"Sul principio dell'ultimo moltiplicatore e suo use come nuovo principio generale di meccanica," Giornale Arcadico di Scienze, Lettere ed Arti **99** (1844), 129-146; *Gesammelte Werke*, v. IV, pp. 511-522.

On the principle of the last multiplier and its use as a new general principle of mechanics

By

PROFESSOR C. G. J. JACOBI

Translated by D. H. Delphenich

I.

We begin by proving a lemma from integral calculus that is important because of its applications to the integration of systems of ordinary differential equations, and mainly the ones whose integration will allow one to determine the motion of a system of material points.

Lemma.

Let $X, X_1, X_2, \ldots, X_n$ *be arbitrary functions of the variables* $x, x_1, \ldots, x_n$*, and let* M *and* u *be two other functions of those same variables that verify the following partial differential equations:*

$$\frac{\partial (M\ X)}{\partial x} + \frac{\partial (M\ X_1)}{\partial x_1} + \cdots + \frac{\partial (M\ X_n)}{\partial x_n} = 0 ,$$

$$X \frac{\partial u}{\partial x} + X_1 \frac{\partial u}{\partial x_1} + \cdots + X_n \frac{\partial u}{\partial x_n} = 0 .$$

Set:

$$u = \alpha ,$$

where α *is an arbitrary constant, and deduce the value of* x_n *from that equation, which one then substitutes in the functions* $X, X_1, \ldots, X_{n-1}$ *and in the quantity:*

$$M_1 = \frac{M}{\dfrac{\partial u}{\partial x_n}} .$$

The function M_1 *of the variables* $x, x_1, \ldots, x_{n-1}$ *verifies an equation that is similar to the one that defines the function M, viz., the equation:*

$$\frac{\partial (M_1 X)}{\partial x} + \frac{\partial (M_1 X_1)}{\partial x_1} + \cdots + \frac{\partial (M_1 X_{n-1})}{\partial x_{n-1}} = 0 .$$

Proof:

The equation to be proved, namely:

$$\frac{\partial (M_1 X)}{\partial x} + \frac{\partial (M_1 X_1)}{\partial x_1} + \cdots + \frac{\partial (M_1 X_{n-1})}{\partial x_{n-1}} = 0 ,$$

can be put into the form:

$$X \frac{\partial \ln M_1}{\partial x} + X_1 \frac{\partial \ln M_1}{\partial x_1} + \cdots + X_{n-1} \frac{\partial \ln M_1}{\partial x_{n-1}} + \frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_{n-1}}{\partial x_{n-1}} = 0 .$$

The quantities $X, X_1, \ldots, X_{n-1}$, and M_1 are regarded as functions of the independent variables $x, x_1, \ldots, x_{n-1}$, but in their original form they also contain the variable x_n, which is given as a function of the other ones by the equation $u = \alpha$. When one regards it in that form, the previous equation can be written in this way:

$$(1) \qquad \left\{ \begin{aligned} & X \frac{\partial \ln M_1}{\partial x} + X_1 \frac{\partial \ln M_1}{\partial x_1} + \cdots + X_{n-1} \frac{\partial \ln M_1}{\partial x_{n-1}} + \frac{\partial \ln M_1}{\partial x_n} \left(X \frac{\partial x_n}{\partial x} + X_1 \frac{\partial x_n}{\partial x_1} + \cdots + X_{n-1} \frac{\partial x_n}{\partial x_{n-1}} \right) \\ & + \frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_{n-1}}{\partial x_{n-1}} + \frac{\partial X}{\partial x_n} \frac{\partial x_n}{\partial x} + \frac{\partial X_1}{\partial x_n} \frac{\partial x_n}{\partial x} + \cdots + \frac{\partial X_{n-1}}{\partial x_n} \frac{\partial x_n}{\partial x} = 0 . \end{aligned} \right.$$

The values of the partial derivatives:

$$\frac{\partial x_n}{\partial x}, \quad \frac{\partial x_n}{\partial x_1}, \quad \ldots$$

are obtained from the equation $u = \alpha$ by means of the formula:

$$\frac{\partial x_n}{\partial x_i} = - \frac{\dfrac{\partial u}{\partial x_i}}{\dfrac{\partial u}{\partial x_n}} .$$

