"Ueber die Biegung und Drillung eines unendlich dünnen elastischen Stabes, dessen eines Ende einem Kräftepaare angegriffen wird," Math. Ann. **23** (1884), 181-212.

On the bending and twisting of an infinitely-thin elastic rod, when a force-couple acts upon one end (*)

By

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(With one lithographed table)

Translated by D. H. Delphenich

The conditions for equilibrium of an elastic rod that is very thin in comparison to its lengths upon one end of which some forces act, while the other end remains fixed were first presented by **Kirchhoff** (**). In that article, **Kirchhoff** had remarked that those conditions possess the same form as the equations that define the motion of a rigid body that rotates around a fixed point. That agreement between the two different problems thus seems to offer not only the possibility of directly evaluating the explicit formulas that were already presented for the various cases of rotation in order to ascertain the equilibrium state of the elastic rod, but also comparing, step-by-step, each process in the rotation of bodies with the corresponding one for the deformation of a rod and perhaps conversely arriving at new questions to ask about the former from the reciprocal relationship between the two problems. Now, an investigation of the equilibrium of a general elastic rod might not connect up with the problem of rotation, since the known works on that subject (***) were concerned with a wire that was equally-flexible in all directions. Therefore, I have addressed the question more closely and indeed initially under the assumption that only a force-couple acts upon the free end of the rod. The analogue of that can be found in the rotation of a body about its center of mass, which is a type of motion that can be treated in a sufficiently-clear way by the coupling of the

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(*) An excerpt of this study appeared in the Sitzungsberichten der bayer. Akad. d. Wiss. (1883), 82-110.

(**) "Ueber das Gleichgewicht und die Bewegung eines unendlich dünnen elastischen Stabes," Crelle's J., Bd. 56, pp. 285-313.

(***) Among them, one must, above all, include the investigations of the "elastic line" by **Bernoulli** and **Euler** [*De curvis elasticis*, Lausanne and Geneva, 1744, 4th ed., and the treatises of **Binet** [Comptes rendus, **18**, pp. 1115-1119], **Wantzel** [*ibid*., pp. 1197-1201], *et al.* Cf., the bibliography in **Navier**, *De la résistance des corps solides*, XII-XXII, Paris, Dunod, 1874, 8th ed.

analytical work of **Jacobi** (*) with the known intuitions of **Poinsot** (**), as I sought to show in my doctoral dissertation (***).

The investigation splits into essentially two parts: First the expressions for the magnitudes of the bending and twisting in each individual cross-section are examined. In the case of motion, those quantities will correspond to the angular velocities around the three principal axes of the body through the center of mass. Now, just as one can construct two cones from the latter, according to **Poinsot**, and one can arrive at a clear picture of the rotation from its successive rolling, so can two skew surfaces be generated by the quantities of bending and twisting that will make the transition from the straight and untwisted elastic rod to its bent and twisted equilibrium state more meaningful by their successive bending. Everything that one does with angular velocities (**Poinsot**'s cone, resp.) can be carried over to the deformation quantities of bending and twisting (the skew surfaces, resp.). By contrast, questions that relate to the form of the bent rod cannot be answered by considering the rotational problem. The second half of the treatise is concerned with them; in particular, with the presentation of the equations for the bent elastic centerline, with the different types of the latter, and with special cases of the problem.

The results of the study will be found to be discussed more precisely in the individual paragraphs. It might only be emphasized that the present problem of the bending and twisting of an elastic rod – for the general case, as well as for the special case – is much more extensive than that of motion, mainly on the grounds that the two resistances of the rod to bending are likewise individually different from the resistance to twisting, like bending and twisting themselves, while the three principal moments of inertia of the body that correspond to those resistances to deformation are quantities of the same type. Whereas for the latter, it would suffice to enter into a single arrangement of their magnitudes, in the present problem, the resistance to twisting must be distinguished from the three possibilities for the three resistances, in relation to whether it is numerically smallest, intermediate, or largest.

Defining the problem

1.

We imagine a straight and untwisted elastic rod whose cross-section is very thin (viz., infinitely thin) in comparison to the length L of the rod – i.e., a *wire* – hold the one end fixed and let a force-couple of intensity l whose axis possesses an arbitrary direction act upon the cross-section of the free end. We then bend the straight elastic centerline or axis of the rod (i.e., the connecting line of the centers of mass of the individual cross-sections) into a curve, and at the same time, rotate the cross-section of each point P on the centerline around the tangent to the latter line at P: The rod will appear to be bent and twisted.

(*) *Sur la rotation d'un corps*, Op. II, pp. 139-197.

(**) "Théorie nouvelle de la rotation des corps," Liouville's J. (1851), pp. 9-129, 289-336.

(***) "Das Rollen einer Flächen 2. Grades auf einer invariablen Ebene," Munich, 1880 or Prog. der Kreisrealsch. Munich, 1881.

If one chooses the tangent to be the Z'-axis and the principal axes of inertia of the cross-section to be the X', Y'-axes then X', Y', Z' will define the rectangular coordinate system of the three *principle axes* of the rod. With them, one can think of the resistance by which the rod opposes the intended deformation as a principal resistance to *bending* in the case of X', Y' and a principal resistance to *twisting* in the case of Z'.

Before the action of the force-couple l, the Z'-axes will all fall along the straight rod axis and the X', Y'-axes will be mutually parallel. After deformation, the positions of the three principal axes will become increasingly different, and thus dependent upon the arc-length s, as one increases s from its starting point P in the cross-section at the free end. They will be known as soon as one is given their inclination cosines:

$$a, b, c\ ;\ a', b', c'\ ;\ a'', b'', c''$$

from the three fixed coordinates:

$$X, Y, Z,$$

resp., of a system that is congruent to X', Y', Z' as function of s.

Now, according to **Kirchhoff**, the determination of the a, b, c initially leads to the same equations that appear in the rotation of a rigid body around its center of mass O, namely, to **Euler**'s equations (*):

$$\begin{aligned} A \cdot \frac{dp}{ds} + (B - C)\, q\, r &= 0, \\ B \cdot \frac{dq}{ds} + (C - A)\, r\, p &= 0, \\ C \cdot \frac{dc}{ds} + (A - B)\, p\, q &= 0. \end{aligned} \tag{1}$$

One therefore has the correspondence:

The three principal axes of inertia X', Y', Z' through the center of mass O of the rotating body	The three principal axes X', Y' (bending and Z' (twisting) through a point P of the elastic centerline of the rod
The principal moments of inertia A, B, C of the body about X', Y', Z'	The *principal moments of resistance* A, B, and C of the rod to bending around X', Y' and to twisting around Z'
The force-couple l that produces motion	The force-couple l that acts upon cross-section of the free end

(*) We learn about that from the book by **Clebsch**: *Theorie der Elasticität fester Körper*, 1862, pp. 211. Except for different notations for the quantities A, B, C, p, q, r, our formulas will differ from those of **Clebsch** by the fact that we shall always have to set ds, instead of $-ds$, since we count the arc length s from the free end, while Clebsch thinks of it as being measured from the fixed end. The force-couple is thought to rotate to the right.

The advance ds along time s (s is counted from the starting time $s_0 = 0$)	The advance ds along the arc length s of the centerline (s is measured from the free end $s_0 = 0$)
The angular velocity components p, q, r about the three principal axes X', Y', Z' at the time s	The *curvature* components (*) p, q for bending around X', Y' and r for the twisting around Z' at the arc length s
The components AP, Bq, Cr of the instantaneous exciting force-couple around X', Y', Z' that are precisely suitable to produce the angular velocities p, q, r around those axes	The components Ap, Bq, Cr of the force-couple at a point P of the elastic centerline that is produced by the applied couple l, which indicates the *stress* at that point, so it will be precisely suitable to generate the curvature components p, q, r themselves

It follows from the analogy between the components of the angular velocity and the curvature that the latter components can be composed and decomposed in exactly the same way as the former. For the present problem, that will mean:

$\Theta = \sqrt{p^2+q^2+r^2}$ is the magnitude of the angular velocity at the time s along the instantaneous rotational axis Θ	$\Theta = \sqrt{p^2+q^2+r^2}$ is the magnitude of the curvature; i.e., of the total deformation at a point at an arc length of s along the *instantaneous curvature axis* Θ
$l = \sqrt{A^2p^2+B^2q^2+C^2r^2}$ is the magnitude of the exciting force-couple at the time s along the instantaneous force-couple axis l	$l = \sqrt{A^2p^2+B^2q^2+C^2r^2}$ is the magnitude of the stress that arises at the arc length s along the instantaneous *stress axis* l
$\Theta' = \sqrt{p^2+q^2}$ is the component of the instantaneous angular velocity Θ relative to the principal plane $X'Y'$ at time s	$\Theta' = \sqrt{p^2+q^2}$ is the component of the instantaneous curvature Θ relative to the plane $X'Y'$ of the cross-section; i.e., the magnitude of pure bending at the distance s

If one has solved **Euler**'s differential equations then the relations:

(2) $da = (br - cq) \cdot ds$, $da' = (b'r - c'q) \cdot ds$, $da'' = (b''r - c''q) \cdot ds$, etc.

(*) Here, "curvature" stands for "deformation." The usual concept of "curvature" in the theory of curves is replaced with "bending" here. Cf., **Thomson** and **Tait**, *Handbuch der theor. Physik*, no. 593, *et seq.*, 1874.

will serve to determine the nine inclination cosines a, b, c, …, c''. Moreover, the latter are connected with each other by way of the conditions:

$$(3) \qquad a = b'c'' - b''c', \qquad a' = b''c - b\,c'', \qquad a'' = b\,c' - b'c, \qquad \text{etc.,}$$

which arise from an orthogonal substitution.

