"Zur Theorie der Variations-Rechnung und der Differential-Gleichungen," J. f. d. reine u. angew. Math. **17** (1837), 68-82.

4.
On the theory of the calculus of variations and differential equations.

(By Prof. *C. G. J. Jacobi* in Königsberg in Pr.)

(Excerpt from a paper on the same topic that was presented on 29 November 1836 to Herrn Professor *Enke*, Secretary of the mathematical class of the Academy of Science in Berlin.)

Translated by D. H. Delphenich

It is my intent to fill a large and essential gap in the calculus of variations. Namely, in the problems of the greatest and smallest that depend upon the calculus of variations, one can give no general rule by which one could recognize whether a solution was actually a greatest, or a smallest, or neither. Indeed, one recognizes that a criterion for that would depend upon whether certain systems of differential equations have integrals that remain finite over the entire interval over which the integral that is supposed to be a maximum or minimum is extended. However, one cannot find these integral themselves, and can in no way discuss the situation of whether one knows if they do or do not remain finite inside the given limits. However, I have remarked that this integrals will always be given when one has integrated the differential equations of the problem – i.e., the differential equations that must be fulfilled in order for the first variation to vanish. If one has obtained expressions for the desired functions by integrating these differential equations that contain a number of arbitrary constants then their partial differential quotients with respect to those arbitrary constants will give the integrals for the new differential equations that one must integrate in order to determine the criterion for greatest and smallest.

In order to consider the simplest case, let the given integral be $\int f\left(x, y, \frac{\partial y}{\partial x}\right)\partial x$; y will be determined by the differential equation $\frac{\partial f}{\partial y} - \partial \cdot \frac{\frac{\partial f}{\partial y'}}{\partial x}$ = 0, in which we have substituted y' for $\frac{\partial y}{\partial x}$. The expression for y that is given by integrating these equations will include two arbitrary constants that I would like to call a and b. If $w = \delta y$, $w' = \partial w / \partial x$ then the second variation will be:

$$\int\left(\frac{\partial^2 f}{\partial y^2}ww+2\frac{\partial^2 f}{\partial y\,\partial y'}ww'+\frac{\partial^2 f}{\partial y'^2}w'w'\right)\partial x\,,$$

and in order for this to be a maximum or minimum, it would be necessary for $\partial^2 f / \partial y'^2$ to always have the same sign. However, in order to get the complete criterion for the maximum or minimum, one must still know the complete expression for a function v that is required to satisfy the differential equation:

$$\frac{\partial^2 f}{\partial y'^2}\left(\frac{\partial^2 f}{\partial y^2}+\frac{\partial v}{\partial x}\right) = \left(\frac{\partial^2 f}{\partial y\,\partial y'}+v\right)^2;$$

one can see this in *Lagrange's* theory of functions or *Dirksen's* calculus of variations. (*Ohm's* calculus of variations is not precise in that theory.) I now find the complete expression for v as follows: Let $u = \alpha\frac{\partial y}{\partial a}+\beta\frac{\partial y}{\partial b}$, in which $\frac{\partial y}{\partial a}$, $\frac{\partial y}{\partial b}$ mean the partial differential quotients of y with respect to the arbitrary constants a, b that enter into y, and α, β are new arbitrary constants, so:

$$v = -\left(\frac{\partial^2 f}{\partial y\,\partial y'}+\frac{1}{u}\cdot\frac{\partial^2 f}{\partial y'^2}\frac{\partial u}{\partial x}\right)$$

will be the desired expression for v, which will contain an arbitrary constant β / α.

