"Sur la dynamique absolue des systèmes rhéonomes," Prac. Matemalyczno-Fizycznych (1933), 25-39.

On the absolute dynamics of rheonomic systems

By

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Introduction. – In the paper "Sur les systèmes non holonomes" ([1]), I constructed an invariant theory of non-holonomic systems, and I showed that my theory would also apply to *rheonomic* systems (and non-holonomic systems, in general) if one took time to be a new parameter. In the French edition of my paper, I indicated only the resulting equations in their explicit form, because the invariant (geometric) significance was obvious from what preceded, while in the Czech edition, I also wrote the equations of motion in their condensed form:

$$\overset{*}{I}_k = Q_k , \qquad \overset{*}{I}_{m+1} = Q_{m+1} + Q'_{m+1} , \tag{1}$$

in which the $\overset{*}{I}_k$, $\overset{*}{I}_{m+1}$ denote the components of the absolute change in the quantity of motion, while Q_k , Q_{m+1} are those of the given force, and Q'_{m+1} are those of the reaction.

In a recent paper ([2]), **A. Wundheiler** recalled the study of rheonomic systems. He envisioned the configuration space of such systems to be a deformable ("rheonomic") Riemannian space, and defined the *dilatation tensor* $W^i_{\cdot k}$ and the *absolute centrifugal force* S^i in such a way that he arrived at the equations of motion:

$$\frac{\delta v^i}{dt} + W^i_{\cdot k}\, v^k = Q^i + S^i, \tag{2}$$

in which v^i signifies the *longitudinal* velocity, Q^i, the given force, and δ, the absolute differential in a deformable space. In no. **21** of his paper, **Wundheiler** remarked that my equations (in regard to their clarity) were no better than the older explicit equations. That remark was obviously concerned with the explicit form of my equations, while the form (1) is simpler than **Wundheiler**'s (2).

In what follows, I would like to show that my interpretation of the dynamical equations admits an extension to the most general case of space-time parameters. One

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([1]) Z. Horák, Bull. Int. Acad. Tchéque, **24** (1928), 1-18; this is only an abbreviated edition of a Czech paper that was published simultaneously in: Rozpravy II. tř. České akademie **37**, no. 15 (1928), 1-29.

([2]) **A. Wundheiler**, "Rheonome Geometrie. Absolute Mechanik," Prace Mat. Fiz. **40** (1932), 97-142.

then arrives at *equations* (31), *which are invariant under absolutely arbitrary space-time transformations*, and it should be emphasized that, despite their unlimited generality, those equations demand that one must apply only ordinary Riemannian geometry. In the particular case of scleronomic systems, equations (31) reduce to the form (34), which lends itself to the study of *relative motion*, as I will show by an example at the end of the present paper.

Permit me to take this occasion to draw attention below to the various results of my earlier papers, which have remained unknown for several years and were then rediscovered by some other authors.

It was already in my Thesis ([3]), which was published in 1924, that I interpreted the configuration space of a non-holonomic system as a *non-holonomic manifold* (§ **7**) and also deduced the components of the *acceleration* in such a manifold [formula (29)], as well as the most general non-holonomic parameters, moreover. In the same place (pp. 26), I remarked that it was not difficult to generalize the fundamental notions of infinitesimal geometry to the case of a non-holonomic manifold. I stressed the fact that the non-holonomic systems do not differ essentially from the holonomic ones and deduced formula (40), which translates into a generalization of *Newton's law* to non-holonomic systems that is equivalent to equation (4) in the present paper. In my thesis, I also pointed out that the rheonomic systems can be subordinate to the study of scleronomic systems when one takes time to be a new parameter, but I indicated the corresponding equations only in another paper ([4]) that was published in 1925 [formula (2)].

I dedicated a later paper ([5]) to the non-holonomic geometry that I outlined in my thesis, in which I gave a precise definition of a *general non-holonomic manifold* and established the *formulas for the most general linear connection* in that manifold. In that way, I laid the foundations of the *non-holonomic absolute differential calculus*, which was envisioned to be a theory of the invariants of the group of *non-holonomic transformations*.

