“Sur les équations absolues du mouvement,” Časopis pro pěstovăni matematiky a fysiky **64** (1935), 229-230.

On the absolute equations of motion

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In the paper “Sur la dynamique absolue des systèmes rhéonomes ([1]),” I gave the absolute dynamical equations that would be valid for the space-time parameters. In this communication, I would like to deduce their explicit form without taking into account the absolute differential calculus.

The motion of a system with n parameters x^λ ($\lambda, \mu = 1, \ldots, n$) with a kinetic energy T that is subject to the action of generalized forces X_λ and non-holonomic, rheonomic constraints ([2]):

$$\Phi_\lambda^K dx^\lambda + \Phi_t^K dt = 0 \qquad (K = 1, \ldots, n - m) \tag{1}$$

is determined by means of the equations:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}^\lambda} - \partial_\lambda T = X_\lambda + \Lambda_K \Phi_\lambda^K . \tag{2}$$

Introduce the $m + 1$ space-time parameters q^a (which are generally holonomic) by setting:

$$dx^\lambda = B_a^\lambda \, dq^a , \qquad dt = B_a^t \, dq^a \qquad (a, b = 0, 1, \ldots, m), \tag{3}$$

with the reservation that the q^a are independent, in such a way that:

$$\Phi_\lambda^K B_a^\lambda + \Phi_t^K B_a^t = 0. \tag{4}$$

By reason of (3), one will obtain:

$$\frac{\partial T}{\partial \dot{q}^a} = \frac{\partial T}{\partial \dot{x}^\lambda}\frac{\partial \dot{x}^\lambda}{\partial \dot{q}^a} = B_a^\lambda \frac{\partial T}{\partial \dot{x}^\lambda},$$

$$\partial_a T = B_a^\lambda \partial_\lambda T + \frac{\partial T}{\partial \dot{x}^\mu}\partial_a \dot{x}^\mu = B_a^\lambda \partial_\lambda T + \frac{\partial T}{\partial \dot{x}^\mu}\partial_a B_b^\mu \dot{q}^b .$$

([1]) Prace Mat.-Fiz., Warszawa, **41** (1933), 25-37.

([2]) I shall appeal to the notations $\partial_\lambda = \partial / \partial x^\lambda$, $\partial_t = \partial / \partial t$, $\partial_a = \partial / \partial q^a = B_a^\lambda \partial_\lambda + B_a^t \partial_t$, and I shall suppress all summation signs.

Hence, upon multiplying (2) by B_a^λ and taking the sum, we will have, by virtue of (4):

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^a} - \partial_a T + \frac{\partial T}{\partial \dot{x}^\mu}(\partial_a B_b^\mu - \partial_b B_a^\mu)\,\dot{q}^b = B_b^\lambda\,\partial_\lambda T + \frac{\partial T}{\partial \dot{x}^\mu}\partial_a B_b^\mu\,\dot{q}^b .$$

Therefore, upon multiplying (2) by B_a^λ and taking the sum, we will have, by virtue of (4):

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^a} - \partial_a T + \frac{\partial T}{\partial \dot{x}^\mu}(\partial_a B_b^\mu - \partial_b B_a^\mu)\,\dot{q}^b = B_b^\lambda\,X_\lambda - \Lambda_K \Phi_t^K\,B_a^t .$$

One infers from (1) and (2) that:

$$\frac{dT}{dt} = X_\lambda \dot{x}^\lambda - \Lambda_K \Phi_t^K ,$$

in such a way that when one writes:

$$\partial'_a T = \partial_a T + B_a^t \frac{dT}{dt} = B_a^\lambda \partial_\lambda T + B_a^t \frac{dT}{dt} ,$$

$$Q'_a = B_a^\lambda X_\lambda + B_a^t X_t \quad (X_t = -X_\lambda\,\dot{x}^\lambda),$$

one will get the absolute equations in the form:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^a} - \partial'_a T + \frac{\partial T}{\partial \dot{x}^\mu}(\partial_a B_b^\mu - \partial_b B_a^\mu)\,\dot{q}^b = Q'_a , \tag{5}$$

which is completely analogous to the form of the usual equations of motion.

If one chooses the q^a to also be holonomic for a holonomic system then (5) will become:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^a} - \partial'_a T = Q'_a ,$$

and for a scleronomic system $\left(dT / dt = X_\lambda\,\dot{x}^\lambda\right)$, our equations will reduce to those of Lagrange:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}^a} - \partial_a T = Q_a \qquad (Q_a = B_a^\lambda X_\lambda).$$