One will then have:

$$X\frac{\partial x_n}{\partial x}+X_1\frac{\partial x_n}{\partial x_1}+\cdots+X_{n-1}\frac{\partial x_n}{\partial x_{n-1}} = -\frac{1}{\dfrac{\partial u}{\partial x_n}}\left(X\frac{\partial u}{\partial x}+X_1\frac{\partial u}{\partial x_1}+\cdots+X_{n-1}\frac{\partial u}{\partial x_{n-1}}\right),$$

or, if:

$$X\frac{\partial u}{\partial x}+X_1\frac{\partial u}{\partial x_1}+\cdots+X_n\frac{\partial u}{\partial x_n} = 0$$

then one will have:

$$X\frac{\partial x_n}{\partial x}+X_1\frac{\partial x_n}{\partial x_1}+\cdots+X_{n-1}\frac{\partial x_n}{\partial x_{n-1}} = X_n\,. \tag{2}$$

In addition, one will have:

$$\frac{\partial X}{\partial x_n}\frac{\partial x_n}{\partial x}+\frac{\partial X_1}{\partial x_n}\frac{\partial x_n}{\partial x_1}+\cdots+\frac{\partial X_{n-1}}{\partial x_n}\frac{\partial x_n}{\partial x_{n-1}} = -\frac{1}{\dfrac{\partial u}{\partial x_n}}\left(\frac{\partial X}{\partial x_n}\frac{\partial u}{\partial x}+\frac{\partial X_1}{\partial x_n}\frac{\partial u}{\partial x_1}+\cdots+\frac{\partial X_{n-1}}{\partial x_n}\frac{\partial u}{\partial x_{n-1}}\right). \tag{3}$$

Now when the equation:

$$-\left(X\frac{\partial u}{\partial x}+X_1\frac{\partial u}{\partial x_1}+\cdots+X_{n-1}\frac{\partial u}{\partial x_{n-1}}\right) = X_n\frac{\partial u}{\partial x_n}$$

is differentiated with respect to x_n and divided by $\dfrac{\partial u}{\partial x_n}$, that will give:

$$-\frac{1}{\dfrac{\partial u}{\partial x_n}}\left(\frac{\partial X}{\partial x_n}\frac{\partial u}{\partial x}+\frac{\partial X_1}{\partial x_n}\frac{\partial u}{\partial x_1}+\cdots+\frac{\partial X_{n-1}}{\partial x_n}\frac{\partial u}{\partial x_{n-1}}\right)$$

$$= \frac{\partial X_n}{\partial x_n}+X\frac{\partial \ln\dfrac{\partial u}{\partial x_n}}{\partial x}+X_1\frac{\partial \ln\dfrac{\partial u}{\partial x_n}}{\partial x_1}+\cdots+X_{n-1}\frac{\partial \ln\dfrac{\partial u}{\partial x_n}}{\partial x_{n-1}}+X_n\frac{\partial \ln\dfrac{\partial u}{\partial x_n}}{\partial x_n}\,.$$

Hence, according to formula (3), one will have:

$$\left\{\begin{aligned}&\frac{\partial X}{\partial x_n}\frac{\partial x_n}{\partial x}+\frac{\partial X_1}{\partial x_n}\frac{\partial x_n}{\partial x_1}+\cdots+\frac{\partial X_{n-1}}{\partial x_n}\frac{\partial x_n}{\partial x_{n-1}}\\ &= \frac{\partial X_n}{\partial x_n}+X\frac{\partial \ln\dfrac{\partial u}{\partial x_n}}{\partial x}+X_1\frac{\partial \ln\dfrac{\partial u}{\partial x_n}}{\partial x_1}+\cdots+X_{n-1}\frac{\partial \ln\dfrac{\partial u}{\partial x_n}}{\partial x_{n-1}}+X_n\frac{\partial \ln\dfrac{\partial u}{\partial x_n}}{\partial x_n}\,.\end{aligned}\right. \tag{4}$$