The (purely-mathematical) results of that work have served to assist me in the work that cited to begin with ([1]), in which I returned to the detailed study of non-holonomic, rheonomic systems. I also gave a general form for Hamilton's principle and the canonical equations. In the same paper, I introduced the notion of *covariant variations*, which also played an important role in one of my notes to the Comptes rendus [Acad. Sc. Paris **188** (1929), 614-616]. Now, that variation was defined again by **Wundheiler** in a note to the Rend. Lincei **12** (1930), and it also appears in his paper that was cited above ([2]) (pp. 128, 136).

([3]) **Z. Horák**, *The principle of conservation of energy and the equations of Physics*, Publ. Fac. Sc. Univ. Charles (Prague) **25** (1924), in Czech, with a French summary.

([4]) **Z. Horák**, "Establishing physical laws by means of energetic principles," Časopis. mat. a fys. **45** (1925), 42-60; in Czech, with a French summary.

([5]) **Z. Horák**, "Sur une généralisation de la notion de variété," Publ. Fac. Sc. Univ. Masaryk, Brno **86** (1927), 1-20. That paper, which I presented in October 1926, is completely independent of the notes by **Vranceanu** that were published in November and December of 1926 in C. R. Acad. Sc. Paris and Rend. Lincei.

Notations. – In what follows, I shall consistently suppress the summation sign, and I will appeal to four types of indices:

λ, μ, ν	take the values:	$1, 2, \ldots, n$;
$i, k, l, m; r$	" " "	$1, 2, \ldots, m$;
a, b, c, d	" " "	$0, 1, \ldots, m$;
K	" " "	$1, 2, \ldots, n - m$.

I shall let ∂_λ , ∂_t , ∂_k , ∂_a denote the derivatives $\frac{\partial}{\partial x^\lambda}$, $\frac{\partial}{\partial t}$, $\frac{\partial}{\partial q^k} = B_k^\lambda \frac{\partial}{\partial x^\lambda}$, $\frac{\partial}{\partial q^a} = B_a^\lambda \frac{\partial}{\partial x^\lambda}$, and the difference $A_{kl} - A_{lk}$ by $2A_{[kl]}$.

1. Generalized Newton law. – I shall begin by rapidly recalling the general law that I gave in my papers that were published in 1924 ([3]) and 1928 ([1]):

(I) *The change in the quantity of motion of a system is equal to the resultant force.*

If one considers only *scleronomic* systems (whether holonomic or not) then the precise significance of the law (I) is the following: The configuration space of such a system is, in general, a non-holonomic Riemannian manifold V_n^m (n is the number of holonomic parameters x^λ, which are subject to $n - m$ non-integral constraints); the quantity of motion is a covariant vector whose components are:

$$p_k = \frac{\partial T}{\partial \dot{q}^k},$$

in which T is the semi-*vis viva*:

$$T = \tfrac{1}{2} a_{\lambda\mu} \dot{x}^\lambda \dot{x}^\mu = \tfrac{1}{2} b_{\lambda\mu} \dot{q}^\lambda \dot{q}^\mu,$$

in which b_{kl} is the fundamental tensor of the manifold V_n^m; the q^k are the independent parameters (which are non-holonomic, in general), whose differentials are coupled with the holonomic parameters by the relations:

$$dx^\lambda = B_l^\lambda \, dq^l ; \tag{3}$$

the resultant force is a vector whose covariant components are equal to **Lagrange**'s generalized forces: Q_k . Finally, one intends the term "change" to mean the absolute change, which corresponds to the Riemannian absolute (covariant) differential δ, which is non-holonomic, in general. Therefore, (I) is expressed by the covariant relation:

