"Ueber die physikalische Bedeutung des Princips der kleinsten Wirkung," J. f. d. reine u. angew. Math. **100** (1887), 137-166.

On the physical meaning of the principle of least action

(By *H. von Helmholtz*)

Translated by D. H. Delphenich

When I speak of the principle of least action in this article, I would not like for that term to be understood in just the original form that *P. L. M. de Maupertuis* published in 1744 (*), which received a precise determination for the variational condition and a complete proof only much later, moreover, and mainly by *Lagrange*. Rather, I would desire that this term, as the oldest and best-known one, should subsume the various transformed forms of that theorem that were derived from it later by *Sir W. Rowan Hamilton* (**). The latter presented the two differential equations that *C. G. J. Jacobi* later combined into a single one in which the common source of these transformations lay, along with many other possible ones, while the physical assumptions with which the calculations started were in no way changed as a result.

The aforementioned researcher first applied the principle of least action to only the mechanics of ponderable bodies and represented the motions of a system as being either freely mobile or rigidly coupled to another mass point in a chain. The physical assumptions from which he started were then given essentially by *Newton's* laws of motion and the manner by which one cared to define the phenomenon that would correspond to the action of rigidly-coupled mass points mechanically. However, he further showed that once one first learns how to treat *Maupertius's* integral correctly, the validity of the law of constancy of energy must be assumed (***). At first, this must seem to be a substantial restriction on the domain of validity of the principle of least action, but more recent physical investigations have established that the law of the constancy of energy is also valid in general, and that apparent restriction does not, in fact, restrict anything. However, one must know completely all of the forms that the equivalence of energy can take for a process being examined in order to include them in calculations. On the other hand, it might seem questionable whether other physical processes that might enter in, and which are not treated simply by means of the motions of ponderable masses and *Newton's* laws of motion, and in which one operates with energy quanta, can also be grasped by the principle of least action.

(*) Histoire de l'Acad. des Sciences de Paris, 1744, April 15. – Histoire de l'Acad. Royale de Berlin (1746), pp. 267.

(**) Philosph. Transact. **II** (1834), 247-308; **I** (1834), 95-144.

(***) *Maupertuis* himself did not see that; he held that his principle was more general than that of the conservation of vis vivas. Histoire de l'Acad. de Berlin (1746), pp. 285. – *C. J. G. Jacobi* discussed this point in the beginning of his Lecture VI on dynamics.

I will choose one of *Hamilton's* forms to be the most convenient form for the principle of least action in the investigations that will be carried out here, which is the one that allows external forces that depend upon time to act upon the mechanical system considered, whose internal forces are only conservative ones. If we denote the potential energy of the system by F and the *vis viva* by L then the function (viz., *Hamilton's* principal function) whose time integral will be a minimum for the normal motion between end points will be:

$$H = F - L,$$

while the energy of the system will be:

$$E = F + L.$$

In this, F depends upon only the coordinates, while L is a homogeneous function of second degree of the velocities.

The function H is the one in terms of whose differential quotients *Lagrange* expressed the forces that act upon the moving system from the outside. Since that function plays an important role in all of the problems that will be treated here, I would like to propose the name of *kinetic potential* for it, due to just that relationship with the forces. An entire series of corresponding names for different special chapters of physics has already been proposed. Thus, they include *F. E. Neumann's* potential of two electrical currents and *R. Clausius's* (*) *electrodynamical potential*; *J. W. Gibbs* (**) called the same thermodynamic function that I called "free energy" the *force function for constant temperature*, while *P. Duhem* (***), by contrast, called that function the *thermodynamic potential*. There thus exist sufficient examples for the new choice of terminology.

The principle of least action can then be expressed as follows:

The mean value of the kinetic potential that is calculated for equal time elements is a minimum for the actual path of the system (a limiting value for longer intervals, respectively), in comparison to all other neighboring paths that lead from the initial position to the final positions in the same amount of time.

For a state of rest, the kinetic potential goes to the values of the potential energy (the potential in the sense that was used up to now, respectively). We do not need to take the mean value for them, since the values that were different while in motion will all be equal to each other here. For a state of rest, our theorem then says simply that *the potential energy must be a minimum for equilibrium* (†).

Jacobi showed that the function H can also include time explicitly with making the construction of the variation and differential equations that follow from it impossible. I have employed it in order to add a sum $\sum_\alpha [P_\alpha \cdot p_\alpha]$ to H, in which the p_α are coordinate

(*) This Journal, Bd. 85, pp. 85. Also in *Wiedemann's* Annalen, Bd. I, pp. 36.

(**) Transact. Connecticut Academy III, pp. 108-248; 343-524. *Silliman's* Journal **XVI** (1878), 441-458.

(***) *Le Potential Thermodynamique*, Paris, 1886.

(†) It should be remarked that *Euler* already sought to base the principle of least action in that way, but he took the mean value of F, not that of $(F - L)$. Histoire de l'Acad. Royale de Berlin, 1751, pp. 175.

and the P_α mean the forces that act in the direction of the coordinates, while the latter are taken in a sense that will be discussed in more detail below. The P_α will be considered to be given functions of time that are, however, independent of the coordinates. In this form, the minimum principle will yield the *Lagrange* equations for the forces P_α, and in that way, an entire series of special investigations that are based upon *Lagrange's* equations of motion will also be subsumed by the somewhat-modified principle of least action. Where it is necessary to distinguish that modified principle from the original one, I would like to call it the *law of minimum kinetic potential*. The form that *Lagrange* gave to the equations of motion is, in fact, important, due to the fact that we can apply it to cases in which processes are at work that are no longer rationally resolvable in various ways, such as friction, galvanic resistance, etc., and in which equilibrium must exist between the conservative forces that are included in *Lagrange's* formula.

Now, there are other work equivalents, besides the potential and actual energies of ponderable masses, namely, thermal, electrodynamical, and electromagnetic ones. Up till now, the motion of heat has generally only been considered as an especially complicated form of the motion of exclusively ponderable atoms. However, since ether waves simultaneously radiate from warm bodies, this restriction [which can, in fact, be derived from *Carnot's* law under simpler assumptions, as *Clausius* (*) and *Boltzmann* (**) showed] should be considered to be a hypothesis that is only adequate initially; the action of other forces (e.g., electrodynamical ones) cannot be excluded with any confidence.

I have proved that, by contrast, the known laws of reversible heat processes can, in fact, be expressed in the form of *Lagrange's* equation of motion, and thus also the law of minimum kinetic potential, in my papers on *the statics of monocyclic motions* (***). However, that shows that temperature, which measures the intensity of thermal motion, enters into the function to be integrated in a very complicated form, as do the velocities, when one defines the values of the *vis viva* in the ponderable system. In the cited papers, I showed that the same formulas could also be true, with certain restricting assumptions, for systems of ponderable masses through the elimination of certain coordinates, such that there is no contradiction between the appearance of such complicated forms and the application of *Lagrange's* equations of motion. However, if one would like to learn the general properties of systems that are governed by the principle of least action then it would be necessary to drop the older, narrower assumption that the velocities enter into just the values of the *vis viva*, and indeed in the form of a homogeneous function of degree two, and to examine how things behave when H is a function of the coordinates and velocities of arbitrary form.

The fact that *chemical forces* (whenever we can compel them to act in only a reversible way) follow *Carnot's* law has indeed been confirmed experimentally in just a small number of cases, but they are all the more unacceptable since they exhibit measurable relations between processes of an apparently quite different nature (†).

(*) *Poggendorff* Annalen, **142** (1871), 433-461.

(**) Wiener Sitzungsberichte **LIII** (1866), Abt. II, pp. 195-220.

(***) This Journal, Bd. 97, pp. 112-123.

(†) Cf., my three papers on the thermodynamics of chemical processes. Sitzungsberichte der Berliner Akademie, 1882, 2 Feb., 27 July; 1883, 31 May. – A good survey of the materials that were used up to now is in the book of *P. Duhem* that was cited above.

Finally, observations of the distant electromagnetic and electrodynamic effects of closed, electric currents have led to expressions for the ponderomotive and electromotor forces that are closely linked with the ones that *Lagrange* gave for the mechanics of ponderable bodies. The first to give such a formulation of the laws of electrodynamics was *F. E. Neumann, sr.* (*). For him, the velocities of the electric currents – i.e., the set of all electricity that goes through a surface element in the conductor that is bounded by material particles in a unit time – appeared as velocities. Later, *W. Weber* and *Clausius* gave other forms in which relative or absolute velocities of electrical quanta in space entered in place of current velocities. The consequences of the various formulations agree completely for closed currents; for unclosed currents, they differ. To the extent that facts are known in this latter domain, they show that *Neumann's* law is inadequate when one includes only the motions of electricity in conductors in one's calculations while applying it. Moreover, one must also consider the motions of electricity in insulators that *Faraday* and *Maxwell* considered, which might occur under increasing or diminishing dielectric polarization. The previously-observed effects of unclosed currents also fit into *Neumann's* law, thus-extended.

However, a deviation in the form of the functions from the ones for ponderable masses also exists here. For electrodynamical phenomena, the velocities of electricity appear as functions of degree two whose coefficients, however, will refer to rectangular coordinates, not constants, like the masses in the values of the *vis viva* on ponderable systems. Secondly, linear functions of velocity will enter whenever permanent magnets act.

It was precisely my investigations into the form of the kinetic potential that is required by *Maxwell's* theory of electrodynamics that led me to the present preliminary investigations.

The theory of light has also subsumed all of its main facts under the hypothesis that the ether is a medium with properties that are similar to the solid-elastic ponderable bodies. The known difficulties in the theory of reflection and refraction will be even easier to defeat by *Maxwell's* hypothesis. However one might cleave to one or the other meaning, one would have to consider the principle of least work to be valid for the motion of light, at least to the extent that its phenomena can be explained by that theory.

That already implies that the domain of validity of the principle of least action goes far beyond the limits of the mechanics of ponderable bodies, and that *Maupertuis's* high hopes for its absolutely general validity seem to be approaching fulfillment, so tenuous was the mechanical proof and so contradictory were the metaphysical speculations that the author himself knew about at the time.

Nowadays, it is regarded as highly probable that it is the general law for all reversible natural processes, and as far as the irreversible ones are concerned (e.g., the generation and conduction of heat), their irreversibility does not seem to be in the nature of things, but only rests upon the limitations of our tools that makes it impossible for us to arrange disorganized atomic motions or to make the motion of all atoms that take part in the motion of heat go precisely backwards.

(*) Cf., my papers in the theory of electrodynamics. This Journal, Bd. 72, pp. 57; Bd. 75, pp. 35; Bd. 78, pp. 273.

In every case, it seems to me that the general validity of the principle of least action has been insured to the extent that it can occupy a high position as a heuristic principle and a guide for any attempts to formulate the laws of new classes of phenomena.

In addition, it has the advantage of summarizing all conditions that are influential for the class of phenomena under consideration in the narrowest scope of a formula, and thus giving a complete overview of everything that is essential.

By this state of affairs, I find it useful for the general principle to give an overview of the proof and the general consequences, which can be kept very brief whenever one applies only known methods to somewhat extended assumptions. I have therefore endeavored to highlight the consequences that are concerned with observable behavior, and which, when combined, will, in turn, serve as an indicator of the validity of the principle in the domain in question.

In § 1, the law of minimum kinetic potential is developed with the greatest possible freedom in the nature of the function H, and *Lagrange's* equations of motion are derived from it. The eliminations by which such general forms can also occur for systems of ponderable bodies are discussed.

In § 2, the constancy of energy is derived from our form of the principle, and one learns how to calculate the value of energy from the value of kinetic potential. It is:

$$E = H - \sum_{\alpha} \left[q_\alpha \cdot \frac{\partial H}{\partial q_\alpha} \right],$$

where q_α are the velocities. That will show that the principle of least action is not always valid, conversely, in any case where the constancy of energy is true. The latter then expresses more than the former, and finding what it expresses in addition will be our problem. At the same time, some mechanical and physical processes will be specified in order to be able to refer to them as explanatory examples for the contents of the first two paragraphs, as well as the following ones, and to make the significance of the principle intuitive.