If one substitutes formulas (2) and (4) in formula (1) then one will get:

$$0 = X\frac{\partial \ln M_1}{\partial x} + X_1\frac{\partial \ln M_1}{\partial x_1} + \cdots + X_n\frac{\partial \ln M_1}{\partial x_n} + \frac{\partial X_n}{\partial x_n} + X\frac{\partial \ln \frac{\partial u}{\partial x_n}}{\partial x} + X_1\frac{\partial \ln \frac{\partial u}{\partial x_n}}{\partial x_1} + \cdots + X_n\frac{\partial \ln \frac{\partial u}{\partial x_n}}{\partial x_n}$$
$$+ \frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_{n-1}}{\partial x_{n-1}} + \frac{\partial X_n}{\partial x_n},$$

or, when:

$$M_1\frac{\partial u}{\partial x_n} = M,$$

one will obtain the formula:

$$0 = X\frac{\partial \ln M}{\partial x} + X_1\frac{\partial \ln M}{\partial x_1} + \cdots + X_n\frac{\partial \ln M}{\partial x_n} + \cdots + \frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n},$$

and when that is multiplied by M, it will change into:

$$0 = \frac{\partial (M\ X)}{\partial x} + \frac{\partial (M\ X_1)}{\partial x_1} + \cdots + \frac{\partial (M\ X_n)}{\partial x_n}.$$

Therefore, the equation to be proved comes back to the same equation by which the quantity M was defined. That proves the lemma that was stated. When:

$$X\frac{\partial u}{\partial x} + X_1\frac{\partial u}{\partial x_1} + \cdots + X_n\frac{\partial u}{\partial x_n} = 0,$$

the equation $u = \alpha$ can also be defined to be an integral of the system of ordinary differential equations:

$$dx : dx_1 : \ldots : dx_n = X : X_1 : \ldots : X_n,$$

which is a definition that I shall adopt in what follows.

II.

In the same way that one deduces the function M_1 from M, one can deduce a new function M_2 from M_1, a new function M_3 from M_2, etc., and when the preceding lemma is applied to all of those functions, that will produce partial differential equations that they must satisfy, and the number of independent variables will continually decrease by unity.

Assume that the equation $u = \alpha$ is an integral of the system of ordinary differential equations:

$$dx : dx_1 : \ldots : dx_n = X : X_1 : \ldots : X_n,$$

and that one has:

$$\frac{\partial(M\ X)}{\partial x}+\frac{\partial(M\ X_1)}{\partial x_1}+\cdots+\frac{\partial(M\ X_n)}{\partial x_n} = 0 ,$$

and in addition, that:

$$M_1 = \frac{M}{\frac{\partial u}{\partial x_n}} .$$

The function M_1 satisfied the equation:

$$\frac{\partial(M_1\ X)}{\partial x}+\frac{\partial(M_1\ X_1)}{\partial x_1}+\cdots+\frac{\partial(M_1\ X_{n-1})}{\partial x_{n-1}} = 0 ,$$

in which the variable x_n has been eliminated from the quantities:

$$X, X_1, \ldots, X_{n-1}$$

by means of the equation $u = \alpha$. Let $u_1 = \alpha_1$ be an integral of the differential equations:

$$dx : dx_1 : \ldots : dx_{n-1} = X : X_1 : \ldots : X_{n-1} ,$$

in which α_1 is a new arbitrary constant. Set:

$$M_2 = \frac{M_1}{\frac{\partial u_1}{\partial x_n}} .$$

From the same theorem, one will have:

$$\frac{\partial(M_2\ X)}{\partial x}+\frac{\partial(M_2\ X_1)}{\partial x_1}+\cdots+\frac{\partial(M_2\ X_{n-2})}{\partial x_{n-2}} = 0 ,$$

in which $X : X_1 : \ldots : X_{n-2}$, M_2 are functions of $x, x_1, \ldots, x_{n-2}$, and one eliminates x_{n-1} by means of the second integral $u_1 = \alpha_1$. If α_2 is a third arbitrary constant then let $u_2 = \alpha_2$ be an integral of the differential equations:

$$dx : dx_1 : \ldots : dx_{n-2} = X : X_1 : \ldots : X_{n-2} ,$$

and set:

$$M_3 = \frac{M_2}{\frac{\partial u_2}{\partial x_{n-2}}} = \frac{M_1}{\frac{\partial u_2}{\partial x_{n-2}}\cdot\frac{\partial u_1}{\partial x_{n-1}}} = \frac{M}{\frac{\partial u_2}{\partial x_{n-2}}\cdot\frac{\partial u_1}{\partial x_{n-1}}\cdot\frac{\partial u}{\partial x_n}} .$$