(4) $$\frac{\delta p_k}{dt} = Q_k \,,$$

in which t signifies **Newton**'s absolute time. From the preceding, one will have:

$$p_k = a_{kl}\,\dot{q}^l = v_k \,,$$

in which $v_k = \dot{q}^k$ denote the contravariant components of the *velocity* vector. One can therefore also write:

(4, cont.) $$\frac{\delta v_l}{dt} = Q_l \,,$$

and replace (I) with the law: *The acceleration of a system is equal to the resultant force.*

In order to obtain the explicit form of equations (4), it will suffice to substitute the expression for the absolute differential. If one lets w_l denote a covariant vector on the manifold then one will have [**Horák** ([5, 1]), **Schouten** ([6])]:

(5) $$\delta w_l = dq_l - \Theta^l_{lk}\, w_l\, dq^k \,,$$

in which:

(6) $$\Theta^l_{lk} = \tfrac{1}{2} b^{ij}(\partial_k\, b_{lj} + \partial_l\, b_{jk} - \partial_j\, b_{kl}) + B^l_\lambda\, \partial_{[k}\, B^\lambda_{l]} + b^{ij}(b_{ml}\, B^m_\lambda\, \partial_{[j}\, B^\lambda_{k]} + b_{mk}\, B^m_\lambda\, \partial_{[j}\, B^\lambda_{l]})$$

$$(B^i_\lambda = b^{ij}\, a_{\lambda\nu}\, B^\nu_j).$$

Therefore, the absolute acceleration translates into the formula:

(7) $$\frac{\delta v_l}{dt} = \frac{dv_l}{dt} - \tfrac{1}{2}\partial_l\, b_{lk}\, \dot{q}^l\, \dot{q}^k - 2b_{jl}\, B^j_\lambda\, \partial_{[k} B^\lambda_{l]}\, \dot{q}^l\, \dot{q}^k \,,$$

which I deduced in 1924 ([3]) [pp. 29, formula (29)] in the form:

$$a_l = b_{kl}\, \ddot{q}^k + \begin{bmatrix} i\,k \\ l \end{bmatrix} \dot{q}^l\, \dot{q}^k + a_{\lambda\mu}\, \dot{x}^\mu \left(\frac{\partial B^\lambda_k}{\partial q^l} - \frac{\partial B^\lambda_l}{\partial q^k} \right) \dot{q}^k \,.$$

When one takes the relation (7) into account, one will infer the equations of motion that were given by the author in 1924 ([3]) and 1928 ([1]) from (4, cont.):

(8) $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^l} \right) - \partial_l T + 2\frac{\partial T}{\partial \dot{x}^\lambda}\, \partial_{[l}\, B^\lambda_{k]}\, \dot{q}^k = Q_l \,.$$

([6]) **J. A. Schouten**, "Über nichtholonome Übertragungen in einer L_n ," Math. Zeit. **30** (1929), 149-172.

Those equations (in a slightly more specialized form) were deduced before by **L. Boltzmann** (Wiss. Abh., III, pp. 692).

2. Extension to rheonomic systems. – Consider a holonomic system in n independent holonomic parameters x^λ that moves under the action of generalized forces X_λ whose kinetic energy is a homogeneous quadratic function of the derivatives $\dot{x}^\lambda$:

$$T = \tfrac{1}{2} a_{\lambda\mu} \dot{x}^\lambda \dot{x}^\mu , \tag{9}$$

in which the coefficients $a_{\lambda\mu}$ do not depend upon time. If one prescribes the $n - m$ rheonomic and non-holonomic constraints:

$$\Phi_\lambda^K \, dx^\lambda + \Phi_t^K \, dt = 0 \qquad (K = 1, 2, \ldots, n - m) \tag{10}$$

on the aforementioned system, in which the Φ_λ^K, Φ_t^K are functions of x_λ and t, then it will be possible to express the dx^λ by means of the m differentials of the non-holonomic parameters q^λ :

$$dx^\lambda = B_l^\lambda \, dq^l + B_t^\lambda \, dt . \tag{11}$$

Hence, the dq^l satisfy the conditions:

$$\Phi_\lambda^K B_l^\lambda \, dq^l + (\Phi_\lambda^K B_t^\lambda + \Phi_t^K) \, dt = 0,$$

and I suppose that the B_l^λ, B_t^λ are chosen in such a manner that:

$$\Phi_\lambda^K B_l^\lambda = 0, \qquad \Phi_\lambda^K B_t^\lambda + \Phi_t^K = 0 ; \tag{12}$$

i.e., the dq^l are independent.