The case in which higher-order differentials enter under the integral sign is more difficult. Let $\int f(x, y, y', y'')\,\partial x$ be the expression that must take on a maximum or a minimum, in which, once more, $y' = \frac{\partial y}{\partial x}$, $y'' = \frac{\partial^2 y}{\partial x^2}$, so y will be the integral of the differential equation:

$$\frac{\partial f}{\partial y}-\partial\cdot\frac{\frac{\partial f}{\partial y'}}{\partial x}+\partial^2\cdot\frac{\frac{\partial f}{\partial y''}}{\partial x^2} = 0,$$

which contains four arbitrary constants a, a_1, a_2, a_3 . If, once more, $\delta y = w$, $\delta y' = w'$, $\delta y'' = w''$ then the second variation will become:

$$\int\left(\frac{\partial^2 f}{\partial y^2}ww+2\frac{\partial^2 f}{\partial y\,\partial y'}ww'+2\frac{\partial^2 f}{\partial y\,\partial y''}ww''+\frac{\partial^2 f}{\partial y'^2}w'w'+2\frac{\partial^2 f}{\partial y'\,\partial y''}w'w''+2\frac{\partial^2 f}{\partial y''^2}w''w''\right)\partial x\,.$$

In order for this to be a maximum or minimum, $\frac{\partial^2 f}{\partial y''^2}$ must always have the same sign. However, in order to get the complete criterion, one must integrate the following system of differential equations, as one can see from *Lagrange's* theory of functions:

$$\left(\frac{\partial^2 f}{\partial y^2}+\frac{\partial v}{\partial x}\right)\left(\frac{\partial^2 f}{\partial y'^2}+\frac{\partial v_2}{\partial x}+2v_1\right) = \left(\frac{\partial^2 f}{\partial y\,\partial y'}+v+\frac{\partial v_2}{\partial x}\right)^2,$$

$$\frac{\partial^2 f}{\partial y''^2}\left(\frac{\partial^2 f}{\partial y^2}+\frac{\partial v}{\partial x}\right) = \left(\frac{\partial^2 f}{\partial y\,\partial y''}+v_1\right)^2,$$

$$\frac{\partial^2 f}{\partial y''^2}\left(\frac{\partial^2 f}{\partial y'^2}+\frac{\partial v_2}{\partial x}+2v_1\right) = \left(\frac{\partial^2 f}{\partial y'\,\partial y''}+v_2\right)^2.$$

The three functions v, v_1, and v_2 are determined from these three first-order differential equations, which present a truly daunting sight, and whose complete expression must contain three arbitrary functions. I have found their general integrals, as follows:

Let:

$$u = \alpha\frac{\partial y}{\partial a}+\alpha_1\frac{\partial y}{\partial a_1}+\alpha_2\frac{\partial y}{\partial a_2}+\alpha_3\frac{\partial y}{\partial a_3}, \qquad u_1 = \beta\frac{\partial y}{\partial a}+\beta_1\frac{\partial y}{\partial a_1}+\beta_2\frac{\partial y}{\partial a_2}+\beta_3\frac{\partial y}{\partial a_3},$$

or let u, u_1 be linear expressions in the partial differential quotients of y with respect to the arbitrary constants that it contains. The eight constants α, α_1, α_2, α_3, β, β_1, β_2, β_3 are not taken to be entirely arbitrary, but a certain condition must exist between the six quantities $\alpha\beta_1 - \alpha_1\beta$, $\alpha\beta_2 - \alpha_2\beta$, $\alpha\beta_3 - \alpha_3\beta$, $\alpha_2\beta_3 - \beta_2\alpha_3$, $\alpha_3\beta_1 - \alpha_1\beta_3$, $\alpha_1\beta_2 - \alpha_2\beta_1$ that are composed from them, although I would not like to go into that in detail here. Thus, the general expressions for v, v_1, v_2 that I have found will be the following ones:

$$v_2 = -\frac{\partial^2 f}{\partial y'\,\partial y''}-\frac{\partial^2 f}{\partial y''^2}\cdot\frac{u\dfrac{\partial^2 u_2}{\partial x^2}-u_1\dfrac{\partial^2 u}{\partial x^2}}{u\dfrac{\partial u_2}{\partial x}-u_1\dfrac{\partial u}{\partial x}},$$