If one eliminates x_{n-2} from the functions $X : X_1 : \ldots : X_{n-2}$, and M_3 then one will have:

$$\frac{\partial (M_2\, X)}{\partial x} + \frac{\partial (M_2\, X_1)}{\partial x_1} + \cdots + \frac{\partial (M_2\, X_{n-3})}{\partial x_{n-3}} = 0 .$$

If one continues in that way then one will successively find the integrals:

(5) $$u = \alpha , \qquad u_1 = \alpha_1 , \qquad \ldots, \qquad u_{n-2} = \alpha_{n-2} ,$$

in which α, α_1, …, α_{n-2} are the arbitrary constants, and in which $u_i = \alpha_i$ is the equation in the variables x, x_1, x_2, …, x_{n-i} that served to eliminate x_{n-i} . In addition, set:

(6) $$M_{n-1} = \frac{M}{\dfrac{\partial u}{\partial x_n} \cdot \dfrac{\partial u_1}{\partial x_{n-1}} \cdots \dfrac{\partial u_{n-2}}{\partial x_2}} ,$$

and eliminate all of the variables x_2, x_3, …, x_n from the functions X, X_1, and M_{n-1} . A repeated application of the lemma that was proved will give:

(7) $$\frac{\partial (M_{n-1}\, X)}{\partial x} + \frac{\partial (M_{n-1}\, X_1)}{\partial x_1} = 0 .$$

Now when one has eliminated the variables x_2, x_3, …, x_n by means of the integrals (5), which are of the integrals of the problem except one, all that will remain to be integrated is the first-order differential equation for the two variables x and x_1 :

(8) $$X_1\, dx - X\, dx_1 = 0 ,$$

and the formula (7) will prove that the quantity M_{n-1} is the *multiplier* of that differential equation. That multiplier, which makes the left-hand side of (8) a complete differential, will reduce the integration of the equation to just a quadrature. That gives the following theorem, and due to its importance and fecundity, I have deemed it appropriate to give a special name:

Suppose that one has the differential equations:

$$dx : dx_1 : \ldots : dx_n = X : X_1 : \ldots : X_n ,$$

and let M be an arbitrary quantity that satisfies the equation:

$$\frac{\partial (M\, X)}{\partial x} + \frac{\partial (M\, X_1)}{\partial x_1} + \cdots + \frac{\partial (M\, X_n)}{\partial x_n} = 0 .$$

Furthermore, suppose that one has found all of the successive integrals of the system of differential equations except for one:

$$u = \alpha, \qquad u_1 = \alpha_1 , \qquad \ldots, \qquad u_{n-2} = \alpha_{n-2} ,$$

in which α, α_1, … *are arbitrary constants. One employs each integral to eliminate one variable, so* $u_i = \alpha_i$ *will be the equation in the variables* x, x_1, …, x_{n-i} *that serves to eliminate* x_{n-i} . *The multiplier of the last differential equation:*

$$X_1\, dx - X\, dx_1 = 0$$

will be:

$$\mu = \frac{M}{\frac{\partial u}{\partial x_n} \cdot \frac{\partial u_1}{\partial x_{n-1}} \cdots \frac{\partial u_{n-2}}{\partial x_2}} ,$$

in which the quantities X, X_1, μ *are expressed in terms of the variables* x *and* x_1 *by means of the integrals that were found.*

That proves the principle of the last multiplier, namely, that *when one knows the quantity* M, *the last integration can always be carried out with only a quadrature.*

When one has:

(9) $$\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n} = 0 ,$$

one can make $M = 1$. It then follows that:

If one is given a system of differential equations:

$$dx : dx_1 : \ldots : dx_n = X : X_1 : \ldots : X_n ,$$

in which:

$$\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n} = 0 ,$$

and one has found all of the complete integrals except for one then the last differential equation can always be integrated by just a quadrature.

The stated principle is developed extensively in Crelle’s Journal. Here, it will suffice to give an application to the problems of mechanics.

III.

PRINCIPLE OF THE LAST MULTIPLIER IN MECHANICAL PROBLEMS.