The tangent Z' to a point P of the elastic centerline defines angles with the fixed coordinate axes X, Y, Z whose cosines are c, c', c'', resp. One will then have the differential equations:

$$(4) \qquad dx = c \cdot ds, \qquad dy = c' \cdot ds, \qquad dz = c'' \cdot ds,$$

for the coordinates x, y, z of the point P relative to that system.

If one integrates them then one will get the coordinates x, y, z as functions of s ; i.e., the equations of the bent centerline. However, if the quantities p, q, r of the bending and twisting are known for a point P of the latter, and furthermore, the positions of the three principal axes X', Y', Z' that go through it will be given in terms of a fixed coordinate system X, Y, Z, and if each coordinate of P relative to that system is expressed in terms of s, moreover, then the equilibrium state of the elastic rod can be considered to be essentially known.

Curvature ratios of the individual cross-sections

2.

One can derive the following integral equations from **Euler**'s equations, upon multiplying by Ap, Bq, Cr ; a, b, c ; a', …, c'' and adding each time, while considering the relation (2):

$$(5) \qquad A^2 p^2 + B^2 q^2 + C^2 r^2 = \text{const.} = l^2,$$

$$(6) \qquad \left\{\begin{aligned} Apa + Bqb + Crc &= \text{const.} = l_1, \\ Apa' + Bqb' + Crc' &= \text{const.} = l_2, \\ Apa'' + Bqb'' + Crc'' &= \text{const.} = l_3 = \sqrt{l^2 - l_1^2 - l_2^2}\,. \end{aligned}\right.$$

They imply the theorem for rotation:

The force-couple l that acts upon the body remains constant in intensity and location during the entire duration of the motion. Its plane is an invariable plane and its axis is an invariable axis in space.

When this theorem is adapted to the rod, it reads:

The stress that appears in each cross-section as a result of the force-couple l that acts on the cross-section at the free end is represented by a force-couple that possesses a constant magnitude and position, along with l. The plane and axis of the latter will then possess an invariable position and direction in space.

If one multiplies **Euler** equations by p, q, r, resp., and adds them then one will get:

$$A\,p^2 + B\,q^2 + C\,r^2 = \text{const.} = h, \tag{7}$$

which can also be written:

$$\Theta \cdot \cos(\Theta\, l) = \text{const.} = \frac{h}{l}. \tag{8}$$

For the rotational problem, that means:

The components of the instantaneous angular velocity Θ along the invariable axis of the applied force-couple l remains constant for the entire duration of the motion.

Direct translation will imply that:

The components of the curvature Θ of a cross-section along the invariable direction of the stress l in it is constant for all cross-sections.

Naturally, it cannot be concluded from this that the projection of the elastic centerline onto the invariable plane that is perpendicular to the l axis is a circle. The curvature component h / l that acts around l is composed of bending and twisting, so it is not the magnitude of a pure bending. Since the bending axis always falls in the cross-section through a point along the centerline, bends can also be composed and decomposed only around axes that lie in the cross-section, and are thus perpendicular to the arc length ds, using the method of projection. If one would like to know the magnitude of a pure bending that takes place along an arbitrary line then a simple multiplication by the cosine of the angle of inclination would not suffice, but one would have to successively apply the two easily-proved theorems:

The bending of the projection of a space curve onto a plane that subtends an angle of α *with the osculating plane is equal to the bending of the space curve, multiplied by* $\cos\alpha$ *or* $\sec^2 \alpha$*, according to whether the plane is parallel to the arc-length element ds or the principal normal, resp., at the curve point considered.*

If one regards the quantities p, q, r of the angular velocity of the rotating body, which vary with time s, as the coordinates of a point P relative to the three principal axes X', Y', Z', when one carries the instantaneous rotational axis of the angular velocity Θ of rotation as a line segment from the center of mass O in the direction of each instantaneous rotational axis, then equations (5) and (7) will represent two ellipsoids, the latter of which is **Poinsot**'s *central ellipsoid*. Its axes are identical to the three principal axes of inertia, their lengths are proportional to the square roots of the principal moments of inertia A, B, C. The fourth-order space curve that is defined by the simultaneous existence of the aforementioned equations is **Poinsot**'s *polhode* and the second-degree cone that implies, viz.:

$$A\,p^2\,(Ah - l^2) + Bq^2\,(Bh - l^2) + Cr^2\,(Ch - l^2) = 0 \tag{9}$$

is the *cone of the polhode*.

If one likewise constructs the momentary curvature axis Θ from the curvature components p, q, r, which vary with arc length s, then all of the axes will be parallel to the generators of a second-degree cone (9) and their totality will define a skew surface, namely, the surface of the polhode (*). If one carries the magnitude of the curvature Θ along each axis Θ as a line segment from the point P of the centerline then the end points of all such lines lie along a transcendental curve, namely, the polhode (*). See paragraph **5** on that topic.

3.

The **Poinsot** cone (9) of the polhode is always real, so the three differences that appear in its equation cannot all possess the same signs. If we assume that $A > B > C$, which is allowed by the fact that the moments of inertia all have the same type, then it will follow that:

$$Ah - l^2 > 0, \quad Ch - l^2 < 0, \quad Bh - l^2 \neq 0\;;$$

i.e.:

The cone of the polhode is never described around the axis of the middle moment of inertia, but only around that of the largest or smallest moments.

Naturally, one cannot take $A > B > C$ arbitrarily for the rod, because the resistance of the rod to twisting depends upon other influences, just as one of the resistances to bending (**) does. One must then distinguish whether the former is the smallest, middle, or largest of the three principal resistances. For that reason, we assemble:

$$A > B > C\;; \quad B > C > A\;; \quad C > A > B.$$

Since the cone (9) is also always real here, the first of these cases will imply that:

$$Ah - l^2 > 0, \quad Ch - l^2 < 0, \quad Bh - l^2 \neq 0\,,$$

as above, and analogously for the other two assumptions. That means:

The second-degree cone whose generators are parallel to the curvature axes in the rest position of the rod never seems to be described around the axis of the numerically

(*) While preserving **Poinsot**'s terminology.
(**) **Clebsch**, *loc. cit.*, pp. 196.

middle of the three principal resistances, but only around the largest or smallest of the resistances.

If one constructs the axis of the force-couple l that acts momentarily relative to the body, which is assumed to be at rest, from the components Ap, Bq, Cr in the same way that one constructs the instantaneous rotational axis Θ from p, q, r then one will once more get a second-degree cone, namely, the cone of the force-couple axes. Once again, it is only described around the principal axes of the largest or smallest moments of inertia, which is why one suitably assumes that the initial position of the axis of the applied couple is in the principal plane of the central ellipsoid that is constructed from the principal axes of largest and smallest moments. The initial position of the instantaneous axis of rotation then falls in that plane.

If one constructs the axis of the stress l that appear at each point P of the undeformed elastic centerline of the rod from the stress components Ap, Bq, Cr then they will once more be parallel to the generators of a second-degree cone, and their totality will again define a skew surface, namely, the skew surface of the force-couple axes. Since the cone that was referred to is always described around the principal axis of the largest or smallest principal resistance, the axis of the force-couple l that acts upon the cross-section at the free end can be, at best, assumed to be in plane that is defined by those two axes. For example, it will fall in the free cross-section when the resistance to torsion is numerically the middle of the three resistances.

The inequalities that are implied by establishing the orders of magnitude $A > B > C$, namely:

$$Ah - l^2 > 0, \qquad Ch - l^2 < 0, \qquad Bh - l^2 \neq 0 ,$$

give one information about the position of the second-order cone whose generators are defined by the instantaneous rotational axes (are parallel to the instantaneous curvature axes, resp.). Now, the same thing can also be explained differently: For the rotation, they mean:

The distance from the invariable plane to the midpoint of the central ellipsoid can never be larger than the largest semi-axis of the ellipsoid nor smaller than the smallest one, but only larger or smaller than the middle semi-axis.

One can also accept a similar interpretation for the problem of the rod when one describes the ellipsoid (7) that corresponds to the central ellipsoid for a point P of the elastic centerline. However, one can explain the inequalities in a sense that is more suited to the essence of elastic rods when one writes:

$$\frac{Ap}{l} > \frac{p/h}{l}, \quad \text{etc.,} \qquad \frac{h}{l}\cdot\frac{Ap}{l} > p, \text{ etc.,}$$

instead of $Ah - l^2 > 0$, etc.

Two theorems follow from a consideration of the relation (8):

The ratios of the stress components along the principal axis of largest, smallest, middle resistance to the unvarying stress l along the force-couple axis is greater, smaller, *greater or smaller, resp., for all points than the ratio of the curvature components that are estimated along that same axis.*

The projection of the constant curvature h / l that appears in the invariable direction of the stress axes onto the axis of greatest, smallest, middle resistance will be greater, smaller, greater or smaller, resp., than the curvature component that comes to light along it.

