Principles of static, monocyclic systems ([*])

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I understand the term *monocyclic systems* to mean those mechanical systems in whose interior one or more stationary, closed motions are present, but which, when there are several, have velocities that depend upon only one parameter. I further assume that only conservative forces act between the individual bodies that define the system, which consist of relatively fixed constraints, while the external forces that must be added in do not necessarily need to be conservative. I refer to the problems that I will treat as *static* whenever it is assumed that the variations that the state of the system experiences come about with such slight velocities that the system never deviates noticeably from those states in which it can continually abide under them.

The main interest of such investigations lies in the fact that the motion of heat, at least in its externally-observable effects, also exhibits the essential peculiarities of a monocyclic system, and in fact, the restricted mobility of the equivalent work that is ignored in the form of heat is added to the monocyclic systems under certain conditions. Indeed, the motion of heat is not monocyclic in the strict sense of the word. Every individual atom obviously changes in the course of its motion, and first takes on the character of a monocyclic motion when all possible stages for the motion of an unimaginably large number of atoms are represented, if one also has that every individual stage starts from first this and then that atom.

In the theoretical examinations of heat motion − to the extent that they have been carried out, up to now − we must continually calculate with mean values of a sequence of values over time for a particle. Those laws of motion that can thus be omitted, despite the fluctuation of the individual values, cannot be invalid in such a way that the mean value is taken over nothing by equal individual values for monocyclic systems. In this sense, the present study connects with the theory of heat.

§ 1. Recapitulation of the laws of heat.

We assume that we can completely characterize the state of a body or systems of bodies whose parts all have equal temperature by the absolute temperature ϑ and a certain number of parameters p_a that are chosen so that the variation of temperature with

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[*] The first three paragraphs are essentially unchanged versions of my communications to the Akademie der Wiss. zu Berlin on 6 and 27 March 1884; the later ones include new generalizations, and are completely revamped.

changes in the quantities p_a requires the absorption or exertion of no other forms of work than quantum heat. In this case, the parameters p_a must be spatial dimensions, taken in the broader sense. It is quite customary that the total or partial volumes appear in them, but they can also express how much of a particular substance or how much electricity is found in a certain space.

I let $P_a \cdot dp_a$ denote the freely convertible work – hence, the work that is not converted into heat – that the system in question takes on from the outside when the parameter p_a goes to the value $(p_a + dp_a)$. The quantity P_a is then the *force moment* of the internal forces that act to increase the parameter p_a. It seems to me that there would not, on the other hand, be any objection to referring to P_a as the *force in the direction of p_a*, as one already does in many examples in the applications. Each of the quantities P_a is, in general, a function of the ϑ and all of the p_a. How one is to find and distinguish the components of given external forces that maintain the individual P_a in equilibrium is treated sufficiently in the textbooks.

We further let U denote the total internal energy of the system and let S be its entropy; both quantities are likewise functions of ϑ and the p_a. Finally, we let dQ refer to the heat that enters the system during a vanishingly small change in the quantities ϑ and p_a, as measured by its work equivalent. As is known, one then has:

(1)
$$\begin{cases} dQ = dU + \sum (P_a \cdot dp_a) \\ \quad = \vartheta \cdot dS. \end{cases}$$

These two equations define the foundations of the mechanical theory of heat. From them, it follows in a well-known way that only a part of the heat dQ_1 that enters into the body at the temperature ϑ_1 can be converted into freely-convertible work. If so much heat dQ_a at the temperature ϑ_a is given that ultimately the original state of the body can be again presented in a completely reversible process then one has:

$$\frac{dQ_1}{\vartheta_1} = \frac{dQ_a}{\vartheta_a},$$

and the quantum:

$$dQ \cdot \frac{\vartheta_1 - \vartheta_a}{\vartheta_1} = dQ_a \cdot \frac{\vartheta_1 - \vartheta_a}{\vartheta_a}$$

is then converted into other work.

Allow me to make the following remarks in regard to what follows: The essential physical meaning of the temperature ϑ is that its equality or inequality between two bodies will decided whether, and in which direction, heat can flow from one of them to the other one. Two bodies with equal temperatures that are in mutual contact do not affect the motion of heat in each other. As long as complete equality of the temperatures between them can be found, they define, in turn, a single composite system of bodies to

which equations (1) can be applied. If we distinguish the quantities that relate to the individual subsystems by the indices 1 and 2 then we have for simultaneous changes:

$$dQ_1 = dU_1 + \sum (P_\mathfrak{a} \cdot dp_\mathfrak{a}) = \vartheta \cdot dS_1 \, ,$$
$$dQ_2 = dU_2 + \sum (P_\mathfrak{a} \cdot dp_\mathfrak{a}) = \vartheta \cdot dS_2 \, ;$$

thus, if we add them:

$$d(Q_1 + Q_2) = d(U_1 + U_2) + \sum (P \cdot dp) = \vartheta \cdot d(S_1 + S_2).$$

In the last equation, the sum of the force moments is to be taken over both systems; $(U_1 + U_2)$ is the total energy of the combined system, $d(Q_1 + Q_2)$ is the total heat supplied, and the equation shows that $(S_1 + S_2)$ is the entropy of the combined systems.

Without a heat supply – so $dQ = 0$ – one also has $dS = 0$ in each individual system, or S is constant for all reversible processes.

This is true for S_1 and S_2 as long as both bodies are isolated, but it is also true for the sum $(S_1 + S_2)$ if they are combined at equal temperatures. The conclusion that was reached by *Clausius* follows from this – viz., that the sum of the entropy values S can change under reversible processes, either separately or in combination.

As is easily seen, the same thing is true for arbitrarily many bodies that can be arbitrarily separate and combined at equal temperatures, and this consequence follows entirely from the two equations (1). Since this theorem is true for unrestricted changes of the parameters $p_\mathfrak{a}$ and $p_\mathfrak{b}$, it is also true in the case where fixed couplings of the two bodies introduce restrictions on the variability of the p.

The determination of the direction of heat flux or the existence of thermal equilibrium could also be resolved from the inequality or equality of the values of an arbitrarily-chosen function of the temperature that determines its value uniquely. From the fact that:

$$\vartheta_1 = \vartheta_2 \, ,$$

it also follows that:

$$f(\vartheta_1) = f(\vartheta_2).$$

As is known, the various thermometer scales of mercury, alcohol, and air thermometers give different functions of ϑ according to the type of thermometric body.

On the other hand, if one understands s to mean a function of S then one would have to set:

$$dQ = \vartheta \cdot \frac{\partial S}{\partial s} \cdot ds,$$

and if one denotes:

$$\vartheta \cdot \frac{\partial S}{\partial s} = \eta$$

then one gets:

(1^a)
$$dQ = \eta \cdot ds.$$

As a function of S, s is, like it, a function of the $p_\mathfrak{a}$ and ϑ, which vanishes, from equation (1^a). Since η, from its definition above, is a function of ϑ and the $p_\mathfrak{a}$, ϑ can be determined from η and $p_\mathfrak{a}$, and η can be introduced in place of ϑ as an independent variable in the values of U, $p_\mathfrak{a}$, and s. The equations:

(1^b)
$$\begin{cases} dQ = dU + \sum (P_\mathfrak{a} \cdot dp_\mathfrak{a}) \\ \quad = \eta \cdot ds \end{cases}$$

then have the same form as equations (1), but there is an important difference, namely, that η is no longer the quantity whose equality indicates the thermal equilibrium between two bodies. This property relates to only one $\eta = \vartheta$ in the entire set of functions that are given by equation (1^a).

