"Über die *curvatura integra* geschlossener Hyperflächen," Math. Ann. **95** (1927), 340-367.

On the *curvatura integra* of closed hypersurfaces

By H. Hopf, in Berlin

Translated by D. H. Delphenich

Introduction

It is well-known in ordinary differential geometry that the theorem of the invariance of the *curvature integra* of a closed surface follows from the Gauss-Bonnet theorem [1]); it may be expressed as follows: "If an everywhere regular metric is defined on a closed surface by an element of arc length ds, and if K is the Gaussian curvature, which is computed from the coefficients of ds^2 in a well-known way, then the integral of K over the entire surface is equal to the product of the area 4π of the boundary surface of a unit sphere with a whole-number topological invariant of the surface." Thus, in order for this theorem to be valid it is not necessary to restrict oneself to surfaces that lie in three-dimensional Euclidian space and carry a metric that they inherit from it. In the present investigation of the *curvature integra* of higher-dimensional manifolds, we shall nonetheless assume the aforementioned restriction: We consider n-dimensional closed hypersurfaces (which are usually allowed to be self-intersecting) that are embedded in $(n+1)$-dimensional Euclidian space, and ask just what values that the *curvature integra* of such hypersurfaces can provide that are the "model" for one and the same manifold. From the Gaussian definition of the curvature by means of the normal map it emerges, with no further assumptions, that the *curvature integra* of a hypersurface m is equal to the product of the boundary surface volume of the n-dimensional unit sphere and the "degree" of the map from the manifold that represents the model m that takes the normals to m onto the "direction sphere" of $(n+1)$-dimensional space. This mapping degree, which we would like to call – by abuse of notation – the *curvature integra* of m thus defines the center of the investigation [2]).

As a result, this is closely linked with the theory of mapping degree, which is based on the works of Brouwer; in particular, the paper "Über Abbilding von Mannigfaltigkeiten," [3]), as well as parts of the paper "Über Jordansche Mannigfaltigkeiten" [3]). In what follows, we shall frequently employ his concept

---

[1]) See, e.g., Blaschke, Vorles. über Diff.-Geom. **1** (Berlin, 1921), § 64.

[2]) Kronecker, in his treatise "Über Systeme von Funktionen mehrerer Variabeln" (Monatsber. d. Kgl. Preuss. Akad. d. Wiss. zu Berlin 1869, 2. Abhandl.), has shown that the *curvature integra* of the surface $F(x, y, z) = 0$ agrees with the "characteristic" of the system of functions F, $\partial F/\partial x$, $\partial F/\partial y$, $\partial F/\partial z$, multiplied by 4π; this is identical with the degree of the map considered in the text. On this, one confers Hadamard "Note sur quelques applications de l'indice de Kronecker," printed in Tannery, *Introduction à la théorie des fonctions II,* 2nd ed., (1910), as well as Dyck, "Beiträge zur Analysis Situs I," Math. Annalen **32** (1888).

[3]) Math. Annalen **71** (1912).

definitions, methods of proof, and terminology – particularly in §§ 1, 2 – so that we do not always have to refer to him in the text, and his results must be assumed as known.

The contents of this paper is briefly summarized as follows: In § 1, the notion of the "index" of a singularity of a continuous vector field, which was essentially introduced by Poincaré [4]) and plays an important role in Brouwer's investigations of fixed points, will be extended in such a way that one can equip a point of an n-dimensional region that goes to the same image point under two different maps, while such an agreement does not enter into its neighborhood at all, with a "coincidence index;" on this, some things will be proved that will prove useful for later purposes – in particular, its invariance under topological transformations, and thus its independence of the coordinate system. In § 2, two maps of an n-dimensional manifold m onto the n-dimensional ball will be considered; it will be shown that the sum of the coincidence indices – assuming that the maps have only finitely many points of coincidence – is, independently of the topological properties of m, equal to the sum of the two mapping degrees for even n and the difference for odd n; this is a generalization of the theorem of Poincaré-Bohl [5]), which says, *inter alia*, that the two mapping degrees differ by the factor $(-1)^{n+1}$, in the event that no points of coincidence appear. The fact that one can determine the one degree and the other from the index sum using the stated theorem will be used for the examination of the *curvature integra* in § 3. There, two hypersurfaces that model m will be considered; a classification follows that was introduced into topology by Antoine [6]) in another connection, and we distinguish whether the map between the two models on μ can be extended to a map of elements (that contain the model), or only to a neighborhood (that contains the model), or whether the possibility of such an extension is not known. It is shown that for *even n* the *curvatura integra* is a topological invariant of μ; i.e., that it remains unchanged under the maps of all three classes. For *odd n*, by comparison, the *curvature integra* still remains indeed unchanged under the maps of the first class (at least, under certain differentiability assumptions), but no longer for those of the second class; here, there are indeed homeomorphic Jordan hypersurfaces, i.e., ones that are intersection-free, that have different values of the *curvature integra* (and not only up to sign). Obviously, the question remains here of whether (for odd $n \geq 3$) one can prescribe arbitrary values for the *curvature integra* of a – not necessarily Jordan – model for a given manifold, and whether the Jordan model of the n-dimensional ball always has the *curvature integra* $\pm$ 1; in this paper, the latter question will be resolved only under the simplifying assumption that the model is the boundary of an element, an assumption for which it is not whether that represents a restriction.

On the other hand, in another direction, we will arrive at the result: In the course of the investigations of § 3, it results that the index sum of the singularities of a vector field that is tangential to a hypersurface must have an *even* degree, namely, for *odd n*, it must be zero, and for *even n*, it must be twice the *curvatura integra* of the hypersurface. From this fact, a necessary condition for an n-dimensional manifold to possess a hypersurface in $(n+1)$-dimensional Euclidian space as a model will be derived in § 4. On the basis of this condition it is found in § 5 that the totality of the complex points of the $2k$-

[4]) Poincaré, "Sur les courbes définies par les equations différentielles (3[iéme] partie)," Chap. 13 (Journ. de Math. (4) **1** (1885), (4) **2** (1886).

[5]) Hadamard, loc. cit., pp. 476 et seq.

[6]) Antoine, "Sur l'homéomorphie de deux figures et de leurs voisinages," Journ. de Math. (8) **4** (1921).

dimensional projective space, which is a $4k$-dimensional, simply-connected, closed manifold, may not be represented by a hypersurface in $(4k+1)$-dimensional Euclidian space, even when one ignores self-intersections.

§ 1.

The index of a coincidence point of two maps

Let a right-angled Cartesian coordinate system $\xi_1, \ldots, \xi_n$ be introduced in the neighborhood of an n-dimensional region Γ that contains a point Ω. We make the following assumptions relative to the indicatrix, which are valid for all considerations in this paper, unless expressly stated to the contrary: The positive indicatrix is defined by the sequence of vertices $(0, \ldots, 0)$; $(1, 0, \ldots, 0)$; $(0, 1, \ldots, 0)$; …; $(0, 0, \ldots, 1)$ of the simplex S that they determine; the positive indicatrix of the boundary of S is thus simultaneously established by this. As the "natural" orientation of a Jordan manifold m, we mean the one for which the interior has the order + 1, and we determine the order of a point A by projecting μ from A onto a simplex χ around A that has sides parallel to those of S and is oriented correspondingly. If m is an oriented $(n-1)$-dimensional surface piece that possesses an $(n-1)$-dimensional tangent space ϑ_Π at each point Π that varies continuously with Π then the positive indicatrix of ϑ_Π is to be chosen in such a way that the topological map that is carried out by perpendicular projection of a sufficiently small region of m that includes Π onto ϑ_Π takes the positive indicatrix of m to that of ϑ_Π. We say that a ray that originates at Π and is not tangential to m is directed towards the "positive" side of m when the positive boundary indicatrix that is defined by the natural orientation of an n-dimensional simplex that is defined by a point of the ray and an $(n-1)$-dimensional simplex of ϑ_Π is the positive indicatrix of ϑ_Π. Thus, the interior normals of a naturally oriented $(n-1)$-dimensional sphere have a positive direction. The orientation of ϑ_Π is therefore the one whose midpoint lies on the positive normal of m, whether or not one regards ϑ_Π as a tangent space to m or to a sphere.

Let H_1 and H_2 be two maps [7]) of Γ onto point sets of the region G, which possesses coordinates $x_1, \ldots, x_n$ and is likewise oriented in the required manner; let Ω be an isolated point of coincidence of H_1 and H_2; let $H_1(\Omega) = H_2(\Omega) = 0$, but $H_1(\Pi) \neq H_2(\Pi)$ when $\Pi \neq \Omega$.

