and these conditions will propagate in the course of time by means of the field equations. These auxiliary conditions are valid in quantum electrodynamics, not as q-number relations, but in the same sense as equation (25), which we will derive later. **Fermi** then arrived at his quantum-electrodynamical equations when he employed the Fourier decomposition for the electromagnetic field and configuration space for the matter field (cf., § 7 of this paper). The question of the relativistic invariance of the C. C. R. or that of the corresponding operator method was not particularly examined by **Fermi**, although it follows with no further assumptions from paper I or from § 4 of the present paper.

$$\frac{1}{2}(\mathfrak{E}^2 - \mathfrak{H}^2) = -\frac{1}{4}\left(\frac{\partial \Phi_\mu}{\partial x_\nu} - \frac{\partial \Phi_\nu}{\partial x_\mu}\right)^2 .$$

However, in this Lagrangian function we consider only the Φ_i (i = 1, 2, 3) to be variables, while we regard Φ_4 as an arbitrarily given function that commutes with all other variables. In particular, e.g., Φ_4 can simply be set equal to zero. That would correspond to the state of affairs in classical theory, in which one of the four components is indeed completely arbitrary, due to (13). Once Φ_4 is established, the invariance (13) will then still exist only for time-independent functions χ. Thus, let χ be an arbitrary function of the three spatial coordinates that vanishes at infinity to a sufficient degree, and look for the integrals that belong to (13).

One should observe that now only the three spatial components of Maxwell's equations follow by variation of the Φ_i in the Lagrangian function, while the equation:

$$\operatorname{div} \mathfrak{E} = \rho \tag{14}$$

does not need to be fulfilled.

Instead of (13), we consider the infinitesimal transformation:

$$\Phi_i \to \Phi_i + \delta \frac{\delta\chi}{\delta x_i} , \qquad \psi_\rho \to -\frac{2\pi i}{h}\frac{e}{c}\delta\chi \cdot \psi_\rho. \tag{15}$$

A functional F of Φ_i and ψ_ρ will go to:

$$\begin{aligned} F &\to F + \delta \int dV \left(\frac{\delta F}{\delta \Phi_i}\frac{\partial \chi}{\partial x_i} - \frac{2\pi i}{h}\frac{e}{c}\frac{\delta F}{\delta \psi_\rho}\psi_\rho \chi \right) \\ &= F - \delta \int dV \left(\frac{\partial}{\partial x_i}\frac{\delta F}{\delta \Phi_i} - \frac{2\pi i}{h}\frac{e}{c}\frac{\delta F}{\delta \psi_\rho}\psi_\rho \right)\chi . \end{aligned} \tag{16}$$

The operator:

$$\int dV \chi \left(\frac{\partial}{\partial x_i}\frac{\delta}{\delta \Phi_i} - \frac{2\pi i}{h}\frac{e}{c}\psi_\rho \frac{\delta}{\delta \psi_\rho} \right) \tag{17}$$

will then correspond to the transformation (15) and commute with the Hamiltonian function.

Therefore:

$$\int dV \chi \left(-\frac{1}{c}\frac{\partial \mathfrak{E}_i}{\partial x_i} - \frac{e}{c}\sum_\rho \psi_\rho^* \psi_\rho \right) \tag{18}$$

is a constant. Since this is true for all arbitrary spatial functions, it will then follow that:

$$\operatorname{div} \mathfrak{E} + e \sum_\rho \psi_\rho^* \psi_\rho = \text{const.} = C. \tag{19}$$

Therefore, only (19) will follow from the formulation that is carried out here in place of (14), in which C represents an arbitrary spatial function. However, it must now be observed that every system of values for C represents a single system of terms that does not combine with the remaining terms, and changes of C are completely impossible. Any sort of interaction terms or perturbations of the Hamiltonian function will leave the invariance under (15) unaffected. Extra terms of the type of ε and δ terms in I are then generally inadmissible; however, it seems justified to assume that only quantities that are invariant under (13) have any physical meaning. Following **Weyl**, we call such quantities gauge-invariant.

The commutation rules for the quantities C will be formulated most simply with the help of the quantities:

$$\bar{C} = \int \chi(\operatorname{div}\mathfrak{E} + e\sum_{\rho}\psi_{\rho}^{*}\,\psi_{\rho})\,dV = \int \chi\cdot C\,dV\,, \tag{20}$$

in which χ once more means an arbitrary spatial function. From I, (47) and (57), one finds that:

$$[\bar{C},\,\psi_\rho] = -\,e\chi\,\psi_\rho, \quad [\bar{C},\psi_\rho^{*}] = e\chi\,\psi_\rho^{*}, \qquad [\bar{C},\,\Phi_k] = \frac{hc}{2\pi i}\frac{\partial\chi}{\partial x_k}. \tag{21}$$

However, that means that the transformation (15) will be mediated by infinitesimal variation of χ according to:

$$f \to f + \frac{2\pi i}{hc}[\delta\bar{C},\,f\,], \tag{22}$$

when f is replaced with any of the quantities ψ_ρ, ψ_ρ^{*}, Φ_k. Therefore, (22) is also true for the variation of an arbitrary quantity f under (15). Let it also be mentioned that this relation generalizes to:

$$f \to e^{\frac{2\pi i}{h}\bar{C}}\cdot f\cdot e^{-\frac{2\pi i}{h}\bar{C}} \tag{22$'$}$$

for the finite transformation (13). In particular, for gauge-invariant quantities – for them, it is mainly:

$$F_{\mu\nu}, \quad \psi_\rho^{*}\psi_\sigma, \qquad \psi_\rho^{*}\left(\frac{hc}{2\pi i}\frac{\partial\psi_\sigma}{\partial x_\mu} + e\,\psi_\sigma\Phi_\mu\right), \qquad \left(\frac{hc}{2\pi i}\frac{\partial\psi_\rho^{*}}{\partial x_\mu} - e\,\Phi_\mu\psi_\rho^{*}\right)\psi_\sigma \tag{23}$$

that comes under consideration – it follows that:

$$[\bar{C},\,F] = 0,$$

and thus, also:

$$[C,\,F] = 0; \tag{24}$$

i.e., they commute with C. If one represents the variables of the system of matrices then the gauge-invariant quantities will include no elements that correspond to transitions of C, although the other non-gauge-invariant quantities will probably include such matrix elements. Since directly-measurable quantities are always gauge-invariant, one can give a numerical value to the constant C. In particular, if one chooses:

$$C = 0 \tag{25}$$

then the fourth component of **Maxwell**'s equations will also be true; indeed, it that will not generally be true as a q-number relation, but probably for all gauge-invariant relationships. $C = 0$ means that the operator (17) will give zero when applied to the Schrödinger functional $F(\psi_\rho, \Phi_i)$ of any stationary state of the system; i.e., the solutions for which the Schrödinger functional is likewise invariant under (15) is singled by $C = 0$.

One can give a number of independent gauge-invariant C. R., from which, all other gauge-invariant C. R. are derivable. In essence, they are the quantities (23), and they must be identical with the ones that can be derived from paper I. They also propagate in time according to I, equation (21). It is therefore quite convenient to employ C. R. between Φ_i and $\mathfrak{E}_i$, and thus, in-gauge-invariant quantities. In the many-body problem of point mechanics, that will correspond to the fact that the equations $p_k\, q_l - q_l\, p_k = \frac{h}{2\pi i}\,\delta_{kl}$ will used for the derivation of $p_k = \frac{h}{2\pi i}\frac{\partial}{\partial q_k}$, although ultimately such C. R. cannot even be defined in the chosen antisymmetric system.

The relativistic invariance of the schema that we just wrote down seems doubtful, at first, since Φ_4 would be singled out by the Φ_k . Before we investigate that question (§ 5), we shall first treat the Lorentz group by a method that is analogous to the one that was employed for the other groups up to now in the case of a relativistically-invariant Lagrangian function C. C. R. (e.g., the Lagrange function that was endowed with ε-terms and was used in I).

