"Über ein Verallgemeinerung der Geometrie, welche in der Quantenmechanik nützlich sein kann," Dokl. Akad. Nauk, U.S.S.R **4** (1929), 73-78.

On a generalization of geometry that can be useful in quantum mechanics

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§ 1. As is well-known, the relative simplicity of the Schrödinger wave equation consists in the fact that it represents a quantum-mechanical generalization of the relativistic relation between the direction cosines u_i :

$$u_i^2 + 1 = 0. \tag{1}$$

This is a purely geometric formula, in which the velocities u_i pay the role of cosines. The problem of the introduction of operators – i.e., the quantum-mechanical interpretation of the u_i – as:

$$u_i = \frac{1}{m} p_i = \frac{1}{m}\left(\frac{h}{i}\frac{\partial}{\partial x_i} + c\varphi_i\right)$$

(in which all quantities are to be regarded as operators) is an autonomous problem of geometry that is distinct from the problem of geometrization that will be of interest to us here. If the translation of the u_i into operators is known, once and for all, then we can interpret the relation (1) and all of geometry as operator geometry. Thus, the usual geometry (we pass over the interesting question of five-spaces, in which one can write u_i^2 = 0, instead of (1)) already suffices for the geometrization of the Schrödinger equation. (The N-body problem will be solved, in which one introduces a configuration space for all N particles.)

We know that Dirac arrived at the exhibition of a system of equations that generalized the Schrödinger equation, namely:

$$S\psi = (u^2 + 1)\psi = 0,$$

in a natural way that yielded a complete accounting of spin phenomena. These equations read:

$$D\psi = (\gamma_i u_i + 1)\ \psi = 0. \tag{2}$$

Here, the:

$$\gamma_i = \left|\gamma_i^{\alpha\beta}\right|$$

are four-rowed matrices; there are also four:

$$\psi = \psi_{1,\,2,\,3,\,4}\ .$$

(We can certainly also understand ψ to be a matrix with 16 components in which, however, only the diagonal elements are non-zero.) The problem of the geometrization of the Dirac equation then arises. It hardly seems to be an idle mathematical problem, since, if we specify the fundamental quadratic form:

$$(ds^2 = g_{ik}\, dx^i\, dx^k; \qquad 1 = -\, g_{ik}\, \frac{dx_i}{ds}\frac{dx_k}{ds}; \qquad u^2 + 1 = 0)$$

then the solution of the N-body problem is given completely by that. Namely, one must only recall that the Hamiltonian function of the interaction is quadratic in the momenta, such that we can write the fundamental quadratic form of the configuration space for it. The solution is fictitious as long as we disregard certain relativistic interaction terms. The inclusion of these relativistic terms is, however, a new independent problem that has also not been solved up to now in classical quantum mechanics. As a starting point, we choose the quantities γ_i and look for their geometric meaning. G. Breit (*) has shown that if we would like to regard the D-equation as the quantum-mechanical analogue of the linear classical equation for momenta then these γ_i are the quantum-mechanical analogous of the classical velocities. In this way, Breit wrote the quantum current that has $\psi\bar{\psi}$ as its density as simply $(\gamma_i\psi\bar{\psi})$. We also write the tensor:

$$T_{ik} = \rho\, u_i\, u_k$$

as

$$\tfrac{1}{2}\,(\psi\, \gamma_i\, p_k\, \psi^* + \psi^*\gamma_k\, p_i\, \psi).$$

The typical term $e^2/r\ u_i u_i'$ for the interaction energy becomes $e^2/r\ \gamma_i\gamma_i'$.

We would now like to treat γ_i as the direction cosines of the new quantum-mechanical matrix. From the formula:

$$1 = -\, g_{ik}\, u_i\, u_k\ ,$$

we get:

(3) $$(\gamma_i\gamma_k)_{\text{symmetric}} = \frac{\gamma_i\gamma_k + \gamma_k\gamma_i}{2} = g_{ik}\ ,$$

because g_{ik} is symmetric and means $u_i^2\gamma_i$.