In order to make it possible to apply the law (I) itself to rheonomic systems, I shall introduce a new parameter q^0, with the supplementary condition:

$$q^0 = t . \tag{13}$$

That condition can be regarded as a new constraint, and as a result, replaced with a constraint force whose components are Q_l', Q_0'. Having done that, all of the parameters $q^0, q^1, \ldots, q^m$ are independent, and one will have:

$$\begin{gathered} dx^\lambda = B_a^\lambda \, dq^a , \\ T = \tfrac{1}{2} b_{ab} \, \dot{q}^a \dot{q}^b , \qquad p_a = b_{ab} \, \dot{q}^b , \qquad b_{ab} = B_a^\lambda B_b^\mu a_{\lambda\mu} . \end{gathered} \tag{14}$$

One can then treat the system as a non-holonomic, scleronomic system with $m + 1$ degrees of freedom that moves in the non-holonomic configuration space V_n^{m+1} under the action of the given force and the reaction Q'_a. From that standpoint, the law (4) can be applied to it, which will give:

$$\frac{\delta p_a}{dt} = Q_a + Q'_a, \tag{15}$$

in which Q_a denotes the projection of the given force onto the space V_n^{m+1}:

$$Q_a = B_a^\lambda X_\lambda . \tag{16}$$

Equations (15), which are $m + 1$ in number, represent a relation between the covariant vectors of the space V_n^{m+1}, and with the aid of (13), they determine the motion of the system completely, because, due to the particular form of the constraint (13), the force Q'_a will reduce to the single component Q'_0 ($Q'_k = 0$), in such a way that (15) will imply the $m + 1$ independent equations:

$$\frac{\delta p_l}{dt} = Q_l , \qquad \frac{\delta p_0}{dt} = Q_0 + Q'_0, \tag{17}$$

which, along with (13), suffice to calculate the $m + 1$ parameters q^a and the reaction Q'_0 as functions of time. In order to determine the single motion of the system, it will suffice to consider the m spatial equations:

$$\frac{\delta p_l}{dt} = Q_l , \tag{17, cont.}$$

in which one sets $q^0 = t$.

It is easy to get the explicit form of (17):

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}^l}\right) - \partial_l T + 2\frac{\partial T}{\partial \dot{x}^\lambda}\partial_{[l}\, B_{k]}^\lambda\, \dot{q}^k + 2\frac{\partial T}{\partial \dot{x}^\lambda}\partial_{[l}\, B_{t]}^\lambda = Q_l , \\ &\frac{d}{dt}(b_{l0}\, \dot{q}^l + b_{00}) - \partial_t T + 2\frac{\partial T}{\partial \dot{x}^\lambda}\partial_{[t}\, B_{k]}^\lambda\, \dot{q}^k = Q_0 + Q'_0 . \end{aligned} \tag{18}$$

Those equations appear in my earlier papers ([4]), ([1]), whereas their implicit form (17) is found in only the Czech edition of the second paper.