Consider the dynamical formulas that relate to the motion of k material points. The $3k$ rectangular points of those k points are:

$$x, \qquad x_1, \qquad x_2, \qquad \ldots, \qquad x_{3k-1},$$

and let:

$$\frac{dx}{dt} = x_{3k}, \qquad \frac{dx_1}{dt} = x_{3k+1}, \quad \ldots, \qquad \frac{dx_{3k-1}}{dt} = x_n, \tag{10}$$

in addition, in which $n = 6k - 1$. Suppose that the forces are applied to the material points along directions that are parallel to the coordinate axes and are functions of only the coordinates $x, x_1, \ldots, x_{3k-1}$, without depending upon time or velocity, and that the system of points is entirely free. The motion of the points will be given by the integration of a system of ordinary differential equations of the form:

$$\frac{dx_{3k}}{dt} = X_{3k}, \quad \frac{dx_{3k+1}}{dt} = X_{3k+1}, \qquad \ldots, \qquad \frac{dx_n}{dt} = X_n, \tag{11}$$

in which $X_{3k}, X_{3k+1}, \ldots, X_n$ are functions of the quantities $x, x_1, \ldots, x_{3k-1}$. For greater conformity with the formula of the preceding articles, set:

$$x_{3k} = X, \qquad x_{3k+1} = X_1, \qquad \ldots, \qquad x_n = X_{3k-1}.$$

Formulas (10) and (11) can be combined into the system of first-order differential equations:

$$dx : dx_1 : \ldots : dx_n = X : X_1 : \ldots : X_n. \tag{12}$$

When that is integrated and one has expressed $X = x_{3k}$ in terms of x by means of the integrals that were found, one will finally have the time:

$$t = \int \frac{dx}{X} + \text{const.} \tag{13}$$

Therefore, as one sees, in mechanical problems, the last integration, which gives the expression for time in terms of one coordinate, can be obtained by just a quadrature. However, I say that the last *two* integrations can always be obtained by the path of just quadratures because, in addition to equation (13), which contains only one quadrature, the last integration of the system (12) can also be reduced to a quadrature by means of the principle of the last multiplier. Indeed, if the quantities

X_{3k}, X_{3k+1}, …, X_n are functions of only x, x_1, …, x_{3k-1} then one will see that no function X contains the variable x and that consequently for each value of i, one has:

$$\frac{\partial X_i}{\partial x_i} = 0 ,$$

so

$$\frac{\partial X}{\partial x} + \frac{\partial X_1}{\partial x_1} + \cdots + \frac{\partial X_n}{\partial x_n} = 0 .$$

One then has the case of $M = 1$. Meanwhile, if one has found all of the integrals of equations (12) except for one then the multiplier of the last differential equation will be provided by the principle that was stated in the previous section when one substitutes $M = 1$.

The same principle will give the last two integrations, even in the case of systems of material points that are not free. In order to make that obvious, take the dynamical formulas in a convenient form, as below.

Let $3k - m$ be the number of constraint equations for the system of k material points. Express all of their $3k$ coordinates x, x_1, …, x_{3k-1} in terms of m independent quantities:

$$q_1 , q_2 , \ldots, q_m .$$

Therefore, set:

$$\frac{\partial q_i}{\partial t} = q_i' ,$$

while T will express one-half the *vis viva* of the system of material points in terms of the quantities:

$$q_1 , q_2 , \ldots, q_m , \qquad q_1', q_2', \ldots, q_m' .$$

Suppose that one has the equations:

$$\frac{\partial T}{\partial q_1'} = p_1 , \qquad \frac{\partial T}{\partial q_2'} = p_2 , \qquad \ldots, \qquad \frac{\partial T}{\partial q_m'} = p_m ,$$

which are linear in the $q_1', q_2', \ldots, q_m'$, and when one solves them, one will obtain the values of $q_1', q_2', \ldots, q_m'$, as expressed in terms of the p_1, p_2, …, p_m. When one substitutes those values in T, T will become a function of the $2m$ quantities:

$$q_1 , q_2 , \ldots, q_m , \qquad p_1 , p_2 , \ldots, p_m ,$$

which are the ones that will serve to establish the differential equations of dynamics. In order to obtain them, suppose that x_i is one coordinate of a point whose mass is m_i and that the point is acted upon by the force X_{3k+i}, whose direction is parallel to the x_i coordinate axis. When one substitutes the values of the coordinates $x, x_1, \ldots, x_{3k-1}$, when expressed in terms of $p_1, p_2, \ldots, p_m$, one will get:

$$m X_{3k}\, dx + m_1 X_{3k+1}\, dx_1 + \ldots + m_{3k-1} X_n\, dx_{3k-1} = Q_1\, dq_1 + Q_2\, dq_2 + \ldots + Q_m\, dq_m .$$

