"Étude géométrique sur la coubure des pseudosurfaces," Bull. Soc. Math. France. **17** (1889), 84-101.

Geometric study of the curvature of pseudo-surfaces

By the Abbot ISSALY

Translated by D. H. Delphenich

Preliminaries.

1. – In our paper on *line congruences* ([1]), it resulted that a pseudo-surface is nothing but a geometric locus of a new kind that is produced by two systems of variable curves that intersect only up to second-order infinitesimals.

As a matter of fact, such a locus cannot be represented by a finite equation in three variables, such a x, y, z, but it can be represented, at least implicitly, by a differential equation of the form:

$$dz = \varphi(x, y)\, dx + \psi(x, y)\, dy,$$

when combined with the non-integrability condition:

$$\frac{\partial \psi}{\partial x} \neq \frac{\partial \varphi}{\partial y}.$$

From that viewpoint, any surface can be considered to be a particular case, or better yet, as the common limit of a double series of infinitely-close pseudo-surfaces.

One concludes from this that, for example, no matter what expression one adopts for the curvature of a surface at each of its points, one must be able to deduce that expression from the one that pertains to any neighboring pseudo-surface by simply setting $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$ or $\frac{1}{\rho_1''} = \frac{1}{\rho_2''}$, according to the notation that was adopted. Since we know *a priori* that the latter expression does not depend upon any of the Gauss parameters E, F, G, one can assert that the former, for its own part, will not depend upon them, either, which is at the least an absolute necessity.

In support of that deduction, we recall that the *original* form for the equations of the remarkable lines on a surface, when taken in general, is obtained without appealing to those parameters, and the necessity of introducing them will be felt only for a variety of applications.

([1]) Bulletin de la Société mathématique de France, t. XVI.

2. – In the same paper, it also resulted that one can imagine two auxiliary osculating surfaces F_μ and F_ν at any point M of a pseudo-surface, one of which corresponds to the arithmetic mean $\frac{1}{2}\left(\frac{1}{\rho_1''}+\frac{1}{\rho_2''}\right)$ of the curvatures that correlate with the coordinate lines, while the other corresponds to their geometric mean $\frac{1}{\sqrt{\rho_1''\rho_2''}}$.

The first of those surfaces is closed linked with the principal planes of the pseudo-surfaces that are tangent to it, while the second one is attached to their focal planes. Now, one knows that the principal planes are always real, while the focal planes can be imaginary. That is what motivates us to employ preferentially the surface F_μ everywhere in all of what follows.

I.

Extension of the principal formulas proposed to measure the curvature of surface to pseudo-surfaces.

3. – First, recall the equation of the indicatrix of the surface F_μ, which is also the equation of the corresponding pseudo-surface [I, no. **19** ([1])], namely:

$$\frac{X^2}{r_1''}+\left(\frac{1}{\rho_1''}+\frac{1}{\rho_2''}\right)X\,Y+\frac{Y^2}{r_2''} = 1. \tag{1}$$

That conic will have the type of an ellipse, hyperbola, or parabola according to whether the quantity:

$$k_\mu \sin^2\theta = \frac{1}{r_1' r_2''}-\frac{1}{4}\left(\frac{1}{\rho_1''}+\frac{1}{\rho_2''}\right)^2$$

is positive, negative, or zero, resp.

One deduces that the equation of the principal radii R_1'' and R_2'' is:

$$\frac{\sin^2\theta}{R''^2}-\left[\frac{1}{r_1'}+\frac{1}{r_2''}-\left(\frac{1}{\rho_1''}+\frac{1}{\rho_2''}\right)\cos\theta\right]\frac{1}{R''}+\left[\frac{1}{r_1' r_2''}-\frac{1}{4}\left(\frac{1}{\rho_1''}+\frac{1}{\rho_2''}\right)^2\right] = 0. \tag{2}$$

When referred to the axes of its figure, the indicatrix will then take the reduced form:

$$\frac{X^2}{R_1''}+\frac{Y^2}{R_2''} = 1, \tag{3}$$

([1]) This abbreviation refers to no. **19** of our first paper.

and one will note that with that choice of coordinates, the expression for the first vertical curvature of the arbitrary line S can be written:

$$\frac{1}{r''} = \frac{\cos^2\theta_1}{R_1''} + \frac{\sin^2\theta_1}{R_2''}. \tag{4}$$

4. – Presently, we have described a sphere with its center at the origin M and a variable radius $\sqrt{R''}$. Therefore, consider the two second-order curves in the horizontal plane $T_1\,M\,T_2$:

$$X^2 + 2XY\cos\theta + Y^2 = R'',$$

$$\frac{X^2}{r_1''} + \left(\frac{1}{\rho_1''} + \frac{1}{\rho_2''}\right) X\,Y + \frac{Y^2}{r_2''} = 1,$$

in which one recognizes a section of the sphere by the horizontal plane, along with the indicatrix (1).