In general, one has:

(4) $$\gamma_i\gamma_k = (\gamma_i\gamma_k)_{\text{sym.}} + (\gamma_i\gamma_k)_{\text{antisym.}} = g_{ik} + a_{ik}\ ,$$

where g_{ik} is a matrix that is equivalent to the usual g_{ik}, and a_{ik} is the anti-symmetric part of the product $\gamma_i\gamma_k$. As a generalization of the known representation:

(*) G. Breit, Proc. Nat. Acad. Sci. **14** (1928), 553.

$$g_{ik} = h_i^{\alpha} h_k^{\beta}$$

(where the h_i^{α} are unit vectors), we let the quantities:

$$\gamma_i \gamma_k = g_{ik}$$

represent the basic tensor of matrix mechanics.

One further has:

$$ds^2 = \gamma_i \gamma_k ; \qquad ds = \gamma_i \, dx_i ,$$

and this suggests the restriction to linear forms.

We thus make a two-step conversion of the usual metric. First, we introduce the $\gamma_i^{\alpha\beta}$ – i.e., the matrix components – in place of the quantities h_i; our metric then becomes asymmetric. Let it be remarked that in his recent work Einstein has preferred to characterize the usual metric, not by g_{ik} , but by h_i^{α}. Matrix geometry perhaps corresponds to four "star-like" organized worlds. The asymmetric metric was also, as is known, tested for the construction of relativistic electrodynamics.

With this Ansatz, we immediately write down the second-order equation (not the first-order one of Dirac) as the operator translation of the fundamental geometric relation:

$$(\gamma_i \, \gamma_k \, u_i \, u_k + 1)\,\psi = 0. \tag{5}$$

Now, "1" + symmetric part of $\gamma_i \, \gamma_k$ times the symmetric part of $u_i \, u_k$ gives simply:

$$1 + g_{ik} \, u_i \, u_k = S ;$$

i.e., the diagonal terms of the operator of the usual Schrödinger equation. However, the product of the anti-symmetric parts of:

$$\gamma_i \, \gamma_k \text{ and } u_i \, u_k \qquad \left((u_i u_k)_{\text{ant.}} = \frac{h}{i} \frac{e}{m^2 c^2} F_{ki} \right)$$

adds precisely nine spin terms; i.e., the magneto-electric moment times the electromagnetic field. With a suitable normalization, we thus write (5) as:

$$(S + \mu \, F)\,\psi = 0 \tag{5a}$$

With that, one also acquires the meaning of the a_{ik} as the quantum-mechanical analogues of the magneto-electric moments μ. The anti-symmetric part of the metric is then the moment, while in the usual asymmetric metric (not the matrix metric), according to Einstein's investigations, it was the electromagnetic field $dh \, F_{ik}$ itself. The extension of the results to the N-body problem obviously comes about by the introduction of more general matrices $\left|\gamma_i^{\alpha\beta}\right|$. We consider all N particles, if α, β run, not from 1 to 4, but from

1 to n, where n is the number of degrees of freedom. The Dirac matrices are then extensions of the Pauli spin matrices in precisely this sense. The Pauli equations are special cases of the general equations for α, β = 1, 2 and the Dirac equations, for i, α, β = 1, …, 4. We would like to remark that this method, which seems to be a generalization of the Pauli-Dirac work, has a certain parallel to the research of Darwin, Frenkel (*), and others. Namely, one can seek to use ψ-tensors instead of ψ-components (as Dirac and Pauli did). Anti-symmetric tensors $\psi_{ik\dots}$ of rank N are necessary for the N-body problem. Raising the rank of ψ corresponds, in some way, to extending the ranks of the γ_i matrices. We would not like to go further into the question of the equivalence of the two methods here (which should clearly be answered in the affirmative).