We associate each point Π of Γ that is different from Ω with the vector $v(\Pi)$ that points from $H_1(\Pi)$ to $H_2(\Pi)$, as well as the point of the "direction sphere R" of G that belongs to $v(\Pi)$; i.e., the intersection point of a fixed, naturally oriented sphere K that "represents R" with the ray that originates from the midpoint and is parallel to $v(\Pi)$. We call the degree of the thus-defined map of a naturally oriented Jordan manifold μ that lies in Γ and includes Ω in its interior onto R the "degree $a_{12}(\mu)$ of μ;" it is obviously independent of the choice of sphere that represents the direction sphere; it is independent of the choice of μ. χ is then a sphere around Ω, Π_1 is the intersection point of χ with the ray $\Omega\,\Pi_0$, where $\Pi_0 = \Pi$ is a point of μ, Π_t is the point on the line segment $\Pi_0\,\Pi_1$, that

---

[7]) The term "map" will always be understood to mean a single-valued and continuous map.

divides it with the ratio $t : (1 - t)$, and if one considers the maps α_t of μ onto R that are mediated by the vectors $v(\Pi_t)$, when t increases from 0 to 1, then the map of degree $a_{12}(\mu)$ goes continuously over to the map α_1, and thus with conservation of the degree. However, this combines the map of μ onto χ by means of projection of Ω, which, by the definition of "order" and our orientation prescription, has degree + 1, and the map of degree $a_{12}(\chi)$ of the sphere χ onto R. Since the degree gets multiplied when two maps are composed, it then follows that $a_{12}(\mu) = a_{12}(\chi)$, from which the independence of the degree a_{12} of the choice of μ is proved.

We call a_{12} the "index of coincidence" of H_1 and H_2 at Ω. – If one exchanges H_1 and H_2 then the vectors v go to their opposites, from which, $a_{21} = (-1)^n a_{12}$. – If one changes the orientation of Γ or that of G then a_{12} changes sign, since μ (R, resp.) then becomes a map of degree – 1; if one changes both indicatrices then a_{12} remains unchanged.

Let the points Π that differ from Ω be associated with a continuous family of simple, closed, continuously differentiable curve segments $s(\Pi)$ that link $H_1(\Pi)$ to $H_2(\Pi)$, and whose tangent vectors $w(\Pi)$ at $H_1(\Pi)$ change continuously with Π. Just like the vectors v, the vectors w also define an "index" for Ω: It is equal to a_{12} . If $H(\Pi, t)$ denotes the point that divides $s(\Pi)$ with the ratio $(1 - t) : t$ and $v(\Pi, t)$ denotes the vector that points from $H_1(\Pi) = H(\Pi, 1)$ to $H(\Pi, t)$ then as t runs continuously from 0 to 1 the vector field of $v(\Pi) = v(\Pi, 0)$ the goes continuously to $w(\Pi) = v(\Pi, 1)$. – The initial directions of the curves s can also be employed for the determination of a_{12} instead of the directions of the line segments that go from $H_1(\Pi)$ to $H_2(\Pi)$.

If Γ were taken to an oriented region Γ' by a topological map φ then two maps $H_1'(\Pi') = H_1'\varphi^{-1}(\Pi')$ and $H_2'(\Pi') = H_2'\varphi^{-1}(\Pi')$ are defined in it that have an isolated coincidence point at $\Omega' = \varphi(\Omega)$. Under φ, the natural orientation of a Jordan manifold μ goes to the natural orientation of $\mu' = \varphi(\mu)$ or its opposite, according to whether φ does or does not preserve the indicatrix. From the definition of indicatrix, it follows immediately that:

The index of an isolated coincidence point of two maps H_1, H_2 remains unchanged under a topological map φ of the domain of definition of H_1 and H_2 or picks up a factor of – 1, *according to whether φ preserves the indicatrix or inverts it.*

The analogous theorem is true for topological maps of *G:*

The index of coincidence is – at best, up to sign, as before – invariant under topological maps of the domain of definition.

Proof: Let f be a topological map of a neighborhood G of O onto the neighborhood G' of the point $O' = f(O)$, let k be a sphere about O that lies in the domain of definition of f, let μ be sufficiently small that $H_1(\mu)$ and $H_1(\mu)$ lie in the interior of k, and let $H(\Pi)$ be the intersection point of k with the ray $H_1(\Pi)$ $H_2(\Pi)$. We continuously carry over the vectors $H_1(\Pi) \to H(\Pi)$ that determine the coincidence index to the vectors $O \to H(\Pi)$ by letting their starting points run from $H_1(\Pi)$ to O in time 1, and in a uniform, rectilinear way. If we now choose a sphere about O as direction sphere R then the desired index of the degree of the map of μ onto R, which is mediated by the projection of $H(\mu)$ onto O, then equal to the "order of O relative to $H(\mu)$."

We transform the vectors that point from $fH_1(\Pi)$ to $fH_2(\Pi)$ that define the coincidence index of fH_1 and fH_2 under examination continuously into the vectors that point from O' to $fH(\Pi)$, during which we:

1. Let their endpoints run uniformly (in the sense of the geometry of G) to $fH(\Pi)$ along the images of the line segments $H_2(\Pi)$ $H(\Pi)$ that are provided by f.

2. Let their final points run uniformly to O along the images of the line segments $H_1(\Pi)O$.

If we choose a sphere around O' as the direction sphere R' then we recognize that the index in question is equal to the "order of $f(O)$ relative to $fH(\mu)$."

The assertion of the invariance of the coincidence index thus leads us back to the *invariance of the order of O relative to H*(μ) *under the topological map f*. We show the validity of this assertion when we prove it for a sequence $H(\mu)$ of uniformly approximating simplicial maps $H^{(\nu)}(\mu)$ and a sequence f of uniformly approximating simplicial maps f_λ ; thus, for the construction of the basic simplexes for f_1, as well as the image simplexes for f_λ and $H^{(\nu)}$, the Euclidian metric of G (G', resp.) is fundamental.

If $K = H^{(\nu)}$ is one of the approximations to H then image set $K(\mu)$ is composed of a finite number of $(n - 1)$-dimensional simplexes. We draw a ray through a point A that does not belong to $K(\mu)$ and that does not meet any $(n - 2)$-dimensional side of one of these simplexes. If p and p' are the numbers of sub-simplexes of K, which is the basis for the decomposition of μ, whose image simplexes intersect the ray in the positive (negative, resp.) sense then $p - p'$ is the order of A relative to $K(\mu)$. Let $q - q'$ be the corresponding number for ray that goes from A to infinity that meets no $(n - 2)$-dimensional side of an image simplex, and on which, no corner point lies on $K(\mu)$. We assert that $q - q' = p - p'$. – This is proved when it is shown that the corresponding number for each closed, oriented polygon is equal to 0. If W is such a polygon then from now on we can assume that position of its vertices has been modified in such a way that the extension of no side meets an $(n - 2)$-dimensional side of $K(\mu)$, At each vertex of W, we add the two rays that define the extension of the sides that come together there, and understand them to have the directional sense that is determined by the sides in question. The difference to be examined is increased by the order of the vertex under the addition of a ray, while it is decreased by this order number under the addition of the other rays; in total it thus remains unchanged. The system of line segments and rays that now lies before us decomposes into a finite number of directed lines. For each line, the difference of the numbers of positive and negative intersections is 0, which one sees when one considers the lines as composed of two rays. Thus, the desired number is, in fact, equal to 0. It is thus shown that the difference $q - q'$, which is determined by an arbitrary line segment from A to infinity, is equal to the order of A [8]). From this, it further follows that for a line segment from A to B this difference is equal to the order of A minus the order of B.

We now assume that K is fixed and show that when f_λ is a sufficiently good approximation to f, the order of O relative to $K(\mu)$ changes under the map f_λ at most by a

[8] Cf., Brouwer, Über Jordansche Mannigfaltigkeiten, pp. 323, as well as Hadamard, loc. cit.

sign. Since this change comes about under a reflection, for the sake of convenience, we can assume f preserves the indicatrix, and then we must prove that the sign of the order also remains unchanged. – First, let a simplicial approximation f'_λ of f be given; we must refine it, in a suitable way, to a map f_λ of the desired type. The simplicial decomposition of G that is based on f'_λ fulfills the following assumptions: O is a corner point of a basic simplex, such that one then has $f'_\lambda(O) = f(O) = O'$. Let t be a ray that emanates from O and meets no $(n-2)$-dimensional side of $K(\mu)$. Let each intersection point of t with $K(\mu)$ be the interior points of a basic simplex. If $P_1, P_2, \dots, P_r$ are the points of μ for which $K(P_\rho)$ lies on t then we can assume that $K(P_{\rho_1}) \neq K(P_{\rho_2})$ for $\rho_1 \neq \rho_2$, since that can be arranged by an arbitrarily small modification of K. Let s_ρ be the basic simplexes of μ that contain the P_ρ, and let S_ρ be the basic simplexes of G that contain the $K(P_\rho)$, which are chosen such that the points $K(P_\rho)$ lie in their interiors. Let the S_ρ be so small that no points of $K(\mu)$ besides those of $K(s_\rho)$ lie at each S_ρ. $K(s_\rho)$ divides S_ρ into two parts S_ρ^1, S_ρ^2; let A_ρ^1, A_ρ^2 be two points of t that lie in the interior of S_ρ^1 (S_ρ^2, resp.), and let the direction $A_\rho^1 \to A_\rho^2$ emanate from O. – We now thicken the current simplicial decomposition of G to a decomposition ζ_λ with the following properties:

1. The $K(P_\rho)$ lie in the interior of the basic simplexes.

The simplicial map f_λ associated with ζ_λ approximates f so well that:

2. If t_ρ denotes the line segment $A_\rho^1 A_\rho^2$, w_ρ, the circumference of S_ρ, and μ_ρ, the subset of μ for which $K(\mu_2)$ lies in S_ρ then $f_\lambda(t_\rho)$ is disjoint from $f_\lambda(w_\rho)$ and $f_\lambda(\mu - \mu_\rho)$, while $f_\lambda(t - \sum_\rho t_\rho)$ is disjoint from $f_\lambda(\mu - \sum_\rho \mu_\rho)$,

and that:

3. If u_ρ is the circumference of S_ρ^1 then the orders of $f_\lambda(A_\rho^1)$ and $f_\lambda(A_\rho^2)$ relative to $f_\lambda(u_\rho)$ are equal to the orders of $f(A_\rho^1)$ ($f(A_\rho^2)$, resp.) relative to $f(u_\rho)$.