The pair of common secants that intersect at the origin have the equation:

$$\left(\frac{1}{R''} - \frac{1}{r_1''}\right) X^2 + 2\left[\frac{\cos\theta}{R''} - \frac{1}{2}\left(\frac{1}{\rho_1''} + \frac{1}{\rho_2''}\right)\right] X\,Y + \left(\frac{1}{R''} - \frac{1}{r_2''}\right) Y^2 = 0,$$

or rather:

$$A\,X^2 + 2B\,XY + C\,Y^2 = 0,$$

and if one sets:

$$\delta\sin^2\theta = AC - B^2$$

then it will become:

$$\delta = \left(\frac{1}{R''}\right)^2 - J_1\left(\frac{1}{R''}\right) + J_2 = \left(\frac{1}{R''} - \frac{1}{R_1''}\right)\left(\frac{1}{R''} - \frac{1}{R_2''}\right);$$

the coefficients J_1 and J_2 are nothing but the invariants:

$$J_1 = \frac{\dfrac{1}{r_1''} + \dfrac{1}{r_2''} - \left(\dfrac{1}{\rho_1''} + \dfrac{1}{\rho_2''}\right)\cos\theta}{\sin^2\theta},$$

$$J_2 = \frac{\dfrac{1}{r_1'' r_2''} - \dfrac{1}{4}\left(\dfrac{1}{\rho_1''} + \dfrac{1}{\rho_2''}\right)^2}{\sin^2\theta}.$$

Having said that, it is easy to see that *when they are generalized* – i.e., when they are applied to pseudo-surfaces – the main formulas that have been presented at various times as a measure of the curvature of a surface are nothing but symmetric functions of the roots $1/R_1''$, $1/R_2''$ of the equation $\delta = 0$, and as a result, they will be rational functions of the invariants J_1 and J_2. Indeed, upon considering the formulas for the transformation of coordinates, one will have:

$$k_\mu = \frac{1}{R_1'' R_2''} = J_2 \qquad \text{(Gauss)},$$

$$k_{\mu_1} = \frac{1}{R_1''^2} + \frac{1}{R_2''^2} = J_1 - 2\,J_2\,,$$

$$k_{\mu_2} = \frac{1}{R_1''} + \frac{1}{R_2''} = J_1 \qquad \text{(Sophie Germain)},$$

$$k_{\mu_3} = \frac{1}{2}\left(\frac{1}{R_1''} + \frac{1}{R_2''}\right) = \tfrac{1}{2} J_1\,,$$

$$k_{\mu_4} = \frac{1}{R_1''^2} + \frac{2}{3R_1'' R_2''} + \frac{1}{R_2''^2} = J_1^2 - \tfrac{4}{3} J_2 \qquad \text{(Bourget and Housel)}.$$

5. – We shall summarize the convenience of those values from the standpoint of measuring the curvature of pseudo-surfaces:

1. $k_\mu = \dfrac{1}{R_1'' R_2''}$. – That ratio can generally be taken to be acceptable, as long as R_1'' and R_2'' have the same or opposite signs; however, it is the opinion of a great number of people that, without fail, it will cease to be so when one of the radii becomes infinite, which will happen when F_μ is a developable surface.

2. $k_\mu = \dfrac{1}{R_1''^2} + \dfrac{1}{R_2''^2}$. – When one adopts this expression, one implicitly replaces the normal curvature $1 / r''$ with the square $1 / V_2$ of the vertical deviation relative to a pencil of normals [MN] to the surface F_μ, or if one prefers, replaces the indicatrix (3) by the *deviator:*

$$\frac{X^2}{R_1''^2} + \frac{Y^2}{R_2''^2} = 1$$

of that surface. Now, that will render any means of distinguishing the concavity of F_μ around the point M from its convexity impossible, which is, one can say without contradiction, an essential choice when one is dealing with curvature.

3. $k_{\mu_2} = \dfrac{1}{R_1''} + \dfrac{1}{R_2''}$. – The third formula breaks down when F_μ is a *minimal* surface, and when one recalls the invariants above, one can add, when the pseudo-surfaces that they replace are minimal.

4. $k_{\mu_3} = \dfrac{1}{2}\left(\dfrac{1}{R_1''} + \dfrac{1}{R_2''}\right)$. – This is equivalent to taking the measure of the curvature of the surface F_μ to be its spherical – or mean – curvature, but one will then meet up with some inconveniences in regard to k_{μ_2}.