$$v_1 = -\frac{\partial^2 f}{\partial y\,\partial y''}+\frac{\partial^2 f}{\partial y''^2}\cdot\frac{\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial^2 u_2}{\partial x^2}-\dfrac{\partial u_2}{\partial x}\cdot\dfrac{\partial^2 u}{\partial x^2}}{u\dfrac{\partial u_2}{\partial x}-u_1\dfrac{\partial u}{\partial x}},$$

$$v = -\frac{\partial v_1}{\partial x}-\frac{\partial^2 f}{\partial y\,\partial y''}-\frac{\partial^2 f}{\partial y''^2}\cdot\frac{\left(u\dfrac{\partial^2 u_2}{\partial x^2}-u_1\dfrac{\partial^2 u}{\partial x^2}\right)\left(\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial^2 u_2}{\partial x^2}-\dfrac{\partial u_2}{\partial x}\cdot\dfrac{\partial^2 u}{\partial x^2}\right)}{\left(u\dfrac{\partial u_2}{\partial x}-u_1\dfrac{\partial u}{\partial x}\right)^2}.$$

Since one identical equation exists between the six quantities $\alpha\beta_1 - \alpha_1\beta$, etc., in addition to which, one condition is given between them, and only their ratios enter into the expressions for v, v_1, v_2, they represent the effect of three arbitrary constants, which was desired.

The general theory, in which one includes differentials of y up to any order under the integral sign, will be derived with no difficulty from a remarkable property of a special class of linear differential equations. These $2n^{\text{th}}$-order linear differential equations have the form:

$$0 = Ay + \frac{\partial \cdot A_1 y'}{\partial x} + \frac{\partial^2 \cdot A_2 y''}{\partial x^2} + \frac{\partial^3 \cdot A_3 y'''}{\partial x^3} + \cdots + \frac{\partial^n \cdot A_n y^{(n)}}{\partial x^n} = Y,$$

in which $y^{(m)} = \dfrac{\partial^m y}{\partial x^m}$, and A, A_1, etc., are given functions of x. If y is any integral of the equation $Y = 0$, and one sets $u = t\,y$ then the expression:

$$y\left(\frac{\partial \cdot A_1 y'}{\partial x} + \frac{\partial^2 \cdot A_2 y''}{\partial x^2} + \frac{\partial^3 \cdot A_3 y'''}{\partial x^3} + \cdots + \frac{\partial^n \cdot A_n y^{(n)}}{\partial x^n}\right) = y\,U,$$

in which $u^{(m)} = \dfrac{\partial^m u}{\partial x^m}$, will become integrable; i.e., one can give its integral without knowing t, and that integral will again have the form of Y, except that n is 1 smaller; namely, one will have:

$$\int yU\,\partial x = B\,t' + \frac{\partial \cdot B_1 t''}{\partial x} + \frac{\partial^2 \cdot B_2 t'''}{\partial x^2} + \cdots + \frac{\partial^{n-1} \cdot B_{n-1} t^{(n)}}{\partial x^{n-1}},$$

in which $t^{(m)} = \dfrac{\partial^m t}{\partial x^m}$, and the functions B can generally be given in terms of u and the functions A and their differentials. The proof of this theorem presents no difficulty. I have found the general expressions for the functions B; for the application that we have posed, it will suffice to just prove that $\int yU\,\partial x$ has the given form, without it being necessary to know the functions B themselves.