Finally, one must direct one's attention to a situation that I have already stressed in my first thermodynamic treatise of 2 February 1882. No rapidly-changing forces enter into equations (1) as work done on the parts of the system; i.e., it is assumed that the changes $dp_\mathfrak{a}$ happen so slowly that the rapidly-changing forces that affect the masses in orderly motion vanish compared to the remaining equivalent works. It is also further assumed that the variations of the individual parts of the system happen so slowly that a complete equalization of their temperatures comes about. Therefore, equations (1) are true only for *variations $d\vartheta$ and $dp_\mathfrak{a}$ of vanishing velocity*. In this sense, the equations of thermodynamics that we spoke of are therefore also to be regarded as laws of the statics of thermal systems in the sense described above, and we have only to look for the analogue in the laws of statics of monocyclic systems. Moreover, both of the *laws of virtual velocities*, which relate to completely immobile systems, are the same for those systems, since this also encompasses the conversions of work that are obtained for slow motion.

The substitution of the functions that enter into equations (1) into the differential quotients of a single function, as *Massieu* ([*]) first did, can, as *Gibbs* already pointed out ([**]), can also be performed on the general formula (1^b), if one employs η and the $p_\mathfrak{a}$ as independent variables. One must then set:

$$dU = \frac{\partial U}{\partial \eta} \cdot d\eta + \sum_\mathfrak{a} \left[\frac{\partial U}{\partial p_\mathfrak{a}} \cdot dp_\mathfrak{a} \right],$$

$$ds = \frac{\partial s}{\partial \eta} \cdot d\eta + \sum_\mathfrak{a} \left[\frac{\partial s}{\partial p_\mathfrak{a}} \cdot dp_\mathfrak{a} \right],$$

and when we set:
(1^c) $H = U - \eta \cdot s,$

([*]) Mémoires des Savants étrangers, t. XXII. Journal de Physique par *d'Almeida*, t. VI, pp. 216.
([**]) Transactions Connecticut Acad. III, pp. 108-248; 343-524. *Silliman's* Journal 1878, XVI, pp. 441-458.

it follows from equations (1^b) that:

$$\frac{\partial U}{\partial p_a} + P_a = h \cdot \frac{\partial s}{\partial \eta},$$

or

(1^d)
$$\begin{cases} P_a = -\dfrac{\partial H}{\partial p_a}, \\[2mm] s = -\dfrac{\partial H}{\partial \eta}, \\[2mm] U = H - \eta \cdot \dfrac{\partial H}{\partial \eta}. \end{cases}$$

These are the three relations that I used as the basis for the thermodynamic arguments in my earlier communications, where I restricted myself to the narrower case of $\eta = \vartheta$. The function H is denoted by $\mathfrak{F}$ there and called the *free energy*.

Naturally, the value of H differs with the choice of η. The product:

$$\eta \cdot s = \vartheta \cdot s \cdot \frac{\partial S}{\partial s}$$

that enters into its value (1^c) then varies with the choice of s as a function of S. These various values of H are all force functions, in the *Jacobi* sense, for the mechanical behavior of the system then, but they correspond to the distinguishing condition that their special variable η must be constant. The force function that I used is the isothermal one; however, as *Gibbs* showed, one can also define an adiabatic one, etc.; as equations (1^b) show immediately, the latter coincides with the function U when one introduces p_a and s as independent variables in it.

The fact that we can express the work equivalent of the heat quantity:

$$dQ = dU + \Sigma \, (P_a \cdot dp_a)$$

in the form:

$$dQ = \eta \cdot ds,$$

says nothing characteristic about the physical quantities that enter into it. In any event, this conversion can then be accomplished for any sort of dependency between the functions U, P_a, and p_a, when only the P_a vary continuously with the U and p_a. One must always be able to define the integral of the equation:

$$dQ = 0$$

in the form:

$$s = \text{const.}$$

or

$$S = f_{(s)} = \text{const.},$$

which expresses the law of adiabatic variations. Therefore, if one has:

$$0 = dU + \sum [P_{\mathfrak{a}} \cdot dp_{\mathfrak{a}}]$$

for variations of the U and $p_{\mathfrak{a}}$ then one also always has:

$$0 = \frac{\partial s}{\partial U} \cdot dU + \sum \left[\frac{\partial s}{\partial p_{\mathfrak{a}}} \cdot dp_{\mathfrak{a}} \right],$$

equations that follow from each other for arbitrary values of dU and $dp_{\mathfrak{a}}$, moreover, only when:

$$\frac{\partial s}{\partial U} = \frac{1}{P_{\mathfrak{a}}} \cdot \frac{\partial s}{\partial p_{\mathfrak{a}}}.$$

If we then multiply the first of equations (1^b) by $\partial s / \partial U$ then we get:

$$\frac{\partial s}{\partial U} \cdot dQ = ds,$$

and if we define η by the equation:

$$\eta \cdot \frac{\partial s}{\partial U} = 1$$

then this yields:

$$dQ = \eta \cdot ds.$$

The characteristic feature of the physical properties of the heat quantity is then not the fact that the expression for dQ can be brought into the latter form, but is only found in the fact that one of the possible integrating factors η of the equation $dQ = 0$ must have the same value for any two bodies between which thermal equilibrium exists.

§ 2. The general equations of mechanics for polycyclic systems

We next assume an arbitrarily-composed mechanical system, between whose individual parts only conservative forces act, relatively fixed constraints exist, and whose momentary position is determined completely by a number of general coordinates $p_{\mathfrak{a}}$ (that must therefore necessarily be measurements of spatial quantities). By contrast, as before, we let denote $(-P_{\mathfrak{a}})$ the moments of external forces, which do not need to be conservative, and which serve to increase the coordinates $p_{\mathfrak{a}}$, such that:

$$P_{\mathfrak{a}} \cdot dp_{\mathfrak{a}}$$

is the work done by the internal forces of the systems in overcoming any external force moment during the change dp_a . We further set, to abbreviate, the differential quotients:

$$(2) \qquad \frac{dp_a}{dt} = q_a$$

of the potential energy of the system equal to Φ and the continual (lebendige) force equal to L. Under the stated conditions, the former – viz., Φ – is a function of only the coordinates p_a , while the latter – viz., L – is a homogeneous function of second degree in the quantities q_a whose coefficients are functions of the p_a . They will be regarded as independent variables in the definition of the partial differential quotients of L with respect to the p_a and q_a , and the relation between them that is expressed by equation (2) is not considered. As is known, from the given behavior of the function L, it follows that:

$$(2^a) \qquad 2L = \sum_a \left[q_a \cdot \frac{\partial L}{\partial q_a} \right].$$

According to *Lagrange*, under these conditions, the equations of motion of the system are of the form:

$$(2^b) \qquad P^a = - \frac{\partial}{\partial p_a} [\Phi - L] - \frac{d}{dt}\left[\frac{\partial L}{\partial q_a} \right].$$