5. $k_{\mu_4} = \dfrac{1}{R_1''^2} + \dfrac{2}{3R_1''R_2''} + \dfrac{1}{R_2''^2}$. – Only this last formula seems to be particularly worthy of our attention: We shall not hesitate to take it as the starting point for our present study by adopting it to pseudo-surfaces. However, we must nonetheless explain its origin.

6. – Suppose that one traces out lengths $1 / r'' = \tau''$ along the normal MN that starts from the point M that are equal to the curvatures of the various normal sections to the surface F_μ. Fold (*rabattons*) each of those lengths along the intersection of the horizontal plane $T_1\,M\,T_2$ with the normal plane that contains the length. Their extremities will determine an arc of the curve whose form will already suffice to make the variation of the curvature of the successive normal sections meaningful, and in that way, that of the surface F_μ around the chosen point.

Upon agreeing to replace θ_1 with ψ from now on (due to the ambiguity that inevitably comes to pass), the equation of the curve in terms of r'' and ψ whose construction we just indicated would become:

$$\frac{1}{r''} = \frac{\cos^2\psi}{R_1''} + \frac{\sin^2\psi}{R_2''}. \tag{5}$$

It is a fourth-order curve. Its transform by reciprocal vector radii:

$$\tau'' = \frac{\cos^2\psi}{R_1''} + \frac{\sin^2\psi}{R_2''} \tag{6}$$

has order six. Let us pause for a moment on that fact.

One sees that it presents three varieties according to whether R_1'' and R_2'' have the same sign, opposite sign, or one of the two (R_2'', for example) is infinite, respectively.

We call it the *(planar) indicatrix of curvature*, in order to distinguish it from the indicatrix, properly speaking.

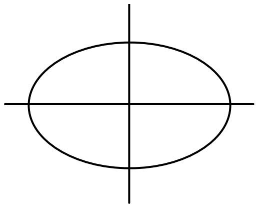

Figure 1.

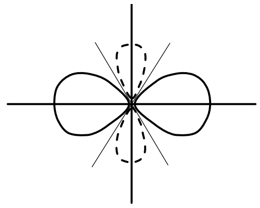

Figure 2.

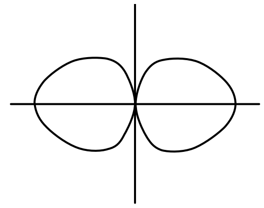

Figure 3.

Upon denoting its area by A_η , in a general manner, one will have:

$$(7)\qquad \begin{cases} \tau'' = \dfrac{\cos^2\psi}{R_1''} + \dfrac{\sin^2\psi}{R_2''}, \\[2mm] \tau'' = \dfrac{\cos^2\psi}{R_1''} - \dfrac{\sin^2\psi}{R_2''}, \\[2mm] \tau'' = \dfrac{\cos^2\psi}{R_1''}, \end{cases}$$

$$(8)\qquad \begin{cases} A_{\eta_1} = \dfrac{\pi}{8}\left(\dfrac{3}{R_1''^2} + \dfrac{2}{R_1'' R_2''} + \dfrac{3}{R_2''^2}\right), \\[2mm] A_{\eta_2} = \dfrac{\pi}{8}\left(\dfrac{3}{R_1''^2} - \dfrac{2}{R_1'' R_2''} + \dfrac{3}{R_2''^2}\right), \\[2mm] A_{\eta_3} = \dfrac{\pi}{8}\dfrac{3}{R_1''^2}. \end{cases}$$

We propose to measure the curvature of a pseudo-surface at each of its points, up to first approximation, by the area of the reciprocal curve (6), following the method that Bourget and Housel introduced.

7. – We now pose the following question:

Is it not possible to replace either the contour or the area of the reciprocal curve (6) [i.e., the indicatrices (7), whose artificial construction provides only an approximate solution to the problem] with a convex surface that measures the curvature of the surface F_μ at M with full exactitude, first geometrically by itself and then numerically by its area, and in turn, the curvatures of all infinitely-close pseudo-surfaces?

One can do that in effect. However, we immediately assert that, from the theoretical viewpoint, those surfaces must give us an *adequate* geometric expression for the curvature, since in practice the numerical calculation of their areas will present serious difficulties to the point that one will, so to speak, regret that one has abandoned the planar indicatrices. We will say immediately that such regret is pointless, since one will see from the following paragraph that it is always possible, and even simple with the aid of a slight transformation (which is a true *criterion* with respect to the first method) to make the convex indicatrices take on an area that is rigorously equal to that of the planar indicatrices that they correspond to.

Since the stated surfaces come down to a general type that enjoys the indicated property, and some others that are no less remarkable, we believe that we must make know that type itself, as well as those properties.

II.

General theorems relating to hemicyclides.