§ 3 then treats the opposite problem, namely, that of deriving H from E. That must involve integrating the aforementioned differential equation, and that will introduce arbitrary integration constants that must be homogeneous functions of degree one in the q_α . This step is meaningful, insofar as it will then become possible to find the kinetic potential from a complete knowledge of the dependency of the energy upon the coordinates and velocities, and thus all of the laws of motion for the system, assuming that the principle of least action is valid. One will be able to find the terms that are linear in q_α, which correspond to "hidden motions," with no difficulties, for the most part.

§ 4 treats the interdependencies between the forces that are simultaneously exerted upon the system in different directions and its accelerations and velocities. That subsumes a series of interesting connections with physical phenomena, such as, e.g., the one between *Ampère's* electromagnetic and electrodynamic laws, on the one hand, and the law of induction, on the other; a series of thermodynamic laws – e.g., the relationship between the raising of the pressure on a contained mass by raising its temperature and the raising of temperature by compression, and the corresponding behavior for thermo-electric and electro-chemical processes. Finally, one can also prove, in turn, that the principle of least action is valid whenever the interdependencies between the forces that were detailed in § 4 exist. That proof is, however, deferred to a later communication.

In § 5, *Hamilton's* theorem, in its general form, will be recapitulated, and in § 6, the reciprocity theorem that flows from it for the changes in the forward and reverse motion that result from small impulses after a certain time has elapsed will be given. Some reciprocal relationships that I myself have verified for sound and light in previous papers, but only for systems at rest, fall within that scope.

Finally, in § 7, the moment of motion shall be introduced in place of the velocities, which will yield another form of the variational problem, and along with the already-known altered representations of the values of the force, another reciprocity law for forward and backward motion, as well.

§ 1.

Formulation of the principle.

I assume that the instantaneous state of the system of bodies in question is given completely by a sufficient number of mutually-independent coordinates $p_\mathfrak{a}$; I denote the velocities of evolution by:

$$q_\mathfrak{a} = \frac{dp_\mathfrak{a}}{dt}. \tag{1}$$

Furthermore, I let $P_\mathfrak{a}$ denote the force with which the system of moving bodies acts upon the change in the coordinate $p_\mathfrak{a}$, such that $(-P_\mathfrak{a})$ will be the external force that must act upon the system in the direction of the coordinate $p_\mathfrak{a}$ in order for the assumed motion of the system to be able to happen in the assumed way.

The forces $P_\mathfrak{a}$, which were introduced by *Lagrange*, are, in general, aggregates of force components that themselves can act upon different parts of the system, and thus their magnitudes and composition are defined in such a way that $(P_\mathfrak{a} \cdot dp_\mathfrak{a})$ is the work that the force $P_\mathfrak{a}$ does outwardly when the coordinate changes from $p_\mathfrak{a}$ to $(p_\mathfrak{a} + dp_\mathfrak{a})$, whereas $P_\mathfrak{a}$ will perform no work when $p_\mathfrak{a}$ remains unchanged while arbitrary variations of the remaining coordinates $p_\mathfrak{a}$ occur.

In the sequel, we shall assume that the quantities $P_\mathfrak{a}$ are functions of time, but they are independent of the coordinates during the given period from $t = t_0$ to $t = t_1$. Let H be a function of the coordinates and velocities, of which, we will first demand only that it must have finite first and second differential quotients with respect to $p_\mathfrak{a}$ and $q_\mathfrak{a}$ at all positions of the path that evolves during the stated time period. Moreover, we define the integral:

$$\Phi = \int_{t_0}^{t_1} dt \cdot \left\{ H + \sum_\mathfrak{a} [P_\mathfrak{a} \cdot p_\mathfrak{a}] \right\}, \tag{1^a}$$

in which the $p_\mathfrak{a}$ are varied in such a way that their variations $\delta p_\mathfrak{a}$ will be zero for $t = t_0$ and $t = t_1$, but they will be arbitrarily-differentiable functions of time at the intermediate times. It then follows from known methods of the variational calculus that one will have:

$$\delta\Phi = 0 \tag{1^b}$$

when one has:

$$0 = P_\mathfrak{a} + \frac{\partial H}{\partial p_\mathfrak{a}} - \frac{d}{dt}\left[\frac{\partial H}{\partial q_\mathfrak{a}}\right] \tag{1^c}$$

during the duration of the motion.

As is known, these are the equations of motion for the system in the form that was given by *Lagrange*.

Elimination of coordinates. In the original applications of the principle to the motions of a free system of material points, as I already remarked in the introduction, H had the form:

$$H = F - L,$$

in which F shall be a function of only the $p_\mathfrak{a}$, and L shall be a homogeneous function of degree two of the $q_\mathfrak{a}$ whose coefficients depend upon the $p_\mathfrak{a}$. For a free system, the number of coordinates $p_\mathfrak{a}$ is three times as large as the number of mass points that are present.

However, in many cases, a decrease in the number of coordinates can occur without changing the form of the representation that was given in equations (1^a), (1^b), and (1^c).

Among these cases, the one that has been treated best up to now is then one in which the degrees of freedom of the system are restricted by so-called *fixed constraints*, which can be expressed mathematically as equations in the coordinates. The composition of the function H from F and L that was given above and the behavior of the last two functions will not change as a result of that, but the number of variable coordinates can be reduced substantially.

Another noteworthy reduction in the number of coordinates occurs when individual ones of them, which we would like to denote by the index $\mathfrak{b}$, appear in the values of H only through their differential quotients $q_\mathfrak{b}$, so the corresponding forces $P_\mathfrak{b}$ need to be equal to zero. Under these circumstances, equations (1^c), which express the value of the $P_\mathfrak{b}$, will reduce to:

$$0 = \frac{d}{dt}\left[\frac{\partial H}{\partial q_\mathfrak{b}}\right] \tag{2}$$

or

$$\frac{\partial H}{\partial q_\mathfrak{b}} = - c_\mathfrak{b} \, . \tag{2^a}$$

One can employ these equations, which are linear in the $q_{\mathfrak{a}}$ ($q_{\mathfrak{b}}$, respectively), to express the $q_{\mathfrak{b}}$ in terms of the remaining velocities and the $p_{\mathfrak{a}}$, and then eliminate them from the value of *H*. We denote the expression for the value of *H* that arises from this elimination by $\mathfrak{H}$. One will then have:

$$\frac{\partial \mathfrak{H}}{\partial p_{\mathfrak{a}}} = \frac{\partial H}{\partial p_{\mathfrak{a}}} + \sum_{\mathfrak{b}} \left[\frac{\partial H}{\partial q_{\mathfrak{b}}} \cdot \frac{\partial q_{\mathfrak{b}}}{\partial p_{\mathfrak{a}}} \right].$$

Thus, with consideration given to (2^a):

$$\frac{\partial H}{\partial p_{\mathfrak{a}}} = \frac{\partial}{\partial p_{\mathfrak{a}}} \left\{ \mathfrak{H} - \sum_{\mathfrak{b}} [c_{\mathfrak{b}} \cdot q_{\mathfrak{b}}] \right\}.$$

If we set:

$$(2^b) \qquad \mathfrak{H} - \sum_{\mathfrak{b}} [c_{\mathfrak{b}} \cdot q_{\mathfrak{b}}] = H'$$

then we will find that:

$$\frac{\partial H}{\partial p_{\mathfrak{a}}} = \frac{\partial H'}{\partial p_{\mathfrak{a}}},$$

and likewise:

$$\frac{\partial H}{\partial q_{\mathfrak{a}}} = \frac{\partial H'}{\partial q_{\mathfrak{a}}},$$

$$(2^c) \qquad p_{\mathfrak{a}} = -\frac{\partial H'}{\partial p_{\mathfrak{a}}} + \frac{d}{dt}\left[\frac{\partial H'}{\partial q_{\mathfrak{a}}}\right].$$

The function H' thus enters in this case, which is free of the $q_{\mathfrak{b}}$ and $p_{\mathfrak{b}}$, but also includes terms that are linear in the $q_{\mathfrak{a}}$ and originate in the values of the $q_{\mathfrak{b}}$, and it completely replaces the original function in the definition of the equations of motions (2^c).

Examples of this are the rotations of a top around its symmetry axis, when its direction, but not its angular velocity, can change around that axis, and furthermore, the motion of a system that is referred to a rectangular coordinate system that is rotating; e.g., the Earth.

Corresponding to this analogy that is given by the mechanics of ponderable bodies, we would meanwhile also like to refer to other cases of physical processes in which the function *H* includes terms that are linear in the velocities as *cases with hidden motion*, although at the moment there are cases in which the existence of such a hidden motion is not confirmed beyond a doubt, such as for the interaction between magnets and electrical currents. It is known that it was adopted for the magnet by *Ampère*; it also showed its influence in the electromagnetic rotation of the plane of polarization of light, as Sir *W. Thomson* remarked, even when no perceptible electric current acts upon it.

This case is distinguished from the one in which H includes the velocities only in terms of degree two essentially by the fact that the motion cannot go backwards under the same conditions unless the hidden motions are simultaneously reversed.

Other eliminations can bring about even more complicated forms for the function H, at least for restricted classes of motion; I have discussed such cases in my first treatise on the monocyclic motions (*). We can choose the conditions of that elimination somewhat differently here. One assumes that a group of coordinates $p_\mathfrak{c}$ is present whose corresponding $q_\mathfrak{c}$ enter into the values of the *vis viva* only multiplied by each other, but not combined into products with the $q_\mathfrak{b}$ in this group, such that all $\frac{\partial^2 H}{\partial q_\mathfrak{c}\,\partial q_\mathfrak{b}} = 0$; moreover, one assumes that the forces $P_\mathfrak{c}$ always remain equal to zero. Under these circumstances, motions of the system are possible for which the $p_\mathfrak{c}$ remain continually constant, so the $q_\mathfrak{c} = 0$. The equations of motion for this class of motions simplify due to the fact that when all $q_\mathfrak{c} = 0$, one will also have all:

$$\frac{\partial H}{\partial q_\mathfrak{c}} = 0.$$

We thus get from (1^c):

$$0 = \frac{\partial H}{\partial q_\mathfrak{c}}, \tag{3}$$

$$P_\mathfrak{b} = -\frac{\partial H}{\partial p_\mathfrak{b}} + \frac{d}{dt}\left[\frac{\partial H}{\partial q_\mathfrak{b}}\right]. \tag{3^a}$$

Now, if equations (3), whose number is equal to that of the $p_\mathfrak{c}$, succeed in expressing these quantities as functions of the $p_\mathfrak{b}$ and $q_\mathfrak{b}$ then one can eliminate the $p_\mathfrak{c}$ from H by means of the values thus-obtained, where, in general, H will be a complicated function of the $q_\mathfrak{b}$ that we would like to denote by $\mathfrak{H}$. From the principles of differential calculus, one will then have:

$$\frac{\partial \mathfrak{H}}{\partial p_\mathfrak{b}} = \frac{\partial H}{\partial p_\mathfrak{b}} + \sum_\mathfrak{c}\left[\frac{\partial H}{\partial p_\mathfrak{c}} \cdot \frac{\partial p_\mathfrak{c}}{\partial p_\mathfrak{b}}\right],$$

$$\frac{\partial \mathfrak{H}}{\partial q_\mathfrak{b}} = \frac{\partial H}{\partial q_\mathfrak{b}} + \sum_\mathfrak{c}\left[\frac{\partial H}{\partial p_\mathfrak{c}} \cdot \frac{\partial p_\mathfrak{c}}{\partial q_\mathfrak{b}}\right];$$

thus, due to equations (3), one will have, in turn:

$$\frac{\partial H}{\partial p_\mathfrak{b}} = \frac{\partial \mathfrak{H}}{\partial p_\mathfrak{b}} \quad \text{and} \quad \frac{\partial H}{\partial q_\mathfrak{b}} = \frac{\partial \mathfrak{H}}{\partial q_\mathfrak{b}}. \tag{3^b}$$

(*) This Journal, Bd. 97, pp. 120-122.