The metaphysics of the results that were found, if I might appeal to a French expression, rests upon roughly the following considerations: As is known, one can give the first variation the form $\int V\,\delta y\,\partial x$, in which $V = 0$ is the equation to be integrated. The second variation thus takes the form $\int \delta V\,\delta y\,\partial x$. If the second variation is to not change sign then it must not be capable of vanishing, or the equation $\delta V = 0$, which is linear in δy, can have no integral δy that fulfills the conditions that δy is subject to according to the nature of the problem. One sees from this that the equation $\delta V = 0$ plays a key role in this investigation, and in fact, one soon recognizes its connection with the differential equations that must be integrated for the criterion of max. or min. In addition, one sees immediately that a value of δy that fulfills the differential equation $\delta V = 0$ will be that partial differential quotient of y with respect to which one of the arbitrary constants is taken that include y as an integral of the equation $V = 0$. One will then obtain the general expression for the integral δy of the differential equation $\delta V = 0$ when one defines a linear expression with all of those partial differential quotients. However, the equation $\delta V = 0$, whose integrals one knows in that way, can, as one can see, be brought into the form of the equation $Y = 0$ above when one writes δy for y in it, and by means of the given properties of that kind, arrives at equations that transform the second variation $\int \delta V\,\delta y\,\partial x$ into another expression by continued partial integration that contains a complete square under the integral sign that is just the transformation of the second variation that one has thus hoped to achieve. If, e.g., the integral above $\int f(x, y, y', y'')\,\partial x$ is given, and one preserves the given meaning of u and u_1 for this case, then δV will take on the form:

$$\delta V = A\,\delta y + \frac{\partial \cdot A_1 \delta y'}{\partial x} + \frac{\partial \cdot A_2 \delta y''}{\partial x^2},$$

and one will have $\delta V = 0$ for $\delta y = u$. If one sets $\delta y = u\,\delta' y$ then, from the general theorem above, one will get:

$$\int \delta V\,\delta y\,\partial x = \int u\,\delta V\,\delta' y\,\partial x$$

$$= \left(B\,\delta' y' + \frac{\partial \cdot B_1\,\delta' y''}{\partial x}\right)\delta' y - \int \left(B\,\delta' y' + \frac{\partial \cdot B_1\,\delta' y''}{\partial x}\right)\delta' y\,\partial x\,.$$

If one now sets the last integral equal to $\int V_1\,\delta' y'\,\partial x$ then the equation $V_1 = 0$ will be fulfilled when one sets $\delta' y = u_1 / u$, so $\delta' y' = \frac{u u_1' - u_1 u'}{u^2}$. One then continues to use the same method when one sets $\delta' y' = \frac{u u_1' - u_1 u'}{u^2} \cdot \delta'' y$, with which, by the same theorem:

$$\int V_1\,\delta' y'\,\partial x = \int V_1 \left(\frac{u u_1' - u_1 u'}{u^2}\right)\delta'' y\,\partial x = C\,\delta'' y \cdot \delta'' y' - \int C(\delta'' y')^2\,\partial x\,,$$

and in the last transformation, the arbitrary variation enters under the integral sign only as a square. One easily sees, moreover, that $B_1 = u^2 A_2$, $C = \left(\frac{u u_1' - u_1 u'}{u^2}\right)^2 B_1$, and thus $C = \left(\frac{u u_1' - u_1 u'}{u^2}\right)^2 A_2$.

Furthermore, $A_2 = \frac{\partial^2 f}{\partial y''^2}$, so C will always have the same sign as $\frac{\partial^2 f}{\partial y''^2}$, which is always positive for a minimum, but must always be negative for the maximum. As is known, one must now examine whether $\partial'' y'$ cannot become infinite between the limits of the integration, where one will be put into that position when one knows the functions u, u_1, which one will know as soon as y or the complete integral of the equation $V = 0$ is given.

If the analysis that was suggested in the foregoing requires rather deep speculations into integral calculus then the criterion that is derived from them of whether a solution gives a maximum or a minimum at all becomes very simple. I would like to consider the case in which y, along with its differentials up to the n^{th}, are present under the integral sign, and the boundary values of $y, y', \ldots, y^{(n-1)}$ are given, along with the boundary itself. If one substitutes these boundary values in the $2n$ integral equations, with their $2n$ arbitrary constants, then the arbitrary constants will determined. However, because the solution of equations is necessary for that, there will be, as a rule, several ways of doing that determination, such that one will get several curves that obey the same boundary conditions and the same differential equation. If one has chosen one of them then one will consider the one boundary point to be fixed, and go from it to the following point along the curve. If one takes one of these following points to be the other boundary point then, from what was said above, it will be possible for one to lay other curves through it and the first one, for which $y', \ldots, y^{(n-1)}$ will have the same values at these two limits, and which will satisfy the given differential equation. Now, as long as one arrives at a point when one proceeds along the curve for which one of the other curves coincides with it (or, as one can also say, comes infinitely close to it), it will be the limit up to which, or beyond which, one cannot extend the integration if one is to find a maximum or minimum. However, when one does not extend the integral up to these limits, a maximum or a minimum will always be found if one assumes that $\frac{\partial^2 f}{\partial y^{(n)2}}$ always has the same sign between the limits.