8. – We say *hemicyclidal surface*, or simply *hemicyclide*, to mean a surface that is generated by a semicircle under the following conditions:

Let AB be an algebraic or transcendental plane curve.

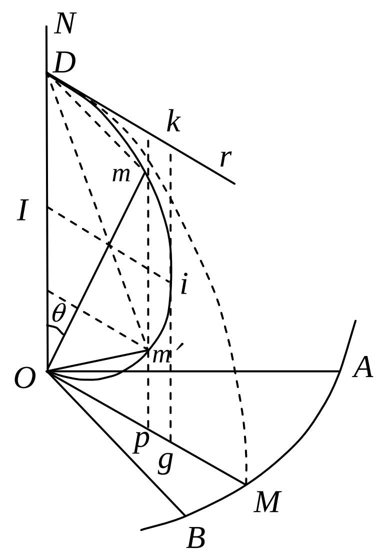

Figure 4.

Let AB be an arc of the algebraic or transcendental plane curve, and let AOB be the sector that corresponds to a given pole O. Draw each of the radius vectors from the plane that is drawn through the pole that end at the various points of the arc in order to form the diameter OD of a semicircle that is situated in the plane of the radius considered and the normal:

Theorem I. – *The area of the given sector is equal to exactly that of the transformed surface that one obtains by replacing each of the sectoral (fusiforme) elements of the hemicyclide that was constructed with the spherical sector (fuseau) that is its projection onto the sphere of equal diameter.*

Indeed, let $F(r, \psi) = 0$ be the equation of the given plane curve, when referred to the point O as its pole and any polar axis.

To be more precise, take the branch of that curve that is defined by the following determination of r :

$$r = f(\psi)\,.$$

Upon letting r denote an arbitrary radius vector Om and letting θ denote the angle that it makes with ON, one will have:

$$\rho = r \cos \theta$$

for the equation of the generating semicircle, in which θ varies only from 0 to $\pi / 2$, and as a result, the equation of the hemicyclide that corresponds to the branch considered will be:

$$\rho = f(\psi) \cos \theta\,.$$

Set:

$$u = \rho \sin \theta.$$

The area of the portion of the transformed surface that corresponds to the angle $AOM = \psi$ will be:

(8) $$A_\upsilon = 2\int_0^\psi d\psi \int_0^{r/2} \frac{u\,du}{\cos 2\theta}\,,$$

and since:

$$\sin 2\theta = \frac{2u}{r}\,,$$

that will become:

$$A_\upsilon = 2\int_0^\psi d\psi \int_0^{r/2} \frac{u\,du}{\sqrt{1-\frac{4u^2}{r^2}}}\,.$$

However:

$$\int \frac{u\,du}{\sqrt{1-\frac{4u^2}{r^2}}} = -\frac{r^2}{4}\sqrt{1-\frac{4u^2}{r^2}} + c\,;$$

hence:

$$A_\upsilon = \tfrac{1}{2}\int_0^\psi r^2 d\psi = A_\eta\,,$$

which proves the property.

9. – As for the exact area A of the hemicyclide, if one lets ψ represent the angle that the XY-plane makes with the tangent plane that is drawn through any of its points m then one will have:

$$A = 2\int_0^\psi d\psi \int_0^{r/2} \frac{u\,du}{\cos\gamma} = 2\int_0^\psi d\psi \int_0^{\pi/2} d\theta \sqrt{dr^2 + r^2 \sin^2 2\theta\, d\psi^2}\,,$$

which is a formula in which r and dr must be replaced with their values as functions of ψ before any calculations are done. One sees from this that the first of the integrations that must be performed (namely, the one over θ) will already depend upon elliptic functions, which justifies what we said in no. **7** about the inherent complications in the calculation of a similar area.

10. – Up to now, it has been essential to observe that if the mode of transformation adopted alters the finite expression for the area of the surface then the same thing will not be true for the expression for the volume, since the sum of the solid elements that one neglects is infinitely small with respect to the volume itself.

11. Theorem II:

The portion of the volume of the right cylinder that circumscribes a hemicyclide that is found between the latter surface and its projection onto the horizontal plane is exactly one-fourth of the corresponding tab (onglet) if one limits the cylinder to its contact curve. It will be one-half of that when one stops the cylinder only at the conoid that is formed by the horizontal tangents of the hemicyclide.