The quantities $X_{3k}, X_{3k+1}, \ldots$, are functions of only $x, x_1, \ldots$, and the quantities $Q_1, Q_2, \ldots$ are functions of only $q_1, q_2, \ldots, q_m$. Once one has found those functions, the differential equations for the variables $q_1, q_2, \ldots, q_m, p_1, p_2, \ldots, p_m$ will be the following ones:

$$(14) \qquad \left\{ \begin{array}{ll} \dfrac{dq_1}{dt} = \dfrac{\partial T}{\partial p_1}, & \dfrac{dp_1}{dt} = -\dfrac{\partial T}{\partial q_1} + Q_1, \\ \dfrac{dq_2}{dt} = \dfrac{\partial T}{\partial p_2}, & \dfrac{dp_2}{dt} = -\dfrac{\partial T}{\partial q_2} + Q_2, \\ \quad \cdots & \quad \cdots \\ \dfrac{dq_m}{dt} = \dfrac{\partial T}{\partial p_m}, & \dfrac{dp_m}{dt} = -\dfrac{\partial T}{\partial q_m} + Q_m. \end{array} \right.$$

The proof of those general formulas can be obtained from the proof that **Hamilton** gave in the case where:

$$m X_{3k}\, dx + m_1 X_{3k+1}\, dx_1 + \ldots + m_{3k-1} X_n\, dx_{3k-1}$$

is a complete differential (see the two papers by that author that were included in the Philosophical Transactions in 1834 and 1835). Separate the element dt and put the differential equation into the form of a proportion:

$$(15) \qquad \left\{ \begin{array}{l} dq_1 : dq_1 : \cdots : dq_m : dp_1 : dp_1 : \cdots : dp_m \\ = \dfrac{\partial T}{\partial p_1} : \dfrac{\partial T}{\partial p_2} : \cdots : \dfrac{\partial T}{\partial p_m} : -\dfrac{\partial T}{\partial q_1} + Q_1 : -\dfrac{\partial T}{\partial q_2} + Q_2 : \cdots : -\dfrac{\partial T}{\partial q_m} + Q_m . \end{array} \right.$$

One must first integrate equations (15) and then the time t will be found as a function of one of the quantities $q_1, q_2, \ldots$ by means of just one quadrature. Now look for the quantity M that corresponds to the system of equations (15). When one assumes the differential equations:

$$dx : dx_1 : \ldots : dx_n = X : X_1 : \ldots : X_n ,$$

one can differentiate each of the quantities X, X_1, etc., with respect to the variable whose differential it is proportional to. If the sum of all of the n partial differentials thus-obtained vanishes then the last integration will reduce to a quadrature. Hence, when one has posed the differential equations (15), one will have to differentiate the quantities:

$$\frac{\partial T}{\partial p_1}, \quad \frac{\partial T}{\partial p_2}, \quad \ldots, \quad \frac{\partial T}{\partial p_m}$$

with respect to the variables:

$$q_1, q_2, \ldots, q_m,$$

and the quantities:

$$-\frac{\partial T}{\partial q_1} + Q_1, \quad -\frac{\partial T}{\partial q_2} + Q_2, \quad \ldots, \quad -\frac{\partial T}{\partial q_m} + Q_m$$

with respect to the variables:

$$p_1, p_2, \ldots, p_m.$$

Now the sum of all those $2m$ partial differentials will vanish because when one combines them pair-wise, one will have:

$$\frac{\partial \frac{\partial T}{\partial p_i}}{\partial q_i} + \frac{\partial\left(-\frac{\partial T}{\partial q_i} + Q_i\right)}{\partial p_i} = \frac{\partial Q_i}{\partial p_i} = 0$$

for any value of the index i. Therefore, since the proposed differential equations (15) correspond to a system of material points that is not free, one can set $M = 1$, and therefore their last integration will reduce to a quadrature.

When the expressions for the forces contain time t explicitly, one cannot obtain time by just one quadrature, as in the previous case. However, even in that case, the new principle will imply that the integration of the last first-order differential equation for t and one coordinate will depend upon just a quadrature.

The same principle also applies to the motion of a comet in a resisting medium, and to any other special cases in which the applied forces are forces of resistance.

Rome, 16 March 1844.