Since the cone of the instantaneous rotational axes seems to be described around the axis Z' of the principal moment of inertia C for $Bh - l^2 > 0$, around the X'-axis of the moment A for $Bh - l^2 < 0$, the component r can never be zero in the former case, and the component p can never be zero in the latter. That must be the case, since **Euler**'s equations allow one to represent p, q, r as being proportional to sin am u, cos am u, Δ am u ($u = n \cdot s$, n constant), r (p, resp.) must be proportional to Δ am u in those cases [†]. Since the aforementioned initial position of the force-couple axis falls in the $X'Z'$-plane, moreover, one must have $B\, q_0 = 0$, so $q_0 = 0$; i.e., q must be proportional to sin am u, and indeed $Bh - l^2 \neq 0$, for both cases. By substituting the elliptic functions in equations (1), (5), and (7), values for p, q, r will easily follow that are found in **Jacobi**'s paper "Sur la rotation." Moreover, one can combine the two distinguished cases $Bh - l^2 < 0$ and $Bh - l^2 > 0$ into a single one by assuming the sequence $A > B > C$ for $Bh - l^2 < 0$ and $A < B < C$ for $Bh - l^2 > 0$. Of course, two different central ellipsoids will be determined in that way. If one would like to assume $A > B > C$, and thus choose the first central ellipsoid, then under the transition from $Bh - l^2 < 0$ to $Bh - l^2 > 0$, one would have to switch p and r, A and C, X' and Z' in **Jacobi**'s formulas.

Since entirely the same considerations relative to the cone whose generators are parallel to the instantaneous curvature axes pertain to the problem of the rod, only **Jacobi**'s expressions for p, q, r when $A > B > C$ need to be written down. One will get the formulas for $B > C > A$, $C > A > B$ by cyclically permuting the quantities A, B, C and p, q, r. If we choose the axis to be that of the force-couple that acts upon the cross-section of the free end $s = 0$ in the quadrants that are defined by the positive halves of the principal axes of largest and smallest resistances (which does not coincide with the initial position that **Jacobi** chose for the rotation, moreover) then p, q, r will be expressed as follows:

I. $A > B > C$. 1. $Bh - l^2 > 0$.

$$p = \sqrt{\frac{l^2 - Ch}{A(A-C)}} \cdot \cos \operatorname{am} u = \frac{l}{A} \sin \varphi_0 \cdot \cos \operatorname{am} u\,,$$

$$q = \sqrt{\frac{l^2 - Ch}{B(B-C)}} \cdot \sin \operatorname{am} u = \frac{l}{A} \cdot \sqrt{\frac{A(A-C)}{B(B-C)}}\, \sin \varphi_0 \cdot \sin \operatorname{am} u\,,$$

[†] Translator's note: The notation "am" refers to elliptic functions, as will become apparent shortly.

$$r = \sqrt{\frac{Ah - l^2}{C(A-C)}} \cdot \Delta \operatorname{am} u = \frac{l}{C} \cos\varphi_0 \cdot \Delta \operatorname{am} u \,.$$

2. $Bh - l^2 < 0$.

$$p = \sqrt{\frac{l^2 - Ch}{A(A-C)}} \cdot \Delta \operatorname{am} u = \frac{l}{C} \cos\varphi_0 \cdot \Delta \operatorname{am} u \,,$$

$$q = \sqrt{\frac{Ah - l^2}{B(A-B)}} \cdot \sin \operatorname{am} u = \frac{l}{C} \cdot \sqrt{\frac{C(A-C)}{B(A-B)}} \sin\varphi_0 \cdot \sin \operatorname{am} u \,,$$

$$r = \sqrt{\frac{Ah - l^2}{C(A-C)}} \cdot \cos \operatorname{am} u = \frac{l}{C} \cos\varphi_0 \cdot \cos \operatorname{am} u \,.$$

II. $B > C > A$. 1. $Ch - l^2 > 0$.

$$p = \sqrt{\frac{Bh - l^2}{A(B-A)}} \cdot \Delta \operatorname{am} u = \frac{l}{A} \cos\varphi_0 \cdot \Delta \operatorname{am} u \,,$$

$$q = \sqrt{\frac{l^2 - Ah}{B(B-A)}} \cdot \cos \operatorname{am} u = \frac{l}{B} \sin\varphi_0 \cdot \cos \operatorname{am} u \,,$$

$$r = \sqrt{\frac{l^2 - Ah}{C(C-A)}} \cdot \sin \operatorname{am} u = \frac{l}{B} \cdot \sqrt{\frac{B(B-A)}{C(C-A)}} \sin\varphi_0 \cdot \sin \operatorname{am} u \,.$$

(10)

2. $Ch - l^2 < 0$.

$$p = \sqrt{\frac{Bh - l^2}{A(B-A)}} \cdot \cos \operatorname{am} u = \frac{l}{A} \cos\varphi_0 \cdot \cos \operatorname{am} u \,,$$

$$q = \sqrt{\frac{l^2 - Ah}{B(B-A)}} \cdot \Delta \operatorname{am} u = \frac{l}{B} \sin\varphi_0 \cdot \Delta \operatorname{am} u \,,$$

$$r = \sqrt{\frac{Bh - l^2}{C(B-C)}} \cdot \sin \operatorname{am} u = \frac{l}{A} \cdot \sqrt{\frac{A(B-A)}{C(B-C)}} \cos\varphi_0 \cdot \sin \operatorname{am} u \,.$$

III. $C > A > B$. 1. $Ah - l^2 > 0$.

$$p = \sqrt{\frac{l^2 - Bh}{A(A-B)}} \cdot \sin \operatorname{am} u = \frac{l}{A} \cdot \sqrt{\frac{A(B-A)}{C(B-C)}} \sin \varphi_0 \cdot \sin \operatorname{am} u ,$$

$$q = \sqrt{\frac{Ch - l^2}{B(C-B)}} \cdot \Delta \operatorname{am} u = \frac{l}{B} \cos \varphi_0 \cdot \Delta \operatorname{am} u ,$$

$$r = \sqrt{\frac{l^2 - Bh}{C(C-B)}} \cdot \cos \operatorname{am} u = \frac{l}{C} \sin \varphi_0 \cdot \cos \operatorname{am} u .$$

2. $Ah - l^2 > 0$.

$$p = \sqrt{\frac{Ch - l^2}{A(C-A)}} \cdot \sin \operatorname{am} u = \frac{l}{B} \cdot \sqrt{\frac{B(C-B)}{A(C-A)}} \cos \varphi_0 \cdot \sin \operatorname{am} u ,$$

$$q = \sqrt{\frac{Ch - l^2}{B(C-B)}} \cdot \cos \operatorname{am} u = \frac{l}{B} \cos \varphi_0 \cdot \cos \operatorname{am} u ,$$

$$r = \sqrt{\frac{l^2 - Bh}{C(C-B)}} \cdot \Delta \operatorname{am} u = \frac{l}{C} \sin \varphi_0 \cdot \Delta \operatorname{am} u .$$

The modulus κ and the constant n in $u = n \cdot s$ are:

I. $A > B > C$. 1. $Bh - l^2 > 0$.

$$\kappa = \sqrt{\frac{(A-B)(l^2 - Ch)}{(B-C)(Ah - l^2)}} = \sqrt{\frac{C(A-B)}{A(B-C)}} \tan \varphi_0 ,$$

$$n = \sqrt{\frac{(B-C)(Ah - l^2)}{ABC}} = \frac{1}{C} \cdot \sqrt{\frac{(B-C)(A-B)}{AB}} \cos \varphi_0 .$$

2. $Bh - l^2 < 0$.

$$\kappa = \sqrt{\frac{(B-C)(Ah - l^2)}{(A-B)(l^2 - Ch)}} = \sqrt{\frac{A(B-C)}{C(A-B)}} \cot \varphi_0 ,$$

$$n = \sqrt{\frac{(A-B)(l^2 - Ch)}{ABC}} = \frac{1}{A} \cdot \sqrt{\frac{(A-B)(B-C)}{BC}} \sin \varphi_0 .$$

In this, φ_0 means the angle that the axis of the applied couple, which certainly falls in the plane of the principal axes of largest and smallest resistances, makes with the principal axis of smallest resistance. For $A > B > C$ and $Bh - l^2 > 0$, one will then have $Bq_0 = 0$, $Ap_0 = l \sin \varphi_0$, $Cr_0 = l \cos \varphi_0$ for the cross-section at the free end, and as a result of equations (5), (7):

$$Ah - l^2 = \frac{l^2}{C}(A - C)\cos^2 \varphi_0 , \qquad l^2 - Ch = \frac{l^2}{A}(A - C)\sin^2 \varphi_0 .$$

If one generally takes the angles between the force-couple axis and the three principal axes X', Y', Z' (μ, ν, ρ, resp.) then since it will follow from (5), (7) that:

$$Bh - l^2 = Ap_0^2 (A - B) - Cp_0^2 (B - C) = \cos^2 \mu \, \frac{A-B}{A} - \cos^2 \rho \cdot \frac{B-C}{C} ,$$

instead of:

$$Bh - l^2 \neq 0,$$

one can write:

$$\frac{\cos\mu}{\cos\rho} \neq \sqrt{\frac{A(B-C)}{C(A-B)}} .$$

The direction of the force-couple axis will then determine whether one has $Bh - l^2 > 0$ or $Bh - l^2 < 0$.

If that direction lies in one of the two planes that go through the principal axis Y' of the middle bending and possess the equation:

$$\frac{x'}{y'} = \pm\sqrt{\frac{A(B-C)}{C(A-B)}}$$

then one will have $Bh - l^2 = 0$ and one will enter into a special case (cf., § **10**).

The same arguments will apply to the other five conventions.

4.