Indeed the volume of the angle ψ of the hemicyclide will be expressed by:

$$V = \tfrac{1}{3}\int_0^{\psi} d\psi \int_0^{\pi/2} r^3 \cos^3\theta \sin\theta \, d\theta = \tfrac{1}{12}\int_0^{\psi} r^3 d\psi \, . \tag{9}$$

On the other hand, that of the circumscribed cylinder that is bounded by the contact curve is:

$$U = \int_0^{\psi} d\psi \int_0^{r/2} z\, u\, du \, , \tag{10}$$

with the condition:

$$u^2 = z\,(r - z) \, .$$

Solving that for z, substituting the smallest root in the expression for U, and integrating will give:

$$U = \tfrac{1}{48}\int_0^{\psi} r^2 d\psi = \tfrac{1}{4} V \, .$$

Finally, upon doubling the result, since the contact curve is diametral line, one will have the total volume of the cylinder that is external to the tab and bounded by the conoid that forms its upper base; i.e., $\frac{1}{2} V$.

Q. E. D.

Those two theorems constitute an advanced generalization of the theorems of Archimedes that relate to the ratio that exists between the sphere and its circumscribed cylinder.

12. Theorem III:

The transform by reciprocal radius vectors of any hemicyclide is a conoid that is parallel to the one that is formed by its horizontal tangents.

One proves that effortlessly by remembering that the inverse of a circumference that passes through the pole of inversion is a line.

III.

Applying the preceding theorems to the convex indicatrix of the curvature of a pseudo-surface.

13. – The three varieties (7) of the planar indicatrix of the curvature correspond, in polar and rectangular coordinates, to the three hemicyclides:

$$\text{(11)} \qquad \left\{ \begin{aligned} \rho &= \left(\frac{\cos^2\psi}{R_1''} + \frac{\sin^2\psi}{R_2''} \right) \cos\theta, \\ \rho &= \left(\frac{\cos^2\psi}{R_1''} - \frac{\sin^2\psi}{R_2''} \right) \cos\theta, \\ \rho &= \frac{\cos^2\psi}{R_1''} \cos\theta, \end{aligned} \right.$$

$$\text{(11}'\text{)} \qquad \left\{ \begin{aligned} X^2 + Y^2 + Z^2 &= \frac{(R_2'' X^2 + R_1'' Y)\,Z}{R_1'' R_2'' (X^2 + Y^2)}, \\ X^2 + Y^2 + Z^2 &= \frac{(R_2'' X^2 - R_1'' Y)\,Z}{R_1'' R_2'' (X^2 + Y^2)}, \\ X^2 + Y^2 + Z^2 &= \frac{X^2 Z}{R_1'' (X^2 + Y^2)}. \end{aligned} \right.$$

Those are three closed surfaces. In the first one, the diameters of the normal sections vary between a maximum of $1 / R_1''$ and a minimum of $1 / R_2''$. In the second one, the vertical planes that go from the origin along the tangents to the corresponding plane indicatrix are tangent to it and divide it into four pair-wise symmetric parts that are situated on one side and the other of the horizontal plane. Finally, the third surface is derived from the second one insofar as the tangent planes that were just spoken of coincide.

With the aid of formula (8), one can verify that once those surfaces are transformed – or better yet, *sphericized* – they will indeed be equivalent to the plane indicatrices (7).

14. Volumes of the indicatrices. – Since the volumes of the convex indicatrices (11) are each the geometric locus of all orthogonal components of the curvatures of the various normal sections that are made around the point M on the surface F_μ , that will itself represent (we shall not say "measure") the curvature of the surface at that point.

Let us go through those volumes in turn; their expressions are integrable:

1. The volume of the first of the indicatrices (11), when calculated by means of formula (9), will be expressed by:

$$V_1 = \frac{\pi}{90}\left(\frac{5}{R_1''^2} - \frac{2}{R_1''R_2''} + \frac{5}{R_2''^2}\right)\left(\frac{1}{R_1''} + \frac{1}{R_2''}\right).$$

2. As far as the second one is concerned, since the integral that expresses it changes sign along the directions with $\tan \psi_1 = \pm\sqrt{\frac{R_2''}{R_1''}}$, which are those of the tangents at the origin, one must then use the doubled integral:

$$\tfrac{1}{3}\int_0^{\psi_1} r^3\, d\psi - \tfrac{1}{3}\int_{\psi_1}^{\pi/2} r^3\, d\psi\,.$$

Upon setting:

$$\psi_1 = \frac{\pi}{4} + \psi_1',$$

$$P = \frac{1}{144}\left(\frac{3\cdot 5}{R_1''^2} - \frac{2\cdot 7}{R_1''R_2''} + \frac{3\cdot 5}{R_2''^2}\right)\frac{1}{\sqrt{R_1''R_2''}},$$

$$Q = \frac{1}{48}\left(\frac{5}{R_1''^2} + \frac{2}{R_1''R_2''} + \frac{5}{R_2''^2}\right)\left(\frac{1}{R_1''} - \frac{1}{R_2''}\right),$$

for more symmetry, one will find that the total volume is:

$$V^2 = 2P + 2Q_1\psi_2',$$

and the partial volumes are:

$$V'^2 = P + Q\left(\frac{\pi}{4} + \psi_1'\right),$$

$$V''^2 = P + Q\left(\frac{\pi}{4} - \psi_1'\right).$$

The first one is situated above the horizontal plane and the second is below it.