In order to make this clear with an example, I would like to consider the principle of least action for the elliptical motion of a planet.

As *Lagrange* believed, the integral that is considered in the principle of least action can never be a maximum. However, it will also in no way always be a minimum, but certain restrictions on its limits would be necessary for that, which would be given by the general rule above, failing which, the integral would be either a maximum or a minimum. The planet (Fig. 1) begins to move from a, where a lies between the perihelion and the aphelion. Let the other endpoint be b; if $2A$ is the major axis then f is the Sun. As is known, one will then get the other focal point of the ellipse as the intersection of two circles that are described by the centers a and b and the radii $2A - af$, $2A - bf$. The two intersection points of the circles give two different solutions of the problem that can coincide in just one only when the circles contact; i.e., when ab goes through the other focal point. If one then draws the chord of the ellipse aa' from a through the other focal point of the ellipse f' then as a result of the given rule, the other limit point b must lie between a and a_1 if the ellipse is to actually make the integral that is considered in the principle of least action become smallest. If b falls upon a_1 then the second variation of the integral can indeed not become negative, but 0, such that the variation of the integral can be of the third order, and this positive, as well as negative. If the starting point a lies between the aphelion and the perihelion then the external point a' will be determined by the chord of the ellipse that one draws from a through the Sun f. Thus, if a and a' (Fig. 2) are the limit points then one will obtain infinitely many solutions of the problem by rotating the ellipse around afa'. Thus, if the second limit point in the latter case lies above a' then it will give a *courbe à double courbure* between the two given limits, for which $\int v\partial s$ will be smaller for the ellipse.

I would like to take this opportunity to say a few words about the variation of the double integral, whose theory is capable of taking on greater elegance than it already has from the work of *Gauss* and *Poisson*. In order to give a presentation of the kind that would seem appropriate to me in order to express the variation of the double integral, I would like to assume the simplest case, in which one considers $\delta\iint f(x, y, z, p, q)\,\partial x\,\partial y$, where $p = \partial z / \partial x$, $q = \partial z / \partial y$.

Let w be the variation of z, so one will have:

$$\delta\iint \partial x\cdot\partial y\; f = \iint \partial x\cdot\partial y\left(\frac{\partial f}{\partial z}w+\frac{\partial f}{\partial p}\cdot\frac{\partial w}{\partial x}+\frac{\partial f}{\partial q}\cdot\frac{\partial w}{\partial y}\right).$$

The method that is applied to simple integrals consists of dividing the expression under the integral sign into two parts, one of which is multiplied by w, while the other of which is the element of an integral. The former must be set equal to zero under the integral if the variation is to vanish; the latter can be integrated, and one can let its integral vanish. I then divide the expression under the double sign into a part that is multiplied by w and another one that is the element of a double integral; that is, if $u = aw$ then I set:

$$\frac{\partial f}{\partial z}w+\frac{\partial f}{\partial p}\cdot\frac{\partial w}{\partial x}+\frac{\partial f}{\partial q}\cdot\frac{\partial w}{\partial y} = A\,w + \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}.$$