It first follows from the equations for p, q, r that the latter are generally periodic in u with period $4K$, up to sign, but they will also assume the same values for $2K \pm u$, such that the absolute values of the bending and twisting only need to be computed for a piece of the rod whose length is established by the values of the parameters u and K, and is therefore K / n. Depending upon which of the six conventions above one chooses, the curvature of the elastic rod will yield different peculiarities. One can see the type and manner of their alteration by means of the easily-understood abbreviations in the following tables, which can be exhibited for $A > B > C$, $Bh - l^2 > 0$ and $Bh - l^2 < 0$:

	$u = 0$	$< K$	$= K$	$< 2K$	$= 2K$	$< 3K$	$= 3K$	$< 4K$	$= 4K$
1.	$+ p_{max}$	$+ p$	0	$- p$	$- p_{max}$	$- p$	0	$+ p$	$+ p_{max}$
	0	$+ q$	$+ q_{max}$	$+ q$	0	$- q$	$- q_{max}$	$- q$	0
	$+ r_{max}$	$+ r$	$+ r_{min}$	$+ r$	$+ r_{max}$	$+ r$	$+ r_{min}$	$+ r$	$+ r_{max}$
2.	$+ p_{max}$	$+ p$	$+ p_{min}$	$+ p$	$+ p_{max}$	$+ p$	$+ p_{min}$	$+ p$	$+ p_{max}$
	0	$+ q$	$+ q_{max}$	$+ q$	0	$- q$	$- q_{max}$	$- q$	0
	r_{max}	$+ r$	0	$- r$	$- r_{max}$	$- r$	0	$+ r$	$+ r_{max}$

Other tables are true for the other four conventions. One sees from them that:

If the resistance of the rod to torsion is the middle (II) *of the three principal resistances then the twisting of the rod cannot result continually in the same sense, but there will necessarily exist cross-sections such that the rod seems to have been twisted in the opposite sense on their two sides. Those cross-section themselves will experience no rotation in their planes, and for all other cross-sections that lie at equal distances to the left or right of them, the absolute value of the twisting will be the same.*

If the resistance to torsion is the smallest (I) *or largest* (III) *of the three principal resistances then the sense of the torsion can never change in one of the two sub-cases* [I.1 and III.2 (*)]: *All cross-sections of the rod seem to have been rotated in the same sense, and the magnitude of the torsion will alternate between a minimum and a maximum.*

By contrast, in the other two sub-cases (I.2 and III.1), *it is the direction of the twisting that will again alternate.*

Similar theorems are true for each component of bending. If a resistance to bending is numerically the middle value of the three resistances then the sense of the bending along its axis will always alternate through 0, since wherever the bending component is zero, the principal plane through the critical axis and the torsion axis will be the osculating plane of the elastic centerline. If a resistance to bending is numerically the smallest or the largest resistance then the sense of the bending component can alternate in each of the two sub-cases, but not in the other.

As far as the intensities of the curvature components are concerned, the component that acts along the principal axis of largest or smallest resistance will become larger as that resistance becomes smaller, the applied force-couple becomes stronger, and the inclination of the axis of the force-couple to the principal axis in question becomes smaller. Those theorems are then true for the sub-case that one also decides upon. The components of the curvature along the principal axis of the middle resistance increases in intensity for two of the two different distinguishing sub-cases: For the first one, if the applied force-couple and the inclination of its axis with respect to the axis of smallest

(*) See the previous paragraphs for the geometric meaning of this.

resistance increases then the two extreme resistances will become larger and the middle one will become smaller. For the second one, that will be true when the applied force-couple and the inclination of its axis with respect to the axis of greatest resistance increase and the two extreme resistances become smaller. By contrast, the components in that sub-case will increase with increasing middle resistance only when it is greater than one-half the largest resistance. If the middle resistance is smaller than one-half of the largest one then the component will drop with an increase in the middle resistance.

The total curvature $\Theta = \sqrt{p^2 + q^2 + r^2}$ that is composed of the curvature components *p*, *q*, *r* will be a maximum for the free end and for each point of the rod whose parameter value *u* is an even multiple of 2*K* and a minimum for the points that are determined by odd multiples of *u*. Their intensities will grow larger with increasing strength of the force-couple, and with the decreasing of each of the three resistances and the inclination angle between the axis of the force-couple and the principal axis of smallest resistance.

The magnitude $\Theta' = \sqrt{p^2 + q^2}$ of pure bending at the points of the elastic centerline that are established by the parameter values $u = 0, 2K, 4K, \ldots$, will be a maximum when the resistance to twisting is the middle of the three resistances and a minimum as long as it is the smallest one. If the resistance to twisting is numerically the largest then one will have a maximum or minimum according to whether it is greater or less than the sum of the resistances to bending. For $n = K, 3K, 5K, \ldots$, minima or maxima, resp., will appear. The intensity of the bending will generally increase for an increase in the intensity of the applied force-couple and for a diminishing of the three principal resistances. By contrast, the strength of the bending can increase or decrease for an increasing angle of inclination between the axis of the force-couple and the axis of least resistance.

The transfer of the straight and untwisted elastic rod to its equilibrium position by means of the successive bending of two skew surfaces

5.

During an infinitely-small time interval *ds*, a rigid body that rotates around its center of mass *O* will rotate around the instantaneous axis of rotation $O\Theta$ with an angular velocity Θ_1, so in that way an axis $O\,\Theta_2$ that is close to the axis $O\,\Theta_1$ will be taken to a position $O\vartheta_2$ and will be itself an instantaneous axis of rotation, moreover. The body will rotate around it with an angular velocity of Θ_2. Under repeated rotation, a neighboring position $O\,\Theta_3$ to $O\,\Theta_2$ will be transferred to $O\ \vartheta_3$, in order to be an axis of rotation in the next moment, etc. It seems as if the cone O $\Theta_1\,\Theta_2\,\Theta_3\ldots$ of instantaneous rotational axes, which one can construct from the component *p*, *q*, *r* for the body, which is thought to be at rest, rolls without slipping on another fixed cone $O\vartheta_1, \vartheta_2, \vartheta_3, \ldots$ in space with which it has a generator in common with the latter at each moment, namely, the instantaneous axis of rotation. The cone $O\,\Theta_1\,\Theta_2\,\Theta_3\ldots$ is **Poinsot**'s cone of the polhode, which was mentioned already in paragraph **2**, while the cone O $\vartheta_1\ \vartheta_2\ \vartheta_3\ldots$ is **Poinsot**'s cone of the *herpolhode*. The end points $\Theta_1\,\Theta_2\,\Theta_3\ldots$ ($\vartheta_1\ \vartheta_2\ \vartheta_3\ldots$, resp.) of the instantaneous rotational axes define the curves of the polhode and herpolhode. Of the two, the polhode is a fourth-order space curve that lies on the central ellipsoid, while the herpolhode is a transcendental plane curve with no inflection points (*) that lies in the invariable plane of the applied force-couple.

If one imagines that the magnitude and direction of the instantaneous curvature Θ at each point P of the elastic centerline of the straight and untwisted rod is determined from the curvature components p, q, r then the line $P\Theta$ that arises will give the axis around which the arc-length element ds will be deformed, as well as the magnitude of the deformation. If one now starts from the free end P_1 and rotates the arc-length element $\overline{P_1 P_2}$ along the curvature axis $P_1\,\Theta_1$ through the prescribed quantity $\Theta_1 \cdot ds$ (when one, say, rotates the arc-length element through the magnitude of bending $\Theta_1' \cdot ds$ around the axis of pure bending and then through the magnitude of twisting $r_1 \cdot ds$ around the rod axis), holds $\overline{P_1 P_2}$ fixed and rotates $\overline{P_2 P_3}$ along $P_2\,\Theta_2$ through $\Theta_2 \cdot ds$, once more holds $\overline{P_2 P_3}$ fixed and rotates $\overline{P_3 P_4}$ along $P_3\,\Theta_3$ through $\Theta_3 \cdot ds$, etc., then those successive rotations will take the straight elastic centerline to its equilibrium position and each cross-section will be rotated in its plane. The curvature axes $P_1\,\Theta_1$, $P_2\,\Theta_2$, $P_3\,\Theta_3$, …, which previously defined a skew surface (namely, the surface of the polhode that was mentioned in § **2**) will now lie on a new skew surface whose guiding line is the bent elastic centerline. Following **Poinsot**'s cone of the herpolhode, we call it the *skew surface of the herpolhode* and the transcendental curves that are analogous to the ones that are defined by the endpoints of the instantaneous curvature axes are the polhodes and herpolhodes. We then see that:

The transfer of a straight and untwisted elastic rod whose one end is acted upon by a force-couple (*) *to its equilibrium position can be rationalized by the bending of a flexible skew surface into a fixed skew surface with which the former has a generator in common at each moment, namely, the instantaneous curvature axis. The guiding lines of the two surfaces are the straight and bent elastic centerlines of the rod.*

In our case of the effect of just one force-couple, the generators of the skew surface are parallel to the generators of a second-degree cone, while those of the fixed skew surface will be parallel to the generators of a transcendental cone. The coordinates of the polhode for the rotation problem relative to the three principal axes of inertia $OX'Y'Z'$ are given directly by:

$$x' = p, \qquad y' = q, \qquad z' = r .$$

The equations of the cone of the polhode are then:

$$x' = \omega \cdot p, \qquad y' = \omega \cdot q, \qquad z' = \omega \cdot r ,$$

in which ω means an arbitrary constant whose elimination, in conjunction with the elimination of u, through which p, q, r is expressed according to (10), will again allow the equation of the cone to take the form (9). If ω runs through a series of values then one

(*) Cf., my dissertation that was cited above.

(*) This theorem is true in general.

will get curves that are similar to the polhode ($\omega = 1$) and each of which can naturally be considered to be the polhode.