§ 4. Lorentz transformation (*). The invariance of the Hamiltonian function under spatial rotations corresponds to the angular impulse law. The method that was employed up to now must be modified somewhat for proper Lorentz transformations, since the Hamiltonian function is not invariant under them, as it and the components of the impulse collectively behave like the components of a four-vector. However, we will see that the proper Lorentz transformation corresponds to three more integrals. Once more, it suffices to consider infinitesimal transformations [I, equation (33′)]:

$$x_\mu \to x_\mu + \varepsilon\, s_{\mu\nu}\, x_\nu \qquad\qquad (s_{\mu\nu} = -\, s_{\mu\nu}). \tag{26}$$

(*) Essential parts of this paragraphs – in particular, the expression (30) for Λ and the proof of the constancy in time of the associated volume integrals (29) – are due to **J. v. Neumann**, to whom, we extend our deepest thanks for informing us of his results.

(Here, and in what follows, equal indices will always be summed over.) The wave functions then change on two grounds: First of all, the Q_α are not scalars, in general, but they transformation in a prescribed way at a well-defined world-point; furthermore, they change the world-point to which Q_α refers. With the relations of I, one will then get [I, equation (34′), (35′), (9)]:

$$Q_\alpha \to Q_\alpha + \varepsilon\, t_{\alpha\beta}\, Q_\beta - \varepsilon\, \frac{\partial Q_\alpha}{\partial x_\mu}\, s_{\mu\nu}\, x_\nu\,, \tag{27a}$$

$$P_{\alpha 4} \to P_{\alpha 4} - \varepsilon\, t_{\beta\alpha}\, P_{\beta 4} - \varepsilon\, \frac{\partial H}{\partial \dfrac{\partial Q_\alpha}{\partial x_k}}\, s_{4k}\, x_k - \varepsilon\, \frac{\partial P_{\alpha 4}}{\partial x_\mu}\, s_{\mu\nu}\, x_\nu\ . \tag{27b}$$

We now seek an operator $\bar{\Lambda}$ such that:

$$Q_\alpha \to Q_\alpha + \varepsilon\, \frac{2\pi}{hc}[\bar{\Lambda}, Q_\alpha]\,. \tag{28a}$$

Such an operator is given by [cf., relation I, (7) between the Hamiltonian and Lagrangian function]:

$$\bar{\Lambda} = \int \Lambda\, dV\,, \tag{29}$$

$$\begin{aligned} \Lambda &= \left(t_{\alpha\beta} Q_\beta - \frac{\partial Q_\alpha}{\partial x_k}\, s_{k\nu}\, x_\nu \right) P_{\alpha 4} - H\, s_{4k}\, x_k\,, \\ &= \left(t_{\alpha\beta} Q_\beta - \frac{\partial Q_\alpha}{\partial x_k}\, s_{k\nu}\, x_\nu \right) P_{\alpha 4} + L\, s_{4k}\, x_k\,. \end{aligned} \tag{30}$$

In fact, if one recalls that:

$$\frac{2\pi}{hc}[\bar{H}, F] = \frac{\partial F}{\partial x_4}\,, \qquad\qquad \frac{2\pi}{hc}[P_{\alpha 4}, Q'_\beta] = \delta_{\alpha\beta}\, \delta(\mathfrak{r}, \mathfrak{r}')$$

then an expression that agrees with the right-hand side of (27a) will follow immediately upon substituting (29), (30) into (28a).

However, with the same $\bar{\Lambda}$, one also has the equation:

$$P_{\alpha 4} \to P_{\alpha 4} + \varepsilon\, \frac{2\pi}{hc}[\bar{\Lambda}, P_{\alpha 4}]\,. \tag{28b}$$

According to I, equation (20), it will then follow that:

$$\frac{2\pi}{hc}[\bar{\Lambda}, P_{\alpha 4}] = -\left(\frac{\partial \Lambda}{\partial Q_\alpha} - \frac{\partial}{\partial x_i}\frac{\partial \Lambda}{\partial \frac{\partial Q_\alpha}{\partial x_i}}\right)$$

$$= - t_{\beta\alpha} P_{\beta 4} - \frac{\partial}{\partial x_i}(s_{i\nu}\, x_\nu\, P_{\alpha 4}) + \frac{\partial H}{\partial Q_\alpha} s_{ik}\, x_k - \frac{\partial}{\partial x_i}\left(\frac{\partial H}{\partial \frac{\partial Q_\alpha}{\partial x_i}} s_{4k} x_k\right),$$

and with the use of the expression for $\partial P_{\alpha 4} / \partial x_4$, it will follow from the field equation that:

$$\frac{2\pi}{hc}[\bar{\Lambda}, P_{\alpha 4}] = - t_{\beta\alpha} P_{\beta 4} - \frac{\partial P_{\alpha 4}}{\partial x_i} s_{i\nu}\, x_\nu - \frac{\partial P_{\alpha 4}}{\partial x_4} s_{4k}\, x_k - \frac{\partial H}{\partial \frac{\partial Q_\alpha}{\partial x_k}} s_{4k},$$

in agreement with (27b).

It follows by generalizing (28a, b) that an arbitrary quantity F that does not include the coordinates explicitly will go to:

$$F \to F + \varepsilon \frac{2\pi}{hc}[\bar{\Lambda}, F] \tag{31}$$

under an infinitesimal Lorentz transformation. For finite Lorentz transformations, it will follow from this that there exists an operator S such that one has:

$$F \to S\, F\, S^{-1} \tag{31′}$$

for it. If one develops S in powers of ε then the term that is linear in ε will be given by:

$$S = 1 + \varepsilon \frac{2\pi}{hc}\bar{\Lambda} + \dots \tag{32}$$

However, we have not succeeded in finding an explicit expression for S for non-infinitesimal transformations. The Schrödinger functions or functional φ will be transformed in a corresponding way under Lorentz transformations according to:

$$\varphi \to S\, \varphi,$$

in which S is regarded as an operator that acts upon the variables that are included in φ.

We must now answer the question of whether $\bar{\Lambda}$ depends upon the time coordinate x_4. We will show that this is not the case, under the assumption that:

$$J_\mu = \int \left(\frac{\partial Q_\alpha}{\partial x_\mu} P_{\alpha 4} - \delta_{\mu 4} L \right) dV$$

define the components of a four-vector (viz., energy-impulse, $J_k = - i\, c\, \mathfrak{E}_k$, $J_4 = \bar{H}$). This means that for the infinitesimal transformation (26), one should have:

$$J_\mu \rightarrow J_\mu + \varepsilon\, s_{\mu\nu}\, J_\nu\,,$$

and in particular:

$$\bar{H} \rightarrow \bar{H} + \varepsilon\, s_{4k}\, J_k\,.$$

A comparison with (31) will then give:

$$\frac{2\pi}{hc}[\bar{\Lambda}, \bar{H}] = s_{4k}\, J_k\,. \tag{33}$$

Moreover, it is easy to calculate $d\bar{\Lambda}/dx_4$. For a quantity F that does not include x_4 explicitly, one would have simply:

$$\frac{\partial F}{\partial x_4} = -\frac{2\pi}{hc}[F, \bar{H}]\,;$$

however, for $F = \bar{\Lambda}$, one must add the term that arises by differentiating $\bar{\Lambda}$ with respect to the symbol x_4 that is included in it explicitly. The second term in (30) makes a contribution to this for $\nu = 4$, and one gets:

$$\begin{aligned} \frac{d\Lambda}{dx_4} &= -\frac{2\pi}{hc}[\bar{\Lambda}, \bar{H}] - \int \frac{\partial Q_\alpha}{\partial x_k} P_{\alpha 4} s_{k4} dV \\ &= -\frac{2\pi}{hc}[\bar{\Lambda}, \bar{H}] + J_k s_{k4}\,. \end{aligned} \tag{34}$$

This will vanish precisely as a result of (33), and we will then have:

$$\bar{\Lambda} = \text{const.} \tag{35}$$

This equation contains six independent integrals, corresponding to the six components $s_{\mu\nu} = - s_{\nu\mu}$ (the $t_{\alpha\beta}$ are determined uniquely by the $s_{\mu\nu}$), and three of them can be interpreted as belonging to s_{ik} as a result of the angular impulse theorem, while the other three that belong to s_{4k} have no such intuitive meaning. It must once more (cf., I) be emphasized that it is indeed essential that one must insure that the temporal constancy of the integral must be independent of the sequence of factors in (4) and (30).