3. Configuration space-time. – Time plays two different roles in our arguments: First of all, it is the independent variable in our differential equations, and secondly, it represents the "temporal parameter" of the system by way of the notation q^0. That suggests that we can call the $(m + 1)$-dimensional Riemannian manifold V_n^{m+1} the *configuration space-time*, as well as put equations (15) into another form. Upon multiplying them by $\dot{q}^a$ and summing, we will get:

$$\dot{q}^a \frac{\delta p_a}{dt} = Q_a \dot{q}^a + Q'_a \dot{q}^a = Q_k \dot{q}^k + Q_0 + Q'_0 .$$

On the other hand (since the absolute differential of the fundamental tensor is zero), that will become:

$$\dot{q}^q \, \delta p_a = \dot{q}^a \, \delta(b_{ab} \dot{q}^b) = b_{ab} \dot{q}^a \, \delta \dot{q}^b = \tfrac{1}{2} \delta(b_{ab} \dot{q}^a \dot{q}^b) = dT ;$$

hence, one infers that:

(19) $$\frac{dT}{dt} = Q_k \dot{q}^k + Q_0 + Q'_0 .$$

If one then defines the vector with the $m + 1$ components Q_l, $\frac{dT}{dt} - Q_k \dot{q}^k$ to be the *space-time force*, which is denoted by $\bar{Q}_a$, then one will arrive at the space-time extension of the law (I):

(20) $$\frac{\delta p_a}{dt} = \bar{Q}_a ,$$

in which:

$$\bar{Q}_k = Q_k , \qquad \bar{Q}_0 = \frac{dT}{dt} - Q_k \dot{q}^k .$$

The vector $\bar{Q}_a$ recalls **Minkowski**’s quadri-force, which is defined in an analogous manner in the theory of relativity. Equations (20) are obviously covariant with respect to non-holonomic transformations of the type:

(21) $$dq^k = B^k_r \, dq^r + B^k_t \, dt , \qquad dt = dt,$$

which leave time t invariant. That is the group of *kinematical* transformations that was considered by **Wundheiler**, and as a result, equations (20) are absolute in the same sense as **Wundheiler**’s equations (2), which are meanwhile quite complex.

However, the method that I just presented admits a generalization that presents equations of motion that are covariant under more general space-time transformations. In order to show that, represent the rheonomic system that was envisioned in no. **2**. If one replaces the constraints (10) with a constraint force with components X'_λ then, as one knows, the equations of motion can be written in the form ([7]):

(22) $$\frac{\delta I}{dt} = X_\lambda + X'_\lambda ,$$

([7]) That will also come about by specializing equations (4) for a holonomic system.

in which $I_\lambda = \partial T / \partial \dot{x}^\lambda$. In order to distinguish the two roles of time, which enters as a temporal parameter, on the one hand, and an independent variable, on the other, denote time by x^0 in the former case. The constraints will then translate into the relations:

$$\Phi^K_\lambda \, dx^\lambda + \Phi^K_0 \, dx^0 = 0,$$

and if one takes those equations into account then the differentials dx^λ, dx^0 can be expressed linearly in terms of the $m + 1$ differentials dq^a :

(23) $$dx^\lambda = B^\lambda_a \, dq^a , \qquad dx^0 = B^0_a \, dq^a ,$$

which implies that:

$$(\Phi^K_\lambda \, B^\lambda_a + \Phi^K_0 \, B^0_a) \, dq^a = 0,$$

and we shall choose the B^λ_a , B^0_a in such a fashion that one has:

(24) $$\Phi^K_\lambda \, B^\lambda_a + \Phi^K_0 \, B^0_a = 0.$$

Now, since the virtual work done by the force X'_λ, which realizes the constraints (10), is zero, its components will have the form:

$$X'_\lambda = \Lambda_K \, \Phi^K_\lambda ,$$

in which the Λ_K denote **Lagrange**'s indeterminate coefficients. If one substitutes those values into equations (22) and multiplies them by B^K_λ then one will have:

$$B^\lambda_a \frac{\delta I_\lambda}{dt} = B^\lambda_a \, X_\lambda + \Lambda_K \, \Phi^K_\lambda \, B^\lambda_a .$$