For the straight elastic rod, the equations of the polhode – i.e., the curve of the endpoints of the instantaneous curvature axes – relative to the system of principal axes X', Y', Z' that is established in the cross-section at the free end are:

$$x' = p, \qquad y' = q, \qquad z' + s = r. \tag{12}$$

In these equations, $s = u / n$, and the values of the quantities p, q, r are obtained from the equations (10). Since the substitution can be accomplished in six ways, there will also be six different types of polhodes, which can, however, be reduced to three essentially-different ones, which are characterized by the fact that in the equation $z' + s = r$ for the magnitude of twisting r, one can first introduce the function sin am u, then cos am u, and thirdly, Δ am u. The equations will show that (†):

The polhode for the elastic rod is a transcendental curve that lies on a second-degree cylinder that is described around the straight rod axis. Its projection onto the plane of the cross-section $X'Y'$ – i.e., the profile of the cylinder – is a complete ellipse, as long as the resistance to torsion is the smallest or largest and at the same time the sense of the twist never alternates through zero [I.1 and III.2 of (10) (*)]. *The projection will be an arc of an ellipse as long as the resistance to torsion is again the smallest or largest, but cross-sections can exist that are not rotated in their planes* (I.3 and III.1). *The projection is an arc of a hyperbola whenever the resistance to torsion is the middle of the three principal resistances* (II). *The projection of the polhode onto the other two principal planes $Y'Z'$ and $Z'X'$ are wave-like when the running curves have the type of sinusoids. They either do not cut the rod axis at all* [*for the plane $Y'Z'$ in the cases* II.2 *and* III.1, Fig 3] *or they will cut the axis. Therefore, all of the inflection point can be found on the same side of the axis* [I.2 *and* II.1, Fig. 2] *or on different sides of the latter* [I.1 *and* III.2, Fig. 1].

The equations of the flexible skew surface or the surface of the polhode can be easily derived from the equations (12) of the polhode, with the addition of a proportionality factor ω:

$$x' = \omega \cdot p, \qquad y' = \omega \cdot q, \qquad z' + \frac{u}{n} = \omega \cdot r. \tag{13}$$

By varying ω, one will obtain a whole family of curves that are similar to the polhode, and each of which can be regarded as the polhode. By eliminating the quantity ω and the variable u that appear in p, q, r, one will get the equation for the transcendental surface in **Cartesian** coordinates.

Constructing the equation of the herpolhode and its skew surface requires that one must know the equations of the bent elastic centerline. If the coordinates of a point P of the latter relative to the fixed coordinate system X, Y, Z (x, y, z, resp.) are those of the

(†) Translator: The cited figures were not available at the time of translation.
(*) For the geometric meaning, see § **3**.

corresponding point of the herpolhode – i.e., the endpoint of the curvature axis Θ (ξ, η, ζ, resp.) that exists at P – then the equations of the herpolhode will be:

$$\xi - x = a\,p + b\,q + c\,r,$$
$$\eta - y = a'p + b'q + c'r,$$
$$\zeta - z = a''p + b''q + c''r = \text{const.} = \frac{h}{l},$$

in which $a, b, c, \ldots, c''$ mean the inclination cosines of the coordinate systems X, Y, Z and X', Y', Z'. The equations of the skew surface of the herpolhode can again be obtained from the foregoing ones by the addition of a factor ω. The functions that appear in both systems of equations can, moreover, seem to be transcendental and so complicated that their actual presentation as functions of u would seem pointless.

The curvature of the entire rod.

6.

In order to find the form of the curve into which the elastic centerline will be bent, we recall the essence of the curvature components p, q, r. They were generally periodic in u with a period of $4K$. We will then have:

The bent elastic centerline of the rod is a periodic curve. All points whose parameter values differ by multiples of the quantity $4K$ *will possess the same curvature ratios, and all curve segments that are bounded by such points will be congruent.*

The magnitude of the bending $\Theta' = \sqrt{p^2 + q^2}$ can never be zero, except when $p = q = 0$. However, from equations (1), that will happen only when the applied force-couple rotates around the torsion axis, so $p = q = 0$ will still be true. If one ignores that special case in which the elastic centerline remains straight, and thus possesses infinitely-many inflection points, then one can say that:

The bent elastic centerline possesses no inflection points.

Of all the projections of the latter curve, the projection onto the invariable plane of the force-couple is the simplest. If, in order to consider it, we take the invariable direction of the force-couple to be the Z-axis, so the plane of the force-couple will be the XY-plane of our fixed coordinate system (whose origin might lie at the free end) then the inclination cosines a'', b'', c'' of the principal axes X', Y', Z' with respect to the Z axis will follow directly:

$$a'' = \frac{Ap}{l}, \qquad b'' = \frac{Bq}{l}, \qquad c'' = \frac{Cr}{l}. \tag{14}$$

If one defines the magnitudes of bending for a point on the projection of the centerline onto the *XY*-plane in accordance with the viewpoint that was taken in the context of the discussion of the composition and decomposition of bends in § **2**, then one will get, after some calculation:

$$\Theta_{xy} = \frac{a''p + b''q}{\left(\sqrt{1-c''^2}\right)^3} = l^2 \cdot \frac{h - Cr^2}{\left(\sqrt{l^2 - C^2r^2}\right)^3}. \tag{15}$$

In order for the bending Θ_{xy} to be constant, r must be constant, and in order for it to be 0, in particular, r would have to equal $\sqrt{h/C}$. The latter assumption would again lead to the case of $p = q = 0$ – i.e., the case of simple torsion. r = constant assumes that $p = 0$, q = constant or conversely. As a result of (1), that can be fulfilled when the two resistances to bending A, B are equal for an arbitrarily-placed force-couple, but when they are unequal, that condition can be fulfilled only for a force-couple that has been rotated around one of the principal axes X, Y of the cross-section. The former case represents the bending of an isotropic rod into a helix, while the latter represents the bending of the general rod onto a circle (cf., *infra*). Except for those special cases:

The projection of the elastic centerline onto the plane of the applied force-couple can never be a circle, not can it possess isolated inflection points.

Since Θ_{xy} is periodic in u with a period of $2K$, due to the square of r, and an examination will point to the impossibility of finding a maximum or minimum of the bending Θ_{xy} between two successive values of the period, one will then have:

The projection of the elastic centerline onto the invariable plane will prove to be a transcendental curve that is always bent in the same sense, its segments that are bounded by two periodic values are congruent, and there will be a maximum of the bending at one end, while there will be a minimum at the other. (Fig. 10)

The projections of the rod curve onto the other fixed coordinate planes *YZ* and *ZX* – and thus onto an arbitrary plane in space – can first be investigated when one has defined the inclination cosines a, b, c, a', b', c' in terms of u. Namely, the magnitudes of bending Θ_{yz}, Θ_{zx}, read entirely like the quantity Θ_{xy}, when one substitutes the cosines a, b (a', b', resp.) in place of a'', b''. That will show:

Assuming the aforementioned cases of circular bending and pure torsion, the elastic centerline cannot be a plane curve. Its projections onto other planes than the invariable plane of the force-couple will not be periodic. In particular, under projection onto a circular cylinder, the elastic centerline can be coiled only in the case of circular (helical, resp.) bending.

7.

Just as in the case of the projections of the bent elastic centerline onto ones in the invariable *XY*-plane, among all the coordinates of a point *P* along that line, the *z*-coordinate is the simplest one, and it gives the distance from the point to that plane. From (14), the cosine c'' of the inclination angle of the tangent PZ' with respect to the axis of the force-couple will be:

$$c'' = \frac{C}{l} \cdot r,$$

which is proportional to the twist *r*, so all of the considerations that could be applied to the behavior of *r* in § **4** will also be true for them. In particular, from (10), c'' can take on six types of values, of which, three are essentially different; for the coordinates (4), they will correspond to:

$$z = \int c'' \cdot ds = \frac{C}{ln} \cdot \int r \cdot du .$$

Three different forms appear according to whether the functions sin am *u*, cos am *u*, $\Delta \operatorname{am} u$, resp., enter into *r*. Now:

$$\int \sin \operatorname{am} u \cdot du = \frac{1}{\kappa} \ln (\Delta \operatorname{am} u - \kappa \cdot \cos \operatorname{am} u),$$

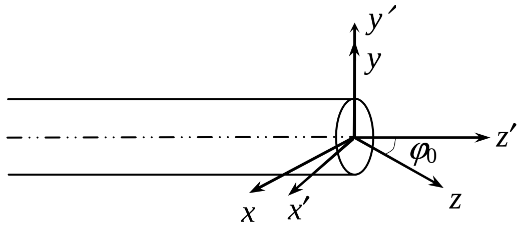
$$\int \cos \operatorname{am} u \cdot du = \frac{1}{\kappa} \arcsin (\kappa \sin \operatorname{am} u),$$

$$\int \Delta \operatorname{am} u \cdot du = \arcsin (\sin \operatorname{am} u) = \operatorname{am} u .$$

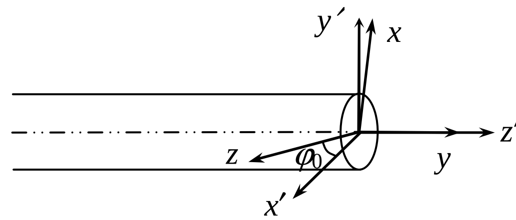
The integration constants that enter into the right-hand side are determined by the special choice of coordinate system. It is known that the axis of the force-couple will be in the plane of the principal axes of smallest and largest resistances. If one calls that axis Z[*] (*), once and for all, so the invariable plane will be *XY*, and one chooses the *Y*-axis such that it coincides with the principal axis of the middle resistance that is drawn in the cross-section at the free end then one will achieve congruence of the coordinate systems *X*, *Y*, *Z* and X', Y', Z' for all three cases. The following figures give information about their relative positions:

(*) One might call the axes *Z*, *X*, *Y*, corresponding to the conventions $A > B > C$, $B > C > A$, $C > A > B$, and achieve many simplifications in that way. However, on the whole, the considerations would probably become more complicated.