The invariance of the C. C. R. under Lorentz transformations follows immediately from (31) or (31′). The proof of invariance that was carried out here is probably somewhat simpler than the one that was given in I. However, it must be stressed that the vector character of J_ν represents a new assumption that cannot be deduced from the

Lorentz invariance of the Lagrangian function alone. By contrast, this assumption always enters into consideration when a differential formulation of the energy-impulse theorem exists in the form of the vanishing of a tensor divergence:

$$\frac{\partial T_{\mu\nu}}{\partial x_\nu} = 0.$$

As would emerge from I, this is always applicable to any physically-important case.

§ 5. Lorentz transformations and gauge invariance. In § 3, we spoke of a process in which one sets $\Phi_4 = 0$ in a special coordinate system and then applies the C. C. R. to it. In it, the equation:

$$C = \operatorname{div} \mathfrak{E} + e\sum_\rho \psi_\rho^* \psi_\rho = 0 \tag{25}$$

is valid only for gauge-invariant quantities as q-number relations, while the other quantities – e.g., the ψ and Φ_μ – do not commute them. However, since C commutes with the energy, it can nevertheless be employed as an auxiliary condition for the Schrödinger functional.

Such a process is not intrinsically relativistically invariant. In another reference system, the C. C. R. will no longer apply to non-gauge-invariant quantities. However, one can show that all statements about gauge-invariant quantities that are obtained in that way will satisfy the requirement of relativistic invariance when one adds the equation (25). To that end, we next establish the gauge invariance of the Hamiltonian function, and above all, the quantity $\bar{\Lambda}$ that was found to be definitive for the Lorentz transformation in the previous paragraphs. According to I, equation (45), (51), (51′), (58′) (when we omit the terms that are endowed with ε, and set $P_{44} \equiv 0$ for the radiation), we will have the following Lagrangian and Hamiltonian functions for the matter and radiation parts, respectively:

$$L^{(m)} = -\left[\psi_\sigma^*\left(\frac{hc}{2\pi}\frac{\partial\psi_\sigma}{\partial x_4} + ei\,\psi_\sigma\Phi_4\right) + \alpha_{\rho\sigma}^k\psi_\rho^*\left(\frac{hc}{2\pi i}\frac{\partial\psi_\sigma}{\partial x_k} + e\,\psi_\sigma\Phi_k\right) + mc^2\alpha_{\rho\sigma}^k\psi_\rho^*\psi_\sigma\right], \tag{36a}$$

$$\begin{aligned} H^{(m)} &= -\frac{hc}{2\pi}\psi_\sigma^*\frac{\partial\psi_\sigma}{\partial x_4} - L^{(m)} \\ &= \alpha_{\rho\sigma}^k\psi_\rho^*\left(\frac{hc}{2\pi i}\frac{\partial\psi_\sigma}{\partial x_k} + e\,\psi_\sigma\Phi_k\right) + mc^2\alpha_{\rho\sigma}^k\psi_\rho^*\psi_\sigma + ei\psi_\sigma^*\psi_\rho\Phi_4, \end{aligned} \tag{37a}$$

$$L^{(s)} = -\tfrac{1}{4}F_{\alpha\beta}F_{\alpha\beta} = \tfrac{1}{2}(\mathfrak{E}^2 - \mathfrak{H}^2), \tag{36b}$$

$$H^{(s)} = -F_{4k}\frac{\partial\Phi_k}{\partial x_4} - L^{(s)} = -F_{4k}\frac{\partial\Phi_4}{\partial x_k} - \frac{1}{2}F_{4k}F_{4k} + \frac{1}{4}F_{ik}F_{ik}\,. \tag{37b}$$

As one sees, $H^{(m)}$ and $H^{(s)}$ are not gauge-invariant, in contrast to $L^{(m)}$ and $L^{(s)}$. On the other hand, the total energy can be transformed by partial integration into:

$$\bar{H} = \int (H^{(m)} + H^{(s)})\, dV =$$

$$\int \left[\alpha^k_{\rho\sigma}\psi^*_\rho\left(\frac{hc}{2\pi i}\frac{\partial\psi_\sigma}{\partial x_k} + e\psi_\sigma\Phi_{k\frac{\partial^2\Omega}{\partial u\partial v}}\right) + mc^2\alpha^4_{\rho\sigma}\psi^*_\rho\psi_\sigma - \tfrac{1}{2}F_{4k}F_{4k} + \tfrac{1}{2}F_{ik}F_{ik} + i\Phi_4 C\right] dV. \quad (38)$$

A similar conversion is true for the total impulse. $\bar{H}$ is then gauge-invariant in the case of $C = 0$, and it is also the time component of a four-vector in only that case.

The calculation of the quantity $\bar{\Lambda}$ that is defined by (29) and (30) takes a similar form. We now understand $t_{\rho\sigma}$ to be quantities that relate to the matter waves, in particular, while the associated $t_{\mu\nu}$ for the Φ_μ will vanish identically, due to their vector character. As a result, one will have:

$$\left.\begin{aligned} &\bar{\Lambda} = \int dV \\ &\times\left[-\frac{hc}{2\pi}\psi^*_\rho\left(t_{\rho\sigma}\psi_\sigma - \frac{\partial\psi_\rho}{\partial x_\mu}s_{\mu\nu}x_\nu\right) + L^{(m)}s_{4k}x_k - F_{4k}\left(s_{k\mu}\Phi_\mu - \frac{\partial\Phi_k}{\partial x_\mu}s_{\mu\nu}x_\nu\right) + L^{(s)}s_{4k}x_k\right]. \end{aligned}\right\} (39)$$

However, one has:

$$\begin{aligned} &\int dV F_{4k}\left(\frac{\partial\Phi_k}{\partial x_\mu}s_{\mu\nu}x_\nu - s_{k\mu}\Phi_\mu\right) \\ &= \int dV F_{4k}\left(F_{\mu k}s_{\mu\nu}x_\nu + \frac{\partial\Phi_\mu}{\partial x_k}s_{\mu\nu}x_\nu + s_{\mu k}\Phi_\mu\right) \\ &= \int dV F_{4k}\left[F_{\mu k}s_{\mu\nu}x_\nu + \frac{\partial}{\partial x_k}(\Phi_k s_{\mu\nu}x_\nu)\right] \\ &= \int dV\left(F_{4k}F_{\mu k}s_{\mu\nu}x_\nu - \frac{\partial F_{4k}}{\partial x_k}\Phi_\mu s_{\mu\nu}x_\nu\right), \end{aligned}$$

where the last step follows by partial integration. In all, one gets:

$$\begin{aligned} \bar{\Lambda} = \int dV &\left[-\frac{hc}{2\pi}t_{\rho\sigma}\psi^*_\rho\psi_\sigma + \psi^*_\rho\left(\frac{hc}{2\pi}\frac{\partial\psi_\rho}{\partial x_\mu} + ie\psi_\rho\Phi_\mu\right)s_{\mu\nu}x_\nu\right. \\ &\left. + L^{(m)}s_{4k}x_k + F_{4k}F_{\mu k}s_{\mu\nu}x_\nu + L^{(s)}s_{4k}x_k - i\,C\,s_{\mu\nu}\Phi_\mu x_\nu\right]. \end{aligned} \quad (40)$$

$\bar{\Lambda}$ will then be gauge-invariant for $C = 0$.

