"Sur la surface dont la courbure moyenne est constant," J. Math. pure et appl. **18** (1853), 163-167.

ON THE SURFACE WHOSE MEAN CURVATURE IS CONSTANT

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Upon seeking the surface of given area that contains the largest volume by the method of variations, one will find, as one knows, that the mean curvature of the desired surface must be constant. If one expresses that condition in terms of partial derivatives then one will have the equation:

$$(A) \qquad (1 + q^2)\, r - 2\, pqs + (1 + p^2)\, t + \frac{2}{a}(1 + p^2 + q^2)^{1/2} = 0.$$

Other than that general equation, the calculus of variations provides boundary conditions that serve to determine the arbitrary functions that are found in the solution. Unfortunately, one cannot further integrate equation (A), which will prevent one from obtaining the general solution of the question that was posed.

Meanwhile, there is one case in which one knows the result that one must obtain in advance. When one seeks the *closed* surface that contains the maximum volume of all of them, some simple considerations will show that the desired surface is a sphere. However, one cannot further prove that result with the calculus of variations, which seems to be a very large gap in that method.

> "One knows [said Delaunay (Journal de l'École Polytechnique, tome XVIII, page 110)], that among the closed surfaces of a given extent, the sphere is the one that contains the largest volume, but that solution cannot be inferred from the equations that calculus of variation provides…It is easy to see that the variation of the integral does not contain boundary terms. The conditions of the absolute maximum then reduce to just equation (A). Hence, in order to prove that the desired surface is a sphere, it will be necessary to show that the sphere is the only closed surface that is included in that equation, and that is something that one cannot do, either."

I have considered that question precisely as Delaunay posed it, and having succeeded in solving it for a very large class of surfaces, I propose to show here that among all of the surfaces whose volume can be expressed by the integral:

$$\int_0^R \int_0^\pi \int_0^{2\pi} r^2 dr \sin\theta \, d\theta \, d\varphi ,$$

the sphere is the only one whose mean curvature is constant. In order to do that, I will first give some theorems that are very remarkable, moreover. In all of what follows, I shall adopt these notations:

Let P be the perpendicular to the tangent plane that is based at the origin, which one supposes to be taken in the interior of a closed surface. Let $d\omega$ be the element of the spherical surface that describes the radius that is parallel to that perpendicular and whose length is unity, in such a way that:

$$d\omega = \sin\theta \, d\theta \, d\varphi,$$

where θ, φ are the polar angles that determine the position of the perpendicular. Let S be the total area of the surface, and let dS be the element of that area. Let I represent the integral:

$$\iint P \, d\omega ,$$

when one supposes that this integral extends over the entire surface. Finally, let R, R' be the radii of principal curvature at the point in question. We have the following theorems:

THEOREM I:

For an arbitrary closed surface:

$$(B) \qquad 2I = \iint \left(\frac{1}{R} + \frac{1}{R'} \right) d\omega ,$$

if one supposes that the integral extends over the entire surface.

Here, it will suffice for me to state the theorem. One will find the proof in my *Calculus of Variations*, pages 351 and 353.

THEOREM II:

For an arbitrary closed surface:

$$(C) \qquad 2S = \iint P \left(\frac{1}{R} + \frac{1}{R'} \right) dS ,$$

when one supposes, as in the preceding theorem, that the integral extends over the entire surface.

Indeed, let x, y, z be the rectangular coordinates of a point on the surface, and set:

$$\xi = \frac{p\,(px+qy-z)}{\sqrt{1+p^2+q^2}} - x\sqrt{1+p^2+q^2}\ ,$$

$$\eta = \frac{q\,(px+qy-z)}{\sqrt{1+p^2+q^2}} - y\sqrt{1+p^2+q^2}\ ,$$

while preserving the usual notation of partial derivatives. Upon differentiating, we will then have:

$$\frac{\partial\xi}{\partial x} = (p\,x + q\,y - z)\left[\frac{(1+q^2)\,r - p\,q\,s}{(1+p^2+q^2)^{3/2}}\right] + s\,\frac{p\,y - q\,z}{\sqrt{1+p^2+q^2}} - \sqrt{1+p^2+q^2}\ ,$$

$$\frac{\partial\eta}{\partial y} = (p\,x + q\,y - z)\left[\frac{(1+q^2)\,t - p\,q\,s}{(1+p^2+q^2)^{3/2}}\right] + s\,\frac{q\,x - p\,y}{\sqrt{1+p^2+q^2}} - \sqrt{1+p^2+q^2}\ .$$