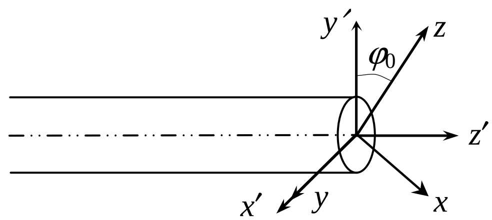
I.



II.



III.



The values of the z-coordinate can be inferred from the following table:

I. $A > B > C$		1. $Bh - l^2 > 0$	2. $Bh - l^2 < 0$
		$G \cdot z = \text{am}\, u$	$G \cdot z = \arcsin(\kappa \cdot \sin \text{am}\, u)$
II. $B > C > A$	(16)	1. $Ch - l^2 > 0$	2. $Ch - l^2 < 0$
		$G \cdot z = \ln \dfrac{\Delta\, \text{am}\, u - \kappa \cdot \cos \text{am}\, u}{1-\kappa}$	$G \cdot z = \ln \dfrac{\Delta\, \text{am}\, u - \kappa \cdot \cos \text{am}\, u}{1-\kappa}$
III. $C > A > B$		1. $Ah - l^2 > 0$	2. $Ah - l^2 < 0$
		$G \cdot z = \arcsin(\kappa \cdot \sin \text{am}\, u)$	$G \cdot z = \text{am}\, u.$

In this, we have set:

$$1 : G = C\sqrt{\frac{AB}{\pm(A-C)(B-C)}},$$

in which the sign in the denominator must be chosen that will make that denominator positive.

As one sees, the expressions for z depend essentially upon the behavior of the resistance to twisting C : If it is numerically the middle of the three resistances then z will be logarithmic, and if it is the smallest or largest then it will be expressed cyclometrically by elliptic functions. The following characteristic fact has decisive importance for the shape of the elastic centerline:

If the resistance of the rod to torsion is the middle of the three principal resistances then the helically-bent elastic centerline will lie completely on the side of the invariable *plane of the force-couple that acts upon the free cross-section, while it will contact the plane at the free end, as well as those locations for which the value of the parameter u takes on multiples of* 4*K*. *All points that have the same arc-length distance to the left or right of the contact points have the same distance from the plane. For u* = 2*K*, 6*K*, …, *the curve contacts a parallel plane whose distance from the plane of the force-couple is equal to:*

$$z_{2K} = \frac{1}{G} \cdot \ln \frac{1+\kappa}{1-\kappa} \qquad \text{(see Fig. 8).}$$

For $u = K$, $3K$, …, one will have:

$$z_K = \tfrac{1}{2}\, z_{2K} \,.$$

If the resistance of the rod to twisting is the smallest or largest of the three principal resistances then, from § **4**, *the sense of the twist will never change in each of the two sub-cases that pertain to each case (namely,* I.1 *and* III.2). *Now, with those assumptions, the z-coordinate will increase continually, so the elastic centerline will lie completely on one side of the invariable plane of the force-couple and will move ever further from it. The shape of the curve is precisely the same between parallel planes through points with the parameter values* 2*K*, 4*K*, 6*K*, …, *and one will have:*

$$z_{\mu,K} = \mu \cdot z_K = \frac{1}{G} \cdot \mu \cdot \frac{\pi}{2} \qquad \text{(Fig. 7)}$$

If the resistance to twisting is the smallest or the largest and the sense of the twist can change (I.2 *and* III.1) *then z will once more be periodic in u and the elastic centerline will intersect the invariable plane of the force-couple for u* = 0, 2*K*, 4*K*, …, *and as it alternates above and below that plane, it will, in due course, contact two parallel planes, each of which possesses a distance of* $z_K = \frac{1}{G} \cdot \arcsin \kappa$ *from the plane of the force-couple.* (Fig. 9)

As far as the proportionality factor $1 : G$ is concerned, it should emphasized that, in contrast to the factors that appeared before, it is independent of h and l (when taken absolutely), so it will be represented by the same expression for all six sub-cases in the classification. That is:

The relative strength $1 : G$ *with which the elastic centerline spirals up the invariable plane of the force-couple does not depend upon the intensity and direction of application of the couple, but only upon the magnitudes of the three resistances to deformation.*

Since the values of z_K depend upon only h and l in the case $B > C > A$, since they enter into κ; the variation of z_K in the other two cases can be attributed to a variation in $1 : G$. That shows that:

If the resistance to twisting is the smallest of the three principal resistances then the spiral into which the elastic centerline is bent will rise ever more steeply from the plane *of the force-couple as that resistance gets smaller and as the resistance to bending gets larger. If the resistance to twisting is the largest then the steepness of the spiral will increase as that resistance increases and the resistance to bending decreases.*

If the resistance to twisting is the middle one then the degree of steepness of the curve will be essentially influenced by inequalities that exist between the resistances A, B, C, except for the fact that the inclination φ_0 of the axis of the force-couple with respect to the principal axis of smallest resistance will enter into κ.

Previously, the distance z from a point on the elastic centerline to the plane of the couple that acts upon the cross-section at the free end was given in the form $z = \int c'' \cdot ds = \frac{C}{l} \cdot \int r \cdot ds$. However, $\int r \cdot ds$ is nothing but the total magnitude T of the torsion for a piece s of the elastic rod (as measured from the free end), so:

$$T = \frac{l}{C} \cdot z . \tag{17}$$

That is: The total torsion of a piece of the elastic centerline of length s (measured from the free end) will be measured by the distance from the endpoint to the invariable plane of the applied force-couple, up to a constant.

In particular, that total torsion will increase continually as long as the twisting cannot change its sense. It will be zero, without becoming negative as long as the twist changes its sense and the resistance to twisting is the middle one, while it will go through zero into negative values as long as the sense of the twisting can change and the resistance to twisting is the smallest or largest.

This total torsion of an arbitrarily-bounded piece of the rod will be measured simply by the difference between the distances from the bounding points to the invariable plane of the force-couple. Moreover, it can also be zero when the resistance to twisting is the middle one.

Since these theorems about the total torsion can be derived from the behavior of the twist r (§ **4**), they will verify the distance z (the three positions, resp.) that the elastic centerline can assume with respect to the invariable plane of the force-couple.

The quantity T is nothing but the angle through which a cross-section will be rotated with respect to the cross-section at the free end. For the point with the parameter value K that will imply:

$$T_K = \frac{l}{C} \cdot z_K .$$

If one would prefer that the cross-section $u = K$ were rotated with respect to the cross-section at the end through a well-defined angle λ then a condition equation would result between λ and the resistances A, B, C that does or does not include the angle φ_0 between the axis of the applied force-couple and the principal axis of smallest resistance.

8.

We shall now construct the expressions for the coordinates x, y of a point P on the elastic centerline (for the inclination cosines c, c' of the tangent PZ' with respect to the axes X, Y of the fixed coordinate system, resp.).

It easily follows from equations (2) and (3) that:

$$c' \cdot \frac{dc}{ds} - c \cdot \frac{dc'}{ds} = a'' \, p + b'' \, q.$$

If one substitutes the values for a'', b'' from (14) in these equations and observes the relation (7) then, since the left-hand side can be extended to a total differential, one can write:

$$-\, c^2 \cdot \frac{d(c'/c)}{ds} = \frac{1}{l} \, (Ap^2 + Bq^2).$$

Now, one has:

$$c^2 + c'^2 = 1 - c''^2 = \frac{1}{l^2} \, (A^2 \, p^2 + B^2 \, q^2),$$

so one might set:

$$c = \frac{1}{l}\sqrt{A^2 p^2 + B^2 q^2} \cdot \cos \psi,$$

$$c' = \frac{1}{l}\sqrt{A^2 p^2 + B^2 q^2} \cdot \sin \psi.$$

If one substitutes these values in the differential equation above and writes u / n for s then one will get:

(18) $$d\psi = -\frac{l}{n} \cdot \frac{Ap^2 + Bq^2}{A^2 p^2 + B^2 q^2} \cdot du.$$

p and q must be replaced with functions of u in these equations. As is known, there exist six different pairs of values for p, q in all, so there will be six distinct solutions. However, in the foregoing sections, we have emphasized that is it essential for the elastic centerline to possess periodicities with no inflection points and always appear to be doubly-curved, and indeed completely independently of the orders of magnitude of the three moments of resistance A, B, C. Those curves will possess essentially the same type in all six cases, and since the characteristic behavior of z has already been discussed, it will suffice to present the cosines c, c' (the coordinate x, y, resp.) for a single assumption, say for:

$$A > B > C, \qquad Bh - l^2 > 0.$$

As one can easily show, the auxiliary angle ψ that was introduced above is the angle between the line of intersection of the plane of the cross-section $X'Y'$ through a point P of the elastic centerline and the invariable XY-plane, on the one hand, and the fixed Y-axis, on the other. It is nothing but the **Euler** angle ψ between the nodal line N of a rigid body that rotates around its center of mass and the fixed Y-axis in the invariable XY-plane. All that one needs to do then it is adapt the value of ψ to **Jacobi**'s work (*). If one sets:

$$\psi = \psi' + n' \cdot u$$

then that will imply:

(19) $$\psi' = \frac{1}{2i} \cdot \log \frac{\Theta(u+ia)}{\Theta(u-ia)} .$$

In these expressions, ia and n' mean constants that are defined by the equations:

$$\frac{1}{i} \cdot \sin \operatorname{am} ia = \sqrt{\frac{C(Ah-l^2)}{A}} ,$$

$$n' = \frac{l}{An} + \frac{\partial}{\partial(ia)} \log \Theta\,(ia).$$

ψ' is obviously the angle between the line of intersection of $X'Y'$ and XY and a new axis in the XY-plane that does not, however, lie fixed in that plane, like Y, but seems to move with an angular velocity of n' / n in the negative sense. Now, one can just as well refer the coordinates of the point P of the centerline to the fixed coordinate system X, Y, Z as to a system that consists of the invariable Z-axis, the moving axis (Y), and a moving axis (X) that is perpendicular to them. The angle ψ' will be periodic for the latter system, so when one looks for the projection of the point P on the rod onto the invariable plane, it would probably be best to proceed by using the position of the line (Y) for the values of s (i.e., u) considered and calculate the rectangular coordinates with respect to the latter. According to **Jacobi** (*), that will imply that the inclination cosines (c), (c') of the principal axis Z' (which is the tangent to the elastic centerline, here) with respect to the moving axes (X), (Y) are:

(20)
$$(c) = \frac{\mathrm{H}_1(0)}{2 \cdot \mathrm{H}_1(ia) \cdot \Theta(u)} \cdot [\Theta\,(u + ia) + \Theta\,(u - ia)],$$

$$(c') = \frac{\mathrm{H}_1(0)}{2i \cdot \mathrm{H}_1(ia) \cdot \Theta(u)} \cdot [\Theta\,(u + ia) + \Theta\,(u - ia)] .$$

One can obtain the inclination angles c, c' of the tangent PZ' with respect to the fixed axes X, Y as soon as one substitutes the values of (c), (c') in the equations:

(*) **Jacobi**, *loc. cit.*, pp. 157-159.
(*) *Loc. cit.*, pp. 162.

$$(21) \qquad \begin{aligned} c &= (c) \cdot \cos n' u + (c') \cdot \sin n' u, \\ c' &= -(c) \cdot \sin n' u + (c') \cdot \cos n' u. \end{aligned}$$

The inclination cosines (c), (c') are periodic in u of period $2K$, while c, c' have no periodicity. One will obtain expressions analogous to (17) for the inclination cosines (a), (a') [(b), (b'), resp.] of the axes X' and Y' with respect to the moving axes (X), (Y). At the same time, they can be taken from **Jacobi**'s work, but here they are less interesting than the inclination cosines (c), (c') of the tangent to the rod Z'.

9.

The Θ-functions that appear in formulas (20) and (21) can be replaced with infinite series. That must happen as soon as one would actually like to define the coordinates (x), (y), and x, y of a point P on the elastic centerline relative to the moving system (X), (Y) and the fixed system X, Y by integrating the equations:

$$\frac{d(x)}{ds} = (c), \qquad \frac{d(y)}{ds} = (c'),$$

$$\frac{dx}{ds} = c, \qquad \frac{dy}{ds} = c'.$$

If we also replace the closed expressions that were obtained for the z-coordinates in paragraph **8** with infinite series (at least, for $A > B > C$ and $Bh - l^2 > 0$) then the coordinates will take the following forms:

I. Moving coordinate system (X), (Y), (Z).

By performing the integration of the **Jacobi** series (*), upon introducing the abbreviations (**):

$$n \cdot \frac{\pi}{K} \cdot s = u = \frac{\pi}{K} \cdot v,$$

$$\frac{a}{K} = b, \qquad e^{-\pi\cdot(K'/K)} = q,$$

$$\frac{2l}{C} \cdot \sqrt{\frac{(A-C)(B-C)}{AB}} = 2l \cdot G = D,$$

(*) Cf., the beginning of the work "Sur la rotation."

(**) In the treatise in the Münch. Ber., one will find, incorrectly, $q = l\ldots$ printed instead of $q = e\ldots$ and $(A - B) \cdot C^2$, instead of $A \cdot B \cdot C^2$.

one will get the coordinates (*x*), (*y*), (*z*):

$$D\cdot(x) = \frac{2q^{b/2}}{1-q^b}\cdot v - 2\left(q^{-b/2}-q^{b/2}\right)\cdot\sum_{\mu=1}^{\infty}\frac{q^{\mu}(1+q^{2\mu})\cdot\sin\mu v}{\mu(1-q^{2\mu-b})(1-q^{2\mu+b})}$$

(22) $$D\cdot(y) = 4\left(q^{-b/2}+q^{b/2}\right)\cdot\sum_{\mu=1}^{\infty}\frac{q^{\mu}(1+q^{2\mu})\cdot\sin^2\frac{\mu v}{2}}{\mu(1-q^{2\mu-b})(1-q^{2\mu+b})},$$

$$D\cdot(z) = v + 4\cdot\sum_{\mu=1}^{\infty}\frac{q^{\mu}\cdot\sin\mu v}{\mu(1+q^{2\mu})}.$$

II. Fixed coordinate system *X*, *Y*, *Z*.

If, in addition to the previous abbreviations, one also introduces:

$$m = \frac{K}{\pi}\cdot n'$$

then one will have:

$$D\cdot x = \frac{2q^{b/2}}{m(1-q^b)}\cdot\sin mv + 2q^{b/2}\sum_{\mu=1}^{\infty}\frac{q^{\mu}\cdot\sin(m-\mu)v}{(m-\mu)(1-q^{2\mu+b})}$$

$$-\ 2q^{-b/2}\sum_{\mu=1}^{\infty}\frac{q^{\mu}\cdot\sin(m-\mu)v}{(m+\mu)(1-q^{2\mu-b})},$$

(23) $$D\cdot y = \frac{4q^{b/2}}{m(1-q^b)}\cdot\sin^2\frac{mv}{2} + 4q^{b/2}\sum_{\mu=1}^{\infty}\frac{q^{\mu}\cdot\sin^2\left(\frac{m-\mu}{2}\right)v}{(m-\mu)(1-q^{2\mu+b})}$$

$$-\ 4q^{-b/2}\sum_{\mu=1}^{\infty}\frac{q^{\mu}\cdot\sin^2\left(\frac{m+\mu}{2}\right)v}{(m+\mu)(1-q^{2\mu-b})},$$

$$D\cdot z = D\cdot(z).$$

Special cases that arise from special locations of the applied force-couple

10.

If force-couple that acts upon a rigid body rotates around the principal axis of the largest or smallest moment of inertia then the motion will proceed continually around that axis with uniform angular velocity. The magnitude of the latter is equal to (*intensity of the force-couple*) : (*moment of inertia*). The cones of the polhode and herpolhode will coincide with the permanent axis of rotation. The state of motion is *stable* – viz., small perturbations of the rotational axis will produce only minor oscillations about its rest position.

If the force-couple that acts upon the cross-section at the free end of the rod rotates around the principal axis of largest or smallest resistance to deformations, and if that principal axis the torsion axis then the elastic centerline of the rod will remain straight, and the rod will be only uniformly twisted. The magnitude of the twisting is equal to (intensity of the force-couple) : *(resistance to torsion). The flexible and fixed skew surface, as well as the curves of the polhode and herpolhode that lie on it, coincide in the axis of the rod. If the principal axis is one of pure bending then mere bending without twisting into a circle will be produced. The curvature of the latter is (intensity of the force-couple)* : *(resistance to bending). The skew surfaces of the polhode and herpolhode are parallel to the axis of the force-couple in the circular cylinder, so the curves will coincide in a circle. The equilibrium state is* ***stable*** *in both cases: viz., if one pushes the rod from its equilibrium position slightly then it will again seek to return to it.*

If the force-couple that acts upon a body ($A > B > C$) rotates around a line that belongs to one of the two planes that go through the principal axis Y' of the middle moment of inertia and have equations relative to the principal axis system X', Y', Z':

$$\frac{x'}{y'} = \pm\sqrt{\frac{A(B-C)}{C(A-B)}}\,,$$

then (cf., the concluding remarks in § **3**) $Bh - l^2 = 0$, and the elliptic functions that appear in p, q, r will reduce to logarithmic ones. The rolling cone of the instantaneous rotational axis will likewise go to two *planes* through the Y-axis:

$$\frac{x'}{y'} = \pm\sqrt{\frac{C(B-C)}{A(A-B)}}\,,$$

and the *polhode* will go to two *ellipses.* The *herpolhode* will become a *spiral*, and the *cone* of the herpolhode will be wound correspondingly. The sub-cases of $Bh > l^2$ and $Bh < l^2$ will coincide.

Of course, the assumption that $Bh - l^2 = 0$ also admits the singular solution $p = 0$, $r = 0$, so from **Euler**'s equations (1), one will have $q = \text{const.} = q_0$. That is, when the applied force-couple acts around the Y'-axis of the middle moment of inertia B itself, the rotation around that axis will proceed continually and with constant velocity. In this case, the state of motion is *labile:* viz., it might take only a minor perturbation to immediately upset the central ellipsoid and allow the phenomena that were described before to come about.