If one compares the terms that are multiplied by w, $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$ then one will obtain:

$$\frac{\partial f}{\partial z} = A + \frac{\partial a}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial a}{\partial y}\cdot\frac{\partial v}{\partial x}, \qquad \frac{\partial f}{\partial p} = a\frac{\partial v}{\partial y}, \qquad \frac{\partial f}{\partial q} = -\,a\frac{\partial v}{\partial x},$$

from which it will follow that:

$$A = \frac{\partial f}{\partial z}-\frac{\partial\left(\dfrac{\partial f}{\partial p}\right)}{\partial x}-\frac{\partial\left(\dfrac{\partial f}{\partial q}\right)}{\partial y},$$

which will give the known partial differential equation when it is set equal to zero that is derived here in a completely symmetric way. The function v must fulfill the equation $\frac{\partial f}{\partial p}\cdot\frac{\partial v}{\partial x}+\frac{\partial f}{\partial q}\cdot\frac{\partial v}{\partial y} = 0$. If one sets $A = 0$ then one will have:

$$\delta\iint \partial x\partial y\; f = \iint \partial x\partial y\left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}\right) = \iint \partial v\,\partial u\,,$$

which must vanish when it is taken at the given limits. When z is given at the limits, w will vanish at the limits, and thus $u = w$, as well, and therefore one will have $\iint \partial v \, \partial u = 0$. If the limiting values of z are completely arbitrary then v must vanish at the limits, or if $v = 0$ means the limit curve then the arbitrary functions that enter into the integral of equation $A = 0$ will be determined such that $\frac{\partial f}{\partial p} \cdot \frac{\partial v}{\partial x} + \frac{\partial f}{\partial q} \cdot \frac{\partial v}{\partial y} = 0$, etc.

Let us get back to the maximum and minimum. It is unfortunate that so much confusion arises in the use of those words. One says: "let an expression be a maximum or a minimum" when one would like to say merely that its variation vanishes, regardless of whether a maximum or minimum is present. One says: "let a quantity be a maximum" when one would like to say only that it is not a minimum. That is what *Poisson* said in his mechanics: For closed surfaces, the shortest line between two given points can be a maximum, regardless of whether one understands that to mean that one can make a long path even longer by bends that can be infinitely small. Admittedly, the shortest path gives a relative minimum when the condition that is posed by my general rule above is fulfilled, namely, that there do not exist two other points along the curve between the two endpoints, between which, one can draw a second infinitely close shorter curve. However, in other cases, the length is not a maximum, but either a maximum or a minimum. I have proved that the shortest line really is a shortest line between any two points of a surface that has two opposite curvatures at each point.

The investigations that were suggested above on the criterion for the greatest and smallest in the isoperimetric problem fill an essential gap in one of the most beautiful parts of mathematics; in addition, they are noteworthy in the integral calculus that is applied in them. However, the following investigations would be more far-reaching in all of science, so I shall allow myself to give a brief hint of them.