If these conditions are fulfilled then the only intersection points of the line segment $f_\lambda(t)$ with $f_\lambda K(\mu)$ are the intersection points of the $f_\lambda(t_\rho)$ with the $f_\lambda K(\mu_\rho)$, and for each ρ the difference in the numbers of positive and negative intersection points is equal to the order of $f(A_\rho^1)$ relative to $f(u_\rho)$ minus the order of $f(A_\rho^2)$ relative to $f(u_\rho)$. The indicatrix of u_ρ is thus established through that of μ_ρ.

Now, the order of A_ρ^2 relative to u_ρ is equal to 0, that of A_ρ^1 relative to u_ρ is equal to ± 1, according to whether the circulation in $K(P_\rho)$ is positive or negative, respectively. With that, our theorem comes down to a simple special case, namely, the assertion that under the topological map f the order of a point relative to the *Jordan manifold* u_ρ

changes by ± 1 for the interior points and remains 0 for the exterior ones. The validity of this assertion, however, follows from well-known theorems of Brouwer [9]).

Thus, the topological invariance of the order of a point relative to $H(\mu)$ – in any event, ignoring the sign – is proved, and likewise the same is true for the coincidence index of two maps. This fact may also be expressed by saying that these numbers are independent of the coordinate system for a correct consideration of the sign.

An application of the previous results that will of use to later is the following one: If H_1, H_2 are maps of Γ onto a region G that belongs to a sphere then for the determination of the coincidence index it is irrelevant whether we employ the great circle arc or the circular arc through an arbitrary fixed point A of the sphere as the curve $s(\Pi)$ that links $H_1(\Pi)$ to $H_2(\Pi)$, through whose starting direction the coincidence index is to be determined (cf., *supra*), as well as whether we carry over the Cartesian coordinate system in G that we need for the determination of the index by stereographic or any other projection of a planar space onto G, as long as the indicatrix remains the same.

The following consideration leads to still more important applications:

If Γ is identical with G and H_1 is the identity map then Ω is an isolated *fixed point* of H_2 and $a = a_{12}$ is its "index;" the foregoing statements remain valid, but one must observe that if $\Gamma = G$ is a topological map that inverts the indicatrix then this inversion will be present in Γ, as well as G, so the sign of a_{12} will *not* change; the theorem that emerges is thus:

The index of a fixed point is a topological invariant.

If $v(\Pi)$ is a continuous vector field in Γ with an isolated singularity given at Ω then we understand the "index" a of this singularity to mean the degree of the map of a sphere χ that surrounds Ω onto the direction sphere, which will be established by the vectors of the field that are brought to χ. It is equal to the index of the fixed point Ω of that map H_2 that displaces each point Π in the direction of its vector $v(\Pi)$ along a line segment $\varepsilon(\Pi)$, where $\varepsilon(\Pi)$ is a continuous function that vanishes at Ω and is positive everywhere else. If Γ were subjected to a differentiable map f with a non-vanishing functional determinant then the vector field would go to a new vector field $v' = fv$ and H_2 would go to fH_2 under this map. We determine the index of the fixed point $f(\Omega)$ under the map fH_2 by the starting direction $f\,v(\Pi)$ of the curve $s(\Pi)$ that leads from $f(\Pi)$ to $fH_2(\Pi)$, which is the image of the line segment $\overline{\Pi H_2(\Pi)}$. It is, on the one hand, equal to a, and on the other hand, equal to the index of the singularity $f(\Omega)$ of the vector field fv. It is thus shown that:

The index of a singularity of a continuous vector field does not change under a map with a non-vanishing functional determinant [10]).

The most important application of this theorem for us is the following one:

---

[9]) Brouwer, Über Jordansche Mannigfaltigkeiten, §§ 4, 5.

[10]) This fact is also easy to prove without our theorem on the invariance of the index of a fixed point.

By a "model" for a closed, n-dimensional manifold μ, we understand this to mean a point set m that lies in an $(n + 1)$-dimensional Euclidian space, onto which m is mapped in a single-valued and continuous way such that this map F is also uniquely invertible in the small; i.e., that there is a neighborhood U_P around each point P that is topologically related to its image set $F(U_P)$. – We consider models that are "hypersurfaces;" this means that to each point P of μ there exists an n-dimensional planar tangent space ϑ_P on $F(U_P)$ that varies continuously with P. There, for each P there is a neighborhood U_P^* that is mapped topologically onto ϑ_P by perpendicular projection of $F(U_P^*)$ by means of the map $S\,F(U_P^*)$.

Let a continuous field of tangent vectors with finitely many singularities be given on m; i.e., with finitely many omissions $P_1, P_2, \ldots, P_r$, there is at each point P of μ, a vector $v(P)$ associated with $F(P)$ that is tangential to $F(U_P)$ and varies continuously with P. Each of the points $F(P_\rho)$ $(\rho = 1, \ldots, r)$ possesses a well-defined index a_ρ, which can be determined by – say – projecting the coordinate system of ϑ_P onto $F(U_P^*)$. $(r \geq 0.)$

We define a continuous function $\varepsilon(P)$ on μ that vanishes at the points P_ρ and is positive everywhere else. We associate each point P with that point P_t' of the tangent ray that is determined by $v(P)$, and which has the distance $t \cdot \varepsilon(P)$ from $F(P)$, where t is a parameter. There is a number $t_0 > 0$ such that for all P and for $0 < t \leq t_0$ the point P_t' lies in $S\,F(U_P^*)$; let it be the image of the point P_t under the map $S\,F$. The map $P_t = h(P, t)$ is single-valued and continuous on all of μ and for all t $(0 < t \leq t_0)$ and possesses the r fixed – i.e., independent of t – points P_ρ with the indices a_ρ.

If there is now a second model $m' = F'(\mu)$ that is likewise a hypersurface, so it is continuous and possesses a planar tangent space ϑ_P', then we choose t sufficiently small that none of the chords $F'(P)\,F'(P_t)$ to $F'(P)$ are perpendicular to $F'(U_P')$, and project these chords onto the tangent space ϑ_P' in the direction of the normal at $F'(P)$. A continuous tangent vector field on m' will then be generated that is singular only at the points $F'(P_\rho)$, and the indices of the singularities are again equal to the indices a_ρ of the fixed points of the transformation h that is defined in μ. – It is thus shown that:

If there is a continuous, tangent vector field on m with r singularities whose indices are $a_1, \ldots, a_r$ *then there is a continuous, tangent vector field on any other model m′ for the same manifold* μ *that has the same number of singularities and the same indices.* $(r \geq 0)$ [11])

The coincidence locus of two maps onto the sphere will be of particular interest for us later on, and is defined as follows:

Let Γ be an n-dimensional, continuously differentiable, orientable surface patch that lies in $(n + 1)$-dimensional space. At the points of Γ, let there be given two continuous

[11]) In the event that the relation between m and m' that is mediated by μ fulfills suitable differentiability assumptions, the proof of our theorem is obviously essentially simpler to complete than was done in the text.

distributions $\mathfrak{C}_1$ and $\mathfrak{C}_2$ of $(n + 1)$-dimensional vectors with the property that all of the vectors $\mathfrak{C}_1$ have negative directions relative to Γ, and the vectors $\mathfrak{C}_1(\Omega)$ and $\mathfrak{C}_2(\Omega)$ coincide for one and only one point Ω of Γ. The maps H_1 and H_2 of Γ onto the direction sphere R that are mediated by $\mathfrak{C}_1$ and $\mathfrak{C}_2$ thus agree at Ω and only there; let the associated index be a_{12} .

At each point Ω of Γ that differs from the point Π, the vectors $\mathfrak{C}_1(\Pi)$, whose opposite vector is $\bar{\mathfrak{C}}_1(\Pi)$, and $\mathfrak{C}_2(\Pi)$ determine a half-plane that, since $\mathfrak{C}_1(\Pi)$ is not tangential, cuts the planar tangent space ϑ_P at a vector $v_{12}(\Pi)$. The field of v_{12} , which we call the *tangent field generated by* $\mathfrak{C}_1$, $\mathfrak{C}_2$, has a singularity at Ω, which we will show has an index of a_{12} :

Let R be represented by a sphere that contacts ϑ_Ω at Ω, whose center A lies on the positive normal to Γ, and thus, in a part of space that does not belong to $\mathfrak{C}_1(\Omega)$. We perform the central projection Z from A onto ϑ_Ω on the half that lies between ϑ_Ω and the space through A that is parallel to it; thus (cf., *supra*), the positive indicatrix of R goes over to the positive indicatrix of ϑ_Ω, and we obtain two maps $Z\,H_1 = H_1'$ and $Z\,H_2 = H_2'$ of Γ onto ϑ_Ω whose coincidence at Ω likewise has the index a_{12} . The vector that points from $H_1'(\Pi)$ to $H_2'(\Pi)$ is parallel to the vector $w(\Pi)$, at which the ϑ_Ω that is constructed from the half-plane in Π that includes $\mathfrak{C}_1(\Pi)$, $\bar{\mathfrak{C}}_1(\Pi)$, $\mathfrak{C}_2(\Pi)$, $v_{12}(\Pi)$ is intersected; we take the $w(\Pi)$, which we can therefore use for the determination of a_{12}, from a neighborhood of Ω continuously over to the vectors $w'(\Pi)$: We take the $\mathfrak{C}_1(\Pi)$, while preserving its initial point, over to the vector parallel to $\mathfrak{C}_1(\Omega)$, by means of the angle defined by the directions $\mathfrak{C}_1(\Pi)$ and $\mathfrak{C}_1(\Omega)$, which happens in the vicinity of Ω without having to go through ϑ_Ω .