If we set:

$$(2^c) \qquad \Phi - L = H$$

and remark that since Φ is independent of q_a , one has:

$$\frac{\partial L}{\partial p_a} = - \frac{\partial H}{\partial q_a},$$

then we can therefore also write the aforementioned equations of motion as:

$$(2^d) \qquad P_a = - \frac{\partial H}{\partial p_a} + \frac{d}{dt}\left[\frac{\partial H}{\partial q_a} \right].$$

The total energy is then:

$$U = \Phi + L$$

$$(2^e) \qquad = H - \sum \left[q_a \cdot \frac{\partial H}{\partial q_a} \right].$$

We would like to introduce only the following restrictions on this most general expression for the equations of motion:

1. For a special group of coordinates, which we would like to distinguish by the index $\mathfrak{b}$, and which we regard as rapidly-varying, we will assume that their variation corresponds to a type of motion that is closed in itself, and that neither Φ nor L changes noticeably during this motion such that both quantities can indeed be independent of the $q_{\mathfrak{b}}$, but not the $p_{\mathfrak{b}}$. With this assumption, equation (2^d) becomes:

$$ (3) \qquad\qquad P_{\mathfrak{b}} = + \frac{d}{dt}\left[\frac{\partial H}{\partial q_a} \right]. $$

If we multiply both sides of this equation by $q_{\mathfrak{b}} \cdot dt$ and set:

$$ P_{\mathfrak{b}} \cdot q_{\mathfrak{b}} \cdot dt = - dQ_{\mathfrak{b}} $$

then $dQ_{\mathfrak{b}}$ is the external work that is employed in the acceleration of the motion $q_{\mathfrak{b}}$. If, to abbreviate the notation, we then set:

$$ (3^a) \qquad\qquad \frac{\partial H}{\partial q_{\mathfrak{b}}} = - s_{\mathfrak{b}} $$

then (3) yields:

$$ (3^b) \qquad\qquad dQ_{\mathfrak{b}} = q_{\mathfrak{b}} \cdot ds_{\mathfrak{b}} , $$

in which the $ds_{\mathfrak{b}}$ denotes the complete differential of the quantity $s_{\mathfrak{b}}$.

An example of such motions is the top moving without friction in a locus of axes that are defined symmetrically about the rotational axis. If we set the parameter $q_{\mathfrak{b}}$ equal to the angular velocity then $s_{\mathfrak{b}}$ will be the moment of the rotational motion; i.e., the product of the moment of inertia and the angular velocity.

Another example is the flow of an inviscid fluid in a closed channel with elastic walls, so it is extensible in its cross-section and flexible and extensible in its length. If we use $q_{\mathfrak{b}}$ to denote the volume of fluid that flows per second through each cross-section ω, the $s_{\mathfrak{b}}$ will be:

$$ s_{\mathfrak{b}} = \int u \cdot dx, $$

in which dx denotes a unit length of the axis of the channel and $u = 1/\omega \cdot q$ denotes the velocity of the fluid particle in the direction of dx.

2. Moreover, we would like to assume (since this always happens in the aforementioned equations of the mechanical theory of heat) that the changes in all of the other parameters $p_{\mathfrak{a}}$, and likewise the quantities $q_{\mathfrak{a}}$, come about with vanishing velocity, such that all of the expressions that are multiplied by $q_{\mathfrak{a}}$, $dq_{\mathfrak{a}} / dt$, or $dq_{\mathfrak{b}} / dt$ are to be treated as vanishing quantities of first order.

This assumes that the external forces that act on the system nowhere deviate far from the values that they must have in order to make p_a and q_b constant, such that all of the differential quotients with respect to time that enter into equations (2^d) become very small, and the system is continually found very close to a stationary state in which it can persist for an arbitrarily long time. We would like to call a system that satisfies these conditions *polycyclic*. For such a system, equations (2^d) reduce to:

$$(3^a) \qquad\qquad P_a = - \frac{\partial H}{\partial p_a},$$

$$(3^b) \qquad\qquad dQ_b = q_b \cdot ds_b,$$

$$(3^c) \qquad\qquad s_b = - \frac{\partial H}{\partial q_b}.$$

Among the assumptions that were made up to now, one has that the function H that enters into equations (3) to (3^c) will always have the value that was given in (2^c):

$$H = \Phi - L,$$

$$2L = - \sum \left[q_a \cdot \frac{\partial H}{\partial q_a} \right],$$

and L will be an entire homogeneous function of second degree in the velocities q_a. However, a variation can enter into the latter relation when one or more forces, which we would like to distinguish by the index c, are continually equal to zero, and we employ the corresponding equations:

$$(4) \qquad\qquad 0 = \frac{\partial H}{\partial p_c}$$

in order to eliminate the quantities p_c. If we denote the expression for the value of H that one gets by eliminating the p_c by $\mathfrak{H}$ then one has:

$$\frac{\partial \mathfrak{H}}{\partial p_a} = \frac{\partial H}{\partial p_a} + \sum_c \left[\frac{\partial H}{\partial p_c} \cdot \frac{\partial p_c}{\partial p_a} \right],$$

$$\frac{\partial \mathfrak{H}}{\partial p_b} = \frac{\partial H}{\partial p_b} + \sum_c \left[\frac{\partial H}{\partial p_c} \cdot \frac{\partial p_c}{\partial p_b} \right].$$

Due to equations (4), this reduces to:

$$\frac{\partial \mathfrak{H}}{\partial p_{\mathfrak{a}}} = \frac{\partial H}{\partial p_{\mathfrak{a}}},$$

$$\frac{\partial \mathfrak{H}}{\partial p_{\mathfrak{b}}} = \frac{\partial H}{\partial p_{\mathfrak{b}}}.$$

One still has:

(4^a) $\mathfrak{H} = \Phi - L,$

(4^b) $P_{\mathfrak{a}} = - \dfrac{\partial \mathfrak{H}}{\partial p_{\mathfrak{a}}},$

(4^c) $s_{\mathfrak{b}} = - \dfrac{\partial \mathfrak{H}}{\partial q_{\mathfrak{b}}},$

(4^d) $L = \sum [q_{\mathfrak{b}} \cdot s_{\mathfrak{b}}].$

However, this value of L is no longer an entire, homogeneous function of second degree in the quantities $q_{\mathfrak{b}}$, since equations (4) generally contain terms of second degree in the $q_{\mathfrak{b}}$, and the values of the $p_{\mathfrak{c}}$ that this yields can be developed into functions of the $q_{\mathfrak{b}}$ that make terms of second degree in L enter into the values of Φ and the coefficients by the elimination of the quantities. It is for just that reason then that the $s_{\mathfrak{b}}$ are also not homogeneous linear functions of the $q_{\mathfrak{b}}$, as was the case before the elimination.

An example of such a case is the continual force of a top whose axis has a centrifugal regulator fixed on it whose rising and falling is not caused by a varying external force $P_{\mathfrak{a}}$, but only by continually-acting conservative forces (e.g., gravity, elastic springs), and therefore can be represented as a function of the rotational velocity. In the value of the continual force, which is equal to one-half the product of the moment of inertia and the square of the rotational velocity, the aforementioned moment will then also depend upon this velocity.

Since it will later be necessary to stress this difference, we would like to call the system that includes all $p_{\mathfrak{c}}$ the *complete* one and the one that remains by the elimination of the $p_{\mathfrak{c}}$ the *incomplete* one.

§ 3. Monocyclic system.