The left-hand side of that equation is the projection of the vector $\delta I_\lambda / dt$ onto configuration space-time, which is equal to the vector $\delta p_a / dt$, in which one intends δp_a to mean the absolute differential that corresponds to the non-holonomic Riemannian connection that is induced in the configuration space-time V^{m+1}_n. If one denotes the projection $B^\lambda_a \, X_\lambda$ of the force X_λ onto V^{m+1}_n by Q_a , as before, then the equations of motion will become:

$$\frac{\delta p_a}{dt} = Q_a - \Lambda_K \Phi^K_0 \, B^0_a ,$$

by virtue of (24), and if one sets:

(25) $$\Lambda_0 = \Lambda_K \Phi^K_0$$

then it will follow that:

(26) $$\frac{\delta p_a}{dt} = Q_a + \Lambda_0 B_a^0 .$$

The second term on the right-hand side of that equation is the projection of the constraint force onto the space V_n^{m+1}. In order to calculate the unknown coefficient Λ_0 , multiply equations (26) by $\dot{q}^a$ and sum:

$$\dot{q}^a \frac{\delta p_a}{dt} = Q_a \dot{q}^a + \Lambda_0 B_a^0 \dot{q}^a .$$

I previously showed that:

$$\dot{q}^a \frac{\delta p_a}{dt} = \frac{dT}{dt} ,$$

and therefore, when one considers (23):

(27) $$\frac{dT}{dt} = Q_a \dot{q}^a + \Lambda_0 \dot{x}^0 .$$

Now, due to the notation that was adopted, $x^0 = t$, in such a way that the second relation (23) can be written:

(28) $$dt = B_a^0 dq^a ,$$

and in addition, $\dot{x}^0 = 1$, which will make it possible to infer the value of Λ_0 from equation (27). However, I would first like to modify the notations slightly. From (28), the differential of absolute time is expressed in an invariant manner by the scalar product of the real displacement of the system by the covariant vector B_a^0, which we shall call the *time vector* and henceforth denote by t_a in such a way that we will have:

(29) $$dt = t_a dq^a .$$

Consequently, we also suppress the index 0 in the symbol Λ_0 . Equation (27) then gives:

(30) $$\Lambda = \frac{dT}{dt} - Q_a \dot{q}^a ,$$

and (26) will become:

(31) $$\frac{\delta p_a}{dt} = Q_a + \Lambda t_a .$$

The expression (30) proves that Λ is an invariant, and equations (31) are, in turn, covariant under *arbitrary* transformations of the space-time parameters q^a. With one of the supplementary relations (29) or (30), they will indeed suffice to determine the $m + 1$ parameters and the invariant Λ as functions of time t.

In summary, we have then obtained *equations of motion that are independent of the framing of configuration space-time.* They are valid for any scleronomic or rheonomic system that is, at the same time, holonomic or not. [As far as free systems are concerned, see number **5**.] It remains for us to point out that equations (31) can also be written in the form of equations (20):

$$\frac{\delta p_a}{dt} = \bar{Q}_a$$

if one introduces the space-time force that is defined by the components:

(32) $$\bar{Q}_a = Q_a + \left(\frac{dT}{dt} - Q_a \dot{q}^a \right) t_a .$$

Of course, in the general case, the analogy between and the quadri-force is no longer as pronounced.

4. Scleronomic systems. – Those systems are characterized by the disappearance of all coefficients Φ_t^K, which will ultimately be denoted by Φ_0^K, so in regard to (25), one can infer the necessary condition:

(33) $$\Lambda = 0,$$

which is fulfilled independently of the choice of given force Q_a ([8]). That condition is also sufficient, since it is supposed to be satisfied for any given force. Indeed, if the linear expression (25) must be annulled for all forces – i.e., for an infinite number of values of Λ_K – then all of the Φ_0^K will necessarily disappear. Therefore, in the case of a scleronomic system (holonomic or not), the equations of motion (31) will reduce to the Newtonian law:

(34) $$\frac{\delta p_a}{dt} = Q_a ,$$

which is extended to the most general space-time parameters this time.