One obtains the values of all quantities in the new reference system from (39), according to formula (31), except for the value of Φ_4, when $\Phi_4 = 0$ in the original system, and one assume the C. C. R. However, for non-gauge-invariant quantities, their non-commutation with C and the contribution to the last term in (40) that arise from it must be considered. They are easily inferred by comparison with (21). One can deduce two kinds of conclusions from this state of affairs. First of all, the C. R. for the gauge-invariant quantities in the new reference system follow from their validity in the original reference system independently of what sort of C. R. are true for the remaining quantities. Only the former C. R. are then necessary for the proof of the validity of (31) by gauge invariance. Secondly, one can show that one can also revert to $\Phi_4 = 0$ and the C. C. R. in the new reference system by a change of gauge that involves a suitable function χ. Generally, that χ will be a q-number.

However, it is unnecessary to go into that change of gauge in more detail in order to show the Lorentz invariance of the entire process. Moreover, it will suffice for that to establish that the C. C. R between the quantities ψ_ρ, ψ_σ^*, Φ_k, F_{i4} still remain valid in the new reference system and that the Φ_4 commutes with all Φ_k and ψ_ρ, ψ_σ^*, as one easily verifies. Furthermore, the spatial components of **Maxwell**'s equations are no longer fulfilled as q-number relations in the new reference system; nevertheless, one can choose the eigenvalue zero on their right hand sides by singling out a subsystem of terms that does not combine with the remaining terms, which would correspond to the choice of $C = 0$ in the original reference system. If one further observes that Φ_4 does not enter into the Hamiltonian function at all for $C = 0$, and that the expression $\frac{hc}{2\pi}\frac{\partial \psi_\sigma}{\partial x_4} + ei\,\psi_\rho\,\Phi_4$ in the other equations can be expressed in terms of the ψ_ρ, Φ_k, and their derivatives by means of the matter-wave equation then one will recognize the identity of the computational schema in the new reference system with the one in the initial system.

§ 6. Implementing the schema with no extra terms. We revert to real time $x_4 = i\,c\,t$ and to the usual units for field strengths and the current vector, introduce the quantities:

$$\Pi_k = -\frac{1}{4\pi c}\,\mathfrak{E}_k, \tag{41}$$

which satisfy the C. C. R. $[\Pi_i, \Phi'_r] = \frac{h}{2\pi i}\,\delta_{ik}\cdot\delta(\mathfrak{r}, \mathfrak{r}')$ [cf., I, (60′), (61′)], and set $\Phi_4 = \Phi_0 = 0$ in the coordinate system that was chosen for the treatment. The Hamiltonian function (37a, b) now reads:

$$\bar{H} = \int dV\left[\frac{hc}{2\pi i}\alpha^k_{\rho\sigma}\psi^*_\rho\frac{\partial\psi_\rho}{\partial x_k} + mc^2\alpha^k_{\rho\sigma}\psi^*_\rho\psi_\sigma + \frac{1}{16\pi}\left(\frac{\partial\Phi_i}{\partial x_k} - \frac{\partial\Phi_k}{\partial x_i}\right)^2 + 2\pi^2\Pi_k^2 + e\,\Phi_k\alpha^k_{\rho\sigma}\psi^*_\rho\psi_\sigma\right] \tag{42}$$

The last term mediates the interaction between radiation and matter, and will be considered to be a perturbing term. For the implementation of the method, as in I, it will be convenient to develop the Φ_i in an orthogonal system that will be found by solving the unperturbed problem. In contradiction to the previous methods, only the three spatial components of the **Maxwell** equations will be fulfilled in the unperturbed problem. We again set [cf., I, equation (84)]:

$$\left.\begin{aligned} \Phi_1 &= \sqrt{\frac{8}{L^3}}\, q_1^r \cos\frac{\pi}{L}\kappa_r x \cdot \sin\frac{\pi}{L}\lambda_r y \cdot \sin\frac{\pi}{L}\mu_r z \qquad \text{(and cycl. permutations)}, \\ \Pi_1 &= \sqrt{\frac{8}{L^3}}\, p_1^r \cos\frac{\pi}{L}\kappa_r x \cdot \sin\frac{\pi}{L}\lambda_r y \cdot \sin\frac{\pi}{L}\mu_r z. \end{aligned}\right\} \tag{43}$$

The radiation part of the Hamiltonian function then becomes:

$$\begin{aligned} \overline{H_r^{(s)}} &= 2\pi c^2 \,[(p_1^r)^2 + (p_2^r)^2 + (p_3^r)^2] \\ &+ \frac{\pi}{8L^3} \cdot [(q_1^r \lambda_r - q_2^r \kappa_r)^2 + (q_1^r \mu_r - q_3^r \kappa_r)^2 + (q_2^r \mu_r - q_3^r \lambda_r)^2] \end{aligned} \tag{44}$$

for an eigen-oscillation. If one then sets $q_i^r = b_i^r \sin 2\pi \nu_r t$ in the classical theory then one will obtain three linear equations for the b_i from the three spatial components of **Maxwell**'s equations whose determinant is:

$$\left(\nu'_r = \frac{2L}{c}\nu_r,\ X_r = \kappa_r^2 + \lambda_r^2 + \mu_r^2 - \nu_r'^2\right)$$

$$\begin{vmatrix} X_r - \kappa_r^2 & -\kappa_r\lambda_r & -\kappa_r\mu_r \\ -\lambda_r\kappa_r & X_r - \lambda_r^2 & -\lambda_r\mu_r \\ -\mu_r\kappa_r & -\mu_r\lambda_r & X_r - \mu_r^2 \end{vmatrix}. \tag{45}$$

By setting the determinant equal to zero, one will obtain a double root

$$\nu'_r = \sqrt{\kappa_r^2 + \lambda_r^2 + \mu_r^2}\,,$$

whose associated coefficients b_i^r must satisfy the condition. It then defines an aperiodic solution:

$$q_i^r = b_i^r \cdot t, \tag{47}$$

in which:

$$\frac{b_1^r}{\kappa_r} = \frac{b_2^r}{\lambda_r} = \frac{b_3^r}{\mu_r}.$$

We introduce the coordinates P^r, Q^r of the principal oscillations as a possible schema:

$$\left.\begin{aligned}
\frac{1}{\sqrt{4cL}}q_1^r &= \frac{\lambda_r}{\sqrt{\nu_r'(\kappa_r^2+\lambda_r^2)}}Q_1^r + \frac{\mu_r\kappa_r}{\nu_r'\sqrt{\nu_r'(\kappa_r^2+\lambda_r^2)}}Q_2^r + \frac{\kappa_r}{\nu_r'\sqrt{\nu_r'}}Q_3^r,\\
\frac{1}{\sqrt{4cL}}q_2^r &= \frac{\lambda_r}{\sqrt{\nu_r'(\kappa_r^2+\lambda_r^2)}}Q_1^r + \frac{\mu_r\lambda_r}{\nu_r'\sqrt{\nu_r'(\kappa_r^2+\lambda_r^2)}}Q_2^r + \frac{\lambda_r}{\nu_r'\sqrt{\nu_r'}}Q_3^r,\\
\frac{1}{\sqrt{4cL}}q_3^r &= -\frac{\sqrt{\kappa_r^2+\lambda_r^2}}{\nu_r'\sqrt{\nu_r'}}Q_2^r + \frac{\mu_r}{\nu_r'\sqrt{\nu_r'}}Q_3^r,\\
\frac{1}{\sqrt{4cL}}p_1^r &= \lambda_r\sqrt{\frac{\nu_r'}{\kappa_r^2+\lambda_r^2}}P_1^r + \frac{\mu_r\lambda_r}{\sqrt{\nu_r'(\kappa_r^2+\lambda_r^2)}}P_2^r + \frac{\kappa_r}{\sqrt{\nu_r'}}P_3^r,\\
\frac{1}{\sqrt{4cL}}p_2^r &= -\kappa_r\sqrt{\frac{\nu_r'}{\kappa_r^2+\lambda_r^2}}P_1^r + \frac{\mu_r\lambda_r}{\sqrt{\nu_r'(\kappa_r^2+\lambda_r^2)}}P_2^r + \frac{\lambda_r}{\sqrt{\nu_r'}}P_3^r,\\
\frac{1}{\sqrt{4cL}}p_3^r &= -\frac{\sqrt{\kappa_r^2+\lambda_r^2}}{\sqrt{\nu_r'}}P_2^r + \frac{\mu_r}{\sqrt{\nu_r'}}P_3^r.
\end{aligned}\right\} \tag{48}$$

The radiation part of the Hamiltonian function reads:

$$\bar{H}_s = \sum_r 2\pi\nu_r\left\{\tfrac{1}{2}[(P_1^r)^2+(Q_1^r)^2]+\tfrac{1}{2}[(P_2^r)^2+(Q_2^r)^2]+\tfrac{1}{2}(P_3^r)^2\right\} \tag{49}$$

in the new variables.