Throughout this process, we observe the change that the vector $v(\Pi, t)$ suffers, at which ϑ_Ω will be intersected by the half-plane $\mathfrak{C}_1(\mathrm{P}, t)$, $\bar{\mathfrak{C}}_1(\Pi, t)$, $v_{12}(\Pi)$, where $\mathfrak{C}_1(\Pi, t)$ is the moving vector: The $v(\Pi, t)$ will be continuously transformed from $w(\Pi)$ to the vectors $w'(\Pi)$ that arise from the $v_{12}(\Pi)$ under parallel projection onto ϑ_Ω in the direction of $\mathfrak{C}_1(\Omega)$. The degree of the map of an $(n - 1)$-dimensional manifold surrounding Ω in ϑ_Ω onto the direction sphere r of ϑ_Ω that is mediated by w' is therefore, on the one hand, equal to a_{12}, and on the other, equal to the index of Ω relative to the field of v_{12}, and with this, the assertion is proved. One can therefore replace the determination of a_{12} with that of the index of Ω relative to the field of v_{12} .

If we subject the $(n + 1)$-dimensional neighborhood of Ω to a single-valued, continuously-differentiable map φ with a non-vanishing functional determinant then the $\mathfrak{C}_1(\Pi)$,$\mathfrak{C}_2(\Pi)$, $v_{12}(\Pi)$ go to $\mathfrak{C}_1'$, $\mathfrak{C}_2'$, v_{12}'; we then choose the orientation of the image Γ′ of Γ in such a way that the $\mathfrak{C}_1'$ is also negatively directed. The $\mathfrak{C}_1'$ and $\mathfrak{C}_2'$ have an isolated coincidence point at Ω′, whose index a_{12}' is equal to the index of Ω′ relative to the tangential vector field $v_{12}' = \varphi(v_{12})$ to Γ′, which is obviously identical with the tangent field that is generated by $\mathfrak{C}_1'$ and $\mathfrak{C}_2'$ (since the linear independence of the vectors $\mathfrak{C}_1$, $\mathfrak{C}_2$, v_{12} at each point is preserved by the map). Now, since the index of Ω relative to v_{12} does not change under the map, as was proved above, the same is true for the index a_{12} of the coincidence of $\mathfrak{C}_1$ and $\mathfrak{C}_2$ at Ω. – Conversely, if we impose the aforementioned assumption on the positive indicatrix in such a way that the map of Γ onto Γ' has the degree + 1 then we must distinguish whether the functional determinant D of φ is positive or negative; i.e., whether φ preserves the indicatrix or inverts it: In the first case, the negative side of Γ goes to the negative side of Γ', so one has $a'_{12} = a_{12}$. In the second case, the $\mathfrak{C}'_1$ and $\mathfrak{C}'_2$ are then positively directed, and the coincidence index $(-1)^{n+1}a'_{12}$ of the negatively-directed maps, which are mediated by the vector distributions $\bar{\mathfrak{C}}'_1$, $\bar{\mathfrak{C}}'_2$ that are diametrically opposite to $\mathfrak{C}'_1$, $\mathfrak{C}'_2$, is equal to the index of Ω' relative to the tangential vector field that is generated by $\bar{\mathfrak{C}}'_1$ and $\bar{\mathfrak{C}}'_2$, which is diametrically opposite to v'_{12}, and thus equal to $(-1)^{n+1}a_{12}$, since Ω has the index a_{12} relative to v'_{12}; from $(-1)^{n+1}a'_{12} = (-1)^n a_{12}$, it follows that $a'_{12} = -a_{12}$, and one sees that $a'_{12} = \pm a_{12}$, according to whether $D > 0$ or $D < 0$.

§ 2.

The coincidence number of two maps of a closed, two-sided manifold onto the sphere

Let the closed, two-sided, oriented, n-dimensional manifold μ be mapped onto the n-dimensional sphere $\mathfrak{K}$ that is given by the equation $\sum_{\nu=1}^{n+1} x_\nu^2 = 1$ by the maps f_1 and f_2. f_1 and f_2 shall agree only at finitely many points P_χ ($\chi = 1, \ldots, k$) of μ. The sum of the indices of these coincidences $\sum_{\chi=1}^{k} a_{12}^{(\chi)} = I_{12} = (-1)^n I_{21}$ is called the "coincidence number" of f_1 and f_2.

We construct two essentially simplicial approximating maps for f_1, f_2 with the same coincidence points P_χ (indices $a_{12}^{(\chi)}$, resp.) :

Let ζ be a simplicial decomposition of μ, for which the P_χ are interior points of the simplexes S_χ, and which is sufficiently dense that the images $f_\lambda(S_\chi)$ and $f_2(S_\chi)$ can be separated by spheres $\mathfrak{H}_\chi$ [$\chi = 1, \ldots, k$] whose spherical radii are smaller than $\pi/2$, such that one can connect any two points in them by a unique circular arc, and whose set union still leaves a region $\mathfrak{N}$ of $\mathfrak{K}$ free. If one now lets μ' denote the subset of μ that comes about by omitting the interior region of S_χ from μ and m is the minimum of the spherical distances $\overline{f_1(P) f_2(P)}$ for all points P of μ' for all points P of μ' then we carry out a subdivision ζ' of ζ that is sufficiently dense that for each of the two associated simplicial approximations δ_1, δ_2 of f_1, f_2, the spherical distance $\overline{\delta_i(P) f_i(P)} < m/2$ in all of μ, and that the images $\delta_1(S_\chi)$ and $\delta_2(S_\chi)$ also lie completely in the interiors of the $\mathfrak{H}_\chi$. Thus, the natural coordinates are chosen for the representation of the simplicial approximations as coordinates in $\mathfrak{K}$; i.e., for the center of mass of $n + 1$ masses at the corners x_1^i, x_2^i, …,

x^i_{n+1} [i = 1, …, n – 1] of a spherical complex $\mathfrak{S}$, one considers the point of $\mathfrak{S}$ whose coordinates ξ_ν behave like the n + 1 numbers $\sum_{i=1}^{n+1} x^i_\nu m^i$; the simplicial maps of μ into all of μ will be defined continuously on the basis of these coordinates [12]). Due to the relations $\overline{\delta_i f_i} < m/2$ [i = 1, 2], $\overline{f_1 f_2} \geq m$, δ_1 and δ_2 have no coincidence point on μ' and thus, in particular, on the boundaries of the S_χ ; thus, there can be infinitely many coincidence points in the interior of the S_χ. In order to eliminate them, we replace δ_2 with a map δ_2': Let $\delta_2' = \delta_2$ on μ'. At each S_χ, we consider the pencil of rays that originate from P_χ and the continuous function α, which vanishes only at P_χ and is positive everywhere else, which, at each boundary point P_{R_χ} of S_χ, is equal to the distance $\overline{\delta_1(P_{R_\chi})\delta_2(P_{R_\chi})}$ and decreases on the ray $P_{R_\chi}P_\chi$ in proportion to the distance from P_χ. We now associate the point P of the ray $P_{R_\chi}P_\chi$ with that point $\delta_2'(P)$ of $\mathfrak{K}$ that has the spherical distance α from $\delta_1(P)$, and for which, under stereographic projection of the antipodal point p_χ from the point $\delta_1(P_\chi)$ the vector $\delta_1(P) \to \delta_2'(P)$ is parallel to the vector $\delta_1(P_{R_\chi}) \to \delta_2(P_{R_\chi})$. δ_2' is then continuous in all of μ and identical with δ_2 in μ', and thus, simplicial. The images $\delta_2'(S_\chi)$ likewise lie completely in the sphere $\mathfrak{H}_\chi$; δ_2' agrees with δ_1 only at the P_χ. The index of this coincidence at P_χ is $a_{12}^{(\chi)}$; it may then be determined by basing the Euclidian coordinate system that is provided by the stereographic projection from p_χ by means of the initial directions of the circular arc $\delta_1(P_{R_\chi})\,\delta_2(P_{R_\chi})$ that belongs to the boundary points P_{R_χ}. Inside of $\mathfrak{H}_\chi$, one can continuously transform the totality of these arcs by a uniform motion of their starting and ending points into the totality of great circular arcs $f_1(P_{R_\chi})\,f_2(P_{R_\chi})$, and no arc will degenerate into a point under this transformation, due to the inequality $\overline{\delta_i f_i} < \frac{1}{2}\,\overline{f_1 f_2}$ [i = 1, 2]. As a result, the starting directions of the arcs $\delta_1(P_{R_\chi})\,\delta_2(P_{R_\chi})$ and the arcs $f_1(P_{R_\chi})\,f_2(P_{R_\chi})$ deliver maps of equal degree onto the direction sphere of the coordinate system; i.e., the coincidence index of δ_1 and δ_2' at P_χ is $a_{12}^{(\chi)}$.