As mentioned above, I would like to use these names for systems in which closed internal motions occur that are determined completely by only one parameter σ, along with the coordinates $p_{\mathfrak{a}}$.

The simplest case of such a motion is given when only one velocity q enters in the system of § 2. We then obtain the equations:

$$
(5) \quad
\begin{cases}
P_{\text{a}} = -\dfrac{\partial H}{\partial p_{\text{a}}}, \\[2mm]
dQ = s \cdot dq, \\[2mm]
s = -\dfrac{\partial H}{\partial q}, \\[2mm]
U = H - q \cdot \dfrac{\partial H}{\partial q}.
\end{cases}
$$

From this, it follows that:

$$
dQ = dU + \sum [P_{\text{a}} \cdot dp_{\text{a}}] = s \cdot dq,
$$

These equations are of entirely the same form as the ones presented above for the motion of heat. The velocity q enters in place of the temperature ϑ or the quantity η that it depends upon. Here, as above, the quantity dQ means the work that is directed along the direct increase of the internal motion, except that this internal motion is now of a different type from the heat motion.

Thus, q is an integrating factor of the equation $dQ = 0$, here; however, as above, we can also preserve the form of the equations if we introduce a quantity $(q \cdot \partial s / \partial \sigma)$ along with the p_{a} as a parameter for the intensity of the motion, in which σ means a function of s. However, the function:

$$
\mathfrak{H} = U - q \cdot \frac{\partial s}{\partial \sigma} \cdot \sigma
$$

that is further introduced then loses the meaning that H takes on in (2^c), namely: $H = \Phi - L$.

It is noteworthy that in this simplest case of monocyclic motion the continual force:

$$
(5^a) \qquad L = -\tfrac{1}{2} q \cdot \frac{\partial H}{\partial q} = \tfrac{1}{2} q \cdot s
$$

is an integrating factor of the system. If one defines $\mathfrak{S}$, which is chosen for this purpose, by the equation:

$$
dQ = 2L \cdot d\mathfrak{S} = q \cdot ds
$$

then we get:

$$
d\mathfrak{S} = \frac{ds}{s},
$$
$$
\mathfrak{S} = \log s - \log A,
$$

where A is an integration constant, or:

$$
(5^b) \qquad \mathfrak{S} = \log\left(\frac{s}{A}\right) = \tfrac{1}{2} \log L + \tfrac{1}{2} \log\left(\frac{s}{A^2 \cdot q}\right).
$$

Here, the analogy with the kinetic theory of gases emerges quite clearly. The temperature ϑ is proportional to the continual force, and if we let $\mathfrak{J}$ denote the mechanical heat equivalent, v, the volume of a unit mass, and let c and γ denote the heat capacities for constant pressure and volume, resp., then the entropy of the unit mass is:

$$S = \mathfrak{J} \cdot \gamma \cdot \log \vartheta + \mathfrak{J} \cdot (c - \gamma) \cdot \log v + C.$$

In the present case, the ratio s / q is the moment of inertia of the rotating mass, which depends upon spatial measurements, like the v in the entropy of the gas, and indeed only upon them, as long as, perhaps, the quantities p_c are not eliminated.

§ 4.

The *general case of monocyclic motion* comes about when several velocities q_b are present, which however will all be determined by one of them and the coordinates p_a . As is known, exceptionally manifold and varying relations between rotational velocities can also present themselves in mechanical apparatuses. For example, a friction roller can run around the circumference of a rotating body that has different diameters at different places, and is shifted to and fro by centrifugal forces on the axis around which it rotates.

In general, we can represent such a coupling analytically by $(n - 1)$ equations in the quantities p_a and the n quantities q_b . As in other parts of mechanics where fixed constraints are assumed, we imagine the action of these fixed constraints to be such that they do not at all influence those motions that would come about by themselves under the action of the applied forces that would correspond to the equations of constraint, but that when deviations begin they exert such opposing forces as would hinder the deviation. Thereby, the fixed constraints always act in such a way that the forces that they produce add no contribution to the work that is done by the external forces. The quantity of the force that it exerts in any state of the system is completely determined by this latter condition.

If we apply this to our composite monocyclic system then we will be able to represent the general form of the dependency of the velocities q_b on each other, such that we can express each of them as a function of the p_a and any new variable x that is introduced.

Since the s_b can be represented by means of the system of equations:

$$s_b = - \frac{\partial H}{\partial q_b}$$

as functions of the p_a and the q_b , they can, in any event, be expressed as functions of the p_a and x. We would like to refer briefly to the system, after the introduction of the aforementioned fixed constraint, as the *bound system*.

Under the assumptions that we made, which characterize the problem as a static one, it follows that the forces P_a act continually while the system does not change its state; in contrast, we neglect the changes of P_a that depend upon the velocity of the change of state. On the other hand, the amount of work dQ_b enters in only as long as state changes occur. Since both types of action are completely different, as they correspond to mutually independent time periods, forces P that are given by the fixed constraint, and which we would like to denote by P' in order to distinguish them, also never preserve the equilibrium in the forces that do the work dQ'_b, and conversely; the forces of each class must be separate in equilibrium.

From this, it *first* follows that the total work done by the forces P'_a satisfies:

$$\sum [P'_a \cdot dp_a] = 0.$$

Since the dp_a must remain completely arbitrary when none of the degrees of freedom are lost by the introduction of the fixed constraint, one must have that all $P'_a = 0$ in this case, so the forces P_a also preserve their previous values in the bound system:

(6)
$$P_a = -\frac{\partial H}{\partial p_a}.$$

If we regard the s_b as functions of the p_a and x then this equation becomes:

(6^b)
$$dQ = \sum_a \sum_b \left[q_b \cdot \frac{\partial s_b}{\partial p_a} dp_a \right] + \sum_b \left[q_b \cdot \frac{\partial s_b}{\partial x} \right] dx.$$

If we let U denote the internal energy of the system then:

$$dU = dQ - \sum [P_a \cdot dp_a].$$

U is originally given as a function of the p_a and q_b ; if the latter were expressed in terms of the p_a and x then this the equation:

$$dQ = 0$$

would take on the form:

$$\sum_a \left[\left(\frac{\partial U}{\partial p_a} + P_a \right) dp_a \right] + \frac{\partial U}{\partial x} \cdot dx = 0,$$

whose integral we can write as:

$$\sigma = \text{const.},$$

in which σ must be a function of the p_a and x. As above at the end of § 1, this would yield:

$$\left(\frac{\partial U}{\partial p_a} + P_a\right) = \lambda \cdot \frac{\partial \sigma}{\partial p_a},$$

$$\frac{\partial U}{\partial x} = \lambda \cdot \frac{\partial \sigma}{\partial x}.$$

As a result, equation (6^b) assumes the form:

(6^c) $dQ = \lambda \cdot d\sigma.$

We can therefore refer to the function σ as the entropy of the bound system. As we mentioned above, arbitrary functions of σ can also be chosen instead of this.