As in I, equation (98), one introduce the number of light quanta $M_{r,1}$ ($M_{r,2}$, resp.) in place of P_1^r, Q_1^r, P_2^r, Q_2^r, along with the conjugate angles, as variables.

$$\left.\begin{aligned}
Q_\lambda^r &= \frac{1}{i}\sqrt{\frac{h}{4\pi}}\left(M_{r\lambda}^{1/2}\,e^{\frac{2\pi i}{h}\chi_{r,\lambda}} - e^{-\frac{2\pi i}{h}\chi_{r,\lambda}}M_{r\lambda}^{1/2}\right),\\
P_\lambda^r &= \sqrt{\frac{h}{4\pi}}\left(M_{r\lambda}^{1/2}\,e^{\frac{2\pi i}{h}\chi_{r,\lambda}} + e^{-\frac{2\pi i}{h}\chi_{r,\lambda}}M_{r\lambda}^{1/2}\right),
\end{aligned}\right\}\ \lambda = 1, 2. \tag{50}$$

By contrast, such a substitution would make no sense for P_3^r, since Q_3^r is not present in the unperturbed Hamiltonian function, so P_3^r is itself a constant in the unperturbed system. We them employ the N_r, $M_{r\lambda}$ (λ = 1, 2), and P_3^r as independent variables of the probability amplitude. If one assumes the exclusion principle for matter then the Schrödinger equation will read:

$$\left.\begin{aligned}
&[-E+\sum_s N_s E_s+\sum_{r,\lambda} M_{r,\lambda} h\nu_r+\sum_r \pi\nu_r (P_3^r)^2]\,\varphi(N_1,N_2,\ldots,M_1,\ldots,P_3^1,\ldots)\\
&=-ie\sqrt{\frac{h}{4\pi}}\sum_{s,t,r} N_s(1-N_t)\,\mathcal{V}_s(N_1,\ldots,1-N_s)\mathcal{V}_t(N_1,\ldots,1-N_t)\\
&\cdot\Big\{\sum_{\lambda=1,2}{}' c_{st}^{r\lambda}[M_{r\lambda}^{1/2}\varphi(N_1,\ldots,1-N_s,1-N_t;M_1,\ldots,M_{r,\lambda}-1,P_3^1,\ldots)\\
&-(M_{r\lambda}+1)^{1/2}\varphi(N_1,\ldots,1-N_s,1-N_t;M_1,\ldots,M_{r,\lambda}-1,P_3^1,\ldots)\\
&+c_{st}^{r3}\sqrt{\frac{h}{\pi}}\frac{\partial}{\partial P_3^r}\varphi(N_1,\ldots,1-N_s,1-N_t;M_1,\ldots,M_{r,\lambda}-1,P_3^1,\ldots)\Big\}\\
&-ie\sqrt{\frac{h}{4\pi}}\sum_{s,r}{}' N_s\Big\{\sum_{\lambda=1,2} c_{ss}^{r\lambda}[M_{r\lambda}^{1/2}\varphi(N_1,\ldots;M_1,\ldots,M_{r,\lambda}-1,P_3^1,\ldots)\\
&-(M_{rs}+1)^{1/2}\varphi(N_1,\ldots;M_1,\ldots,M_{r,\lambda}+1,P_3^1,\ldots)]\\
&+\sqrt{\frac{h}{\pi}}c_{ss}^{r3}\frac{\partial}{\partial P_3^r}\varphi(N_1,\ldots,M_1,\ldots,P_3^1,\ldots)\Big\}
\end{aligned}\right\}\tag{51}$$

in these variables.

However, along with this equation, φ must satisfy the further condition that the operator C must give zero when it is applied to φ. It reads:

$$\left.\begin{aligned}
&P_3^r\,\varphi(N_1,\ldots,M_1,\ldots,P_3^1,\ldots)+\sum_{s,t}{}' N_s(1-N_t)\cdot\mathcal{V}_s(N_1,\ldots,1-N_s)\mathcal{V}_t(N_1,\ldots,1-N_t)\\
&\quad\cdot d_{st}^r\,\varphi(N_1,\ldots,1-N_s,1-N_t,\ldots,M_1,\ldots,P_3^1,\ldots)\\
&\quad+\sum_s N_s\,d_{ss}^r\varphi(N_1,\ldots,M_1,\ldots,P_3^1,\ldots)=0.
\end{aligned}\right\}\tag{52}$$

In this, we have set:

$$d_{st}^r=\int u_\rho^{*s}\,u_\rho^t\,v_r^0\,dV\,,\tag{53}$$

where:

$$v_r^0=\frac{4}{\pi}\sqrt{\frac{2}{c\,\nu_r'^3}}\sin\frac{\pi}{L}\kappa_r x\cdot\sin\frac{\pi}{L}\lambda_r y\cdot\sin\frac{\pi}{L}\mu_r z\,.\tag{54}$$

In the unperturbed system, in which the interaction between matter and radiation can be neglected, from (52), one will have:

$$P_3^r=0.\tag{55}$$

All that remains then are the two known principal oscillations 1 and 2. However, the P_3^r must also be considered in the unperturbed system, which brings with it some differences from the previous schema that is due to the continuous eigenvalue spectrum of the P_3^r.

In what follows, as in I, we will recalculate only the electrostatic interaction; in the meantime, the magnetic and retarded effects will be ascertained by the method of **Breit** (†) in I.

For the electrostatic interaction, one expresses the operator P_3^r in (51) most simply by (52). One can then neglect the terms with c_{st}^r in (51) in comparison to the terms with d_{st}^r in the first approximation. Only the temporal mean of $\sum_r \pi\nu_r (P_3{}^r)^2$ remains as the perturbing energy in that approximation, in which P_3^r is replaced with the operator in (51). It will then follow that the perturbation of the eigenvalue is:

$$\Delta E = e^2 \sum_{r,s,t}{}' \pi\nu_r N_s^0 (1 - N_t^0)\, d_{st}^r d_{ts}^r + e^2 \sum_{r,s,t} \pi\nu_r N_s^0 N_t^0\, d_{ss}^r d_{tt}^r\,. \tag{56}$$

(Let N_s^0 be the value of N_s in the unperturbed system.)

In complete analogy to the calculation in I, one will then find that:

$$\Delta E = \frac{e^2}{2}\left[\sum_{s,t}{}' N_s^0 (1 - N_t^0)\, A_{st,ts} + \sum_{s,t} N_s^0 N_t^0 A_{ss,tt}\right], \tag{57}$$

in which $A_{st,\,ts}$ means the exchange integral (I, 114):

$$A_{st,\,nm} = \int dV \cdot dV' \frac{u_\rho^{*s}(P)\, u_\rho^{*t}(P) u_\rho^{*n}(P') u_\rho^{*m}(P')}{r_{PP'}}\,.$$

The u_ρ^s represent the orthogonal system in which the matter eigenfunction is developed.

It emerges from (57) that an infinite interaction of the electron with itself will also result from the method that is followed here that will make the application of the theory impossible in many cases. The only advantage of the method that is described here then consists of the fact that it makes the extra terms in the **Maxwell** equations superfluous.