δ_2 and δ_2' have equal degree, since the region $\mathfrak{N}$ will be covered by both maps of the same points of μ. Thus, if g_1, g_2 are the degrees of f_1, f_2 then they are also the degrees of the simplicial approximations, and therefore also those of δ_1 and δ_2'.

We choose a point O of $\mathfrak{N}$ that does not lie on the boundary of an image simplex of δ_1 or δ_2 ; since $\mathfrak{N}$ will be covered by both maps only by points of μ' and there one has $\delta_2' \equiv \delta_2$, both of these maps are simplicial for all points of m that go to points of the neighborhood of O under one of the maps δ_1 and δ_2'.

[12]) For Brouwer, Über Abbildung von Mannigfaltigkeiten, § 1, the simplicial maps of μ will then be defined only for those basic simplexes of μ whose vertex images belong to the same element of $\mathfrak{K}$; thus, it seems to me that the simplicial maps onto the sphere in § 3 of the Brouwer paper are based on the modification of the definition that was employed in the text above.

Now, we associate the points P of μ with those vectors that are tangential to the circular arc from O leading to $\delta_2'(P)$ over $\delta_1(P)$ [12a]). This association is undetermined at the following points P^*, and only at those points:

1. The points P_χ.
2. The points $A_1, \ldots, A_r$ for which one has $\delta_1(A_r) = O$.
3. The points $B_1, \ldots, B_s$ for which one has $\delta_2'(B_\sigma) = \delta_2(B_r) = O$.

Each of these points P^* lies in the interior of a basic simplex $S^*(P^*)$ of the decomposition ζ'. If S^{**} is sub-simplex of S^* [13]) that contains only one of these singular points then the periphery (Umfang) of S^{**}, through the vectors that are associated with its points, will be mapped onto the direction sphere of the tangential planar space to $\mathfrak{K}$ at $\delta^1(P^*)$ by means of stereographic projection. The degree of this map is called the "index" $I(P^*)$. We determine $\sum_{P^*} I(P^*)$ by the rule:

1. It is $I(P_\chi) = a_{12}^{(\chi)}$, so $\sum_{P^*=P_\chi} I(P^*) = I_{12}$.

2. The index of the singularity $\delta_1(A_\rho) = O$ of the vector field that is defined by the vectors at the points of $\delta_1(S^{**}(A_\rho))$ that are associated with the points of $S^{**}(A_\rho)$ is + 1 [14]); thus, $I(A_\rho) = \pm 1$, according to whether $\delta_1(S^{**}(A_\rho))$ is a positive or negative image simplex of $S^{**}(A_\rho)$. However, since the degree g_1 of f_1 is the number of times that O positively covers the image simplex minus the number of times that O covers it negatively, one has $\sum_{P^*=A_\rho} I(P^*) = g_1$.

3. The index of the singularity $\delta_1(B_\rho)$ of the vector field that is defined by the vectors at the points of $\delta_1(S^{**}(B_\rho))$ that are associated with the points of $S^{**}(A_\rho)$ is ∓ 1, according to whether the simplexes $\delta_1(S^{**}(B_\rho))$ and $\delta_2(S^{**}(B_\rho))$ have equal or unequal signs [14]); thus, $I(B_\sigma) = \mp 1$, according to whether $\delta_2(S^{**}(B_\rho))$ is a positive or negative image simplex. Thus, $\sum_{P^*=B_\rho} I(P^*) = -g_2$.

Thus, one has: $\underline{\sum_{P^*} I(P^*) = I_{12} + g_1 - g_2}$.

We now determine $\sum_{P^*} I(P^*)$ in yet another way:

We choose an arbitrary point Q on $\mathfrak{K}$ that does not belong to any boundary of an image simplex of δ_1, but is covered by at least one such point. It will be positively (negatively, resp.) covered by δ_1 with p' image simplexes $T_1, \ldots, T_p$; $T_1', \ldots, T_{p'}'$; one has $p - p' = g_1$. As mediating maps, for the sake of introducing Euclidian coordinate system for the determination of the indices of P^*, we use basic simplexes for this whose images are the T and T' under stereographic projection from the antipodal point $\bar{Q}$ to Q, and all

[12a]) The rest of this paragraph is merely a modification of the considerations of Brouwer in the treatise "Über Abbildung von Mannigfaltigkeiten."

[13]) An A and a B can then belong to the same S^*.

[14]) Brouwer, Über Abbildung von Mannigfaltigkeiten, § 3.

of the other basic simplexes are stereographically projected from Q. Thus, the boundary of each T_i (T'_j, resp.) will be subjected to a map L_i (L'_j, resp.) onto the direction sphere of ϑ_Q and a map $\bar{L}_i$ ($\bar{L}'_j$, resp.) onto the direction sphere of $\vartheta_{\bar{Q}}$; if the degrees of these maps are:

$$c_i\,,\ c'_j\,; \qquad \bar{c}_i\,,\ \bar{c}'_j\,; \qquad [i = 1, \ldots, p;\ j = 1, \ldots, p']$$

then one has [14]):

$$c_\nu + \bar{c}_\nu = c'_\nu + \bar{c}'_\nu = 1 + (-1)^n.$$

Now, $\sum_{P^*} I(P^*)$ is, however, equal to the sum of the mapping degrees of all peripheries of the basic simplexes of m onto the corresponding direction spheres, and in this summation the two contributions from each $(n-1)$-dimensional face cancel, as long as they do not correspond to the face of a T or a T', and will thus be mapped onto two different direction spheres. One thus has:

$$\underline{\sum_{P^*} I(P^*)} = \sum_i c_1 - \sum_j c'_j + \sum_i \bar{c}_i - \sum_j \bar{c}'_j$$

$$= \sum_{i=1}^{p} (c_i + \bar{c}_i) - \sum_{j=1}^{p'} (c'_j + \bar{c}'_j) = (p - p')\,(1 + (-1)^n) = \underline{g_1 + g_1(-1)^n}\,.$$

Earlier, we found that $\sum_{P^*} I(P^*) = I_{12} + g_1 - g_2$, so [15]):

$$\underline{I_{12} = (-1)^n \cdot g_1 + g_2}\,.$$

§ 3.

On the *curvature integra* of an *n*-dimensional model

Let a hypersurface m be given, which is [cf., § 1] a model of an n-dimensional, closed, two-sided, oriented manifold μ. We choose the positive indicatrix of m such that the map F of U_P has the degree + 1, and determine the positive normal direction from the previously given prescription. – If F is one-to-one in all of μ then we call m a "simple" or "Jordan" model; the positive normal direction of not necessarily directed into it, but depends upon the orientation of μ. –

We understand $C(m)$ – the "*curvature integra*" of m – to mean degree of the map of μ onto the direction sphere R that is mediated by the negative normals to m that belong to

[15]) In the case $n = 2$, the formula $I_{12} = g_1 + g_2$ may be confirmed function-theoretically: If $F(z, w)$ is a complete rational irreducible function in z and w of degree g_1 (g_2, resp.), μ, the Riemann surface of the structure defined analytically by $F(z, w) = 0$, then, by the associated consideration of the infinitely distant points the equation $F(z, z) = 0$ has precisely $g_1 + g_2$ roots – assuming that one does not have $F \equiv k(z - w)$, so that $F(z, z) \equiv 0$.

the points P. If one changes the orientation of μ then the normals must be replaced with their diametrically-opposite vectors, so in order to obtain the new *curvatura integra* $C'(m)$ one must map:

1. μ onto itself with degree – 1.
2. μ onto R with degree C.
3. R onto itself with degree $(-1)^{n+1}$.

This yields:

$$C' = -1 \cdot C \cdot (-1)^{n+1} = (-1)^n C.$$

$C(m)$ is thus independent of the orientation of μ for even n, and for odd n, its sign depends upon it. – Instead of the normals, from the theorem Poincaré-Bohl [cf., Introduction], one may use any other vectors that vary continuously with P and are directed towards the negative side of $F(U_P)$ for the determination of C. For the time being, as was explained in the previous paragraphs on all of μ, we employ a simplicially modified approximation γ_1 to the normal map.