   Up to now, the choice of the function x was completely arbitrary. If we take it to be one of these σ then in (6^b) one would have to set:

$$dQ = \lambda \cdot d\sigma,$$

and (6^b) would then decompose into the sequence of equations:

(6^d) $\displaystyle\sum_b \left[q_b \cdot \frac{\partial s_b}{\partial p_a} \right] = 0,$

(6^e) $\displaystyle\sum_b \left[q_b \cdot \frac{\partial s_b}{\partial \sigma} \right] = \lambda.$

Equations (6^d) are linear, homogeneous equations in the q_b . We assume that the number of p_a that are contained in them can be reduced to its smallest value by a suitable choice of them. Namely, it can happen that n of the quantities p_a enter into the values of the s_b only in the form of less than n functions of them. One would then be able to introduce these functions in place of an equal number of the p_a . If *the number $\mathfrak{A}$ of the indices $\mathfrak{a}$ is equal to the number $\mathfrak{B}$ of indices $\mathfrak{b}$* after this reduction then the functional determinant of the s_b with respect to the variables p_a must be set to zero; i.e., there must exist an equation between the s_a whose coefficients are independent of the p_a that appear in equations (6^d), but which still depend upon the σ. We write this as:

(6^f) $F = \sigma,$

in which F is a function of only the s_b , and possibly also the p that do not enter into the values of the s_b . Therefore, the differentiation of them with respect to the p_a and σ yields:

$$(6^g) \qquad \sum_b \left[\frac{\partial F}{\partial s_b} \cdot \frac{\partial s_b}{\partial p_a} \right] = 0,$$

$$(6^g) \qquad \sum_b \left[\frac{\partial F}{\partial s_b} \cdot \frac{\partial s_b}{\partial \sigma} \right] = 1.$$

If we compare these with equations (6^d) and (6^e) then that comparison would yield:

$$(6^i) \qquad q_b = - \lambda \cdot \frac{\partial F}{\partial s_b}.$$

If F were chosen arbitrarily as a function of the s_b then, since the q_b can be expressed as functions of the p_a and s_b , as we remarked above, equations (6^i) would first yield $\mathfrak{B}$ equations for the determination of the s_b in terms of p_a and λ, which, if they are mutually independent, define the action of the fixed constraint completely. Finally, equations (6^f) determined the σ as a function of the s_b (the p_a and λ, resp.), or also the λ as a function of the p_a and λ.

However, the same form of solution is also true for $\mathfrak{A} > \mathfrak{B}$ and $\mathfrak{A} < \mathfrak{B}$. It is now easy to convince oneself that in both cases equations (6^f) and (6^i) are actually solutions for the system of equations (6^d) and (6^e), and if they are mutually independent then with the help of the equations that give the q_b as functions of the s_b , namely:

$$(6^k) \qquad \frac{\partial H}{\partial q_b} = - s_b ,$$

they would yield values of the s_b that satisfy all of the conditions of the problem.

However, one can also prove that this form for the solution has the required degree of generality ([*]), namely, it also applies to the case where $\mathfrak{A} < \mathfrak{B}$, and the number of equations (6^d) is therefore smaller than the number of quantities s_b , which should be determinate. Namely, if one replaces the q_b in equations (6^d) with their values that are expressed in terms of the s_b and p_a by means of equations (6^k) then equations (6^d) would include only the quantities s_b and p_a . When one takes arbitrary values of the s_b for definite values of the p_a , these equations admit further arbitrary values for all their first

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[*] This problem is treated completely in regard to the case in which equations (6^i) are not independent in the following article of *L. Kronecker*.

differential quotients, up to one of the s, which we would like to denote by s_1 . However, the differential quotients of the s_1 with respect to the p_a are then determined completely by equations (6^d), but not the $\partial s_1 / \partial \sigma$. Thus, as long as the σ are kept constant, it is entirely unrestricted how the remaining s_b should vary with increasing p_a in each of the differential equations, with the exception of s_1 . How the s_1 should vary is, however, given completely. Precisely this freedom with regard to the constraints on the aforementioned function will be represented by the integral equation:

(6^f) $$F = \sigma,$$

if F is an arbitrarily-chosen function of all the s_b, and therefore the given solution in each case is sufficiently general.

The given solution of the problem thus satisfies the demands of the problem when equations (6^i) are not mutually independent; I reserve the aforementioned exceptional case for later treatment. One thus has that all of the quantities q_b , s_b , λ are representable as functions of the p_a and σ, and likewise also the original ones in the unbound state of the system are representable as functions of the values of P_a and U, as well as H, that are given by either the p_a and q_b (the p_a and s_b , resp.).

In the unbound system we can, as was already remarked by *Gibbs* in § 1, write the equation for the constancy of energy as:

$$dU = \sum_b [q_b \cdot ds_b] - \sum [P_a \cdot dp_a],$$

and represent the U, like the q_b and P_a , as functions of the p_a and s_b . This yields the relations:

$$\frac{\partial U}{\partial p_a} = - P_a ,$$

$$\frac{\partial U}{\partial s_b} = q_b .$$

If the s_b in U were replaced by their values in terms of p_a and σ after imposing bounds on the system then we would like to denote the function U by $\mathfrak{U}$. One then has:

$$\frac{\partial \mathfrak{U}}{\partial p_a} = \frac{\partial U}{\partial p_a} + \sum_b \left[\frac{\partial U}{\partial s_b} \cdot \frac{\partial s_b}{\partial p_a} \right],$$

$$\sum_b \left[\frac{\partial U}{\partial s_b} \cdot \frac{\partial s_b}{\partial p_a} \right] = \sum_b \left[q_b \cdot \frac{\partial s_b}{\partial p_a} \right] = 0,$$

from (6^d), so it follows that:

$$P_{\mathfrak{a}} = -\frac{\partial \mathfrak{U}}{\partial p_{\mathfrak{a}}}.$$

Furthermore, one has:

$$\frac{\partial \mathfrak{U}}{\partial \sigma} = \sum_{\mathfrak{b}} \left[\frac{\partial U}{\partial s_{\mathfrak{b}}} \cdot \frac{\partial s_{\mathfrak{b}}}{\partial \sigma} \right] = \sum_{\mathfrak{b}} \left[q_{\mathfrak{b}} \cdot \frac{\partial s_{\mathfrak{b}}}{\partial \sigma} \right];$$

therefore, from (6^e):

$$\frac{\partial \mathfrak{U}}{\partial \sigma} = \lambda.$$

For the bound system, this would then yield the formula:

$$dU = \lambda \cdot d\sigma - \sum_{\mathfrak{a}} [P_{\mathfrak{a}} \cdot dp_{\mathfrak{a}}],$$

or, if we set:

(6^n) $\qquad\qquad\qquad \mathfrak{H} = \mathfrak{U} - \lambda \cdot \sigma$

and make the $p_{\mathfrak{a}}$ and λ into independent variables:

$$d\mathfrak{H} = -\sigma \cdot d\lambda - \sum_{\mathfrak{a}} [P_{\mathfrak{a}} \cdot dp_{\mathfrak{a}}],$$

respectively.

Thus:

$$\frac{\partial \mathfrak{H}}{\partial p_{\mathfrak{a}}} = -P_{\mathfrak{a}},$$

$$\frac{\partial \mathfrak{H}}{\partial \lambda} = -\sigma$$

agree with the formulas that were given for the simple, monocyclic system in the previous paragraphs, except that one does not necessarily find a function of s that has the same distinguished relation for the continual force as the s in equation (5^a) does, and whose associated λ is equal to the continual force. Under what conditions this comes about shall be examined in the next paragraphs.

§ 5. Conditions under which the continual force is an integrating factor.