Those various results simplify singularly when $R_1'' = R_2''$, which is the case (once the – sign has been specified) for the *minimal pseudo-surfaces.*

One will then have:

$$V'^2 = V''^2 = P = \frac{1}{9R_1'^2},$$

which is a value that one can verify directly, moreover, by calculating the corresponding integral:

$$\frac{1}{3R_1'^2}\int_0^{\pi/4}\cos^3 2\psi\, d\psi .$$

3. The third volume is inferred from either of the first two by setting $R_2'' = \infty$. One finds:

$$V_3 = \frac{5}{96}\frac{\pi}{R_1^3}.$$

15. – One can equivalently replace the volumes of the hemicyclides with those of the corresponding *indicator cylinders* (no. **11**), since they are one-fourth the former, because the cylinder stops at the contact curve. That is what gives one the right to appeal to them to *represent* the curvature at the point *M*.

It should be pointed out that the bases of those cylinders are homothetic to the plane indicatrix of Bourget and Housel, and that they are equal to one-fourth of it, moreover. In order to avoid constructing them in an artificial way, one must then take those bases into consideration.

16. Various properties. – We must observe that all of what follows will be applicable to only the case in which the indicatrix of the surface F_μ is elliptic.

I. If one subtracts the area A_σ of the sphere whose radius $\frac{1}{4}\left(\frac{1}{R_1''}+\frac{1}{R_2''}\right)$ is one-half the spherical curvature at the point M from the area $A_{\upsilon_1} = A_{\eta_1}$ of the first of the transformed convex indicatrices then one will find that the excess of the first of those surfaces relative to the second one is:

$$\varepsilon = \frac{A_{\upsilon_1} - A_\sigma}{A_\sigma} = \frac{1}{2}\left(\frac{R_2'' - R_1''}{R_1'' + R_2''}\right)^2 .$$

On the other hand, a comparison of the corresponding volumes (whether or not the first one transformed) will also give:

$$\varepsilon' = \frac{V_1 - V_\sigma}{V_\sigma} = \frac{3}{2}\left(\frac{R_2'' - R_1''}{R_1'' + R_2''}\right)^2 .$$

One can then deduce this simple relation:

$$\frac{\varepsilon}{\varepsilon'} = \frac{1}{3} .$$

II. Consider the infinitesimal curve:

$$(\lambda) \qquad d\lambda = \frac{(R_2'' - R_1'')\sin\psi\cos\psi}{\sqrt{R_1''^2 \sin^2\psi + R_2''^2 \cos^2\psi}}\, ds .$$

It is the projection onto the *XY*-plane of the line of *axial striction* (I, no. **28**) of the generators of the pencil [*MN*] of the (true) normals to the surface F_μ . Now, if one calculates the ratio of its area to that of the infinitesimal circle of radius *ds*, which is the director of the pencil, then one will find that:

$$\frac{\Omega_\lambda}{\Omega} = \frac{1}{2}\left(\frac{R_2'' - R_1''}{R_1'' + R_2''}\right)^2 = \varepsilon .$$

III. – Likewise, if one looks for the volume found between the curve (λ) and conoidal surface that is composed of the curves of shortest distance *IK* then since one can compare that volume to the volume of a cylinder of revolution that has the circle of radius *ds* for its bases and an altitude that is equal to $\frac{R_1'' R_2''}{R_1'' + R_2''} + \frac{1}{4}(R_1'' + R_2'')$, one will see that their ratio is also itself equal to ε.

17. – We hasten to add that the calculations in the preceding integration will become much less simple if, instead of considering the pencil of normals [*MN*] of F_μ , we take the corresponding pencil of pseudo-normals. The radical that enters into equation (λ) would not be independent then of the alternating curvatures $1 / P_1$ and $1 / P_2$ of the coordinate lines that are presently tangent to the axes of the indicatrix, because if one generally has $\frac{1}{P_1} + \frac{1}{P_2} = 0$ in that case for any pencil of circum-axial pseudo-normals of *MN* then one will have $1 / P_1 = 0$ and $1 / P_2 = 0$ only for the single pencil of true normals to the surface F_μ .

IV.

Applying the various complementary geometric loci relating to the curvature of an auxiliary surface F_μ .