Hamilton has shown that the problems of mechanics in which the law of *vis viva* is true can lead back to the integration of a first-order partial differential equation. He actually required the integration of two such partial differential equations; however, one easily shows that it suffices to know any complete integral of one of them. One also extends his results easily to the case in which the force functions – i.e., the function whose partial differential quotients give the force – contains time explicitly. The law of *vis viva* is not true for that case, but the principle of least action still is. It seems as if less can be accomplished by this conversion to a partial differential equation, since from *Pfaff's* method in the treatises of his Academy (and up to now, one knows nothing else about the integration of first-order partial differential equations in more than three variables), the integration of the one partial differential equation to which the dynamical problem comes down is much more difficult than the integration of the system of the originally-given, ordinary differential equations of motion. In fact, if, as one can do with no difficulty, one extends *Hamilton's* investigation to all first-order partial differential equations then it will be, conversely, a meaningful discovery in the theory of first-order partial differential equations that they can always be converted to the integration of a single system of ordinary differential equations, which *Pfaff's* method did not succeed in doing, up to now. That can be important for the integration of the differential equations of mechanics themselves only when one confirms that the system of ordinary differential equations to which the first-order partial differential equations come down is capable of a special treatment that distinguishes them from other differential equations. Whether or not *Hamilton* sought to make many applications of his *new methods*, as he called them, nothing has been said of them since then, and therefore no essential use of his remarkable theorems has been made, either. However, in fact, *Lagrange* already remarked about the first-order partial differential equations in *three* variables that he restricted himself to, and whose integration belongs to his most beautiful and celebrated discoveries, that when one knows *one* integral of the system of three first-order ordinary differential equations in four variables into which the problem has been converted, all that one must integrate are two first-order differential equations, each of which is in two variables. However, in general, one would have to integrate a *second*-order differential equation in two variables that one can thus always reduce to first order for that special system of ordinary differential equations. If the first-order partial differential equation in three variables does not contain the unknown function itself, but only its two differential quotients, then one will have only two first-order differential equations in three variables to integrate, and if one knows *one* integral of them then, from *Lagrange's* method, one must perform only two quadratures, while, in general, there would still be one first-order differential equation to integrate. The latter case occurs in mechanics; viz., the first-order partial differential equations to which the dynamical problem reduces never contain the unknown function itself. Thus, one can already deduce some new, most remarkable, theorems from *Lagrange's* process for three variables. Namely, it follows from it in a completely general way that when any problem of mechanics for which the law of *vis viva* is true depends upon a second-order differential equation, and one know *one* integral from that law, such that the problem comes down to the integration of a first-order ordinary differential equation in two variables then one can always integrate the latter; i.e., one can find its multiplier by a general, well-defined rule. One such mechanics problem is – e.g. – the motion of a body in a plane that is drawn between two fixed centers. *Euler* found a second integral for it, in addition to the *vis viva*, with ease; however, the first-order differential equation to which he arrived was so complicated that his great dauntlessness was shown by the fact that he addressed the integration of it, and his success in that endeavor belongs to his most celebrated masterpieces. However, this integration was achieved by means of the aforementioned general rule with no further artifices. Maybe half a year ago, I lectured to the Paris Academy on formulas that relate to the case of the free motion of a point in a plane, which generally reduce the problem to quadratures when one knows another integral besides the *vis viva*. These formulas can be immediately extended to the motion of a point on a given surface.

However, in order for an application of these considerations to more complicated systems to be possible, it is necessary to extend *Lagrange's* method for the integration of first-order partial differential equation in three variables to any number of variables. *Pfaff*, who saw that as being connected with insurmountable obstacles, felt that on that basis, one should abandon that method entirely. He considered the problem to be a special case of a much more general one whose fortunate solution belongs to the most important ways by which integral calculus has been enriched. However, the problem of the integration of first-order partial differential equations admits some simplifications in comparison to the general problem that *Pfaff* considered that escaped him, and which he could not find using his methods. I have succeeded in removing the difficulties that stand in the path of the generalization of *Lagrange's* method, and have thus founded a new theory of first-order partial differential equations for any number of variables that offers the most essential advantage for their integration and finds immediate application to the problems of mechanics. The following comments might suffice here.