In § 3 we found that the continual force is always an integrating factor for the simple, monocyclic systems. We then ask whether, and under which conditions, this is also the case for the systems of moving parts with fixed constraints that we spoke of in § 4. This question is also important for the analogy with the theory of heat, since in it, from the kinetic theory of gases, the temperature that defines the integrating factor is, in fact, proportional to the continual force of the internal motion, and the hypothesis that was

proposed by *Boltzmann* ([*]) and *Clausius* ([**]), under which this is also the case in all other bodies, at least has a high degree of probability in its own right.

   The continual force L of the system, whether it is bound or unbound, is, from (4^d), given by the equation:

(7) $$2L = \sum_{\mathfrak{b}} [q_{\mathfrak{b}} \cdot s_{\mathfrak{b}}].$$

Since, in place of the value σ for entropy that was found in the previous paragraphs, any function of σ can also come under consideration, we would like to denote the entropy that belongs to $2L$ as an integrating factor by $\log \sigma$. Thus:

(7^a) $$dQ = 2L \cdot \frac{d\sigma}{\sigma},$$
$$= \lambda \cdot d\sigma,$$

so, in this case, one has:
(7^b) $$2L = \lambda \cdot d\sigma.$$

If we substitute the values of $q_{\mathfrak{b}}$ from (6^i) that correspond to the fixed constraint in equation (7) then we get:

(7^c) $$2L = \lambda \cdot \sum \left[s_{\mathfrak{b}} \cdot \frac{\partial F}{\partial s_{\mathfrak{b}}} \right],$$

and due to equation (6^f):
$$F = \sigma$$

(7^b) can be written:
$$2L = \lambda \cdot F.$$

The condition for the continuous force to be an integrating factor is therefore:

$$F = \sum_{\mathfrak{b}} \left[s_{\mathfrak{b}} \cdot \frac{\partial F}{\partial s_{\mathfrak{b}}} \right];$$

i.e., the function F, which gives the value of the entropy of the bound system, *is a homogeneous function of first degree* of the moment of motion $s_{\mathfrak{b}}$ of the unbound system.

From the assumptions that were made above in (7) and (7^a), the quantities λ and σ stand in precisely the same relationship to L and dQ in the unbound system that the quantities q and σ do in the simple monocyclic systems of § 3. We can thus appropriately refer to λ as *the resultant velocity of the internal motion* and σ as *the resultant moment of motion that is associated with it.* Both of them are determined by the given derivatives up to a constant factor; all equations of this paragraph then remain unchanged when we set ($n\sigma$) in place of σ and, at the same time, λ / n in place of λ. Since such a constant factor can take on an arbitrary sort of physical meaning, the unit that is given for σ is therefore also

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([*]) Wiener Sitzungsber. **8**, part II (1866), 195-220.
([**]) *Poggendorff* Annalen **142** (1871), 433-461, § 14 and 15 of the treatise.

determined arbitrarily; if it is determined then that of λ will be determined as a result, since the product $\lambda \cdot \sigma$ refers to a continuous force – i.e., a quantum of work. It becomes preferable, in the sense of the proposed notation for both quantities, to regard λ as a velocity – i.e., the differential quotient of a spatial quantity by time – and σ as a moment of motion (mass times spatial quantity, times velocity). Since the function F is a homogeneous function of first degree and:

$$\frac{1}{\sigma} \cdot F = 1,$$

the latter function can be represented as one of the quantities $\left(\dfrac{1}{\sigma} \cdot s_{\mathfrak{b}} \right)$, which, from what was said above, are pure spatial quantities, and the constant coefficients that are included in the function F will then also need to be only pure spatial quantities.

As far as the quantities $q_{\mathfrak{b}}$ are concerned, they are expressed as functions of the $s_{\mathfrak{b}}$ by means of the equations:

(3^a) $$s_{\mathfrak{b}} = - \frac{\partial H}{\partial q_{\mathfrak{b}}},$$

or, by eliminating some of the $p_{\mathfrak{c}}$:

(4^c) $$s_{\mathfrak{b}} = - \frac{\partial \mathfrak{H}}{\partial q_{\mathfrak{b}}},$$

respectively. In the former case the $s_{\mathfrak{b}}$ are linear, homogeneous functions of the $q_{\mathfrak{b}}$, while in the latter case they are non-linear and inhomogeneous, in general. If one writes equations (6^f) and (6^i) as:

(7^h) $$\begin{cases} \dfrac{q_{\mathfrak{b}}}{\sigma} = \dfrac{\lambda}{\sigma} \cdot \dfrac{\partial F}{\partial s_{\mathfrak{b}}}, \\[2ex] \dfrac{1}{\sigma} \cdot F = 1 \end{cases}$$

then, by complete preservation of the $p_{\mathfrak{a}}$, the quantities $1 / \sigma \cdot q_{\mathfrak{b}}$ are linear functions of the $s_{\mathfrak{b}} / \sigma$ with coefficients that depend upon the parameters $p_{\mathfrak{a}}$. Only the $(\mathfrak{B} + 1)$ quantities $1 / \sigma \cdot s_{\mathfrak{b}}$ and $1 / \sigma \cdot \lambda$ then enter into the system of $(\mathfrak{B} + 1)$ equations (7^h) as unknowns, which are to be found by solving these equations as functions of the parameter $p_{\mathfrak{a}}$. This yields the fact that *when the complete system of parameters $p_{\mathfrak{a}}$ is kept constant, all of the moments of motion $s_{\mathfrak{b}}$ and, accordingly, also all of the velocities $q_{\mathfrak{b}}$ of the system (which is, by the assumptions of this paragraph, bound) that are proportional to the values of σ, as well as the resultant velocity λ, must vary.*

This is, in general, no longer the case when a number of the parameters $p_{\mathfrak{c}}$ are eliminated, because then the condition posed – viz., the constancy of all the parameters –

can no longer be fulfilled, since the necessary forces P_c are lacking. In fact, the q_b are then no longer linear functions of the s_b , and the quantity σ remains conserved in the equations when one seeks to bring the quantities s_b into the form $1 / \sigma \cdot s_b$.

The manner of representing the mechanics of such a system is not merely in the sense (as was described in the conclusion of the previous paragraph for the general bound system) that it gives a complete analogue of the simple, monocyclic system, but one has, in addition, that the continual force has the same relationship to the parameters λ and σ as it does to the q and s in the simple system. One also has that:

$$H + \tfrac{1}{2} \cdot \lambda \cdot \sigma = \Phi$$

is the potential energy of the system.

However, an essential generalization enters into one relation. The q of the simple, monocyclic system is the differential quotient with respect to time of a spatial quantity that determines the instantaneous position of the part that is regarded as in stationary motion, and the integral over time of q thus yields the difference between the initial and final values of this spatial quantity independently of the intermediate states of the system. In the present generalized system, this is no longer the case, and a spatial quantity whose differential quotient with respect to time would be the quantity λ no longer exists.

Finally, one can more precisely characterize the particular kind of fixed constraints between the moving parts of the system that are assumed in this paragraph. It was already explained that the equation:

$$\frac{F}{\sigma} = 1$$

that characterizes the constraint here assumes that F is a homogeneous function of first degree of the quantities s_b and that for just that reason the constant coefficients of this equation only need to be spatial quantities without including masses or time quantities. However, the entire type of constraint is, as the development above has shown, dependent upon the function F, and therefore also on its constants. Only these enter into the values of the s_b and q_b as well-defined for the type of constraint.