By virtue of (30), the relation (33) will translate into the *vis viva* theorem:

(35) $$\frac{dT}{dt} = Q_a \dot{q}^a ,$$

which still remains valid for arbitrary space-time parameters, but only for scleronomic systems. For the *rheonomic* systems, that theorem will take the more general form:

([8]) In general, the value of Λ will depend upon the given force.

$$\frac{dT}{dt} = \bar{Q}_a \dot{q}^a ,$$

in which the space-time force takes the place of the given force. One can assure oneself, moreover, that those two forces coincide in the case of a scleronomic system by comparing equations (32) and (35).

5. Free system. Relative motion. – In the preceding number, we supposed, without saying so expressly, that all of the Φ_t^K were annulled, but that the coefficients Φ_λ^K were not equal to zero simultaneously. Now, if there are no constraints at all – so the system is completely *free* – then the two numbers n and m will be equal, all of the coefficients Φ_λ^K, Φ_t^K will be annulled, and the B_a^λ, B_a^0 will be arbitrary. Under those conditions, equations (23) express only a non-holonomic transformation of the n + 1 parameters x^λ, x^0. However, the most important consequence of the particular supposition that was made above concerns the fundamental tensor. Namely, the kinetic energy is a positive-definite form of the derivatives $\dot{x}^\lambda$, and as a result, the rank of the form:

$$T = \tfrac{1}{2} b_{ab} \dot{q}^a \dot{q}^b$$

will be equal to n, while there are n + 1 parameters q^a. Hence, the rank of the fundamental tensor b_{ab} is smaller by one than the number of dimensions of space-time. That situation excludes the direct application of ordinary Riemannian geometry, which supposes, as one knows, that the determinant $| b_{ab} |$ is non-zero. Nonetheless, it is still possible to generalize the absolute calculus itself for this particular case, as well, as was shown recently by **E. Bortolotti** ([9]).

Without going into the details, I would like to present some of its results that will be useful. Since the rank of the tensor b_{ab} is n, there will exist a system ω^a of solutions to the n + 1 equations:

(36) $$b_{ab} \, \omega^a = 0$$

that are defined up to an arbitrary factor. The ω^a are the contravariant components of a *zero vector* whose covariant components all disappear. In general, the covariant components x_b of a vector are defined uniquely, but the contravariant components, which are determined by the equation:

(37) $$\xi^a \, b_{ab} = \xi_b ,$$

are written:

([9]) **E. Bortolotti**, "Sulle forme differenziali quadratiche specializzate," Rend. Lincei **12** (1930), 541-547; "Calcolo assoluto rispetto a una forma differenziale quadratica specializzata," Rend. Lincei **13** (1931), 19-25.

$$\xi^a = \overset{*}{\xi}{}^a + \omega^a,$$

in which $\overset{*}{\xi}{}^a$ denotes an arbitrary solution of (37). **Bortolotti** defined the **Christoffel** symbols of the second kind in an analogous manner by the relation:

$$\left\{ \begin{matrix} a\,b \\ c \end{matrix} \right\} b_{cd} = \left[\begin{matrix} a\,b \\ d \end{matrix} \right] = \tfrac{1}{2} (\partial_a b_{bd} + \partial_b b_{ad} - \partial_d b_{ab}), \tag{38}$$

which makes it possible to define the absolute differential by the same expression as in the case of ordinary Riemannian space. For example, for a covariant vector p_a , one will then have:

$$\delta p_a = dp_a - \left\{ \begin{matrix} a\,b \\ c \end{matrix} \right\} p_c\, dq^b ,$$

or also, due to (38):

$$\frac{\delta p_a}{dt} = \frac{dp_a}{dt} - \left[\begin{matrix} a\,b \\ d \end{matrix} \right] \dot{q}^b \, \dot{q}^c ,$$

in such a way that the *equations of motion* (34) *are determined unequivocally*. The latter formulas are valid only if the parameters q^a are holonomic. Now, in the case of a free system, that supposition is always admissible, because such a system is necessarily holonomic. Anyway, one can further generalize the aforementioned formulas to non-holonomic parameters by means of non-holonomic transformation.