When we add those equations, we will find that:

$$\frac{\partial\xi}{\partial x} + \frac{\partial\eta}{\partial y} = (p\,x + q\,y - z)\ \frac{(1+q^2)\,r - p\,q\,s + (1+p^2)\,t}{(1+p^2+q^2)^{3/2}} - 2\sqrt{1+p^2+q^2}\ .$$

However, by virtue of the relations:

$$P = \frac{z - p\,x - q\,y}{\sqrt{1+p^2+q^2}},$$

$$\frac{1}{R} + \frac{1}{R'} = \frac{(1+q^2)\,r - p\,q\,s + (1+p^2)\,t}{(1+p^2+q^2)^{3/2}},$$

that equation can be written:

$$\frac{\partial\xi}{\partial x} + \frac{\partial\eta}{\partial y} = \left[P\left(\frac{1}{R} + \frac{1}{R'}\right) - 2\right]\sqrt{1+p^2+q^2}\ .$$

One will then have:

$$2\sqrt{1+p^2+q^2}\; dx\, dy = P\left(\frac{1}{R}+\frac{1}{R'}\right)\sqrt{1+p^2+q^2}\; dx\, dy - \left(\frac{\partial\xi}{\partial x}+\frac{\partial\eta}{\partial y}\right) dx\, dy\,,$$

or, more simply:

$$2\; dS = P\left(\frac{1}{R}+\frac{1}{R'}\right) dS - \left(\frac{\partial\xi}{\partial x}+\frac{\partial\eta}{\partial y}\right) dx\, dy\,.$$

Upon integrating this and letting Σ denote the area of a portion of the surface that is bounded by an arbitrary contour, we will have:

(*D*) $$2\,\Sigma = \iint P\left(\frac{1}{R}+\frac{1}{R'}\right) dS + \int\left(\xi\frac{dy}{dx}-\eta\right) dx\,,$$

in which the double integral extends over the entire portion that one considers, and the simple integral extends over the entire boundary curve. One can deduce several curious consequences of that equation. However, for the sake of our immediate purposes, it will suffice for me to remark that when one extends the integration over the entire surface, the simple integral will obviously disappear. One will then have simply:

$$2\,\Sigma = \iint P\left(\frac{1}{R}+\frac{1}{R'}\right) dS\,. \qquad \text{Q. E. D.}$$

THEOREM III: Now consider the equation:

(*E*) $$\frac{1}{R}+\frac{1}{R'} = \frac{2}{a}\,,$$

which represents a surface whose mean curvature is constant. When one puts that equation into the form:

$$\left(\frac{1}{R}-\frac{1}{R'}\right)^2 = \frac{4}{a^2}-\frac{4}{RR'}\,,$$

multiplying by P / 4 dS, and integrating, one will easily find that:

(*F*) $$M = a^2\, I + \frac{a^2}{4}\iint\left(\frac{1}{R}-\frac{1}{R'}\right)^2 P\, dS\,,$$

while represent three times the volume by M. Having said that, if one multiplies equation (*E*) by dS and integrates then, by virtue of equation (*B*), one will find that:

$$\iint dS\left(\frac{1}{R}+\frac{1}{R'}\right) = 2\iint P\, d\omega = \frac{2S}{a}\,;$$

one will then have:

$$S = a\, I.$$

When one multiplies the new equation (E) by $P\, dS$ and integrates, one will have:

$$S = \frac{M}{a},$$

by virtue of equation (C). Upon eliminating S from those equations, one will have:

$$M = a^2\, I.$$

That condition reduces equation (F) to:

$$\iint \left(\frac{1}{R} - \frac{1}{R'} \right)^2 P\, dS = 0.$$

However, since:

$$P\, dS = r^2 \sin \theta\, d\theta\, d\varphi,$$

one will easily see that for the class of surfaces that one considers, all of the elements that comprise the last integral are essentially positive. The total integral cannot vanish then unless each of the elements becomes zero.

One will then have:

$$\frac{1}{R} - \frac{1}{R'} = 0, \qquad \frac{1}{R} + \frac{1}{R'} = \frac{2}{a}.$$

Therefore:

$$R = R' = a,$$

which shows that the desired surface is a sphere whose radius is a.