§ 7. Transition to configuration space (††). In this section, we will treat the question of how one can calculate (say, for a given energy) the probability that for a given number of light quanta $M_{r,\lambda}$ (λ = 1, 2) and a given $P_{r,3}$ the locations of the N electrons that are present will lie inside of the volume $dq_{i1} \ldots dq_{ip} \ldots dq_{iN}$ around the location $q_{i1} \ldots q_{ip} \ldots q_{iN}$. The index i runs from 1 to 3 and refers to the three spatial coordinates, the index p runs from 1 to N and refers to the different particles. One sees

(†) **G. Breit**, Phys. Rev. **34** (1929), 553.

(††) **R. Oppenheimer** gave us friendly encouragement to elaborate upon this method, and we would like to express our thanks to him at this point.

that the total number of particles present can be assumed to be constant, such that annihilation processes will be excluded at first. We further preserve the Fourier decomposition of the radiation field, in contrast to that of the matter waves, since for the time being that is the only way to eliminate the zero-point energy of the radiation. We will show that probability amplitudes:

$$\varphi_{\rho_1\dots\rho_N}(q_{i1}\dots q_{iN}, M_{r\lambda}, P_{r3})$$

can be defined, in which the indices ρ_p can assume four values for each p, corresponding to the four wave functions of the **Dirac** theory of the spin electron, and from which the desired probability can be calculated from:

$$\sum_{\rho_1\dots\rho_N=1}^{4} |\, \varphi_{\rho_1\dots\rho_N}(q_{i1}\dots q_{iN}, M_{r\lambda}, P_{r3})\,|^2.$$

These functions satisfy simple differential equations, without it being necessary to introduce any sort of omissions or approximations. It is clear that the comparison of the results of the quantum theory of wave fields with those of the non-relativistic **Schrödinger** theory of the many-body problem (viz., waves in configuration space) will be eased by the introduction of such functions. One can also derive those functions along a detour to the functions $\Phi(N_s, M_{r\lambda}, P_{r3})$ that were defined in the previous paragraphs, but we prefer to follow a direct path.

First, we would like to exhibit the **Schrödinger** equation that belongs to the Hamiltonian function (42) and the auxiliary condition for the functional with the variables $N_\rho(x_i) = \psi_\rho^* \psi_\rho$, $M_{r\lambda}$, P_{r3} that corresponds to $C = 0$. The most important part of the argument will then be the transition from $N_\rho(x_i)$ to $q_{i1}, \rho_1, \rho_p, \dots q_{iN}, \rho_N$ as variables.

According to (43), (48), and (50), one will have:

$$\Phi_k = \sum_{\lambda=1,2}\sum_r \sqrt{\frac{h}{4\pi}}\, v_k^{r\lambda}\frac{1}{i}\left(M_{r\lambda}^{1/2} e^{\frac{2\pi i}{h}\chi_{r,\lambda}} - e^{-\frac{2\pi i}{h}\chi_{r,\lambda}} M_{r\lambda}^{1/2}\right) + \sum_r v_k^{r3} Q^{r3}, \tag{58a}$$

$$\Pi_k = \sum_{\lambda=1,2}\sum_r \sqrt{\frac{h}{4\pi}}\, \frac{\nu_r}{2c^2} v_k^{r\lambda}\frac{1}{i}\left(M_{r\lambda}^{1/2} e^{\frac{2\pi i}{h}\chi_{r,\lambda}} + e^{-\frac{2\pi i}{h}\chi_{r,\lambda}} M_{r\lambda}^{1/2}\right) + \sum_r \frac{\nu_r}{2c^2} v_k^{r3} P^{r3}. \tag{58b}$$

In this:

$$v_1^{r\lambda} = c\sqrt{\frac{2}{\nu_r}}\sqrt{\frac{8}{L^3}}\, f_k^{r\lambda}\cdot\cos\frac{2\pi\nu_r}{c}\varepsilon_{r1}x_1\cdot\sin\frac{2\pi\nu_r}{c}\varepsilon_{r2}x_2\,\sin\frac{2\pi\nu_r}{c}\varepsilon_{r3}x_3 \tag{59}$$

(and cyclic permutations),

if f_k^{λ} is set equal to the matrix:

$$
\begin{array}{c|ccc}
{}_k\backslash^{\lambda} & 1 & 2 & 3 \\
\hline
1 & \dfrac{\varepsilon_2}{\sqrt{\varepsilon_1^2+\varepsilon_2^2}} & \dfrac{\varepsilon_1\varepsilon_2}{\sqrt{\varepsilon_1^2+\varepsilon_2^2}} & \varepsilon_1 \\
2 & -\dfrac{\varepsilon_1}{\sqrt{\varepsilon_1^2+\varepsilon_2^2}} & \dfrac{\varepsilon_2\varepsilon_3}{\sqrt{\varepsilon_1^2+\varepsilon_2^2}} & \varepsilon_2 \\
3 & 0 & -\sqrt{\varepsilon_1^2+\varepsilon_2^2} & \varepsilon_3
\end{array}
\tag{48′}
$$

for each r. We see that the $\varepsilon_{r,k}$ are the components of the unit vector in the direction of the wave normal $\left(\sum_k \varepsilon_k^2 = 1\right)$, and for each r, we have set:

$$\kappa = \nu'\varepsilon_1, \qquad \lambda = \nu'\varepsilon_2, \qquad \mu = \nu'\varepsilon_3 \qquad \left(\nu' = \frac{2L}{c}\nu\right).$$

It follows from this that:

$$\operatorname{div}\mathfrak{E} = -4\pi c \operatorname{div}\Pi$$

$$= \sum_r \frac{8\pi^2}{c}\sqrt{\frac{\nu_r^3}{2}}\sqrt{\frac{8}{L^3}}\sin\frac{2\pi\nu_r}{c}\varepsilon_{r,1}x_1\cdot\sin\frac{2\pi\nu_r}{c}\varepsilon_{r,2}x_2\cdot\sin\frac{2\pi\nu_r}{c}\varepsilon_{r,3}x_3\cdot P_{r3},$$

which will yield the equation:

$$\operatorname{div}\mathfrak{E} + 4\pi e\sum_\rho \psi_\rho^*\psi_\rho = 0$$

is solved for P_{r3} by means of the **Fourier** theorem:

$$P_{r3} + e\int \upsilon_{0r}(x_i)\sum_\rho \psi_\rho^*\psi_\rho\, dV = 0, \tag{60}$$

in which υ_{0r} is defined by (54). Furthermore, from (42), when one drops the zero-point energy of the radiation, the Hamiltonian function will be:

$$\begin{aligned}
\bar{H} = {} & \sum_{r,\lambda} M_{r\lambda}h\nu_{r\lambda} + \sum_r \pi\nu_r(P_{r3})^3 + \int dV\left(\frac{he}{2\pi i}\alpha_{\rho\sigma}^k\psi_\rho^*\frac{\partial\psi_\sigma}{\partial x_k} + mc^2\alpha_{\rho\sigma}^k\psi_\rho^*\psi_\sigma\right) \\
& + e\sum_r\sqrt{\frac{h}{4\pi}}\cdot\sum_{\lambda=1,2}\frac{1}{i}\left(M_{r\lambda}^{1/2}\cdot e^{\frac{2\pi i}{h}\chi_{r\lambda}} - e^{-\frac{2\pi i}{h}\chi_{r\lambda}}\cdot M_{r\lambda}^{1/2}\right)\cdot\int \upsilon_k^{r\lambda}\alpha_{\rho\sigma}^k\psi_\rho^*\psi_\sigma dV \\
& + \sum_r Q^{r\lambda}\int \upsilon_k^{r\lambda}\alpha_{\rho\sigma}^k\psi_\rho^*\psi_\sigma dV\,.
\end{aligned} \tag{61}$$