One sees that our equations even apply to free systems that admit the introduction of space-time parameters in this particular case. One will thus arrive at the study of *relative motions*, which translates into equations of the form (34) independently of the motion of the coordinate system.

In order to give an example, consider a unit point-mass that moves in a fixed plane under the action of a force whose components in a fixed rectangular coordinate system Oxy are X, Y. Look for the equations of motion relative to a rectangular coordinate system $O\xi\eta$ that turns around O. The solution is given by equations (34), in which one sets:

$$q^1 = \xi, \qquad q^2 = \eta, \qquad q^0 = \tau \ (= t); \quad n = 2, m + 1 = 3 .$$

Now, one has:

$$x = \xi \cos \alpha - \eta \sin \alpha ,$$

$$y = \xi \sin \alpha + \eta \cos \alpha ,$$

α is a function of τ whose derivative $d\alpha / d\tau$ we shall denote by ω. By differentiation, it will become:

$$\begin{aligned} dx &= \cos \alpha\, d\xi - \sin \alpha\, d\eta - \omega y\, d\tau, \\ dy &= \sin \alpha\, d\xi + \cos \alpha\, d\eta + \omega x\, d\tau. \end{aligned} \tag{39}$$

so one can infer the coefficients B_a^λ that enter into (14). The *vis viva* is:

$$2T = \dot{\xi}^2 + \dot{\eta}^2 - 2\omega\eta\,\dot{\xi}\,\dot{\tau} + 2\omega\xi\,\dot{\eta}\,\dot{\tau} + r^2\,\omega^2\,\dot{\tau}^2,$$

in which:

$$r^2 = x^2 + y^2 = \xi^2 + \eta^2, \qquad \dot{\tau} = 1,$$

and equations (34) will become:

$$(40) \qquad \begin{cases} \dfrac{\delta p_\xi}{dt} = \ddot{\xi} - \dot{\omega}\eta - 2\omega\dot{\eta} - \omega^2\xi = Q_\xi, \\ \dfrac{\delta p_\eta}{dt} = \ddot{\eta} + \dot{\omega}\xi + 2\omega\dot{\xi} - \omega^2\eta = Q_\eta, \\ \dfrac{\delta p_\tau}{dt} = \omega(\xi\ddot{\eta} - \eta\ddot{\xi}) + r^2\omega\omega + 2\omega^2 r\,\dot{r} = Q_\tau. \end{cases}$$

The first two equations determine the relative motion, while the Q_ξ , Q_η are the components of the given force in the moving system. One sees that the relative and comoving accelerations, when combined with the complementary one (which corresponds to the composite centrifugal force), will give the absolute change of the quantity of motion. The fictitious forces are not space-time vectors, but along with the relative acceleration, they form the space-time vector $\delta p_a / dt$. Hence, the notion of absolute acceleration in the sense of Riemannian geometry coincides with that of absolute acceleration, in the mechanical sense, in this case. In order to also understand the significance of the latter equation (40), replace Q_τ by its expression:

$$Q_\tau = B_\tau^x\,X + B_\tau^y\,Y = \omega\,(x\,Y - y\,Z),$$

which gives:

$$\xi\,\ddot{\eta} - \eta\,\ddot{\xi} + \frac{d}{dt}(r^2\omega) = x\,Y - y\,X\,.$$

That amounts to saying that the moment of the given force with respect to the rotational axis is equal to the sum of the moment of the relative acceleration and twice the areal acceleration. One deduces that relation from (39) by taking into account the equations of motion $\ddot{x} = X$, $\ddot{y} = Y$.

It is not perhaps pointless to stress the fact that it is the application of the absolute calculus that makes it possible to summarize the three equations (40) in a space-time relation (34) that translates a general law that is independent of the choice of coordinate system.

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