§ 2. It is not unnatural to pose the following question: What do our matrices and equations yield in the limit $n \to \infty$? The response to this question has an immediate relationship with the problem of quantum electrodynamics. It seems that the reasonable generalization of the theory of an infinite number of degrees of freedom would lead to precisely this objective. However, we must use the greatest care, since the limited mechanical methods were constructed especially for singular points of the general field. With all of that, one cannot deny that that quantum electrodynamics is a generalization of the theory of a finite number of degrees of freedom, such that the generalization that we spoke of seems suitable. Jordan and Pauli, and then also Mie, have already presented equations that could serve as functional extensions of the Schrödinger equation. Ignoring the particular criticism that the results of the aforementioned papers are doubtful, we remark that their equations do not yield the spin effects. The examination of the Dirac equations is then to be simply converted into function space (i.e., Hilbert space), which is already one degree more correct – so to speak – than the aforementioned ones (even though the methods in this notice have less in common with the cited papers). For the sake of intuitiveness, we first take the elementary Einstein geometry with:

$$g_{ik} = \sum h_i^\alpha h_k^\alpha$$

(summed over α). This case can be regarded as degenerate, since all of the geometries in the family collapse to a single one. The electron, with its spin, then arises when the fundamental tensor is asymmetric. If the number of components varies continuously then we must introduce an integration over α instead of a summation. We write:

(6) $$g_{ik} = \int h_i(\alpha)\, h_k(\alpha)\, d\alpha .$$

In ordinary planar space:

$$g_{ik} = \delta_{ik} = \begin{cases} 1 & i = k, \\ 0 & i \neq k, \end{cases}$$

i.e., the fundamental tensor defines an identity tensor. If the Euclidian character of space remains preserved under the passage to the limit then we have:

(*) J. Frenkel, Zeit. f. Phys. **47** (1928), 819.

(6a) $$\int h_i(\alpha)\, h_k(\alpha)\, d\alpha = \delta_{ik} ,$$

which is nothing but the orthogonality condition for the functions h_i . Formula (6) then ascribes a curvature to the function space. The development in orthogonal functions is parallel to the decomposition of vectors into orthogonal components. In our matrix space, where 16 linearly-independent matrices exist (e.g., Dirac, Neumann), we decompose into 16 "matrix unit vectors," which correspond to the 16 "dimensions." One can also exhibit an analogy between the contraction of one pair of indices and operations on orthogonal functions. The generalization of the non-degenerate case – i.e., matrix geometry – is much more difficult, since here all indices *i*, *k* also run to infinity. We thus have a Hilbert space with infinitely many coordinates (the corresponding mathematical tools are poorly constructed; from a purely mathematical standpoint, the treatment of continuously infinite, rather than countable infinite sets is especially risky); an asymmetric metric would yield the desired equation by applying the fundamental form – regarded as an operator – to a quantity ψ.

As a result of the quantum-mechanical interpretation, we can also use the amplitudes $\varphi(r_i)$ for the orthogonal functions (in which, *i* appears as the number of points), which are the so-called quantized wave amplitudes; we must then always write products of φ_i and φ_k^+ (adjoint quantities), and not simply $\varphi_i\ \varphi_k$. The same Hermitization also seems to be required in the fundamental metric form – thus, in formulas (4) and (6) – and we get a link to Jordan's ideas in his construction of quantum electodynamics. Namely, we let the anti-symmetric part of the product be $\varphi_i\ \varphi_k^+$ (not forgetting the intended integration, such that everything is written only symbolically); that is:

$$\varphi_i\ \varphi_k^+ - \varphi_k^+\ \varphi_i = d_{ik} ,$$

the anti-symmetric part of the fundamental tensor of this function space. Along with every fundamental tensor, we shall also define a substitution operator. On the whole, then the anti-symmetric α_{ik} is defined by conditions that are likewise peculiar to the symmetric part (as an identity tensor and a substitution operator). Our α_{ik} is nothing but the Dirac function δ_k^i, because the latter is indeed determined by the same conditions. It is known that:

$$\int dv\ f_i(r)\ \delta_k^i = \int dv\ f(r^i)\ \delta(r^i - r^k) = f(r^k) = f_k(r).$$

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