18. – Instead of folding (*rabattre*) the inverses $1 / r''$ or τ'' of the radii of curvature of the normal sections of the surface F_μ onto the horizontal surface, as we did in no. **6**, we shall fold those radii onto themselves; we will then have the three curves:

$$(12) \qquad \left\{ \begin{aligned} \frac{1}{r''} &= \frac{\cos^2\psi}{R_1''} + \frac{\sin^2\psi}{R_2''}, \\ \frac{1}{r''} &= \frac{\cos^2\psi}{R_1''} - \frac{\sin^2\psi}{R_2''}, \\ \frac{1}{r''} &= \frac{\cos^2\psi}{R_1''}. \end{aligned} \right.$$

One then concludes that for the corresponding hemicyclides (i.e., from Meusnier's theorem, for the locus of the centers of curvature of all normal or oblique sections that are made around the point M on the surface F_μ), one has:

$$(13) \qquad \left\{ \begin{aligned} \rho &= \frac{R_1'' R_2''}{R_2'' \cos^2\psi + R_1'' \sin^2\psi} \cos\theta, \\ \rho &= \frac{R_1'' R_2''}{R_2'' \cos^2\psi - R_1'' \sin^2\psi} \cos\theta, \\ \rho &= \frac{R_1''}{\cos^2\psi} \cos\theta, \end{aligned} \right.$$

$$(13') \qquad \left\{ \begin{aligned} X^2 + Y^2 + Z^2 &= \frac{R_1'' R_2'' (X^2 + Y^2) Z}{R_1'' X^2 + R_1'' Y^2}, \\ X^2 + Y^2 + Z^2 &= \frac{R_1'' R_2'' (X^2 + Y^2) Z}{R_1'' X^2 - R_1'' Y^2}, \\ X^2 + Y^2 + Z^2 &= \frac{R_1'' (X^2 + Y^2) Z}{X^2}. \end{aligned} \right.$$

The first of those surfaces is closed. The last two have infinite sheets: For one of them, they are in the asymptotic directions $\tan \psi_1 = \pm \sqrt{\dfrac{R_2''}{R_1''}}$, while for the other, they are in the direction $\psi_1 = \pi / 2$.

Those surfaces are attached to some others that are homothetic to them, and which can qualify as *surfaces of curvature*, since they are the loci of *circles* of curvature for all normal sections that are made around the point *M*. Since one obtains them by doubling the radius vectors of the preceding ones, it will suffice to address the latter.

19:

1. Applying the first two general theorems in § II will give us:

$$A'_{\upsilon_1} = \pi\left(\frac{R''_1 + R''_2}{2}\right)\sqrt{R''_1 R''_2} \qquad \text{(\textit{area of an ellipse}),}$$

$$V'_1 = \frac{\pi}{48}(3R''_1 + 2R''_1 R''_2 + 3R''_2)\sqrt{R''_1 R''_2},$$

in the first case.

When $R''_1 = R''_2$ (i.e., when the point *M* is an umbilic for each of the points of the surface F_μ), one will recover the known expression for the area or volume of the sphere of radius R''_1.

2. As for the transformed area and volume of the second surface, since they will become infinite for $\psi = \psi_1$, it will be reasonable to calculate them only for a sector or a tab below that limit. One will then find that the transformed area of the sector is:

$$8A'_{\upsilon_2} = \frac{R''_1 R''_2 (R''_1 + R''_2)\sin^2\psi}{R''_2\cos^2\psi - R''_1\sin^2\psi} + (R''_1 - R''_2)\sqrt{R''_1 R''_2}\; L\left(\frac{R''_2\cos\psi + R''_1\sin\psi}{R''_2\cos\psi - R''_1\sin\psi}\right).$$

The case of $R''_1 = R''_2$, which is corresponds to minimal pseudo-surfaces, is particularly interesting.

One first confirms that the limit of the last term in A'_{υ_2} is zero. That will then imply:

$$A'_{\upsilon_2} = \frac{R''^2_1}{4}\tan 2\psi.$$

That shows that in the present case, the total area of the surface that we consider is the same at the point *M* as that of *the band in the horizontal plane that is found between two parallels that are equidistant to the origin by the quantity* R''_1.