We now write the two relations (60) and (61) as operator equations that act upon the function $\varphi\{N_\rho(x_i), M_{r\lambda}, P_{r3}\}$. For that, we consider that $e^{\pm\frac{2\pi i}{h}\chi}$ converts the value M into $M \mp 1$, resp. and that Q^{r3} is replaced with $\frac{ih}{2\pi}\frac{\partial}{\partial P_{r3}}$. We will then have:

$$\left(P_{r3} + e\int \upsilon_{0r}(x_i)\cdot\sum_\rho N_\rho(x_i)\,dV\right)\varphi\{N_\rho(x_i), M_{r\lambda}, P_{r3}\} = 0. \tag{60′}$$

$$\begin{aligned}
&\left[-E + \sum_{r,\lambda} M_{r\lambda} h\nu_{r\lambda} + \sum_r \pi\nu_r (P_{r3})^2\right]\psi\{N_\rho(x_i), M_{r\lambda}, P_{r3}\} \\
&+ \left[\int dV\left(\frac{hc}{2\pi i}\alpha^k_{\rho\sigma}\psi^*_\rho\frac{\partial\psi_\sigma}{\partial x_k} + mc^2\alpha^k_{\rho\sigma}\psi^*_\rho\psi_\sigma\right)\right]\varphi\{N_\rho(x_i), M_{r\lambda}, P_{r3}\} \\
&+ e\sum_r\sqrt{\frac{h}{4\pi}}\cdot i\cdot\sum_{\lambda=1,2}\left(\int \upsilon^{r\lambda}_k\,\alpha^k_{\rho\sigma}\psi^*_\rho\psi_\sigma dV\right) \\
&\times[(M_{r\lambda}+1)^{1/2}\,\varphi\{N_\rho(x_i), \ldots, M_1, M_{r\lambda}+1, \ldots, P_{r3}\} \\
&- M^{1/2}_{r\lambda}\,\varphi\{N_\rho(x_i), \ldots, M_1, M_{r\lambda}-1, \ldots, P_{r3}\}] \\
&+ e\sum_r\frac{ih}{2\pi}\left(\int \upsilon^{r3}_k\,\alpha^k_{\rho\sigma}\psi^*_\rho\psi_\sigma\,dV\right)\frac{\partial}{\partial P_{r3}}\,\varphi\{N_\rho(x_i), \ldots, M_1, \ldots, P_{r3}\} = 0.
\end{aligned} \tag{61′}$$

It is now important to see how operators of the form:

$$\int\sum_{\rho,\sigma} f_{\rho\sigma}(x_i)\psi^*_\rho\psi_\sigma dV \qquad \text{and} \qquad \int\sum_{\rho,\sigma} f_{\rho\sigma}(x_i)\psi^*_\rho\frac{\partial\psi_\sigma}{\partial x_k}dV$$

(the f are c-numbers) act upon a functional $\Phi\{N_\rho(x_i)\}$ when $N_\rho(x_i) = \psi^*_\rho\psi_\rho$, and one has the C. C. R.:

$$[\psi_\rho, \psi'^*_\sigma] = \delta_{\rho\sigma}\,\delta(\mathfrak{r}, \mathfrak{r}');$$

moreover, one wishes to know the result when it acts upon $\varphi(q_{i1}, \ldots, q_{ip}, \ldots, q_{iN})$.

The required transformation theory has been developed several times already (†). However, it is convenient to first replace the $\psi(x_i)$ with step functions, and then to go to configuration space, and only at the end will the functions once more be allowed to become continuous. Thus, let the cells inside of which ψ^* and ψ have equal values be chosen to have then same volumes ΔV, and set:

$$a_{\rho,x_i} = \psi_\rho(x_i), \qquad a^*_{\rho,x_i} = \psi^*_\rho(x_i)\;\Delta V,$$

such that one will have:

(†) **P. A. M. Dirac**, Proc. Roy. Soc. (A) **114** (1927), 243; **P. Jordan** and **O. Klein**, Zeit. Phys. **45** (1927), 751; **P. Jordan**, *ibidem* **45** (1927), 766; **P. Jordan** and **E. Wigner**, *ibidem*, **47** (1928), 631.

$$[\, a_{\rho,x_i}\,,\ a^*_{\sigma,x_i}\,]_\pm = \delta_{\rho\sigma}\delta_{x_i x_{i'}}\,,$$

in which the x_i run through only discrete values. One sees that for a fixed total number N of particles (this assumption is essential at first):

$$N_{\rho,x_i} = a^*_{\rho,x_i}\, a_{\rho,x_i}$$

will possess the eigenvalue:

$$\sum_{p=1}^{N} \delta_{\rho\rho_p}\,\delta_{x_i q_{i_p}}\,,$$

in which several pairs of values ρ_p, q_p can also coincide.

$$N_\rho\,(x_i) = \psi^*_\rho(x_i)\,\psi_\rho\,(x_i) = \lim \frac{1}{\Delta V}\, a^*_{\rho,x_i}\, a_{\rho,x_i}$$

then has the eigenvalues:

$$\sum_{p=1}^{N} \delta_{\rho\rho_p} \cdot \delta(x_i - q_{i_p})\,,$$

in which the **Dirac** δ-function now appears.

The transition to configuration space – i.e., the association of:

$$\varphi\,(\rho_1, q_1, \ldots, \rho_N, q_N) \quad \text{with} \quad \Phi\{N_{\rho,x_i}\}$$

results from the equations:

$$\left.\begin{aligned}
\Phi(1_{\rho_1 q_1}, \ldots, 1_{\rho_2 q_2}, \ldots, 1_{\rho_N q_N}) &= (N!)^{1/2}\,\varphi(\rho_1, q_1, \ldots, \rho_N, q_N),\\
\Phi(1_{\rho_1 q_1}, \ldots, 2_{\rho_\tau q_\tau}, \ldots, 1_{\rho_{N-1} q_{N-1}}) &= \left(\frac{N!}{2!}\right)^{1/2} \varphi(\rho_1 q_1, \ldots, \rho_\tau q_\tau, \rho_\tau q_\tau, \ldots)\\
\Phi(N^{(1)}_{\rho_1,q_1}, N^{(2)}_{\rho_2,q_2}, \ldots) &= \left(\frac{(N!)}{\prod_\tau N^{(2)}!}\right)^{1/2} \varphi(\underbrace{\rho_1 q_1 \ldots \rho_1 q_1}_{N^{(1)}\text{ times}}, \underbrace{\rho_1 q_1 \ldots \rho_1 q_1 \ldots}_{N^{(2)}\text{ times}}).
\end{aligned}\right\} \qquad (62)$$

One sees that all pairs ρ_p, q_p are different from each other in the first row, while two pairs are equal to each other in the second row, and in the last one generally $N^{(1)}$, $N^{(2)}$, … values will coincide. For **Einstein-Bose** statistics, $\varphi(\rho_1\ q_1, \ldots, \rho_N\ q_N)$ is symmetric in this, and for the exclusion principle, it is antisymmetric; in the latter case, only the first row of (62) will be in force.