If the force-couple that acts upon the elastic rod rotates around a straight line that belongs to one of the two *characteristic* planes that go through the principal axis of the middle resistance, and whose equations for $A > B > C$, etc., are:

$$\frac{x'}{y'} = \pm\sqrt{\frac{A(B-C)}{C(A-B)}}, \quad \text{etc.,}$$

relative to the system of three principal axes then the modulus κ of the elliptic functions in p, q, r will be equal to 1, and formulas (10) will go to the following ones:

I. $A > B > C$.

$$p = \frac{l}{A}\sqrt{\frac{A(B-C)}{B(C-A)}}\cdot\frac{2e^u}{e^{2u}+1}, \quad r = \frac{l}{C}\sqrt{\frac{C(A-B)}{B(A-C)}}\cdot\frac{2e^u}{e^{2u}+1},$$

$$q = \frac{l}{B}\cdot\frac{e^{2u}-1}{e^{2u}+1}.$$

II. $B > C > A$.

$$p = \frac{l}{A}\sqrt{\frac{A(B-C)}{C(B-A)}}\cdot\frac{2e^u}{e^{2u}+1}, \qquad q = \frac{l}{B}\sqrt{\frac{B(C-A)}{C(B-A)}}\cdot\frac{2e^u}{e^{2u}+1}, \tag{24}$$

$$r = \frac{l}{C}\cdot\frac{e^{2u}-1}{e^{2u}+1}.$$

III. $C > A > B$.

$$q = \frac{l}{B}\sqrt{\frac{B(C-A)}{C(B-A)}}\cdot\frac{2e^u}{e^{2u}+1}, \qquad r = \frac{l}{C}\sqrt{\frac{C(A-B)}{A(C-B)}}\cdot\frac{2e^u}{e^{2u}+1},$$

$$p = \frac{l}{A}\cdot\frac{e^{2u}-1}{e^{2u}+1}.$$

The sub-cases $Bh - l^2 > 0$ and $Bh - l^2 < 0$, etc, coincide. From what was said above, as well as the formulas, that will imply:

As long as the resistance to torsion is the middle one, the surface of the polhode will go to a pair of planes through the axis of the rod, the polhode $\xi = p$, $\eta = q$, $\zeta = -(u/n) + r$ *will go to a plane curve that lies in one or the other plane according to whether the axis of the applied force-couple lies in one or the other of the two "characteristic" planes that correspond to those planes. The projection of the elastic centerline onto the plane of the* *cross-section will be a line segment. The projections onto the other principal planes will be curves that run asymptotically to the axis of the rod. The curve of the polhode will then approach the axis of the rod asymptotically at the same time* (Fig. 4).

If the resistance to torsion is the largest or smallest of the three principal resistances to deformation then the projection of the polhode onto the plane of the cross-section will be a quadrant of an ellipse, and the projections onto the other two principal planes will be curves that run asymptotically to two lines that are parallel to the axis of the rod. (Fig. 5).

The bending Θ_{xy} (15) of the projection of the elastic centerline onto the invariable plane of the force-couple is known to depend upon r. It will no longer periodic in u here, but will rise or fall continually according to whether r enters into the quotient $\frac{e^{2u}-1}{e^{2u}+1}$ or $\frac{e^{u}}{e^{2u}+1}$. That is:

The projection of the elastic centerline onto the invariable plane of the force-couple no longer possesses periodic curvature, but winds into a spiral,, in complete analogy to the corresponding case of ***Poinsot****'s spiral. The spiral will then run from one fixed circle* (s = 0) *asymptotically to a second one that is concentric to it* ($s = \infty$) *as long as the resistance to torsion is not the middle one* (Fig. 12); *By contrast, it will run from a fixed circle* (s = 0) *asymptotically to its center* ($s = \infty$) *as long as the resistance to torsion is smaller than the one resistance to bending and larger than the other one.* (Fig. 11).

The elastic centerline itself appears to be a curve that loops around the axis of the force-couple. In the first of the aforementioned cases, it will move more and more distant from that axis without exceeding a certain limiting distance, moreover, and in the second case, it will approach the axis closer and closer.

Just as rotation under the assumption that $Bh - l^2 = 0$ ($A > B > C$) yielded the *singular* solution $p = 0$, $r = 0$, $q = q_0$, the same thing will be true here. One will see that:

When the force-couple rotates around the principal axis of the middle moment, mere torsion or circular bending will be produced once again, except that the equilibrium state of the elastic rod will be labile: viz., a small perturbation will suffice to take the elastic centerline, which remains straight or curved into a circular arc, to an entirely new curve with the aforementioned properties.

Special cases that arise from special choices of the three principal resistances

11.

If two principal moments of inertia of a body that rotates around its center of mass are equal to each other then the moment of inertia around the all of the lines that are drawn in the plane of the two principal axes will be equal. The polhode and the herpolhode will go to a circle, their cone will go to a circular cone, and the rotation will be completed continually with uniform velocity around the instantaneous axis of rotation.

If two principal moments of resistance of the elastic rod are equal to each other then one must decide which moments they are. If the resistances A, B to *bending are equal* then the rod will be *isotropic*, so each of the lines of resistance that are drawn in the plane of the cross-section will be *equal* (= $A = B$). If one calls the inclination angle of the axis of the force-couple with respect to the axis of torsion λ then (10) will imply that:

$$p = \frac{l}{A} \cdot \sin\lambda \cdot \cos u,$$

$$q = \frac{l}{A} \cdot \sin\lambda \cdot \sin u, \tag{25}$$

$$r = \frac{l}{C} \cdot \cos\lambda = r_0 \,.$$

The twist r, as well as the bending $\Theta' = \sqrt{p^2 + q^2}$, will be constant, so from (15), one will also have $\Theta_{xy} = \dfrac{l}{A \cdot \sin\lambda}$; i.e.:

The elastic centerline will be bent into an ordinary helix around the invariable direction of the axis of the force-couple under uniform twisting. Its pitch angle is $90^{\mathrm{o}} - \lambda$. *The radius of the circular cylinder that it lies upon is equal to the product of the constant resistance to bending and the sine of the pitch angle, though which, the axis of the force-couple is inclined with respect to the axis of the rod, divided by the intensity of the applied couple. The cylinder will then become narrower as the applied force-couple gets stronger and the smaller that the resistance to bending gets, and the closer that the axis of the force-couple gets to the axis of the rod.*

The consideration of $\xi = p$, $\eta = q$, $\zeta = -(u / n) + r_0$ implies that:

The polhode goes to an ordinary helix around the axis of torsion, while its flexible skew surface will go to an ordinary helicoid. The pitch angle of the helix is $\alpha = \arctan\left(\dfrac{2 \cdot A^2 C}{(A-C) \cdot l^2 \cdot \sin 2\lambda}\right)$, *and the radius of the circular cylinder that it lies on will be* $\dfrac{A}{l \cdot \sin\lambda}$. *The circular cylinder will then become wider as the strength of the applied force-couple increases, the larger the angle between its axis and the axis of the rod becomes, and the smaller that the equal resistance to bending become. The pitch will get larger as that resistance and the resistance to twisting does, and the smaller that the angle between the force-couple and the axis of the rod becomes.*

If the force-couple rotates around an arbitrary principal axis of the cross-section then the constant twist will go to zero, and simply a circular bending will be produced.

If one *resistance to bending* (e.g., B) and the *resistance to twist* C are numerically *equal* then if the case $A > C$ (= B) is satisfied, the curvature components p, q, r will be:

$$p = \frac{l}{A} \cdot \sin \varphi_0 \,,$$

(26)
$$q = \frac{l}{C} \cdot \sin \varphi_0 \cdot \sin u,$$

$$r = \frac{l}{C} \cdot \cos \varphi_0 \cdot \cos u \,.$$

One learns from the fact that $\xi = p$, $\eta = q$, $\zeta = - u / n + r$:

The surface of the polhode goes to a plane that is perpendicular to the principal axis of the unequal principal resistance to bending, so the polhode will go to a plane curve. Their projections onto the two principal planes through that principal axis are straight lines, and the projection onto the three parallel principal planes is a sinusoidal curve whose inflection points lie on the axis of the rod. (Fig. 6)

The elastic centerline itself cannot be represented by equations of a simpler form. Its projection onto the invariable plane of the applied force-couple will again prove to have periodic curvature when:

$$\Theta_{xy} = \frac{l}{AC} \cdot \frac{C \sin^2 \varphi_0 + A \cos^2 \varphi_0 \sin^2 u}{(\sin^2 \varphi_0 + \cos^2 \varphi_0 \sin^2 u)^{3/2}} \,.$$

The period of u is 2π this time.

If the three principal moments of inertia of the rotating body are equal to each other them its central ellipsoid will go to a cone, and every axis around which the force-couple rotation will be a permanent axis of rotation.

If the three *resistances* A, B, C that oppose the deformation of the rod are *equal* to each other (= A) then the three curvature components p, q, r will be constant. The equations $\xi = p_0$, $\eta = q_0$, $\zeta = - s + r_0$ show that the surface of the polhode goes to a plane through the torsion axis, and the polhode goes to a straight line that is parallel to itself. If the force-couple rotates around one of the principal axes of the isotropic cross-sections then mere circular bending will arise, while in any other case, the elastic centerline will be bent into a helix. Therefore, this case of equality of the three moments of resistance A, B, C of the rod does not differ essentially from that of the equality of the resistance to bending A, B, in contrast to the problem of rotation, which will be greatly simplified when one goes from the equality of two principal moments of inertia to the equality of all three.

If the sum of two principal moments of inertia of the rotating body is equal to the third one then the body would go to a plate, and the central ellipsoid would arrive at the characteristic limits that **Poinsot** emphasized. Furthermore, the rotation would proceed essentially the same as it does in the general case.

If the *sum of the two resistances to bending is equal to the resistance to twisting* of the elastic rod then the elastic centerline will be deformed into a curve of *constant bending*, so it will follow from **Euler**'s equations (1) that:

$$p^2 + q^2 = \text{const.}$$

Moreover, it will not possess equations that are essentially simpler than they are in the general case, nor will the polhode and its surface be specialized.

Munich, in July 1883.

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