The equations of motion (3^a) then reduce to:

$$(3^c) \qquad P_{\mathfrak{b}} = -\frac{\partial \mathfrak{H}}{\partial p_{\mathfrak{b}}} + \frac{d}{dt}\left[\frac{\partial \mathfrak{H}}{\partial q_{\mathfrak{b}}}\right],$$

in which only the $p_{\mathfrak{b}}$ and $q_{\mathfrak{b}}$ appear, and generally $\mathfrak{H}$ is no longer the sum of a function of the coordinates and a homogeneous function of degree two of the velocities.

However, if the original H is such a function, and thus no hidden motions will have any influence, then equations (3) will be of degree two in the $q_{\mathfrak{b}}$; the value of the $p_{\mathfrak{c}}$ can then remain unchanged (even when it is multi-valued) when all of the $q_{\mathfrak{b}}$ simultaneously change their signs, from which, it will follow that the total motion can also go backwards in this case.

In the mechanics of ponderable masses, problems in which the function H includes the velocities $q_{\mathfrak{a}}$ in terms of first or higher degree can be referred to as *incomplete problems*, insofar as a part of the possible motions is excluded, and a part of the coordinates that are necessary for the determination of the position of the system does not enter into the function H, and certain forces are constantly set equal to zero, so they can no longer be determined arbitrarily.

Functional determinant of the moment of motion. For the sake of brevity, we would like to denote the quantities $\partial H / \partial q_{\mathfrak{a}}$ that enter into the previous derivatives by:

$$(3^d) \qquad s_{\mathfrak{a}} = -\frac{\partial H}{\partial q_{\mathfrak{a}}},$$

and call the $s_{\mathfrak{a}}$ *moments of motion.* For the motion of a free system that is referred to rectangular coordinates, they correspond to the product of the mass and velocity, whose differential quotient with respect to time is *Newton's* measure of the corresponding force component:

$$X = \frac{d}{dt}\left[m \cdot \frac{dx}{dt}\right].$$

In the cases that were summarized, the influence that the inertia of the moving masses exerts upon a well-defined type of motion was different from that of the position of the masses. Thus, e.g., for a rotational motion of a solid body, the moment of motion is equal to the moment of inertia, multiplied by the angular velocity. In that sense, the quantities $s_{\mathfrak{a}}$ now measure the influence that the inertia of the moving mass has, and its acceleration enlists a corresponding part of the force of motion, as equations (1^c) show.

In the original, complete problems of the mechanics of ponderable bodies, the $s_{\mathfrak{a}}$ are linear, homogeneous functions of the $q_{\mathfrak{a}}$ whose coefficients are functions of the $p_{\mathfrak{a}}$, in general, and one then has a system of linear equations:

$$(3^e) \qquad s_\mathfrak{a} = \sum \left[\frac{\partial s_\mathfrak{a}}{\partial q_\mathfrak{b}} \cdot q_\mathfrak{b} \right],$$

which will represent the $q_\mathfrak{b}$ as linear, homogeneous functions of the $s_\mathfrak{a}$ when they are solved for those variables. That representation would not be possible if the determinant of the quantities $\frac{\partial s_\mathfrak{a}}{\partial q_\mathfrak{b}}$ (the $\frac{\partial^2 H}{\partial q_\mathfrak{a}\, \partial q_\mathfrak{b}}$, resp.) were identically zero. However, the latter case cannot come about without the *vis viva* being zero for certain motions with finite velocities. Namely, since L is an essentially positive, homogeneous function of degree two of the $q_\mathfrak{a}$, one will have:

$$2L = \sum_\mathfrak{a} [q_\mathfrak{a} \cdot s_\mathfrak{a}].$$

If the aforementioned determinant were zero then all of the $s_\mathfrak{a}$, and correspondingly L, as well, could be zero without the $q_\mathfrak{a}$ needing to be zero.

The condition that the determinant of equations (3^e) is not identically zero can also be expressed as follows: *No identity can exist between the quantities $s_\mathfrak{a}$ and $p_\mathfrak{a}$, with the exception of the $q_\mathfrak{a}$*, and for that reason, the $q_\mathfrak{a}$ can always be represented as functions of the $s_\mathfrak{a}$ and the $p_\mathfrak{a}$.

This relationship will not be changed if we set individual $s_\mathfrak{a}$ equal to constants, as in the case of hidden motions, or also set them equal to zero, as in the case of the eliminated $p_\mathfrak{a}$. The value of the remaining $s_\mathfrak{a}$ will not be changed by those variations. Since the same thing is also true for the electrical motions and reversible heat motions, to the extend that their physical laws have been ascertained up to now, there is, up to now, no physical motivation to consider the exceptional cases in which the determinant in equations (3^e) might be equal to zero, and for that reason the assumption will be made from now on that the determinant cannot be identically zero, except for at most special values of the $p_\mathfrak{a}$.

Once that condition has been established, the variational problem can be expressed in such a way that the equations that were singled out in the beginning of this paragraph, namely:

$$(1) \qquad q_\mathfrak{a} = \frac{dp_\mathfrak{a}}{dt},$$

will be assumed in it.

As above, let H be a function of the $p_\mathfrak{a}$ and $q_\mathfrak{a}$, and let the $P_\mathfrak{a}$ be functions of time. One sets:

$$(1^d) \qquad \Phi_1 = \int_{t_0}^{t_1} dt \left\{ H - \sum_\mathfrak{a} \left[\left(q_\mathfrak{a} - \frac{dp_\mathfrak{a}}{dt} \right) \cdot \frac{\partial H}{\partial q_\mathfrak{a}} + P_\mathfrak{a} \cdot p_\mathfrak{a} \right] \right\}$$

and demands that:

$$(1^e) \qquad \delta\Phi_1 = 0$$

must be true for arbitrary variations of the $p_\mathfrak{a}$ and $q_\mathfrak{a}$, which are both to be treated as independent variables. One should have $\delta p_\mathfrak{a} = 0$ at the times t_0 and t_1, while the $dq_\mathfrak{a}$ also remain arbitrary.

The variation of $q_\mathfrak{b}$ yields:

$$(1^f) \qquad 0 = \sum_a \left[\frac{\partial^2 H}{\partial q_\mathfrak{a}\, \partial q_\mathfrak{b}} \cdot \left(\frac{dp_\mathfrak{a}}{dt} - q_\mathfrak{a} \right) \right],$$

which implies equations (1), since the determinant of the $\dfrac{\partial^2 H}{\partial q_\mathfrak{a}\, \partial q_\mathfrak{b}}$ should not vanish identically.

The variation of the $p_\mathfrak{a}$ is performed as above, and will yield the same result.

If one denotes the function of the $p_\mathfrak{a}$ and $q_\mathfrak{a}$ that enters into (1^d) by:

$$E = H - \sum_\mathfrak{a} \left[q_\mathfrak{a} \cdot \frac{\partial H}{\partial q_\mathfrak{a}} \right]$$

(this is the energy, as will be shown in the next paragraph) then one will get:

$$(1^g) \qquad \Phi_1 = \int_{t_0}^{t_1} dt \left\{ E - \sum_\mathfrak{a} \left[s_\mathfrak{a} \cdot \frac{dp_\mathfrak{a}}{dt} + P_\mathfrak{a} \cdot p_\mathfrak{a} \right] \right\}.$$

I cite this form here, since we will encounter an analogous form in the conclusion, and both of them can be regarded as very exotic when one meets up with them in physical investigations before one knows for sure which quantities are to be referred to as $p_\mathfrak{a}$, $q_\mathfrak{a}$, and $s_\mathfrak{a}$.

On the other hand, it is precisely these forms that are included in the complete statement of the problem.

§ 2.

Relationship to the principle of the constancy of energy.

If one multiplies the equations of motion (1^c), in sequence, by $q_\mathfrak{a}$ and adds them then one will get:

$$\begin{aligned} \sum [P_\mathfrak{a} \cdot q_\mathfrak{a}] &= - \sum \left[\frac{\partial H}{\partial p_\mathfrak{a}} \cdot q_\mathfrak{a} \right] + \frac{d}{dt} \sum_\mathfrak{a} \left[q_\mathfrak{a} \cdot \frac{\partial H}{\partial q_\mathfrak{a}} \right] - \sum_\mathfrak{a} \left[\frac{\partial H}{\partial q_\mathfrak{a}} \cdot \frac{dq_\mathfrak{a}}{dt} \right] \\ &= - \frac{d}{dt} \left\{ H - \sum_\mathfrak{a} \left[q_\mathfrak{a} \cdot \frac{\partial H}{\partial q_\mathfrak{a}} \right] \right\}. \end{aligned}$$

If we set:

$$(4) \qquad E = H - \sum_{\mathfrak{a}} \left[q_{\mathfrak{a}} \cdot \frac{\partial H}{\partial q_{\mathfrak{a}}} \right],$$

as we have done up to now, then we can write:

$$(4^a) \qquad \sum [P_{\mathfrak{a}} \cdot q_{\mathfrak{a}}] \cdot dt + \frac{dE}{dt} \cdot dt = 0.$$

The sum that enters into this is the work that the force $P_{\mathfrak{a}}$ does on the environment in the time interval dt, and that will then imply that the quantity E continually increases or decreases according to whether that force does positive or negative work, respectively. It will then follow from this that E denotes the energy supply of the system, expressed in terms of its coordinates $p_{\mathfrak{a}}$ and velocities $q_{\mathfrak{a}}$.

It emerges from this that the principle of least action, when it is taken in the form of § 1, always includes the principle of the constancy of energy.

On the other hand, the principle of least action is not necessarily true in all conceivable cases that are subject to the law of the constancy of energy. One can make many supplements to the system of equations (1^c) that do not at all affect the derivation of equations (4^a), but they will probably cancel the summation in the variational formula. For example, one adds a term ($\varphi \cdot q_{\mathfrak{b}}$) to those equations ($1^c$) that have the index $\mathfrak{a}$ in their terms, and a term ($- \varphi \cdot q_{\mathfrak{b}}$) to the ones that have the index $\mathfrak{b}$, in which φ is any function of the coordinates. If, in order to derive the energy equation, we then multiply the former equation by $q_{\mathfrak{a}}$ and the latter one by $q_{\mathfrak{b}}$ then the extra terms will drop away, and the constancy of energy will not be affected. By contrast, the corresponding variation:

$$\varphi [q_{\mathfrak{b}} \cdot \delta p_{\mathfrak{a}} - q_{\mathfrak{a}} \cdot \delta p_{\mathfrak{b}}]$$

can be considered to be the complete variation of a function of the $p_{\mathfrak{a}}$ and $q_{\mathfrak{a}}$ under the integral only when φ depends upon the variables ($q_{\mathfrak{b}} \cdot p_{\mathfrak{b}}$) and ($q_{\mathfrak{b}} \cdot p_{\mathfrak{a}}$).

If the function φ that enters into the supplementary terms is independent of the velocities then the corresponding motion will not be reversible. However, we can make φ into a linear function of the velocities; the entire motion can then go backwards, as well.

Since one can install such terms in any arbitrarily-chosen pair of equations from the system (1^c), a great multitude of cases are conceivable in which the law of the constancy of energy is valid, but not that of least action.

It then follows from this that when the latter principle is true, it will express a special character of the conservative natural forces that are present that is not already present as a result of their being defined to be conservative forces. Illuminating this idea more clearly will be the objective of the investigation that follows.

Explanatory example:

Since it will be repeatedly desirable to cite examples in the sequel in which the significance of the theorems that are obtained becomes intuitive, I shall allow myself to cite some suitable spaces here to the extent that it is necessary to characterize them, and to which I can refer briefly, not only for the contents of this and the following paragraphs, but also later on.