We now examine how $C(m)$ is obtained for certain types of transitions from m to other models for m:

First, let an element E that contains m in its interior be subjected to a continuously differentiable map φ with non-vanishing functional determinant:

$$x'_\nu = \varphi_\nu(x_1, \ldots, x_{n+1}), \qquad [n = 1, 2; \ldots, n + 1],$$

$$D(x_1, \ldots, x_{n+1}) = \frac{\partial(\varphi_1, \cdots, \varphi_{n+1})}{\partial(x_1, \cdots, x_{n+1})} \neq 0.$$

m then goes over to the model $m' = \varphi(m)$. From our assumption, the orientation of m' is to be chosen in such a way that φ takes the positive indicatrix of m to the positive indicatrix of m'; the vectors $\mathfrak{C}_1$ on m that are mediated by the γ_1 and point in the negative direction to negatively (positively, resp.) directed vectors $\mathfrak{C}'_1$ on m' according to whether φ does or does not change the indicatrix of E; i.e., whether $D > 0$ or $D < 0$. As a result, the degree of the map γ'_1 of μ onto R that is mediated by the $\mathfrak{C}'_1$ is therefore $(\pm 1)^{n+1}C'$, where $C' = C(m')$. We now choose a point A of R and bring to each point of m the parallel vectors $\mathfrak{C}_2$ that correspond to A, which relate to μ by the map $\gamma_2(P) = A$ of degree 0 onto R. $\mathfrak{C}_1$ and $\mathfrak{C}_2$ agree at (at most) finitely many points, and one has (from § 2) $I_{12} = (-1)^n C$. Under φ, the $\mathfrak{C}_2$ go to vectors $\mathfrak{C}'_2$, by means of a map γ'_2 of μ onto R of degree c'_2, and the coincidence number of γ'_1 and γ'_2 is:

$$I'_{12} = (-1)^{n+1} C + c'_2.$$

From § 1 (last paragraph), however, one has $I_{12} = \pm I'_{12}$, so:

$$(\pm 1)^n C' = C \mp c_2'(-1)^n .$$

Now, the totality of *all* parallel vectors that correspond to A and are attached to points of E go to a vector distribution Δ' that is single-valued and continuous on all of $E' = \varphi(E)$ and associated with the $\mathfrak{C}_2'$. Inside of E', since E' is an element, m' may be contracted to a point Q by a single-valued and continuous deformation. Thus, when one, at each moment of the process of deformation, associates the point P of μ with the point of R that belongs to the vector Δ' that is attached to the momentary image point of P, the map γ_2' goes to a map that associates all points of μ with the same point $\Delta'(Q)$. Thus, $c_2' = 0$, $C' = (\pm 1)^n C$, and we have obtained the theorem:

Under a continuously differentiable map with non-vanishing functional determinant D of an element included in m, C(m) always changes by the factor – 1; *indeed, this change comes about when and only when D* < 0 *and n is odd.*

The question now remains whether one can omit the differentiability assumption. The following theorem teaches us that this is possible, at least, for a particularly simple special case:

The boundary m of an element has the curvature integra $(\pm 1)^n$.

Proof: If m has the "natural" orientation then the degree of the map that is mediated by taking the interior normals N to m onto R is $(-1)^{n+1}C$. This field of N projects m continuously onto a neighboring parallel surface m_1 of m, and may be, by contraction of m_1 to an interior point, deformed into a distribution of vectors that point to this point, which has the index $(\pm 1)^n$. Thus, $C = 1$ and for an arbitrary orientation of m, one has $C = (\pm 1)^n$. (Concerning the sign, one can confer the first paragraph of this section.)

We now extend the class of the maps in question: φ no longer needs to be defined on all of E, but only on a neighborhood of m – i.e., in the set union of the neighborhoods of all points of m. Everywhere else, all assumptions and notations remain unchanged. One now has:

$$(\pm 1)^n C' = C \mp c_2'(-1)^n .$$

Along with the vectors $\mathfrak{C}_2$ that belong to A, we also consider the vectors $\bar{\mathfrak{C}}_2$ that are diametrically opposite to them. They go to the vectors $\bar{\mathfrak{C}}_2'$, which is mediated by a map $\bar{\gamma}_2'$ of degree $\bar{c}_2'$, and one has:

$$(\pm 1)^n C' = C \mp c_2'(-1)^n ,$$

so $\bar{c}_2' = c_2'$. Since the vectors $\bar{\mathfrak{C}}_2$ and $\bar{\mathfrak{C}}_2'$ are diametrically opposite to each other, one has, however, that $\bar{c}_2' = (-1)^{n+1} c_2'$; for even n one then also has that $c_2' = 0$, so $C' = C$.

We will show that for *odd n* one can have $c_2' \neq 0$:

We perform a transformation through reciprocal radii with a point Q that does not lie on m for its center; this map fulfills our assumption. φ is then continuously differentiable with a negative functional determinant at all points that differ from Q. φ takes m to a model m' and the infinitely distant points to the point Q. Let the order of Q relative to m' be q. (We understand this to mean, as before, the degree of the map of μ onto k that is provided by projection of the model m' for Q onto a sphere k around Q; it is usually easy to see that it is equal to the order of Q relative to m.) Under φ, the totality of all vectors in the domain of definition for φ that correspond to points A of R goes to a vector field Δ' with the isolated singularity Q; its index is + 2. φ may then be represented as follows: One first projects the entire space stereographically onto an $(n+1)$-dimensional sphere K_{n+1} that lives in $(n+2)$-dimensional space and contacts Q; thus, the vectors that belong to A go to a single-valued, continuous vector field that is singular only at the antipodal point Q^* to Q, and thus has index +2 at Q^* [14]). One then projects K_{n+1} stereographically from Q onto the contact space at Q^* and then back to the original space perpendicularly. Thus, Q^* goes to Q with conservation of the index.

In order to determine c_2', we let the points of m' go uniformly to the rays that emanate from Q onto a sphere k around Q in time 1, and then observe the vectors of Δ' at the running points at each moment: At first, they are the vectors $\mathfrak{C}_2'$, and at the end, they are vectors that sit on k. The map of m onto R that is provided by the latter is composed of the projection of m' from Q onto k, which has the degree q, and the map of k onto R of degree 2 – which makes $c_2' = 2q$ – so:

$$C + C' = 2q.$$

In particular, if m and m' are simple, and if Q lies in the interior of m then Q also lies in the interior of m', since Q and the infinitely distant point through m will be separated from each other, which then makes $q = \pm 1$, $C + C' = \pm 2$, and one sees that:

If, for odd n, the simple model m possesses a curvature integra that is invariant in its absolute value under maps of the type in question then it is equal to ± 1.

We now show that for each odd $n \geq 3$ there is a simple model with $C(m) = 0 \neq \pm 1$. We consider the one-parameter group of motions $x' = f(x;\ \alpha)$ of the $(n+1)$-dimensional space that is given by the equations:

$$\begin{aligned} x'_{2\nu-1} &= f_{2\nu-1}(x_1, \ldots, x_{n+1};\ \alpha) = \quad \cos\alpha \cdot x_{2\nu-1} + \sin\alpha \cdot x_{2\nu} \\ x'_{2\nu} &= f_{2\nu}\ \ (x_1, \ldots, x_{n+1};\ \alpha) = -\sin\alpha \cdot x_{2\nu-1} + \cos\alpha \cdot x_{2\nu} \end{aligned}$$
$$\left[\nu = 1, \cdots, \frac{n+1}{2}\right].$$

The trajectory of any point is a circle; two such circles are disjoint. Then, since $f(f(x;\ \alpha);\ \beta) = f(x;\ \alpha+\beta)$, it follows from the fact that $f(x;\ \alpha) = f(y;\ \beta)$ that for *each* γ one has: $f(x;\ \alpha+\gamma) = f(y;\ \beta+\gamma)$; i.e., that the trajectory circles of x and y coincide. The single fixed point of the motions is the null point; at every other point the vector of the direction of motion is distinguished.

The (n–1)-dimensional sphere K_0: $\sum_{\nu=1}^{n}(x_\nu - 2)^2 = 1$, $x_{n+1} = 0$ does not contain the null point. Furthermore, no trajectory circle will cut it at two points; if one had $y = f(x;\ \alpha)$ with $\alpha \neq 0$ for two points x and y of the sphere then it would follow from:

$$\begin{aligned} y_n &= \cos\alpha\cdot x_n + \sin\alpha\cdot x_{n+1} \\ y_{n+1} &= -\sin\alpha\cdot x_n + \cos\alpha\cdot x_{n+1} \\ x_{n+1} &= y_{n+1} = 0 \end{aligned}$$

that either $x_n = 0$, which is not consistent with the equation of the sphere, or that $\alpha = \pi$ and $y_\nu = -x_\nu$ for $\nu = 1, 2, \ldots, n+1$. However, from the fact that:

$$\sum_{\nu=1}^{n}(x_\nu - 2)^2 = 1 \qquad \text{and} \qquad \sum_{\nu=1}^{n}(-x_\nu - 2)^2 = 1$$

it would follow by addition that $8n + 2\sum x_\nu^2 = 2$, $\sum x_\nu^2 = 1 - 4n < 0$.

Likewise, one obtains each sphere K_a into which K_0 goes under our motions. Thus, the surface r that is described by K_0 – i.e., the totality of all points $f(x;\ \alpha)$, for which $f(x, 0)$ lies on K_0 – has the property that $f(x;\ \alpha) = f(y;\ \beta)$, when and only when $x = y$, $\alpha = \beta$. r is thus a simple model of the product manifold that is composed of a circle and an (n – 1)-dimensional sphere [16]).

From the Poincaré-Bohl theorem, in order to determine $C(r)$ we can employ the r tangential direction vectors of the motion. The map onto R that they define has degree 0, since r may thus be continuously contracted to a point inside the field of the motion vectors without passing through any of the singular null points that lie outside of it. Thus, $C(r) = 0$, and from $C + C' = \pm 2$ it then follows that $C' = C(r) = \pm 2$.

The product manifold μ, whose simple model is r, provides an example that is interesting in another direction. It shows that there can be infinitely many models of a manifold with completely different values for C that are generally not simple. Namely, since the circle is a q-fold unbranched covering space of itself for each positive whole number q, the same is true for each product manifold that contains the circle as a factor. One can thus represent r as a q-fold covering of a model r_q over μ; Q then has the order $\pm q$ relative to $\varphi(r_q)$, so $C(\varphi(r_q)) = \pm 2q$, since $C(r_q) = 0$.

For a simple model m with $C(m) = \pm 1$ – thus, in particular, for the n-dimensional sphere – the transformation through reciprocal radii provides no homeomorphic simple model m' with $C(m') \neq C(m)$. It is also not known to me whether there is a simple model m for the n-dimensional sphere with $C(m) \neq \pm 1$.