For the sake of simplicity, the further calculations will be performed for the **Einstein-Bose** statistics. One will then have:

$$a_{\rho,x_i}^* = N_{\rho x_i}^{1/2} e^{-i\Theta_{\rho,x_i}}; \quad a_{\rho,x_i} = e^{i\Theta_{\rho,x_i}} N_{\rho x_i}^{1/2},$$

and $e^{\pm i\Theta_{\rho,x_i}}$ converts N_{ρ,x_i} into $N_{\rho,x_i} \pm 1$, resp., as an operator. We will then have:

$$\left(\sum_{\rho,\sigma,x_i} f_{\rho,\sigma,x_i} a_{\rho,x_i}^* a_{\sigma,x_i}\right)\Phi\{N_{\rho',x_i'}\}$$

$$= \sum_{\rho,\sigma,x_i} f_{\rho,\sigma,x_i} N_{\rho,x_i}^{1/2}(N_{\sigma,x_i}+1)^{1/2}\Phi\{N_{\rho',x_i'}, -\delta_{\rho\rho'}\delta_{x_i x_i'} + \delta_{\sigma\rho'}\delta_{x_i x_i'}\} \tag{63}$$

For a well-defined ρ, σ, x_i, the argument of Φ on the right-hand side will differ from the one on the left-hand side by the fact that the value of N in the cell ρ, x_i is reduced by one, while the value of N in the cell σ, x_i is increased by one; if the value of N were equal to zero in the cell ρ, x_i then the factor $N_{\rho,x_i}^{1/2}$ would ensure that the right-hand side would vanish. If we replace $N_{\rho',x_i'}$ with the eigenvalue $\sum_\rho \delta_{\rho'\rho_p} \cdot \delta_{x_i' q_{i_p}}$, in particular, and perform the transition to configuration space according to (62) then we will get:

$$\left(\sum_{\rho,\sigma,x_i} f_{\rho,\sigma,x_i} a_{\rho,x_i}^* a_{\sigma,x_i}\right) \varphi(\rho_1\, q_1, \ldots, \rho_N\, q_N)$$

$$= \sum_{\rho,\sigma,x_i} f_{\rho,\sigma,x_i} \sum_p \delta_{\rho\rho_p}\delta_{x_i q_{i_p}} \;\varphi(\rho_1, q_{i\,1}, \ldots, \sigma,\; q_{i\,p}, \ldots, \rho_N, q_{iN})$$
$$= \sum_\sigma \sum_p f_{\rho_p,\sigma,q_{i_p}} \;\varphi(\rho_1\, q_{i\,1}, \ldots, \sigma,\; q_{i\,p}, \ldots, \rho_N\, q_{iN})\,. \tag{64}$$

The factors $N_{\rho,x_i}^{1/2}(N_{\sigma,x_i}+1)^{1/2}$ in (63) thus drop out in comparison to the combinatorial factors that arise in (62). The transition to the continuum can then be completed with no further assumptions. One will have:

$$\varphi_{\rho_1\ldots\rho_N}(q_1, \ldots, q_N) = \lim\,(\Delta V)^{-N/2}\;\varphi(\rho_1, q_1, \ldots, \rho_N, q_N),$$
$$\Phi\{N_\rho(x_i)\} = \lim\,(\Delta V)^{-N/2}\,\Phi\{N_{\rho,x_i}\}.$$

For:

$$N_\rho(x_i) = \sum_{p=1}^N \delta_{\rho,\rho_p}\cdot\delta(x_i - q_{i_p}),$$

we will get the association:

$$\left(\int f_{\rho\sigma}(x_i)\,\psi_\rho^*(x_i)\,\psi_\sigma(x_i)\,dV\right)\Phi\left\{\sum_{p=1}^N \delta_{\rho,\rho_p}\delta(x_i - q_{i_p})\right\}$$

$$\to \sum_{p=1}^{N}\sum_{\sigma_p} f_{\rho_p,\sigma_p}(q_{i_p})\,\varphi_{\rho_1\ldots\sigma_p\ldots\rho_N}(q_{i1},\ldots,q_{iN})\,. \tag{65}$$

In particular, for $f_{\rho,\sigma} = \delta_{\rho,\sigma} f$, it will follow that:

$$\left(\int f(x_i)\,N(x_i)\,dV\right)\Phi\left\{\sum_{p=1}^{N}\delta_{\rho,\rho_p}\delta(x_i-q_{i_p})\right\} \to \sum_{p=1}^{N} f(q_{i_p})\,\varphi_{\rho_1\ldots\rho_N}(q_{i1},\ldots,q_{iN_1})\,. \tag{66}$$

One likewise shows that:

$$\left(\int\sum_{\rho,\sigma} f_{\rho\sigma}(x_i)\,\psi_\rho^*\frac{\partial\psi_\sigma}{\partial x_k}dV\right)\Phi\left\{\sum_{p=1}^{N}\delta_{\rho,\rho_p}\delta(x_i-q_{i_p})\right\}$$

$$\to \sum_{p=1}^{N}\sum_{\sigma_p} f_{\rho_p,\sigma_p}(q_{i_p})\frac{\partial}{\partial q_{k_p}}\varphi_{\rho_1\ldots\sigma_p\ldots\rho_N}(q_{i1},\ldots,q_{iN})\,. \tag{67}$$

As would emerge from the arguments of **Jordan** and **Wigner**, the statements (65), (66), (67) will also remain correct for the case of the exclusion principle when the function φ is assumed to be antisymmetric in the pairs ρ_p, q_p . (The sequence of arguments ρ_1, q_1, …, ρ_N, q_N is thus definitive for the determination of certain signed functions.)

We can immediately rewrite our equations (60′), (61′) in configuration space. We will get:

$$\left[P_{r3} + e\sum_{p=1}^{N}\upsilon_{0r}(q_{ip})\right]\varphi_{\rho_1\ldots\rho_N}(q_{i1},\ldots,q_{iN},M_{r\lambda},P_{r3}) = 0. \tag{68}$$

$$\begin{aligned}
&\left[-E + \sum_{r,\lambda} M_{r\lambda}h\nu_{r\lambda} + \sum_r \pi\nu_r(P_{r3})^2\right]\varphi_{\rho_1\ldots\rho_N}(q_{i1},\ldots,q_{iN},M_{r\lambda},P_{r3}) \\
&+ \sum_{k,p,\sigma_p}\left(\frac{hc}{2\pi i}\alpha^k_{\rho_p,\sigma_p}\frac{\partial}{\partial q_{kp}} + mc^2\alpha^4_{\rho_p,\sigma_p}\right)\varphi_{\rho_1\ldots\sigma_p\ldots\rho_N}(q_{ip},M_{r\lambda},P_{r3}) \\
&\qquad + e\sum_r\sqrt{\frac{h}{4\pi}}\,i\sum_{\lambda=1,2}\sum_{k,p,\sigma_p}\upsilon_k^{r\lambda}(q_{ip})\alpha^k_{\rho_p,\sigma_p} \\
&\quad \times\Big[(M_{r\lambda}+1)^{1/2}\varphi_{\rho_1\ldots\sigma_p\ldots\rho_N}(q_{ip},M_1,\ldots,M_{r\lambda}+1,\ldots P_{r3}) \\
&\qquad - M_{r\lambda}^{1/2}\varphi_{\rho_1\ldots\sigma_p\ldots\rho_N}(q_{ip},M_1,\ldots,M_{r\lambda}+1,\ldots,P_{r3})\Big] \\
&\quad + e\sum_r\frac{ih}{2\pi}\sum_{k,p,\sigma_p}\upsilon_k^{r3}(q_{ip})\,\alpha^k_{\rho_p,\sigma_p}\frac{\partial}{\partial P_{r3}}\varphi_{\rho_1\ldots\sigma_p\ldots\rho_N}(q_{ip},M_{r\lambda},P_{r3}) = 0.
\end{aligned} \tag{69}$$

The extent to which these equations can be approximated by the **Schrödinger** equation in configuration space will be examined more closely by **R. Oppenheimer** in a paper that will appear soon. The self-energy of the electrons will also give rise to complications here.

Let it be mentioned how one is to generalize the process that was applied here for the transition to configuration space for the case in which annihilation processes are present. In that case, the number of particles will no longer remain constant. However, it is possible to work with a system of functions:

$$\varphi(M_{r\lambda}, P_{r3}),\ \varphi(q_{i1}, M_{r\lambda}, P_{r3}),\ \ldots,\ \varphi(q_{i1}, \ldots, q_{iN}, M_{r\lambda}, P_{r3}),\ \ldots$$

in different-dimensional spaces that correspond to the cases in which zero, one, …, N, … particles are present, respectively. These functions will then be linked by a simultaneous system of differential equations for a given theory. It would create no difficulty to exhibit that system of equations for the particular extra terms that were given in § 2, equation (12). However, that should be avoided, since those particular terms hardly admit any physical interpretation.