I. *Example of top.* Let the top be a rotating body on a gimbal mounting. The outer ring *a* might rotate around a vertical axis, and we let α denote the angle of rotation, when measured from a well-defined vertical plane in space. The second ring *b* rotates inside of the first one around a horizontal axis, and I shall let β denote the angle between the planes of the rings *a* and *b*. The rotational axis of the top lies in the ring *b* at right angles to the rotational axis between *a* and *b*. Let the angle between a reference meridian on the top and the plane of *b* be γ, let the moment of inertia of the top around its rotational axis be $\mathfrak{A}$, and let the moment of inertia around one of its equatorial axes by $\mathfrak{B}$; that of the ring itself will be neglected. The *vis viva* on the top will then be:

$$
(4) \qquad \left\{ \begin{aligned} L &= \tfrac{1}{2}\mathfrak{A}\cdot\left[\frac{d\gamma}{dt}+\cos\beta\cdot\frac{d\alpha}{dt}\right]^2 + \tfrac{1}{2}\mathfrak{B}\cdot\left[\sin^2\beta\cdot\left(\frac{d\alpha}{dt}\right)^2+\left(\frac{d\beta}{dt}\right)^2\right], \\ H &= -L. \end{aligned} \right.
$$

That will yield the forces *A*, *B*, *C* that tend to increase the angles α, β, γ, respectively:

$$
(4^a) \qquad A = -\frac{d}{dt}\left\{\mathfrak{A}\cdot\cos\beta\cdot\left[\frac{d\gamma}{dt}+\cos\beta\cdot\frac{d\alpha}{dt}\right]+\mathfrak{B}\cdot\sin^2\beta\cdot\frac{d\alpha}{dt}\right\},
$$

$$
(4^b) \qquad B = -\mathfrak{A}\cdot\sin\beta\cdot\left[\frac{d\gamma}{dt}+\cos\beta\cdot\frac{d\alpha}{dt}\right]\frac{d\alpha}{dt}+\mathfrak{B}\cdot\sin\beta\cdot\cos\beta\cdot\left(\frac{d\alpha}{dt}\right)^2-\mathfrak{B}\cdot\frac{d^2\beta}{dt^2},
$$

$$
(4^c) \qquad C = -\frac{d}{dt}\left\{\mathfrak{A}\cdot\left[\frac{d\gamma}{dt}+\cos\beta\cdot\frac{d\alpha}{dt}\right]\right\}.
$$

The force *A* is simply a rotational moment that makes the circle *a* rotate, and likewise, *B* makes *b* rotate. However, *C* rotates the circle relative to *b*, so it must have its pivot at *b*.

If the force *C* is absent then (4^c) will imply that:

$$
(4^d) \qquad -\mathfrak{A}\cdot\left[\frac{d\gamma}{dt}+\cos\beta\cdot\frac{d\alpha}{dt}\right] = c.
$$

That will yield the value of H', according to (2^b):

$$(4^e) \qquad H' = +\tfrac{1}{2}\frac{c^2}{\mathfrak{A}} - \tfrac{1}{2}\mathfrak{B}\cdot\left[\sin^2\beta\cdot\left(\frac{d\alpha}{dt}\right)^2 + \left(\frac{d\beta}{dt}\right)^2\right] + c\cdot\cos\beta\cdot\frac{d\alpha}{dt},$$

and the values of the forces, when derived in accordance with (4^a), (4^b), and (4^c):

$$(4^f) \qquad \begin{cases} A = \dfrac{d}{dt}\left[c\cdot\cos\beta - B\cdot\sin^2\beta\cdot\dfrac{d\alpha}{dt}\right], \\ B = c\cdot\sin\beta\cdot\dfrac{d\alpha}{dt} + \mathfrak{B}\cdot\sin\beta\cdot\cos\beta\cdot\left(\dfrac{d\alpha}{dt}\right)^2 - \mathfrak{B}\cdot\dfrac{d^2\beta}{dt^2}. \end{cases}$$

The first constant term in the value of H' can be omitted, since it enters into just the arbitrary constant of the value of E; the last term the linear one that is absent from the value of L.

II. *Example: Electrodynamic effect of closed, circular currents on the potential law* (*).

We would like to understand $J_{\mathfrak{b}}$ to mean the current intensity of the $\mathfrak{b}^{\text{th}}$ circular current and $p_{\mathfrak{a}}$ to mean the coordinates of the ponderable masses, whose *vis viva* we shall neglect. The function H has the form:

$$(5) \qquad H = -\tfrac{1}{2}\sum_{\mathfrak{b}}\sum_{\mathfrak{c}}[Q_{\mathfrak{b},\mathfrak{c}}\cdot J_{\mathfrak{b}}\cdot J_{\mathfrak{c}}],$$

in which $Q_{\mathfrak{b},\mathfrak{c}}$ are functions of $p_{\mathfrak{a}}$, and each of the two successive indices $\mathfrak{b}$ and $\mathfrak{c}$ refer to all current circles. The induced electromotive forces, which I would like to denote by $\mathfrak{E}_{\mathfrak{b}}$, are:

$$(5^a) \qquad \mathfrak{E}_{\mathfrak{b}} = -\sum_{\mathfrak{c}}\left[\frac{d}{dt}(Q_{\mathfrak{b},\mathfrak{c}}\cdot J_{\mathfrak{c}})\right],$$

$$(5^b) \qquad P_{\mathfrak{a}} = \tfrac{1}{2}\sum_{\mathfrak{b}}\sum_{\mathfrak{c}}\left[\frac{\partial Q_{\mathfrak{b},\mathfrak{c}}}{\partial p_a}\cdot J_{\mathfrak{b}}\cdot J_{\mathfrak{c}}\right].$$

If a permanent magnet whose position is given by the coordinate p_0 is involved then a series of linear terms will be added to H that I would like to denote by h, and which will have the form:

$$(5^c) \qquad h = \sum_{\mathfrak{b}}[R_{\mathfrak{b}}\cdot J_{\mathfrak{b}}],$$

in which the $R_{\mathfrak{b}}$ are functions of the coordinates $p_{\mathfrak{a}}$ and p_0. The calculation of the forces comes about here by the same methods. The total electrodynamic energy is:

(*) See, among others, my paper in this Journal, Bd. 72, pp. 70-72.

$$E = H - \sum_{\mathfrak{b}} \left[J_{\mathfrak{b}} \cdot \frac{\partial H}{\partial J_{\mathfrak{b}}} \right] = - H.$$

The function h will vanish, since:

$$h - \sum_{\mathfrak{b}} \left[J_{\mathfrak{b}} \cdot \frac{\partial h}{\partial J_{\mathfrak{b}}} \right] = 0.$$

I have shown that the $E = - H$ that enters into this, like the vis viva on ponderable masses, is a necessarily positive quantity for closed currents (*). In addition, E is a homogeneous function of degree two in the $J_{\mathfrak{a}}$, and the same considerations can be applied to it as the ones that were discussed at the end of § 1, in which the determinant of the quantities $\frac{\partial^2 H}{\partial J_{\mathfrak{a}} \cdot \partial J_{\mathfrak{b}}}$ cannot be equal to zero identically.

III. *Example. Thermodynamics*. For a suitable choice of coordinates, the laws of reversible thermal processes can be represented in the form (**):

$$(6) \qquad \begin{cases} P_{\mathfrak{a}} = -\dfrac{\partial}{\partial p_{\mathfrak{a}}}[F - L] - \dfrac{d}{dt}\left[\dfrac{\partial L}{\partial q_{\mathfrak{a}}}\right], \\ \dfrac{dQ}{dt} = -\vartheta \cdot \dfrac{d}{dt}\left[\dfrac{\partial \mathfrak{F}}{\partial \vartheta}\right]. \end{cases}$$

$$(6^a) \qquad E = \mathfrak{F} - \vartheta \cdot \frac{\partial \mathfrak{F}}{\partial \vartheta} + L.$$

In this, $\mathfrak{F}$ is what I called the "free energy," which is a function of the coordinates $p_{\mathfrak{a}}$ and the absolute temperature ϑ, and L is the vis viva of the visible motions of the heavy masses, and thus a function of the $p_{\mathfrak{a}}$ and the $q_{\mathfrak{a}}$ that is homogeneous of degree two in the latter and independent of ϑ. dQ is the amount of heat that enters the body in the element of time dt – i.e., the work that is done by the environment – and only forces that bring about the motion of heat will be exerted.

We will get this form, with different variables, when we set:

$$\frac{\partial \mathfrak{F}}{\partial \vartheta} = - S,$$

then let s denote a function of S, and further set:

(*) This Journal, Bd. 72, pp. 86, *et seq.* and pp. 125.

(**) See this Journal, Bd. 97, pp. 112-117. The vis viva of the visible motions L is added in order to insure the completeness that is desirable here.

$$(6^b) \qquad \begin{cases} \vartheta \cdot \dfrac{\partial S}{\partial s} = \eta, \\ H = \mathfrak{F} - \vartheta \cdot \dfrac{\partial \mathfrak{F}}{\partial \vartheta} - \eta \cdot s. \end{cases}$$

If we now express H and s as functions of the $p_\mathfrak{a}$ and η then the equations above can be written [*loc. cit.*, Abh. I, eq. (1^d)]:

$$(6^c) \qquad \frac{dQ}{dt} = + \vartheta \cdot \frac{dS}{dt} = \eta \cdot \frac{ds}{dt},$$

$$(6^c) \qquad P_\mathfrak{a} = - \frac{\partial}{\partial p_\mathfrak{a}}[H - L] - \frac{d}{dt}\left[\frac{\partial L}{\partial q_\mathfrak{a}}\right],$$

$$(6^c) \qquad E = H - \eta \cdot \frac{\partial H}{\partial \eta} + L.$$

These equations, like the first ones (6), (6^a), have entirely the same form (cf., *loc. cit.*, § 3) as the ones for the motion of a monocyclic system whose kinetic potential is (H – L), and for which η denotes the velocity and s, the moment of motion of the monocyclic motion.

If we let $P_{(\eta)}$ denote the force that is exerted in the direction of the velocity η then we will have:

$$(6^f) \qquad P_{(\eta)} \cdot \eta \cdot dt = - dQ.$$

The analogy with the *Lagrange* expressions thus remains true here, as well, and in that way that entropy S of the motion might depend upon the moment of motion s of the monocyclic motion. In this case, this possibility of the coupling of equally-warm systems of bodies to a larger system, and the kinetic theory of gases lead one to assume that:

$$\vartheta = s \cdot q,$$
$$S = -\frac{\partial \mathfrak{F}}{\partial \vartheta} = C \cdot \log s.$$

Thus, for any given system of bodies, the temperature would be set to be proportional to the vis viva of the heat motion, as *Clausius* and *Boltzmann* apparently already sought to before me (*). The later applications of our example are independent of the question of the relationship between the functions S and s.

(*) In a communication to the Berlin Academy (Sitzungsberichte 8 December 1884), I have expressed this theorem more definitively, but then I recognized that a step in the proof could not be justified without demanding a further restricting condition whose physical meaning I still do not know how to interpret, although I hope this will come to pass. I must therefore accept that the objection that *L. Boltzmann* made against that paper (Wiener Sitzungsberichte (2), Abh. XCII, Bd. 8 October 1885) is justified in that regard.

The motion of heat can apparently be regarded as an especially far-reaching example of the elimination of the coordinates p_c, and for that reason H can be a complicated function of ϑ or η. However, my investigations into the combined monocyclic systems have shown that many combined forms of motions that are already quite similar to the internal molecular motions of warm bodies can also lead to the same laws.

§ 3.

The derivation of the kinetic potential from the value of energy.

For the physical investigations, it is usually easier and more certain to recognize what the factors are that influence the energy supply of any system of bodies, and thus, to determine the value of the function E, than it is to find all of the laws of the variations and determine the kinetic potential from that. We thus come to the examination of the extent to which the latter can be determined from the value of the energy supply.

We assume that the quantity E has been found as a function of the p_a and q_a. For the form of that function, equation (4) implies that:

$$E = H - \sum_a \left[q_a \cdot \frac{\partial H}{\partial q_a} \right]. \tag{4}$$

It follows from this that:

$$\frac{\partial E}{\partial q_b} = - \sum_a \left[q_a \cdot \frac{\partial^2 H}{\partial q_a \, \partial q_b} \right]. \tag{7}$$

For the variation of the function Φ that is given in equation (1^a) that is necessary in order to construct the equations of motion, it must be assumed that the first and second differential quotients of H always remain finite along the path that is traversed by the system. Thus, it follows from equation (7) that when all $q_a = 0$, all $\partial E / \partial q_a$ will also be equal to zero.

Other restrictions of the function E that are implied by the physical interpretation might be mentioned only briefly here.

1) For a free system, the coordinates that are involved refer to only the relative positions of the masses of the system, because the same process of motion must be capable of proceeding for the same relative positions of the masses everywhere in space.