The following consideration serves as an aid for the examination of the behavior of $C(m)$ under maps, for which nothing less than before will be assumed:

---

[16]) See, e.g., Steinitz, Beiträge zur Analysis Situs (Sitz.-Ber. d. Berl. Math. Ges. **7** 1908).

We attach an arbitrary field V of tangent vectors with at most finitely many singular loci to m; i.e., we associate each point P of μ, with finitely many omissions, with a tangent vector at $F(P)$ to m that varies continuously with P (e.g., we can choose V to be the tangential vector field that is generated by the vector fields $\mathfrak{C}_1$ and $\mathfrak{C}_2$ (cf., § 1)). Let s be the sum of the indices of the singularities of V. We define a continuous function $w(P)$ on m that equals 0 at the singular points, is positive and < 1 everywhere else, and then associate each non-singular point P with the vector $H(P)$ that lies in the quadrant spanned by $V(P)$ and the negative normal $N(P)$ and defines the angle $w \cdot \pi/2$ with $N(P)$; for the P_χ, we set $H(P_\chi) = N(P_\chi)$. H is then continuous on all of m and agrees with the N at the P_χ. From § 1, the coincidence number of the map of μ onto R that is provided by N and H is $I_{12} = s$. On the other hand, from the Poincaré-Bohl theorem both maps have the same degree $C = C(m)$, so, from the result of § 2, one has:

$$s = I_{12} = C + (-1)^n C,$$

so for even n one has:

$$s = 2C.$$

Now since, from § 1, a vector field with the same index sum s exists on any arbitrary model m' for μ, the theorem now follows that:

The curvature integra of the model for a closed, two-sided manifold μ of even dimension is an invariant of μ.

§ 4.

Curvatura integra and index sum; conditions for the representability of an n-dimensional manifold in an $(n+1)$-dimensional space

The relation between the index s of the singularities of a tangential vector field to m and the *curvatura integra* $C(m)$:

$$s = C \cdot (1 + (-1)^n)$$

leads to some consequences that, indeed, do not relate immediately to the *curvatura integra*, but shall still be mentioned. First, one can derive the following theorem from it:

The index sum of the singularities of a tangential vector field to a model for μ is a topological invariant of μ – i.e., it is independent of the choice of model, as well as the vector field – and indeed this invariant is always 0 *for odd n.*

This theorem was expressed by Hadamard [17]), and indeed, with the extension that the model in question might lie in a space of arbitrary dimension. Thus, no proof is given in the cited chapter, which possesses only the character of a report, nor is any such proof known to me in the existing literature. It is noteworthy that in the cases in which s has been computed, namely, for $n = 2$ [4]), for odd n, for n-dimensional spheres [14]), and for the

[17]) Loc. cit., pp. 474 et seq.

manifolds that will be considered in the next paragraph, s is equal to the characteristic of the manifold; i.e., it equals $\sum_{k=0}^{n}(-1)^k \alpha_k$, if α_k denotes the number of k-dimensional simplexes that occur in a simplicial decomposition [*]).

We can now make a sharper statement than the theorem on the invariance of the index sum. Namely, we know that:

The index sum of the model for μ is an even number.

Thus, as always, it is assumed that μ is two-sided and that the model lies in $(n+1)$-dimensional Euclidian space. This provides the possibility of answering the question of whether each n-dimensional manifold m possesses such a model: Namely, if m admits a continuous deformation $P' = f(P; t)$, $0 \leq t \leq 1$, $f(P, 0) = P$ that possesses no fixed points except for finitely many fixed – i.e., independent of t – points for $t > 0$, and if σ is the sum of the indices of this fixed point then σ must be even when μ possesses a model. Then, analogously to the process carried out in § 1, a tangential vector field with the index sum $s = \sigma$ may be constructed on the model. We consider a particularly simple sort of fixed point: We call an isolated fixed point a "center" when it lies in a closed, otherwise fixed-point free element that goes to itself. A center always has the index $(-1)^n$, since one can continuously vary the vectors that point from the points P of the boundary of the element to the points $f(P)$ while preserving their initial points into ones that point to a fixed point in the interior. We can thus express the theorem by saying that a manifold possesses no hypersurface as a model when it admits a deformation whose fixed points are centers, and an odd number of them are present. On the basis of this fact, one may prove:

The totality Z_{2k} of complex points of the $2k$-dimensional projective space is a $4k$-dimensional, closed, two-sided manifold that possesses no hypersurface in $(4k+1)$-dimensional Euclidian space as a model.

The proof is obtained from the foregoing as soon as one shows that Z_r is a closed, two-sided manifold and that a deformation of the required type with $(r+1)$ centers exists. This will be proved in the next paragraph, in which the characteristic of Z_r will be computed, in addition. It is likewise $(r+1)$, which, in light of what was said above, is also of interest in other cases where an agreement between the characteristic and the index sum is present.

§ 5.

The complex projective space

Let Z_r denote the totality of all complex points in the r-dimensional projective space – i.e., the totality of all ratios $z_0 : z_1 : \ldots : z_r$ in which the z_ρ are complex numbers do not all

---

[*]) Added by the editor: A proof of the theorem stated by Hadamard, with the addition that the invariant that appears in the index sum is the characteristic, will be published in these Annalen by the author.

vanish. Z_r is $2r$-dimensional, and in the case $r = 1$ it is known to be homeomorphic to the sphere.

Z_r is a closed manifold with *characteristic* $r+1$.

Proof: We decompose Z_r into $r+1$ parts E_ρ: E_ρ is the totality of all points of Z_r for which $|z_\rho| \geq |z_\sigma|$ ($\sigma = 0, \ldots, r$). Since $z_\rho \neq 0$ in E_ρ, we can normalize the coordinates of all points of E_ρ that one always has $z_\rho = 1$. If we then set $z_\sigma = x_\sigma + i\, y_\sigma$ then E_ρ is mapped topologically onto the piece E'_ρ of a $2r$-dimensional Euclidian space that is defined by inequality $x_\sigma^2 + y_\sigma^2 \leq 1$ with $0 \leq \sigma(\neq \rho) \leq r$. We can represent E'_ρ by r circular discs K_σ^ρ ($0 \leq \sigma(\neq \rho) \leq r$) of radius 1, in which we refer to each group of r points A_σ^ρ, to which the point A_σ^ρ of the disc K_σ^ρ belongs, as points of E'_ρ. From the fact that, without changing the topological structure of E_ρ, instead of the circular discs K_σ^ρ, we can also use the square discs that are defined by $|x_\sigma| \leq 1$, $|y_\sigma| \leq 1$ – thus, a $2r$-dimensional cube can be used instead of E'_ρ – it comes out that E_ρ is an *element*. A point of Z_r belongs to both E_{ρ_1} and E_{ρ_2} when and only when $|z_{\rho_1}| = |z_{\rho_2}| \geq |z_\rho|$; i.e., if the point $A_{\rho_2}^{\rho_1}$ of the point group $A_\sigma^{\rho_1}$, in the aforementioned representation of E'_{ρ_1}, lies on the boundary of $K_{\rho_2}^{\rho_1}$. From this, it is apparent that when $\rho_1, \ldots, \rho_s$, are s of the numbers $0, \ldots, r$, the intersection $E_{\rho_1 \cdots \rho_s}$ of the elements $E_{\rho_1}, \ldots, E_{\rho_s}$ is homeomorphic to the product manifold that is composed of $s - 1$ circles and $r - s + 1$ closed circular discs; hence, it has the dimension $s - 1 + 2(r - s + 1) = 2r - s + 1$.

We carry out a decomposition of Z_r into elements, which shows that Z_r is a "manifold" according to Brouwer's [3]) definition, and furthermore has the property that for each s, each of the manifolds $E_{\rho_1 \cdots \rho_s}$ is comprised exclusively of $(2r - s + 1)$-dimensional element faces of the decomposition. We demonstrate that one can present one such decomposition, in the case of $r = 2$ [18]): The intersection E_{012} of the three elements E_0, E_1, E_2 is a product manifold composed of two circles, and thus it is homeomorphic to a toral surface. We can define E_{012} by way of:

$$z_0 = 1, \qquad z_1 = e^{i\varphi_1}, \qquad z_2 = e^{i\varphi_2}; \qquad 0 \leq (\varphi_1, \varphi_2) \leq 2\pi;$$

or:

$$z_0 = e^{i\varphi_1}, \qquad z_1 = 1, \qquad z_2 = e^{i\varphi_2}; \qquad 0 \leq (\varphi_2, \varphi_0) \leq 2\pi;$$

or:

$$z_0 = e^{i\varphi_0}, \qquad z_1 = e^{i\varphi_1}, \qquad z_2 = 1; \qquad 0 \leq (\varphi_0, \varphi_1) \leq 2\pi.$$

E_{012} decomposes the neighborhood of E_0 that is homeomorphic to the three-dimensional sphere into the manifolds E_{01} and E_{02} that are determined by:

$$z_0 = 1, \qquad z_1 = e^{i\varphi_1}, \qquad z_2 = r_2\, e^{i\varphi_2}; \quad 0 \leq (\varphi_1, \varphi_2) \leq 2\pi; \qquad 0 \leq r_2 \leq 1;$$
$$z_0 = 1, \qquad z_1 = r_1\, e^{i\varphi_1}, \qquad z_2 = e^{i\varphi_2}; \qquad 0 \leq (\varphi_1, \varphi_2) \leq 2\pi; \qquad 0 \leq r_1 \leq 1,$$