First-order partial differential equations and the isoperimetric problem, in which, the differential quotients of the unknown functions appear under the integral sign only up to the first order, depend upon the same analysis, such that every such isoperimetric problem can also be regarded as the integration of a first-order partial differential equation. Among these isoperimetric problems, one can also address ones in which the expression that must be a maximum or minimum – or, more generally, whose variation should vanish – are given, not as an integral from the outset, but by a first-order differential equation. Conversely, one can also approach the integration of a first-order partial differential equation as such an isoperimetric problem. As a result of the principle of least action, the motion of a system of mutually-attracting bodies can be considered to be an isoperimetric problem of the stated kind, which can be subject to parallel forces, in addition to forces that are directed from fixed or moving centers, provided that the bodies of the system do not react to the last center, and its motion will be assumed to be otherwise known. Thus, such a mechanical problem can always be posed as the integration of a first-order partial differential equation, as well. That integration will depend upon that of a system of ordinary differential equation that agrees with the known differential equations of mechanics, but certain simplifications will be possible, as they are for a first-order partial differential equation. Namely, one can arrange that every integral that one finds represents the effect of *two* integrations by a special form of the procedure and a special choice of the quantities that one introduces as variables. In order to make this clearer, I would like to say that a system of differential equations has order n when one can convert it into an n^{th}-order ordinary differential equation in two variables by eliminating the remaining variables. For first-order partial differential equations that do not include the unknown function itself, but only its partial differential quotients, as well as for the isoperimetric problems of the stated kind, in which the expression whose first variation should vanish is given as an integral, and therefore, also for the stated mechanical problems, the course of operations to be observed, and the advantage that one gains by it, can now be given as follows: Let the system of ordinary differential equations upon which the problem depends be of order $2n$. One knows an integral of it, so the problem can be converted into a system of differential equations of order $n - 2$ by a certain choice of quantities that one introduces as variables. If one further knows another integral of that system then it can be converted into a system of order $2n - 4$ by a new choice of variables, and so on, until one no longer has any differential equations left to integrate. All of the operations in addition to the ones that are performed are merely quadratures. For the sake of clarity, I remark that I call an equation $U = a$ an *integral* of a system of ordinary differential equations when a is an arbitrary constant that does not enter into U, and U is an expressions such that the differential expression dU will become zero identically when one uses it.

As an example of the general method, I take a mechanical problem that I have already had the honor of presenting to the Academy in a previous paper. Namely, there are cases in the motion of heavenly bodies – such as, e.g., the Moon or a comet that passes close to Jupiter – in which one is so far from elliptic motion that one can establish no process of approximation for the integration of the differential equations of motion that would have any scientific value. It is therefore of great importance to find other motions that are capable of a simple treatment and can better approach the case of nature. For this, one can attempt to choose the motion of a massless point that is attracted to two bodies that rotate uniformly and with the same angular velocity around their common center of mass. For the Moon, one can still assume for the approximation problem that the three bodies move in a plane. One will then have two second-order differential equations that, since the forces include time explicitly, and therefore either the law of areas or the law of *vis viva* will be true, will represent the effect of a fourth-order differential equation in two variables. Even when both the law of areas and the law of *vis viva* are not true, I have still shown that a certain combination of them can also find its place here. However, this integral that I found did not merely convert the problem into one of order three, but the application of the general method to that case showed that, by a suitable choice of variables, one could convert the problem to second-order differential equation in two variables, for which the same method would illuminate the fact that one would again need to know only a single integral. It was then by means of that method that I found that the integration of the fourth-order differential equation was converted by one integral into that of finding a single integral of a second-order differential equation when one requires that all of the remaining integrations should be just quadratures.

The entire course of suggested operations depends upon the integrals that one can find in each case; the choice of variables likewise depends upon them, and also requires integrations of differential equations in its own right, but always in such a way that the system of differential equations can be converted into another one whose order is two lower by *one* integral that has been found. Moreover, the differential equations that allow one to determine the choice of variables will be easy to integrate in many cases. Provided that one does not overlook the simple integrals that one can find, one can be certain of converting the problem in the stated way, if not completely to quadratures, then as far as its nature will make that possible. Moreover, if the differential equations to which one comes cannot be integrated then one will recognize noteworthy properties of them that can be employed to advantage. Thus, one knows that in the problem that was cited, when one also cannot integrate the second-order differential equations to which it is converted, its two integrals can be found from each other by mere quadratures.

You see, most revered professor, that the results that were quoted in the foregoing brief outline establish a new and important chapter in analytical mechanics whose advantage can be deduced from the special form of the differential equations of mechanics for their integration. We owe this form to *Lagrange*, but up to now, in his hands and those of the analysts that followed him, it only served to render the analytical transformations faster and clearer, and to extend the known general mechanical laws, where that was possible. However, this form now takes on much more important meaning when one shows the precisely the differential equations of that particular form are capable of a special treatment that reduces the difficulties in their integration substantially.

29 November 1836.