2) The value of E must have a minimum for finite separations of the masses and finite velocities; otherwise, the supply of work in the system would be infinitely large. Thus, the value of E must necessarily be a positive quantity for infinitely-increasing q_a. I

have sought to show the inadmissible consequences that follow from the opposite assumption for the electrodynamic theory of *W. Weber* (*).

Equation (4) next easily implies that when H can be represented as a sum of homogeneous, entire functions of the $q_{\mathfrak{a}}$ of differing degrees, the same thing will be true for E. If we denote a homogeneous function of degree n of the $q_{\mathfrak{a}}$ by $P_{\mathfrak{n}}$, and if:

$$H = \sum_{\mathfrak{n}} [P_{\mathfrak{n}}] \tag{7^a}$$

then

$$E = \sum_{\mathfrak{n}} [(1 - \mathfrak{n}) \cdot P_{\mathfrak{n}}], \tag{7^b}$$

or:

$$E = P_0 - P_2 - 2P_3 \text{ , etc.}$$

The terms P_1 of first degree drop out from the value of E, P_0 corresponds to the potential energy, which is independent of the motion, which we have denoted by F above, and P_2, to $(-L)$. Higher terms enter into the problems of the mechanics of ponderable bodies only in the altered cases in which one eliminated certain coordinates $p_{\mathfrak{c}}$.

Moreover, the problem that was posed can also be solved *when E is an entirely arbitrary function of the velocities* that satisfies only the condition that was posed above that, from equation (7), all $\partial E / \partial q_{\mathfrak{a}}$ will approach the value zero when the $q_{\mathfrak{a}} = 0$. For our purposes, it will be sufficient to retain the determination that was made above, in which the coefficients in the system of equations (7) should be finite, although there will also be cases in which those integrable coefficients would become infinite.

In order to solve our problem, we would like to set:

$$q_{\mathfrak{a}} = x \cdot \mathfrak{q}_{\mathfrak{a}} \text{ ,} \tag{8}$$

in place of the $q_{\mathfrak{a}}$ in the values of E and H, and understand x to mean a variable factor whose variation will indeed change the absolute values of the $q_{\mathfrak{a}}$, but not their mutual ratios.

After that substitution has been made, I will denote the functions H and E by H' and E'. One then has:

$$\frac{\partial E'}{\partial x} = \sum_{\mathfrak{a}} \left[\mathfrak{q}_{\mathfrak{a}} \cdot \frac{\partial E}{\partial q_{\mathfrak{a}}} \right] . \tag{8^a}$$

Since all $\partial E / \partial q_{\mathfrak{a}} = 0$ when all $q_{\mathfrak{a}} = 0$, from the convention that was made for equation (7), but, from equation (8), that will happen only when $x = 0$, that will imply that:

(*) This Journal, Bd. 72, pp. 85, § 4 to § 7; Bd. 75, pp. 35-62. – *Helmholtz,* Gesammelte Wissenschaftliche Abhandlungen, Bd. I, pp. 684.

$$(8^b) \qquad \frac{\partial E'}{\partial x} = 0, \qquad \text{when} \quad x = 0,$$

and indeed, from our assumptions, for very small x, $\partial E / \partial x$ must become proportional to x itself, if not a higher power of x. On the other hand, we have:

$$\frac{\partial H'}{\partial x} = \sum_{\mathfrak{a}} \left[\mathfrak{q}_{\mathfrak{a}} \cdot \frac{\partial H}{\partial q_{\mathfrak{a}}} \right],$$

so:

$$(8^c) \qquad x \cdot \frac{\partial H'}{\partial x} = \sum_{\mathfrak{a}} \left[q_{\mathfrak{a}} \cdot \frac{\partial H}{\partial q_{\mathfrak{a}}} \right] = - E' + H',$$

and therefore:

$$(8^d) \qquad \frac{\partial E'}{\partial x} = - x \cdot \frac{\partial^2 H'}{\partial x^2}.$$

If yet a second solution exists for the differential equation (8c):

$$(8c) \qquad E' = H' - x \cdot \frac{\partial H'}{\partial x},$$

which we would like to denote by H'', then:

$$0 = H' - H'' - x \cdot \frac{\partial}{\partial x} [H' - H''] ;$$

i.e.:

$$\log (H' - H'') = \log x + \log C,$$

or

$$H' - H'' = x \cdot C,$$

in which C can be a function of the $\mathfrak{q}_{\mathfrak{a}}$. However, if $(H' - H'')$ can also be represented as a function of the $q_{\mathfrak{a}}$, which are free of x, then it can be only homogeneous of first degree.

We now need only to find a particular integral of equation (8^c).

We will obtain one when we first write equation (8^c) for $x = 0$:

$$E_0 = H_0 ,$$

and deduce this from (8^c):

$$(E' - E_0) = (H' - H_0) - x \cdot \frac{\partial}{\partial x} [H' - H_0].$$

Dividing by x^2 will give this:

$$-\frac{1}{x^2}(E' - E_0) = \frac{\partial}{\partial x}\left[\frac{H' - H_0}{x}\right].$$

From the discussion that was made for (8^b), the quantity on the left-hand side is also finite for $x = 0$, and we then obtain by integrating between the limits $x = 0$ and $x = 1$:

$$(8^f) \qquad H' - H_0 = -\int_0^1 \frac{E' - E_0}{x^2} \cdot dx + H_1 ,$$

in which the integration constant H_1, as was remarked, can be any homogeneous function of degree one in the $q_\mathfrak{a}$.

E is then derived uniquely from *H* by means of equation (4), while in the derivation of *H* from *E*, the function H_1, which corresponds to the “hidden” motions, will remain undetermined. Whether such terms of first degree mix with each other will be ascertained in special problems mostly from the conditions under which the motion can take proceed backwards.

If one then knows which physical quantities in the value of *E* in the problem in question are to be treated as coordinates and which, as velocities then one can, as a rule, solve the problem that was posed here. However, the opposite cases can occur in which resolving the state question seems to be uncertain.

§ 4.

The general characteristics of the forces on moving systems.

It is known that the forces that act upon *systems at rest* from the outside, which satisfy the law of the constancy of energy, exhibit certain legitimate relations with each other that are expressed in the equations:

$$\frac{\partial P_\mathfrak{a}}{\partial p_\mathfrak{b}} = \frac{\partial P_\mathfrak{b}}{\partial p_\mathfrak{a}},$$

and that when these equations are fulfilled, the value of the potential energy can be found.

One likewise finds that similar relations that are implied immediately by *Lagrange’s* expressions for the forces can be presented *for moving systems* that are subject to the law of minimum kinetic energy. They are thus not merely to be regarded as functions of the coordinates $p_\mathfrak{a}$, as in the systems at rest, but also as functions of the velocities $q_\mathfrak{a}$ and the accelerations:

$$(9) \qquad q'_\mathfrak{a} = \frac{dq_\mathfrak{a}}{dt}.$$

Equation (1^c):

$$(1^c) \qquad P_{\mathfrak{a}} = -\frac{\partial H}{\partial p_{\mathfrak{a}}} + \frac{d}{dt}\left[\frac{\partial H}{\partial q_{\mathfrak{a}}}\right]$$

immediately yields:

$$P_{\mathfrak{a}} = -\frac{\partial H}{\partial p_{\mathfrak{a}}} + \sum_{\mathfrak{b}}\left[\frac{\partial^2 H}{\partial q_{\mathfrak{a}}\,\partial q_{\mathfrak{b}}}\cdot q_{\mathfrak{b}}\right] + \sum_{\mathfrak{b}}\left[\frac{\partial^2 H}{\partial q_{\mathfrak{a}}\,\partial q_{\mathfrak{b}}}\cdot q'_{\mathfrak{b}}\right].$$

A. Forces and accelerations.

As one sees, when represented in this form, the forces are linear functions of the accelerations. The coefficient of the $q'_{\mathfrak{b}}$ in the value of the force $P_{\mathfrak{a}}$ can thus be written:

$$(9^a) \qquad \frac{\partial P_{\mathfrak{a}}}{\partial q'_{\mathfrak{b}}} = \frac{\partial^2 H}{\partial q_{\mathfrak{a}}\,\partial q_{\mathfrak{b}}} = \frac{\partial P_{\mathfrak{b}}}{\partial q'_{\mathfrak{a}}}\ ;$$

i.e.: *If the acceleration $q'_{\mathfrak{b}}$ makes the force $P_{\mathfrak{a}}$ larger by a certain amount then the same increase in the acceleration $q'_{\mathfrak{a}}$ will make the force $P_{\mathfrak{b}}$ larger by the same amount.* Whether such an influence is present in a given case or not will depend upon whether the quantities $\frac{\partial^2 H}{\partial q_{\mathfrak{a}}\,\partial q_{\mathfrak{b}}}$ are non-zero or equal to zero, respectively. The stated quantities are zero, for example, for the motions of a completely-free system of ponderable masses when they are referred to rectangular coordinates. Every individual force component affects the acceleration only in the direction of the coordinate to which it is refers.

For the top in example I of § 2, we have:

$$\frac{\partial A}{\partial \beta''} = \frac{\partial B}{\partial \alpha''} = 0,$$

$$\frac{\partial A}{\partial \gamma''} = \frac{\partial C}{\partial \alpha''} = -\mathfrak{A}\cdot\cos\beta,$$

$$\frac{\partial C}{\partial \beta''} = \frac{\partial B}{\partial \gamma''} = 0,$$

in which α'', β'', γ'' denote the accelerations of the angles α, β, γ, resp.

In example II for the electrodynamic effects, one has:

$$\frac{\partial P_{\mathfrak{a}}}{\partial J'_{\mathfrak{b}}} = \frac{\partial \mathfrak{E}_{\mathfrak{b}}}{\partial q'_{\mathfrak{a}}} = 0,$$

$$\frac{\partial \mathfrak{E}_{\mathfrak{b}}}{\partial J'_{\mathfrak{c}}} = \frac{\partial \mathfrak{E}_{\mathfrak{c}}}{\partial J'_{\mathfrak{b}}}.$$

The former equation says: Since the ponderomotive force of the circular current does not depend upon the *acceleration* of the current, the induced electromotive force can also not depend upon the *acceleration* of the current conductor (but possibly upon the velocities, in both cases). The latter equation says that when for a given position and form of the circular currents *b* and *c*, a rise in the force $\mathfrak{E}_{\mathfrak{b}}$ that acts upon *b* cause an increase in $J_{\mathfrak{c}}$ by electromagnetic induction, the same rise in the force $\mathfrak{E}_{\mathfrak{c}}$ will produce the same effect on $J_{\mathfrak{b}}$.

This reciprocal relationship is not present in example III for the thermodynamic effects, since the *vis viva L* of heavy masses does not depend upon the temperature, and therefore the product $\vartheta \cdot q_{\mathfrak{a}}$ will not enter into the value of $(\mathfrak{F} - L) = H$.

B. Relations between forces and velocities.

It follows further from equations (7) that:

$$\frac{\partial P_{\mathfrak{a}}}{\partial q_{\mathfrak{b}}} = -\frac{\partial^2 H}{\partial p_{\mathfrak{a}}\,\partial q_{\mathfrak{b}}} + \frac{\partial^2 H}{\partial p_{\mathfrak{b}}\,\partial q_{\mathfrak{a}}} + \frac{d}{dt}\left[\frac{\partial^2 H}{\partial q_{\mathfrak{a}}\,\partial q_{\mathfrak{b}}}\right].$$

Thus:

$$(9^b)\qquad \left\{\begin{aligned} \frac{\partial P_{\mathfrak{a}}}{\partial q_{\mathfrak{b}}} + \frac{\partial P_{\mathfrak{b}}}{\partial q_{\mathfrak{a}}} &= 2\cdot\frac{d}{dt}\left[\frac{\partial^2 H}{\partial q_{\mathfrak{a}}\,\partial q_{\mathfrak{b}}}\right] \\ &= 2\cdot\frac{d}{dt}\left[\frac{\partial P_{\mathfrak{a}}}{\partial q'_{\mathfrak{b}}}\right] = 2\cdot\frac{d}{dt}\left[\frac{\partial P_{\mathfrak{b}}}{\partial q'_{\mathfrak{a}}}\right]. \end{aligned}\right.$$

In the very great number of cases, where:

$$(9^c)\qquad \frac{\partial P_{\mathfrak{a}}}{\partial q'_{\mathfrak{b}}} = \frac{\partial P_{\mathfrak{b}}}{\partial q'_{\mathfrak{a}}} = \frac{\partial^2 H}{\partial q_{\mathfrak{a}}\,\partial q_{\mathfrak{b}}} = \text{const.},$$

it will follow that:

$$(9^d)\qquad \frac{\partial P_{\mathfrak{a}}}{\partial q_{\mathfrak{b}}} = -\frac{\partial P_{\mathfrak{b}}}{\partial q_{\mathfrak{a}}};$$

i.e., *when a rise in the velocity* $q_{\mathfrak{b}}$ *for the same position and acceleration makes the force* $P_{\mathfrak{a}}$ *increase, a corresponding rise in* $q_{\mathfrak{a}}$ *will diminish the force* $P_{\mathfrak{b}}$. The case in which the prerequisite (9^c) is fulfilled have already been remarked in the examples that were cited in *A*. They best show the extended meaning of this theorem, but also the fact that one must control the fulfillment of the prerequisite, before one applies the simpler theorem (9^d), instead of the generally correct one (9^b).