[18]) The method applied here may be carried over to arbitrary r with no further assumptions; the wording of the representation then becomes so complicated that the treatment of the special case $r = 2$ seems to make the situation clearer than the consideration of the general case.

each of which is homeomorphic to the product of a circle and a circular disc, and thus, to an ordinary toral space. Analogously, the neighborhood of E_1 and E_2 will be decomposed by E_{012} into the toral spaces E_{12} and E_{10} (E_{20} and E_{21}, resp.). We now decompose E_{012} into a system of two-dimensional elements that fulfills our requirements, by drawing, for example, the twelve closed curves:

$$\left.\begin{array}{lll} z_0 = 1, & z_1 = e^{ik\pi/2}, & z_2 = e^{i\varphi_2}; \\ z_0 = e^{i\varphi_0}, & z_1 = 1, & z_2 = e^{ik\pi/2}; \\ z_0 = e^{ik\pi/2}, & z_1 = e^{i\varphi_1}, & z_2 = 1; \end{array}\right\} \qquad 0 \leq (\varphi_1, \varphi_2, \varphi_3) \leq 2\pi; \quad k = 0, 1, 2, 3,$$

which define a network of 32 curvilinear triangles on E_{012}. We now decompose E_{01} into the four two-dimensional element manifolds F^a (a = 1, 2, 3, 4):

$$z_0 = 1, \qquad z_1 = e^{ik\pi/2}, \qquad z_2 = r_2 e^{i\varphi_2}; \qquad 0 \leq \varphi_2 \leq 2\pi; \quad k = 0, 1, 2, 3$$

into four three-dimensional elements E_{01}^b (b = 1, 2, 3, 4); the neighborhood of each E_{01}^b will be defined by two of the F^a, as well as a fourth of E_{012}, and this fourth is now composed of elements of the aforementioned subdivision of E_{012}, since both of its boundaries are themselves curves of the network. In each of the two F^a that bound E_{01}^b, we draw the four curves that are defined by:

$$z_2 = r_2 e^{ik\pi/2}; \qquad 0 \leq r_2 \leq 1; \qquad k = 0, 1, 2, 3.$$

They emanate from the same point of F^a and end on the four vertices of the subdivision of E_{012} that lie on the boundary of F^a. In this way, the neighborhood of E_{01}^b is subjected to a decomposition of the desired type that agrees, in the part that belongs to E_{012}, with the decomposition that is already present there. We extend this decomposition of the neighborhood of E_{01}^b to a decomposition of the element E_{01}^b itself, in which we draw curves from an interior point to points of the sides and vertices of the local decomposition that correspond to lines under the topological map of E_{01}^b to the interior of a two-dimensional sphere. If we proceed in this way with all four E_{01}^b then we subdivide all of E_{01} into elements of the required type, and then we do the same thing with E_{12} and E_{02}. If we now extend the existing decomposition of the vicinity of E_0, E_1, E_2 in the required manner to E_0, E_1, E_2 itself then we obtain a decomposition of Z_2 that satisfies all of the demands presented.

The property of the aforementioned decomposition of Z_2 that each manifold $E_{\rho_1\cdots\rho_s}$ is composed of faces of elements may also be expressed as follows: Either a given element face has no interior points or it belongs to an entire element face of E_ρ. This compels us to subdivide the totality of all of the elements of arbitrary dimension of the decomposition in question (i.e., vertices, n-dimensional sides, $2r$-dimensional faces) into r + 1 complexes K_s (s = 1, …, r + 1), which are determined in such a way that K_s contains those elements e_s that belong to precisely s of the E_ρ. K_s has the characteristic $k(K_s) = \beta_s$, i.e., let $\beta_s = \sum_{\nu=0}^{2r}(-1)^\nu \alpha_\nu^{(s)}$, where the number of n-dimensional elements that belong to K_s is denoted by $\alpha_\nu^{(s)}$. We next determine β_m for $m \geq 2$; as a product manifold with a circle for a factor, $E_{\rho_1\cdots\rho_s}$ has the characteristic $k(E_{\rho_1\cdots\rho_s}) = 0$, since the characteristic of a product is equal to the product of the characteristics of the factors [16]), and the circle has characteristic 0 [19]). One then has $S_s = 0$, if one sets $S_s = \sum k(E_{\rho_1\cdots\rho_s})$, where the sum is to be taken over all of the E, which have s indices. An element belonging to K_m that lies in $E_{\lambda_1\cdots\lambda_s}$ appears in all of the $E_{\rho_1\cdots\rho_s}$ whose indices are included among the $\lambda_1, \ldots, \lambda_m$, and will thus be counted precisely $\binom{m}{s}$ times. Thus, one has the relations:

$$S_s = \sum_{m=1}^{r+1}\binom{m}{s}\beta_m = 0 \qquad [s = 2, \ldots, r + 1],$$

which, since $\binom{m}{s} = 0$ for $m < s$, may be written in the form:

$$\sum_{m=s}^{r+1}\binom{m}{s}\beta_m = 0 \qquad [s = 2, \ldots, r + 1].$$

The determinant of this system of equations is 1, since there are nothing but ones in the principal diagonal and beneath it there are zeroes; thus, $\beta_m = 0$ for $m \geq 2$. For $s = 1$, the corresponding equation reads:

$$S_1 = \sum_{\rho=0}^{r} k(E_\rho) = \sum_{m=1}^{r+1} m\beta_m = \beta_1,$$

and since E_ρ, as an element, has characteristic 1, one has $\beta_1 = r + 1$. – The characteristic of Z_r is thus [20]):

$$\underline{k(Z_r)} = \sum_{s=0}^{r+1} k(K_s) = \sum_{s=1}^{r+1}\beta_s = \underline{r+1}, \qquad \text{Q.E.D.}$$

---

[19]) For a product with a circle for a factor, the vanishing of the characteristic can also be derived without using the Steinitz theorem that was employed in the text, since such a manifold is a two-fold unbranched covering of itself; it then follows that $k(\mu) = 2k(\mu)$, $k(\mu) = 0$.

[20]) Herr H. Künneth has written to me that he has determined the Betti numbers of Z_2 and has found that $P_1 = P_3 = 1$; thus, the characteristic $k(Z_2)$ follows from the formula $k = \sum_{i=0}^{n} P_i(-1)^i + \frac{1}{2}(1+(-1)^n)$ (cf., Tietze, Die topologischen Invarianten, …, Wiener Monatshefte **19** (1908), pp. 48) when one considers that $P_0 = P_n = 1$ (Tietze, loc. cit., pp. 35, footnote 5), in agreement with out result: $k(Z_2) = 3$.

Z_r is *simply connected*. This is known for $r = 1$, and we thus need only to confirm the simple connectedness of Z_r when we already know it for Z_{r-1}: If a closed curve is $z_0(t)$, …, $z_r(t)$ then we can, with no loss of generality, assume that the point 0, 0, …, 0, 1 does not lie on it; the points of the curve $z_0(t)$, …, $z_{r-1}(t)$, $\lambda\, z_r(t)$ thus define a closed curve for each λ and also for $\lambda = 0$. If λ runs from 1 to 0 then the given curve K_1 will be continuously transformed into K_0 : $z_0(t)$, …, $z_{r-1}(t)$, 0; it belongs to structure that is defined by the equation $z_r = 0$, which is homeomorphic to Z_{r-1} . Since Z_{r-1} is assumed to be simply connected, K_0 can be continuously contracted to a point in the structure $z_r = 0$, from which the deformation of K_1 to a point is completed.

From simple connectivity, it follows that Z_r is *two-sided.*

Z_r admits a single-valued and continuous transformation that includes the identity:

$$P' = f(P, t); \qquad 0 \leq t \leq 1; \qquad f(P, 0) = P,$$

which possesses $\underline{r+1}$ *centers* that are independent of t, but no fixed points for $t > 0$.

Then, under the deformation:

$$z'_\rho(z_0, \cdots, z_r; t) = e^{\frac{\rho t}{r+1} \cdot 2\pi i} \cdot z_\rho \qquad [r = 0, \ldots, r], \quad [0 < t \leq 1]$$

the $r + 1$ points defined by:

$$1, 0, \ldots, 0; \qquad 0, 1, 0, \ldots, 0; \quad \ldots; \qquad 0, 0, \ldots, 0, 1$$

remain fixed; they are centers, since each E_ρ will be deformed into itself and precisely one of these points is contained in the interior. Other fixed points do not appear, though; if there were such a point then there would be two indices $\rho_1 > \rho_2$ with:

$$z_{\rho_1} \neq 0, \qquad z_{\rho_2} \neq 0, \qquad z'_{\rho_1} : z'_{\rho_2} = z_{\rho_1} : z_{\rho_2}, \qquad \text{thus} \quad e^{\frac{\rho_1 t}{r+1} \cdot 2\pi i} = e^{\frac{\rho_2 t}{r+1} \cdot 2\pi i},$$

$$\frac{(\rho_1 - \rho_2)}{r+1} \cdot 2\pi = 2k\pi \geq 2\pi, \qquad (\rho_1 - \rho_2)\, t \leq r + 1,$$

while $0 < \rho_1 - \rho_2 \leq r$, $0 < t \leq 1$, so $(\rho_1 - \rho_2)\, t \leq r$.

(Received on 7/3/1925.)