Things are different in the general case of § 4 when F is not a homogeneous function of the first degree of the quantities s_b . In this case, the time differential of the quantity $1 / \sigma \cdot F$, which should equal the singular (Einzahl), can vanish only when time quantities also appear in the constants that are measured in term of absolute units. Time itself can not enter in, since the equation should be independent of time, but not differential quotients with respect to time like velocity and forces. However, these must, in fact, exist in the system in order for it to be mechanically efficient. Constant velocities could only be kept constant from the outside, and would thus collide with the assumed independence of the system and its monocyclic character. However, forces that indeed have dt^2 in the denominator can enter in as constants that remove time. Such forces cannot be external forces, because everything in a complete system must be assumed to be variable. On the contrary, constants that determine the efficiency of internal forces, like the constant a^2 in the equation of the elastic force:

$$X = -a^2 \cdot x,$$

can possibly enter in; at least, as far as I know, nothing prevents this from the mathematical standpoint. The mechanical tools that were found up to now for the continual evolution of motion, such as infinite strings, gears, and friction rollers always give only ratios of velocities that are independent of the velocity quantities, but can depend upon the position of the parts; an example of the latter is a friction roller for which the angle of their axis, or also the line on which they move, can vary. Examples of the latter occur in, e.g., the integrating machines. For that reason, I propose that constraints of that kind, which make the equation $F = \sigma$ homogeneous in the s_b and σ, and accordingly the continual force can exist as an integrating factor, can be distinguished as *purely kinematical constraints.* As long as only such constraints are introduced, the continual force remains an integrating factor of the bound system ([*]).

I do not know how to find examples of the general form of the constraint, in which the ratio of the various velocities would be independent of those quantities for unchanging position of all moving parts. The kinetic theory of gases also gives relations between the simultaneously-existing motions of the various parts of the system that comply with the law of purely kinematical constraints.

§ 6. Coupling of any two systems.

If two monocyclic systems that are originally put into a state by a suitable regulation of the external forces P_a that corresponds to the conditions of a definite type of fixed constraint then one can let such a fixed constraint enter between them without, moreover, perturbing the existing motion in so doing, and they can move further from then on while introducing new variations of the forces P_a while preserving this fixed constraint. I would like to refer to this intermittently fixed constraint on states as a *coupling* of the system. An analogous process occurs in heat motion when two bodies of equal temperature can be put into conductive contact without changing their internal motion such that they preserve their equal temperatures under new sufficiently slow changes.

Therefore, it could happen that the two systems in the position that they are given for the purpose of coupling also act upon each other with pressures or distant forces. We would next like to assume that forces of this type, where they occur, can be known completely and the arbitrarily varying forces P_a can be calculated. The coupling that we refer to as such would then only have to convert work quantities dQ from one system to the other one. We can refer to such a coupling as a *pure motion coupling*. Under this assumption, we can regard the coupling of the system as one of the fixed constraints whose law we discussed in the last two paragraphs. Without the coupling, the two bodies are a dicyclic system that will change into a monocyclic system under the coupling.

Here, we once more proceed in complete analogy to the theory of heat. If two bodies at equal temperature are put into heat-conducting contact then they preserve equal temperature from then on, at least for sufficiently slow changes under which the

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([*]) I have since found a further generalization of this theorem, and I hope to publish it soon.

constraint acts to exchange their internal motions. In addition, they can also act upon each other by pressures and attractive forces, but these remain unperturbed in equilibrium with them and the remaining external forces under the simultaneous exchange of temperature. Moreover, none of the forces P_a that act upon each body from the outside changes, due to the fact that the introduction of any change of its volume or molecular structure would now allow other supplies of heat to enter the body or emanate from it, since this would be the case without the constraint. Herein lies an essential difference between the ways that actual and potential energy behave. In the event that a change of state of a body admits the transfer of potential energy from the one body to the other, the force would be changed, by the principle of virtual velocities. Heat does not, by any means, retain its character of actual energy, as was developed in our theorems of §§ 4 and 5.

Of interest here is, in fact, the case in which a mechanical constraint is presented between two systems that have equal values of one of their integrating factors, in such a way that whenever this constraint exists the equality of the aforementioned factors must be preserved. In order to refer to such a type of constraint briefly, I would like to call it an *isomorphic coupling* (ἴσον μόριον, equal factor). Contact between two bodies of equal temperature is an example of this type, since temperature is an integrating factor of both bodies. Another example would be that of two tops whose axes are linked in such a way that they must move with equal orbital velocities. If equal rotation is already present by the introduction of the constraint then the coupling can be exhibited without perturbing the motions that are present. Their axes can carry centrifugal regulators of different kinds that would regulate the location of those axes by means of forces P_a that would also immediately bring the orbital velocity to an arbitrary magnitude.

Two currents in annular pipes can be united into one ring without perturbing either of them when both have equal currents through each cross-section. Stretching the channel reduces the current, while shortening it raises it. In this way, one can give any two of them equal values.

Precisely the same considerations that follow from equations (1) for heat motion, and which I briefly recalled in § 1, are valid. In this way, we get precisely the same laws for the possibility of obtaining work from the forces P_a at the expense of the internal motion of the system. The basic assumption for this is then that we have no other means of acting directly upon the internal motion of the given monocyclic system except for isomorphic couplings with other systems, and that for any one of the systems that can be coupled a single integrating factor exists that allows the unperturbed coupling with equally large factors to another system, as would be the case in the examples presented above of centrifugal regulators or annular currents.

The fact that we cannot directly confront and alter the heat motion of the atom, which is indeed the case, also depends only upon the fact that we cannot introduce our effects on particular isolated atoms in particular directions but must always necessarily address all of a definite spatial region uniformly. It is based upon only the restriction of the methods that we proposed and not upon anything intrinsic to the motion. It is for just that reason that we can also assume similar restrictions of our demands on the problem for the analogous cases that we spoke of here without changing the nature of the problem.

If, as we did at the end of the previous paragraph, we assume purely kinematical constraints on both system, such that the continual force of the coupled system becomes

an integrating factor of this system then it follows further in relation to the possible types of coupling: Let $\eta_1 = \eta$ and $\eta_2 = \eta$ be the two integrating factors of the two systems that will be made equal by the coupling, and let σ_1 and σ_2 be the entropies that are associated with them, so:

$$(8) \qquad \begin{cases} dQ_1 = \eta \cdot d\sigma_1, \\ dQ_2 = \eta \cdot d\sigma_2, \\ dQ = dQ_1 + dQ_2 = \eta \cdot d(\sigma_1 + \sigma_2). \end{cases}$$

η is then also an integrating factor of the coupled system. Under these conditions, the continual forces, since they are also integrating factors of the system in question, are of the form:

$$L_1 = \eta \cdot \Phi_{\sigma_1},$$
$$L_2 = \eta \cdot \Psi_{\sigma_2},$$
$$L_1 + L_2 = \eta \cdot X_{(\sigma_1 + \sigma_2)};$$

thus:

$$(8^a) \qquad X_{(\sigma_1 + \sigma_2)} = \Phi_{\sigma_1} + \Psi_{\sigma_2}.$$

This latter equation, when differentiated with respect to σ_1 and σ_2, gives:

$$X'' = 0.$$

Thus, X is a linear function of $(\sigma_1 + \sigma_2)$ and of the form:

$$(8^b) \qquad \begin{cases} X = a + b + c(\sigma_1 + \sigma_2), \\ \Phi = a + c \cdot \sigma_1, \\ \Psi = b + c \cdot \sigma_2. \end{cases}$$