Example I. If a force that increases the angle β – i.e., the axis of the top tends to move from the vertical – causes a greater precessional motion α then a force that causes the precessional motion to accelerate will bring the axis to the vertical line.

Example II. Electromagnetic induction, according to Lenz. The motion of two circular currents with respect to each other that is produced by ponderomotive, electrodynamical forces will bring about electromotive, induced forces that act against the currents.

The corresponding relationship will be true for the motion of a magnet relative to a current conductor.

Example III. Thermodynamics. When rises in temperature raise the pressure of a system of bodies, compression of them will raise the temperature.

For this case, we can write equation (9^d), after multiplying both sides by η, using the notations and explanations of § 2 for this example:

$$(9^e)\qquad \left\{\begin{array}{l} \dfrac{\partial}{\partial q_a}[P_{(\eta)}\cdot\eta] = -\dfrac{\partial P_a}{\partial\log\eta}, \\ \text{or, from } (6^f): \\ \dfrac{\partial}{\partial q_a}\left[\dfrac{dQ}{dt}\right] = +\dfrac{\partial P_a}{\partial\log\eta}. \end{array}\right.$$

Now, from (6^c), one has:

$$\frac{dQ}{dt} = \eta\cdot\frac{ds}{dt} = \eta\cdot\sum_a\left[\frac{\partial s}{\partial p_a}\cdot q_a\right] + \eta\cdot\frac{\partial s}{\partial\eta}\cdot\frac{d\eta}{dt}.$$

Thus, one has:

$$(9^f)\qquad \frac{\partial}{\partial q_a}\left[\frac{dQ}{dt}\right] = \eta\cdot\frac{\partial s}{\partial p_a}.$$

From (6^d), one had:

$$P_a = -\frac{\partial}{\partial p_a}[H-L] - \frac{d}{dt}\left[\frac{\partial L}{\partial q_a}\right],$$

and since L is independent of h, one will have:

$$(9^g)\qquad \frac{\partial P_a}{\partial\eta} = -\frac{\partial^2 H}{\partial p_a\,\partial\eta} = \frac{\partial s}{\partial p_a}.$$

by which, in conjunction with (9^f), the validity of equation (9^c) will be confirmed, and thus, also the applicability of our general theorem. Thus, any of the functions η in equation (6^b) can be regarded as the velocity, except that $d\eta / dt$ must then correspondingly mean the acceleration. Also, the temperature ϑ, in turn, belongs to the integrating denominator η such that one also has:

$$\frac{\partial}{\partial q_a}\left[\frac{dQ}{dt}\right] = \frac{\partial P_a}{\partial\log\vartheta}.$$

Since one must have $d\vartheta / dt = 0$ in this application, $\frac{\partial}{\partial q_{\mathfrak{a}}}\left[\frac{dQ}{dt}\right]$ will be the velocity with which the heat enters when the parameter $p_{\mathfrak{a}}$ increases with the velocity, while ϑ remains constant. This will give the formulation of the theorem that was given above.

The same considerations can also be applied to the reversible parts of thermoelectric and electrochemical processes.

Peltier's phenomenon: If warming at one place in a closed conductor brings about an electrical current then the same current will produce cooling there (ignoring the formation of heat by the resistance of the conductor.)

Electrochemistry: If warming of a constant galvanic element raises the electromotive force then the current in it will make the heat latent (*).

However, the formulas above not only exhibit the sense of the change, but also, at the same time, they give one information about how one is to deal with the quantities.

C. Relations between the forces and coordinates.

Finally, it follows from equation (9) that:

$$(9^h)\qquad \left\{\begin{aligned} \frac{\partial P_{\mathfrak{a}}}{\partial p_{\mathfrak{b}}} - \frac{\partial P_{\mathfrak{b}}}{\partial p_{\mathfrak{a}}} &= \frac{d}{dt}\left[\frac{\partial^2 H}{\partial q_{\mathfrak{a}}\,\partial p_{\mathfrak{b}}} - \frac{\partial^2 H}{\partial q_{\mathfrak{b}}\,\partial p_{\mathfrak{a}}}\right] \\ &= \tfrac{1}{2}\frac{d}{dt}\left[\frac{\partial P_{\mathfrak{a}}}{\partial q_{\mathfrak{b}}} - \frac{\partial P_{\mathfrak{b}}}{\partial q_{\mathfrak{a}}}\right]. \end{aligned}\right.$$

For the case of rest, where the right-hand side will be zero, this will yield the general law of conservative forces:

$$(9^i)\qquad \frac{\partial P_{\mathfrak{a}}}{\partial p_{\mathfrak{b}}} = \frac{\partial P_{\mathfrak{b}}}{\partial p_{\mathfrak{a}}}.$$

However, the same condition is also fulfilled when the motion proceeds temporarily in such a way that the right-hand side of (9^h) is equal to zero. Thus, we can also apply the law (9^i) in order to define a force function for the forces of warm bodies (monocyclic systems, resp.), in the event that only one of the functions η in equation (6^b) remains constant during the motion. If we therefore neglect the *vis viva* L on the associated motions then from equation (6^b) then we will have simply:

$$P_{\mathfrak{a}} = -\frac{\partial H}{\partial p_{\mathfrak{a}}},$$

(*) See my treatises on the thermodynamics of chemical processes. Sitzungsberichte der Berliner Akademie, 1882, 2 Feb., pp. 24-26; 1882, 27 July, pp. 825-835.

so our equation (9^i) will be fulfilled. However, we will almost always be in this case when we are concerned with the mechanics of terrestrial bodies that contain more or less heat. Even when the bodies are in a state of violent internal motion, we can, e.g., define force functions for the molecular forces for their elastic effects by means of the law that was proved here, and apply them as if their state of equilibrium were one of stable equilibrium in absolute rest.

Here, I would like to remark that the reciprocal relationships that were expressed in the equations:

(9^a)
$$\frac{\partial P_{\mathfrak{a}}}{\partial q'_{\mathfrak{b}}} = \frac{\partial P_{\mathfrak{b}}}{\partial q'_{\mathfrak{a}}},$$

(9^b)
$$\frac{\partial P_{\mathfrak{a}}}{\partial q_{\mathfrak{b}}} + \frac{\partial P_{\mathfrak{b}}}{\partial q_{\mathfrak{a}}} = 2\,\frac{d}{dt}\left[\frac{\partial P_{\mathfrak{b}}}{\partial q'_{\mathfrak{a}}}\right],$$

(9^h)
$$\frac{\partial P_{\mathfrak{a}}}{\partial p_{\mathfrak{b}}} - \frac{\partial P_{\mathfrak{b}}}{\partial p_{\mathfrak{a}}} = 2\,\tfrac{1}{2}\frac{d}{dt}\left[\frac{\partial P_{\mathfrak{a}}}{\partial q_{\mathfrak{b}}} - \frac{\partial P_{\mathfrak{b}}}{\partial q_{\mathfrak{a}}}\right],$$

in conjunction with the fact that the $P_{\mathfrak{a}}$ are linear functions of the $q'_{\mathfrak{a}}$, which one can write as:

(9^k)
$$\frac{\partial^2 P_{\mathfrak{a}}}{\partial q'_{\mathfrak{b}}\,\partial q'_{\mathfrak{c}}} = 0,$$

and with the previously-given definitions:

(1)
$$q_{\mathfrak{a}} = \frac{dp_a}{dt},$$

(9)
$$q'_{\mathfrak{a}} = \frac{dq_a}{dt},$$

are sufficient to prove that a *kinetic potential* exists such that the force $P_{\mathfrak{a}}$ could be expressed in terms of the differential quotients of it in the way that *Lagrange* gave, and that the equations of motion could be reduced to the principle of least action.

The relationships between the forces that were summarized here thus include a way of completely characterizing the motions that that are subject to the principle of least action.

The proof of this theorem can be given immediately with the previously-prepared tools of analysis for the case in which no more than three coordinates $p_{\mathfrak{a}}$ are present.

However, theorems from the theory of potentials functions in spaces of three dimensions will be used for that. If one would wish to go on to more coordinates p_a then one would need the corresponding theorems for a larger number of coordinates. They can be defined to the extent that they are necessary for our proof. However, since that is something that is interesting in its own right, it seems to me that it would not be suitable to go into that peripherally, and for that reason I would prefer to give the stated proof on another occasion.

Other general characteristics of the motions that take place under the principle of least action will be described in the next paragraphs.

“Ueber die physikalische Bedeutung des Princips der kleinsten Wirkung,” J. f. d. reine u. angew. Math. **100** (1887), 213-222.

On the physical meaning of the principle of least action.

(Continuation of the paper on page 137 of this volume)

(By *H. von Helmholtz*)

Translated by D. H. Delphenich

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§ 5.

Generalization of *Hamilton's* differential equation.

Hamilton has taught us how to represent the function Φ that he defined, under somewhat restricting assumptions, to be:

$$\Phi = \int_{t_0}^{t_1} (F - L) \cdot dt$$

as a function of time $t = t_1 - t_0$ and the values of the coordinates at the time t_1 and t_0. We would like to denote the moments of motion for time t_1 by $p_\mathfrak{a}$ and $s_\mathfrak{a}$ and the ones for time t_0, by $\mathfrak{p}_\mathfrak{a}$ and $\mathfrak{s}_\mathfrak{a}$, resp. It is assumed that the changes in the $p_\mathfrak{a}$ during the time interval ($t_1 - t_0$) result from the laws of motion. The value of the integral that is denoted by Φ can then be calculated as a function of the $p_\mathfrak{a}$, $\mathfrak{p}_\mathfrak{a}$, and t, and for this kind of representation, we will have:

$$(10) \qquad \begin{cases} \dfrac{\partial \Phi}{\partial p_\mathfrak{a}} = -s_\mathfrak{a}, \\ \dfrac{\partial \Phi}{\partial \mathfrak{p}_\mathfrak{a}} = \mathfrak{s}_\mathfrak{a}, \\ \dfrac{\partial \Phi}{\partial t} = E \end{cases}$$

or

$$(10^a) \qquad d\Phi = E \cdot dt - \sum_\mathfrak{a} [s_\mathfrak{a} \cdot dp_\mathfrak{a}] + \sum_\mathfrak{a} [\mathfrak{s}_\mathfrak{a} \cdot d\mathfrak{p}_\mathfrak{a}].$$

This entire conversion can also be performed under the extended assumptions that were made in § 1 (*). For the present purposes, it will suffice to do this under that assumption

(*) As *C. G. J. Jacobi* has already remarked in his lectures on dynamics. Lecture XIX.

that $P_\mathfrak{a} = 0$. Moreover, the function H might be an arbitrary function of the $p_\mathfrak{a}$ and $q_\mathfrak{a}$, as long as it fulfills the continuity condition that were discussed above.