Passing to the volume of the tab that corresponds to $\psi < \psi_1$ and setting:

$$P' = [R''_2(5R''_1 - 3R''_2)\cos^2\psi + R''_1(5R''_2 - 3R''_1)\sin^2\psi]\frac{R''_1 R''_2 (R''_1 + R''_2)\sin^2\psi}{(R''_2\cos^2\psi - R''_1\sin^2\psi)^2},$$

$$Q' = (3R_1''^2 - 2R_1''R_2'' + 3R_2''^2)\sqrt{R_1''R_2''}\,L\left(\frac{R_2''\cos\psi + R_1''\sin\psi}{R_2''\cos\psi - R_1''\sin\psi}\right),$$

to abbreviate, one will have the relation:

$$192\ V_2 = P' + Q'.$$

When $R_1'' = R_2''$, that will give:

$$V_2' = \frac{R_1''^3}{48}\left[\frac{\tan 2\psi}{\cos 2\psi} + L\tan\left(\frac{\pi}{4} + \psi\right)\right].$$

3. Finally, the third surface also takes on an infinite area and volume for $\psi_1 = \pi / 2$; however, for a tab and a sector of angle less than $\pi / 2$, one will have:

$$A'_{\upsilon_2} = \frac{R_1''\tan\psi}{6}\left(2 + \frac{1}{\cos^2\psi}\right),$$

$$V_3' = \frac{R_1''^3\tan\psi}{180}\left(8 + \frac{4}{\cos^2\psi} + \frac{3}{\cos^4\psi}\right).$$

20. Conoidal transformations. – From Theorem III (no. **10**), any transform by reciprocal radius vectors of a hemicyclide will be a conoid with horizontal generators. Let us look for the equations of those conoids, first for the geometric locus that we are dealing with, and then for the convex indicatrices of curvature.

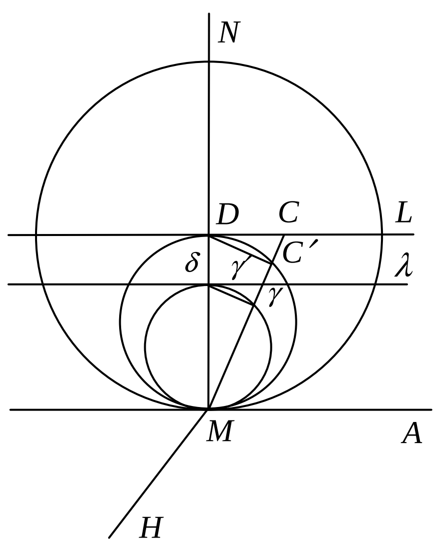

Figure 5.

1. Let $MD = r''$, $M\delta = 1 / r'' = \tau''$. Let C' denote the center of curvature of the oblique section (Fig. 5) that is made in the surface F_μ by the plane *CMH*. Since one has $MC' \cdot M\gamma = 1$, the transform by reciprocal radius vectors of the first of the surface (13′) is the conoid:

$$Z = \frac{R_2'' X^2 + R_1'' Y^2}{R_1'' R_2'' (X^2 + Y^2)},$$

which is composed of the horizontal tangents to the first of the convex indicatrices (11′), since the modulus of the transformation is unity.

When that modulus becomes equal to $R_1'' R_2''$, the equation of the transform will become:

$$Z = \frac{R_2'' X^2 + R_1'' Y^2}{X^2 + Y^2}. \tag{14}$$

That is the conoid of *axial striction* (I, no. **29**) of the pencil [*MN*] of normals to the surface F_μ.

In the last case, the intersection of the two surfaces belongs to the sphere:

$$X^2 + Y^2 + Z^2 = R_1'' R_2'',$$

and its projection onto the horizontal plane will be the curve:

$$r^2 = (R_2'' - R_1'')(R_1'' \sin^4 \psi - R_2'' \cos^4 \psi).$$

2. Return to the convex indicatrix of curvature (11) or (11′). Since one has, similarly, $MC \cdot M\gamma = 1$, the transform of the first of the surfaces (11′) by reciprocal radii will be the conoid:

$$Z = \frac{R_1'' R_2'' (X^2 + Y^2)}{R_2'' X^2 + R_1'' Y^2},$$

which is the locus of horizontal tangents to the first of the surfaces (13′), since the constant is equal to unity.

When that constant becomes equal to $\dfrac{1}{R_1'' R_2''}$, one will have the conoid:

$$Z = \frac{X^2 + Y^2}{R_2'' X^2 + R_1'' Y^2},$$

which differs only by the change of Z into $1 / Z$ of the conoid (14).

Here, the intersection of the surface will take place on the sphere:

$$X^2 + Y^2 + Z^2 = \frac{1}{R_1'' R_2''},$$

and its projection onto the horizontal plane will be the curve:

$$r^2 = \left(\frac{1}{R_1''} - \frac{1}{R_2''}\right) \frac{R_1'' \sin^4 \psi - R_2'' \cos^4 \psi}{(R_1'' \sin^2 \psi + R_2'' \cos^2 \psi)^2}.$$

Some considerations of the same type will be applicable to the *deviator* that we spoke of in no. **5**, but they are useless for everything that is at least concerned with the curvature of surfaces or pseudo-surfaces at each of their points.