If we denote the resultant moments of motion of both systems by s_1 and s_2, as they appear in the original problem, then we will have:

$$dQ_1 = 2L_1 \cdot \frac{ds_1}{s_1} = \eta \cdot d\sigma_1,$$

$$dQ_2 = 2L_2 \cdot \frac{ds_2}{s_2} = \eta \cdot d\sigma_2.$$

Thus:

$$(8^c) \qquad \begin{cases} 2\dfrac{ds_1}{s_1} = \dfrac{d\sigma_1}{\Phi_{\sigma_1}} = \dfrac{d\sigma_1}{a + c\sigma_1}, \\[2mm] 2\dfrac{ds_2}{s_2} = \dfrac{d\sigma_2}{\Phi_{\sigma_2}} = \dfrac{d\sigma_2}{b + c\sigma_2}. \end{cases}$$

Since σ_1 is a function of only s_1 and σ_2 is a function of only s_2, when this is integrated, that gives:

(8^d)
$$\begin{cases} \log(s_1^{2c}) = \log(a+c\sigma_1)+\log(a^{2c}), \\ \log(s_2^{2c}) = \log(a+c\sigma_2)+\log(\beta^{2c}), \end{cases}$$

where α and β denote integration constants, or:

(8^e)
$$\begin{cases} \Phi_{\sigma_1} = a+c\sigma_1 = \left(\dfrac{s_1}{\alpha}\right)^{2c}, \\[2ex] \Phi_{\sigma_2} = a+c\sigma_2 = \left(\dfrac{s_2}{\beta}\right)^{2c}, \end{cases}$$

and

(8^f)
$$\begin{cases} \eta_1 = L_1 \cdot \left(\dfrac{\alpha}{s_1}\right)^{2c}, \\[2ex] \eta_2 = L_2 \cdot \left(\dfrac{\beta}{s_2}\right)^{2c}. \end{cases}$$

From this, it follows that a purely kinematical coupling can be performed isomorphically only when those integrating factors are set equal to each other that are the products of the continual force, an arbitrary constant, and a power of the resultant moment of motion, powers that have the same exponent on both sides of the equality.

In the examples above – viz., coupling of the rotational axes of tops and the annular currents – one has:

$$q = \frac{2L}{s}.$$

In the kinetic theory of gases, ϑ is proportional to L itself.

§ 7. Equilibrium of the internal motion for three monocyclic systems.

Finally, we have to look for the analogues of that characteristic property of heat motion that it makes possible to speak of the temperature of a body as a quantity, and which is summarized in the theorem: *If each of two bodies is in thermal equilibrium with a third one then they are in equilibrium with each other.*

The corresponding condition for three monocyclic systems can be formulated thus: One will find that the coupling equation between (2) and (3) is fulfilled whenever it is fulfilled by (1) and (2), on the one hand, and (1) and (3), on the other.

From this it follows, as is analytically uncomplicated to prove, that the coupling equation must be brought into the form:

(9) $$\varphi_1 = \psi_2 = \chi_3 \, ,$$

where φ_1 is a function of the parameters of the first system, ψ_2 is such a function for the second system, and χ_3 is one for the third system. These three quantities would then represent the analogues of temperature for monocyclic systems.

Of the previously-discussed examples, the ones that fall under the condition that is imposed here are the top with equal rotational velocity of their coupled axes and the current in annular channels whose currents through the cross-sections must coincide.

Among these circumstances is the fact the general coupling equations (6^i) must have the form (9); i.e., the two equivalent expressions:

(9^a)
$$\left\{ \frac{q_1}{\dfrac{\partial F}{\partial s_1}} = \frac{q_2}{\dfrac{\partial F}{\partial s_2}} \right.$$

can contain s_1 and s_2 simultaneously in both sides only in a common factor, which is omitted, and indeed, this factor must appear in the differential quotients of F, since q_1 contains only parameters of the first body, while q_2 only includes parameters of the second one. One must then have:

(10)
$$\left\{ \begin{array}{l} \dfrac{\partial F}{\partial s_1} = \varphi_{s_1} \cdot \chi, \\[2ex] \dfrac{\partial F}{\partial s_2} = \psi_{s_2} \cdot \chi, \end{array} \right.$$

in which χ is a function of s_1 and s_2. From this, it follows that:

$$\frac{\partial^2 F}{\partial s_1 \, \partial s_2} = \varphi \cdot \frac{\partial \chi}{\partial s_2} = \psi \cdot \frac{\partial \chi}{\partial s_1} .$$

Comparing the last equation with (10) shows that:

(10^a)
$$\frac{\partial F}{\partial s_1} \cdot \frac{\partial \chi}{\partial s_2} - \frac{\partial F}{\partial s_2} \cdot \frac{\partial \chi}{\partial s_1} = 0;$$

i.e., χ is a function of only F. If we write:

$$\frac{1}{\chi} = \frac{d\mathrm{X}}{dF} \, ,$$

$$\varphi \;=\; \frac{d\Phi}{ds_1},$$

$$\psi \;=\; \frac{d\Phi}{ds_2}$$

then equations (10) yield the integral:

(10^b) $\qquad\qquad\qquad\qquad X = \Phi_{s_1} + \Psi_{s_2} + C \,.$

   Since the value of F must be equal to the entropy σ of the combined system, one can now regard X as a function of σ.
   On the other hand, the coupling equation (9^a) now becomes:

(10^c) $\qquad\qquad\qquad\qquad \dfrac{q_1}{\varphi_{s_1}} = \dfrac{q_2}{\psi_{s_2}} \,.$

If, as we have assumed, q_1 is an integrating factor for the first system and q_2 is one for the second system then every product of a q with an arbitrary function of the associated entropy is also an integrating factor for the system in question. The quantities that were set equal to each other in equation (10^c) above are then integrating factors that belong to the entropy values Φ_{s_1} and Ψ_{s_2}, the coupling is an isomorphic one, and the mobility of the internal motion is, as was discussed in § 5, subject to restrictions that correspond to the ones on heat according to *Carnot's* law.
   This then shows in full generality that if monocyclic systems admit only those constraints amongst themselves for which the stated peculiarities of heat motion are valid, then the third essential peculiarity of heat that is expressed by *Carnot's* law is also valid, which restricts mobility. The stated two characteristics of the coupling are:

   1. The external forces of each system depend upon only the instantaneous state of the system and not on the intrusive or adhesive linking with other systems. The coupling is then a pure motion coupling and generates a new monocyclic system.

   2. As long as the conditions of exchange of the internal motion between two or more systems are known, the equilibrium of the internal motion between them depends upon the fact that a well-defined function of the parameters of one of the individual motions has the same values as the corresponding functions of the other.

   This well-defined function that plays the role of decreasing temperature in the theory of heat must then necessarily be an integrating factor of the system, from which the restricted mobility of the internal work follows. On the other hand, it must be stressed that the first condition must also be fulfilled when any coupling is isomorphic and corresponds to the second of the conditions that were just proposed.