As is known, the first two systems of equations (10) are obtained by carrying out the partial integrations that convert the variation of Φ by the $\delta q_\mathfrak{a}$ into a variation by $\delta p_\mathfrak{a}$. The differential quotient with respect to time $\partial\Phi / \partial t$ that enters into the third of equations (10) will be obtained for unvaried values of the $p_\mathfrak{a}$ and $\mathfrak{p}_\mathfrak{a}$ when we look for the change in the value of Φ that occurs as a result of an actual motion for a lengthening of the time by dt. Thus, $p_\mathfrak{a}$ will increase by $q_\mathfrak{a} \cdot dt$, and on the other hand, equation (1^a) shows that the stated variation of Φ equals the final value of $H \cdot dt$. Thus:

$$H \cdot dt = \left\{\frac{\partial\Phi}{\partial t} + \sum\left[\frac{\partial\Phi}{\partial p_\mathfrak{a}} \cdot q_\mathfrak{a}\right]\right\} \cdot dt,$$

or, from the first equations (10) and (4):

$$\frac{\partial\Phi}{\partial t} = E.$$

The following relations between the quantities $s_\mathfrak{a}$, $\mathfrak{s}_\mathfrak{a}$, E now result from equations (10) when those quantities are represented as functions of the $p_\mathfrak{a}$, $\mathfrak{p}_\mathfrak{a}$, and t:

$$(10^b) \qquad \begin{cases} \dfrac{\partial s_\mathfrak{a}}{\partial p_\mathfrak{b}} = \dfrac{\partial s_\mathfrak{b}}{\partial p_\mathfrak{a}}, \\[2ex] \dfrac{\partial s_\mathfrak{a}}{\partial \mathfrak{p}_\mathfrak{b}} = -\dfrac{\partial \mathfrak{s}_\mathfrak{b}}{\partial p_\mathfrak{a}}, \\[2ex] \dfrac{\partial \mathfrak{s}_\mathfrak{a}}{\partial \mathfrak{p}_\mathfrak{b}} = \dfrac{\partial \mathfrak{s}_\mathfrak{b}}{\partial \mathfrak{p}_\mathfrak{a}}, \end{cases}$$

$$(10^c) \qquad \begin{cases} \dfrac{\partial E}{\partial p_\mathfrak{a}} = -\dfrac{\partial s_\mathfrak{a}}{\partial t}, \\[2ex] \dfrac{\partial E}{\partial \mathfrak{p}_\mathfrak{a}} = \dfrac{\partial \mathfrak{s}_\mathfrak{a}}{\partial t}. \end{cases}$$

When these conditions are fulfilled:

$$(10^d) \qquad E \cdot dt - \sum_\mathfrak{a}[s_\mathfrak{a} \cdot dp_\mathfrak{a}] + \sum_\mathfrak{a}[\mathfrak{s}_\mathfrak{a} \cdot d\mathfrak{p}_\mathfrak{a}] = d\Phi$$

will be the complete differential of a function of the $p_\mathfrak{a}$, $\mathfrak{p}_\mathfrak{a}$, and t.

Moreover, if the quantities E, $s_\mathfrak{a}$, and $\mathfrak{s}_\mathfrak{a}$ that enter into the differential equation (10^d) are to correspond to the energy and moment of motion for a possible motion of the system that is not acted upon by external forces then they will not be completely independent of each other. In fact, as *Hamilton* has shown already, the equations of motion of the system can be represented by the system of equations:

$$\mathfrak{s}_a = \text{const.} \tag{10^e}$$

Since the $\mathfrak{s}_a$ are functions of the p_a, of t, and of the $\mathfrak{p}_a$, in general, the p_a will become functions of time from this, and the $\mathfrak{p}_a$ and $\mathfrak{s}_a$, which represent integration constants, can be determined, so the position of the system will be given for later instants. Now, if one substitutes the value of p_a that is thus obtained in the value of E for a conservative system then it would be converted into a function of the $\mathfrak{p}_a$ and $\mathfrak{s}_a$ that cannot, however, be dependent upon time for any longer. If we revert to equations (10) then this will say that:

$$\frac{\partial \Phi}{\partial t} = E\left(\mathfrak{p}_a, \frac{\partial \Phi}{\partial \mathfrak{p}_a}\right), \tag{10^f}$$

or that *a first-order differential equation must exist between the differential quotients* $\partial\Phi / \partial t$ *and* $\partial\Phi / \partial\mathfrak{p}_a$ *of the function* Φ *whose coefficients depend upon only the* $\mathfrak{p}_a$.

However, we can likewise traverse the path of a system backwards from a certain final position, in which we will have to treat the values of the p_a and s_a as if they were constant. The equations:

$$s_a = \text{const.}$$

will then yield the quantities $\mathfrak{p}_a$ as functions of t and the fixed values of s_a and p_a. When these values of the $\mathfrak{p}_a$ are substituted into the function E, it will prove to be a function of the s_a and p_a, from which, t must be absent. It follows from this that there must be a *second differential equation* for the function Φ:

$$\frac{\partial \Phi}{\partial t} = G\left(p_a, \frac{\partial \Phi}{\partial p_a}\right), \tag{10^g}$$

between the differential quotients $\partial\Phi / \partial t$ *and* $\partial\Phi / \partial\mathfrak{p}_a$ whose coefficients depend upon only the p_a .

Hamilton gave a specific form to these two differential equations for the function Φ, since he considered the two components components of the electrokinetic potential to be given, and indeed in the older, more restricted, form, while here we seek only the most general character of those equations that simultaneously corresponds to the principle of the conservation of energy and that of least action.

This also comes down to saying that any pair of associated s_a and $\mathfrak{s}_a$ should be values of the same function at the beginning and the end of the time interval t. If we apply the differential equation (10^d) to very small time intervals t then for actual motions, the quantities will have to be set to:

$$p_{\mathfrak{a}} - \mathfrak{p}_{\mathfrak{a}} = q_{\mathfrak{a}} \cdot t, \tag{10^h}$$

and these $q_{\mathfrak{a}}$ will approach the values of the velocity all the more closely as t becomes smaller. Likewise, however, the difference ($s_{\mathfrak{a}} - \mathfrak{s}_{\mathfrak{a}}$) must also approach zero with decreasing t. If the differential equation (10^d) and these auxiliary conditions are fulfilled then the *variational problem* will also be fulfilled.

To that end, one needs only to hold the $\mathfrak{p}_{\mathfrak{a}}$ constant, and vary the $p_{\mathfrak{a}}$ in the way that they would change when one varies the time interval dt for an unperturbed traversal of the motion; thus:

$$d\mathfrak{p}_{\mathfrak{a}} = 0 \qquad \text{and} \qquad dp_{\mathfrak{a}} = q_{\mathfrak{a}} \cdot dt.$$

From (10^d), this will imply that:

$$d\Phi = \{E - \sum [s_{\mathfrak{a}} \cdot q_{\mathfrak{a}}]\} \cdot dt,$$

or

$$\Phi = \int_0^t dt \cdot \{E - \sum_{\mathfrak{a}} [s_{\mathfrak{a}} \cdot q_{\mathfrak{a}}]\} \tag{10^i}$$

in which, one must take E, $s_{\mathfrak{a}}$, and $q_{\mathfrak{a}}$ under the integral sign to have the actual values that they have at time t, which is the start of the corresponding motion. That is the previous representation of the function Φ, and the differential equation (10^d) implies that this value of Φ must satisfy the minimum condition for the actual path of the system that is retraced. Namely, when we think of the path of the system from the position that is denoted by 0 to the one that is denoted by 2 as being divided by an intermediate, variable position, which we will denote by 1, then from (10^i):

$$\Phi_{0,2} = \Phi_{0,1} + \Phi_{1,2}\,.$$

If we now vary the coordinates of the intermediate position then, from (10^d), we will have:

$$\delta\Phi_{0,2} = -\,\delta\Phi_{1,2} = -\sum_{\mathfrak{a}} [s_{\mathfrak{a}} \cdot \delta p_{\mathfrak{a}}],$$

and as a result:

$$\delta\Phi_{0,2} = 0.$$

As is easily seen, this can be carried over to an arbitrary subdivision of the path into arbitrarily many pieces, and that will imply that the integral $\Phi_{0,2}$ will not vary when makes any sort of small changes to the intermediate positions.

The minimal theorem depends upon the fulfillment of the differential equation (10^d), and for the extended form of H, as well as for the original, more restricted form that *Lagrange* and *Hamilton* started with.

The conditions that exist between the quantities that enter into the differential (10^d) that were discussed in this paragraph reduce to one equation that gives E as a function of the $p_\mathfrak{a}$ and $s_\mathfrak{a}$ when one employs *C. G. J. Jacobi's* (*) conversion.

§ 6.

Reciprocity for the forward and reverse motions.

I call the motion of a system *reversible* when the sequence of positions that it passed through during its forward motion can also be traversed by the reverse motion without the action of other forces, and with the same intermediate times for each pair of equal positions. The reverse motion will be possible when the values of the kinetic potential is not changed when one changes the signs of all $q_\mathfrak{a}$. However, if products and powers of the $q_\mathfrak{a}$ of odd degree occur, which happen, e.g., for the interference of hidden motions (§ 1), then the motion will be reversible only when it is mechanically possible to also make some of the constants (viz., the velocities of the hidden motions) negative in such a way that the quantity H does not change in value under a simultaneous setting to negative values of these constants and all $q_\mathfrak{a}$. This is easily obtained from a consideration of the equations of motion (1^c) when one considers that dt must also assume the opposite sign under reversal.

Law of reciprocity.

In my acoustic investigations (**), I proved a law of reciprocity that I easily extended to small oscillations around a stable equilibrium position of an arbitrary, oscillating, mechanical system in my lectures. However, it is more general, and true for any moving system that is subject to the law of least action and has a reversible motion.

The original motion A will be unchanged when one keeps all initial positions at time t_0 unchanged, but increases one of the moments of motion $\mathfrak{s}_1$ by $d\mathfrak{s}_1$. In that way, the coordinate p_2 must increase by dp_2 at time t. If one then changes the moment of motion s_2 in the reversed motion when it goes through the value $p_\mathfrak{a}$ of the coordinates by the same amount that one changed $\mathfrak{s}_1$ then the coordinate $\mathfrak{p}_1$ will be changed by just as much as p_2 after the time interval $t = t_1 - t_0$.

Since all dt and $d\mathfrak{p}_\mathfrak{a}$ must be zero, we will have:

$$d\mathfrak{s}_\mathfrak{a} = \sum_\mathfrak{b} \left[\frac{\partial s_\mathfrak{a}}{\partial p_\mathfrak{b}} \cdot dp_\mathfrak{b} \right]. \tag{11}$$

(*) *Jacobi, loc. cit.*, Lecture XX.
(**) "Theorie der Luftschwingungen in Röhren mit offenen Enden," this Journal, Bd. 57, pp. 27-30.

Of these, only $d\mathfrak{s}_1$ should be non-zero. For the sake of brevity in our notation, we would like to write:

$$(11^a) \qquad \sigma_{\mathfrak{a},\mathfrak{b}} = \frac{\partial \mathfrak{s}_\mathfrak{a}}{\partial p_\mathfrak{b}} = -\frac{\partial s_\mathfrak{b}}{\partial p_\mathfrak{a}}.$$

From (10^b), the quantities $\sigma_{\mathfrak{a},\mathfrak{b}}$ are the same as the $\sigma_{\mathfrak{b},\mathfrak{a}}$. We denote the determinant of the quantities $\sigma_{\mathfrak{a},\mathfrak{b}}$ by $D_{(\sigma)}$. If these are not identically zero then, from equations (11), with the restriction that was made, we will have:

$$(11b) \qquad dp_2 = \frac{\partial \log D_{(\sigma)}}{\partial \sigma_{1,2}} \cdot d\mathfrak{s}_1 .$$

By contrast, if we demand that all $dp_\mathfrak{a} = 0$, and likewise all $ds_\mathfrak{a} = 0$, with the exception of ds_2, then we will get, with consideration given to (11^a), the corresponding equation:

$$d\mathfrak{p}_1 = \frac{\partial \log D_{(-\sigma)}}{\partial(-\sigma_{1,2})} \cdot d\mathfrak{s}_2$$

for the forward motion. For the reverse motion, the signs of the moments of motion are inverted, and thus, those of the $\sigma_{\mathfrak{a},\mathfrak{b}}$, as well; thus, one will have:

$$(11^c) \qquad d\mathfrak{p}_1 = \frac{\partial \log D_{(\sigma)}}{\partial \sigma_{1,2}} \cdot ds_2$$

for them. It will then follow from the combination of equations (11^b) and (11^c) that:

$$dp_2 : d\mathfrak{s}_1 = d\mathfrak{p}_1 : ds_2 ,$$

with which, the theorem that was expressed above will be proved.

As far as the *exceptional case* is concerned, in which the determinant $D_{(\sigma)}$ is equal to zero identically, in that case, the $dp_\mathfrak{b}$ would not necessarily be equal to zero when all $d\mathfrak{s}_\mathfrak{a}$ are also equal to zero, without exception. Now, since the motion of the system must be determined completely, and therefore, the values of the $p_\mathfrak{a}$ would not be double-valued during the course of time t if the initial positions $\mathfrak{p}_\mathfrak{a}$ and the initial velocities were given at the start of the time interval t, this exceptional case could occur only if the $\mathfrak{q}_\mathfrak{a}$ were not determined completely by the values of the $\mathfrak{s}_\mathfrak{a}$, which we excluded in the concluding remarks of § 1. It is therefore unnecessary to pay any attention to that exceptional case.

The sudden changes in the values of the $\mathfrak{s}_\mathfrak{a}$ and the $s_\mathfrak{a}$ here, under which, the coordinates themselves should suffer no changes in their values, would come about mechanically in such a way that one lets forces $P_\mathfrak{a}$ act during a very small time interval, but with corresponding intensity. In that way, the various rising degrees of velocities can be traversed without also having to change the position of the time at which the greatest velocity is attained notably. For such an assumption, it follows from equation (1) that:

$$- \int P_{\mathfrak{a}} \cdot dt = s_1 - s_0 \, .$$

Since $P_{\mathfrak{a}}$ denoted the force that was exerted upon the moving system from the outside, in the notation that was used there, $(- P_{\mathfrak{a}})$ will be the opposite external force that is required in order to bring about the desired change in motion.

Following Sir *W. Thomson*, we would like to refer to such a force effect as a *push in the direction of the coordinate* $p_{\mathfrak{a}}$. In that regard, one must remark that, in general, the forces $P_{\mathfrak{a}}$ are aggregates of components that act upon different parts of the system, and are distributed in such a way that the aggregate of forces $P_{\mathfrak{a}}$ performs no work under any variation of the remaining coordinates, except for $p_{\mathfrak{a}}$. Moreover, since we have to distinguish between forward and reverse motion, it would be preferable to consider values of the $d\mathrm{s}_{\mathfrak{a}}$ that increase the forward moment $\mathfrak{s}_{\mathfrak{a}}$, as well as displacements $dp_{\mathfrak{a}}$ that increase the distance $(p_{\mathfrak{a}} - \mathfrak{p}_{\mathfrak{a}})$, to be positive for the forward motion, while for the reverse motion, values of the $ds_{\mathfrak{a}}$ that increase the reverse moment $(- s_{\mathfrak{a}})$ and displacements $(- d\mathfrak{p}_{\mathfrak{a}})$ that increase the distance $(\mathfrak{p}_{\mathfrak{a}} - p_{\mathfrak{a}})$ are considered to be negative, and to treat them as equivalent to the positive changes in $d\mathfrak{s}_{\mathfrak{a}}$ and $dp_{\mathfrak{a}}$ for the forward motion.

The reciprocity theorem can then be expressed as:

If a push that increases only the moment $\mathfrak{s}_{\mathfrak{a}}$ *by* $d\mathfrak{s}_{\mathfrak{a}}$ *at the initial position of the forward motion has increased the coordinate of the final position* $p_{\mathfrak{b}}$ *by* $dp_{\mathfrak{b}}$ *after the time interval t then an equivalent reverse push that increases the reverse moment* $(- \mathfrak{s}_{\mathfrak{b}})$ *by the same amount at the earlier final position would provoke the equivalent reverse change in the coordinate* $\mathfrak{p}_{\mathfrak{a}}$ *after time t.*

§ 7.

Introducing the moment of motion as an independent variable, in place of velocity.

The differential equation (10^d) gives many opportunities for the transformation of values by a different choice of independent variables, which *Hamilton* (*) has already partially employed. However, since he assumed that the *vis viva* was a homogeneous function of second degree in the velocities there, here, I shall allow myself to carry out those of these conversions under which the action of external variable forces does not

(*) Philosph. Transact., 1835, pt. I, pp. 98-100. – See also *Jacobi, loc. cit.,* Lect. XIX.

need to be excluded for the general form of the problem. One will get them when one replaces the velocities $q_\mathfrak{a}$ in the values of H (E, resp.) with the moments of motion $s_\mathfrak{a}$.

We have considered the kinetic potential H to be a function of the $p_\mathfrak{a}$ and $q_\mathfrak{a}$. Thus:

$$dH = \sum \left[\frac{\partial H}{\partial p_\mathfrak{a}} \cdot dp_\mathfrak{a} + \frac{\partial H}{\partial q_\mathfrak{a}} \cdot dq_\mathfrak{a} \right]. \tag{12}$$

We have denoted:

$$-\frac{\partial H}{\partial q_\mathfrak{a}} = s_\mathfrak{a} .$$

It then follows from this that:

$$dE = d\,[H + \sum_\mathfrak{a} (s_\mathfrak{a} \cdot q_\mathfrak{a})\,] = \sum \left[\frac{\partial H}{\partial p_\mathfrak{a}} \cdot dp_\mathfrak{a} + q_\mathfrak{a} \cdot ds_\mathfrak{a} \right]. \tag{12^a}$$

If the determinant of the quantities $\partial s_\mathfrak{a} / \partial q_\mathfrak{b}$ is not equal to zero then we can introduce the $p_\mathfrak{a}$ and $s_\mathfrak{a}$ as variables, in place of the $p_\mathfrak{a}$ and $q_\mathfrak{a}$, and, from (12^a), that will yield:

$$\frac{\partial H}{\partial p_\mathfrak{a}} = \frac{\partial E}{\partial p_\mathfrak{a}},$$
$$q_\mathfrak{a} = \frac{dp_\mathfrak{a}}{dt} = \frac{\partial E}{\partial s_\mathfrak{a}}.$$

In this, H is assumed to be a function of the $p_\mathfrak{a}$ and the $q_\mathfrak{a}$ for the partial differentiations, but E is assumed to be a function of the $p_\mathfrak{a}$ and $s_\mathfrak{a}$.

That will imply the value of force that was given in (1^c):

$$\left\{ \begin{aligned} P_\mathfrak{a} &= -\frac{\partial E}{\partial p_\mathfrak{a}} - \frac{ds_\mathfrak{a}}{dt}, \\ \frac{dp_\mathfrak{a}}{dt} &= \frac{\partial E}{\partial s_\mathfrak{a}}, \end{aligned} \right. \tag{12^b}$$

$$H = E - \sum_\mathfrak{a} \left[s_\mathfrak{a} \cdot \frac{\partial E}{\partial s_\mathfrak{a}} \right]. \tag{12^c}$$

The corresponding variational problem will take on a somewhat different form from the one that *Hamilton* gave to it:

$$\Psi = \int_{t_0}^{t_1} dt \cdot \left\{ E + \sum_\mathfrak{a} \left[p_\mathfrak{a} \left(P_\mathfrak{a} + \frac{ds_\mathfrak{a}}{dt} \right) \right] \right\}. \tag{12d}$$

In this, $P_{\mathfrak{a}}$ are regarded as functions of only time, and E, as a function of the $p_{\mathfrak{a}}$ and $s_{\mathfrak{a}}$. One varies the $p_{\mathfrak{a}}$ and the $s_{\mathfrak{a}}$ independently of each other and demands that the $\delta s_{\mathfrak{a}} = 0$ at the limits of the time interval. The condition:

$$\delta\Psi = 0$$

will then give the two systems of equations of motion (12^b) with no other auxiliary equations.

In this manner of representation, we do not at all generally need to know the kinetic energy, but we must know the quantities $s_{\mathfrak{a}}$ that we can derive from the $q_{\mathfrak{a}}$ for the general form of the E only by means of H.

We will obtain the corresponding form of the differential equation (10^d) for the case in which the P are equal to zero when we add to both sides of that equation, namely:

$$d\Phi = E \cdot dt - \sum (s_{\mathfrak{a}} \cdot dp_{\mathfrak{a}}) + \sum (\mathfrak{s}_{\mathfrak{a}} \cdot d\mathfrak{p}_{\mathfrak{a}}), \tag{10^d}$$

the term:

$$d\,\{\sum (s_{\mathfrak{a}} \cdot dp_{\mathfrak{a}}) - \sum (\mathfrak{s}_{\mathfrak{a}} \cdot d\mathfrak{p}_{\mathfrak{a}})\}.$$

That will give:

$$\left\{\begin{aligned} &d\left\{\Phi + \sum_{\mathfrak{a}}[s_{\mathfrak{a}} \cdot p_{\mathfrak{a}}] - \sum_{\mathfrak{a}}[\mathfrak{s}_{\mathfrak{a}} \cdot \mathfrak{p}_{\mathfrak{a}}] = E \cdot dt + \sum_{\mathfrak{a}}[p_{\mathfrak{a}} \cdot ds_{\mathfrak{a}}] - \sum_{\mathfrak{a}}[\mathfrak{p}_{\mathfrak{a}} \cdot \mathfrak{s}_{\mathfrak{a}}]\right\} \\ &\qquad = d\Psi. \end{aligned}\right. \tag{12^c}$$

Thus, if E, the $p_{\mathfrak{a}}$, and the $\mathfrak{p}_{\mathfrak{a}}$ are represented as functions of time t and the $s_{\mathfrak{a}}$ and $\mathfrak{s}_{\mathfrak{a}}$ then:

$$\left\{\begin{aligned} \frac{\partial p_{\mathfrak{a}}}{\partial s_{\mathfrak{b}}} &= \frac{\partial p_{\mathfrak{b}}}{\partial s_{\mathfrak{a}}}, \\ \frac{\partial \mathfrak{p}_{\mathfrak{a}}}{\partial \mathfrak{s}_{\mathfrak{b}}} &= \frac{\partial \mathfrak{p}_{\mathfrak{b}}}{\partial \mathfrak{s}_{\mathfrak{a}}}, \\ \frac{\partial p_{\mathfrak{a}}}{\partial \mathfrak{s}_{\mathfrak{b}}} &= -\frac{\partial \mathfrak{p}_{\mathfrak{b}}}{\partial s_{\mathfrak{a}}}, \\ \frac{\partial E}{\partial s_{\mathfrak{a}}} &= \frac{\partial p_{\mathfrak{b}}}{\partial t}, \\ \frac{\partial E}{\partial \mathfrak{s}_{\mathfrak{a}}} &= -\frac{\partial \mathfrak{p}_{\mathfrak{a}}}{\partial t}. \end{aligned}\right. \tag{12^f}$$

The middle equation of this system can be employed, again, like (10^b), in order to define a *second reciprocity theorem*, by which, a displacement $\delta\mathfrak{p}_1$ is performed at the start of the time interval t, while all other $\mathfrak{p}_{\mathfrak{a}}$ and all $\mathfrak{s}_{\mathfrak{a}}$ remain unchanged; let s_2 be changed by δs_2 after the time interval t. Under the reverse motion, only p_2 will be changed by δp_2, and the moment $\mathfrak{s}_1$ will be changed by $\delta\mathfrak{s}_1$ after the time interval *t*. One will then have, once more:

(12[g])
$$\delta\mathfrak{p}_1 : \delta\mathfrak{s}_2 = \delta\mathfrak{p}_2 : \delta\mathfrak{s}_1 \,,$$

assuming that the determinant of the equations:

$$\delta p_{\mathfrak{a}} = \sum_{\mathfrak{b}} \left[\frac{\partial p_{\mathfrak{a}}}{\partial \mathfrak{s}_{\mathfrak{b}}} \cdot d\,\mathfrak{s}_{\mathfrak{b}} \right]$$

is not equal to zero. If it is then the two positions will be reciprocal foci of the motion.

Berlin, April 1886.