"Über die Polarisation der Lichtquanten," Zeit. f. Phys. **44** (1927), 292-300.

On the polarization of light quanta

By **P. JORDAN**, presently in Copenhagen

(Received on 16 June 1927)

Translated by D. H. Delphenich

It will be shown that a quantum-mechanical description of the polarization properties of a single light quantum can be carried out in a way that is formally equivalent to the Pauli theory of the magnetic electron.

Ever since the beginning, polarization has defined a special complication for the corpuscular theory of light. Attempts to understand polarization in the context of the theory of light quanta are often carried out in such a way that the light quanta would, in any event, describe a certain definite polarization state. However, the impossibility of understanding such a concept is related to the fact that these experiments have found no general explanation. The problem still remains today, and for that reason it might not be unwelcome to show that the static conception of physical quantities, as were obtained in the more recent development of quantum mechanics, might deliver a clarification of this question that is satisfactory in the sense of this theory.

The subject is, however, meaningful in another respect. The fact that in statistics the polarization of the light quantum or the proper magnetic moment of the electron allows one to multiply the number of phase cells by the number 2 strongly suggests that there might be an intrinsic similarity between these phenomena. C. G. Darwin [1], on the basis of the fact that the magnetic electron can be represented by polarizing the de Broglie waves, succeeded in deriving a generalization of the Schrödinger equation that is suitable for the magnetic electron. Pauli [2] made an essential advance on this by means of considerations that were based in the general statistical interpretation of quantum mechanics. Thus, Pauli made no use of the concept of polarized Schrödinger waves, and his results can be regarded as contradicting this notion. In the following, we will see that one can describe the polarization properties of light quanta precisely by carrying over the Pauli conceptual structure – with that, in turn, the possibility of converting between the two phenomena emerges in a very surprising way.

We assume that the main points of the Pauli theory are known here, along with some extending remarks of the author [3].

§ 1. Observable quantities for light quanta. We shall concern ourselves solely with light quanta whose frequency and direction of motion are given, so – otherwise

[1] C. G. Darwin, Nature, March, 1927.
[2] W. Pauli, Jr., Zeit. Phys. (to appear).
[3] P. Jordan, Zeit. Phys. (to appear).

speaking – we are concerned with harmonic waves that might perhaps run parallel to the z-axis. One must first set down how one can measure the polarization properties of a single light quantum, and how one can describe them by quantum-mechanical "quantities." It likewise shows that naturally one cannot utilize the classical concept of a physical quantity, but only the quantum-mechanical concepts of quantities that are given in the statistical formulation of quantum mechanics, and are explained in an especially clear manner by Pauli's arguments regarding the magnetic electron.

If we put a Nicol prism in the path of a purely harmonic light wave that can be elliptically polarized then it will split into a transmitted linearly polarized component and a reflected polarized component that is perpendicular to it. If we experiment with only single light quanta then such a light quantum will either go through the Nicol prism or be reflected, and one must assume that in the former case the light quantum once more goes through a Nicol apparatus in such a way that it goes through it a second time, while in the latter case it is reflected a second time. The different possible placements of the Nicol are to be referred to in terms of an angle ψ in the interval $0 \leq \psi < \pi$. Any such measurement determines a special mechanical quantity for the light quantum: We say that the quantity $(0, \psi)$ has the value $+\frac{1}{2}$ when the light quantum is transmitted through the Nicol with the angle ψ, and we say that $(0, \psi)$ is $-\frac{1}{2}$ when it is reflected. If one has:

$$| \psi_1 - \psi_2 | = \frac{\pi}{2} \tag{1}$$

then the measurement of $(0, \psi_2)$ certainly delivers the value $+\frac{1}{2}$ when a previous measurement of $(0, \psi_1)$ gave the value $-\frac{1}{2}$, and conversely. For that reason, we must say that:

$$(0, \psi_1) = -(0, \psi_2) \qquad \text{for} \qquad | \psi_1 - \psi_2 | = \frac{\pi}{2} \tag{2}$$

in the case (1).

Theoretically, one can also split a purely periodic light wave into circularly polarized partial waves with a positive and negative sense of rotation, rather than into two perpendicular linearly-polarized components. Thus, the energy of the combined waves is also equal to the sum of the energies of the components, and for that reason, it would be allowable for us to assume that we can also carry out this splitting by means of an analyzer that – analogously to the Nicol – transmits an arbitrary polarized wave with a positive circular polarized component and reflects the negative component.

However, we would like to go a step further. We would like to characterize the general elliptically-polarized wave:

$$\left.\begin{aligned} x &= A \sin t, \\ y &= B \sin(t + \delta) \end{aligned}\right\} \tag{3}$$

(with a suitably chosen zero point and normalization for time) by the associated ellipse:

$$\frac{x^2}{A^2}+\frac{y^2}{B^2}-2\frac{xy}{AB}\cos\delta = \text{const.} \tag{4}$$

with the addition of a + or – sign for a positive or negative sense of rotation, respectively. If ψ is the angle that the major axis of the ellipse (4) defines with the x-axis and h denotes the ratio of the minor axis to the major axis then we can characterize the wave (3) by three symbols: η, ψ, $\pm$. It is more convenient for the later formulas if we calculate with only two numbers $\sigma = \pm\,\eta$ and ψ; one then has:

$$-1 \leq \sigma \leq 1, \qquad 0 \leq \psi < \pi. \tag{5}$$

The angle ψ will be undetermined in the case of $\sigma = \pm\, 1$. As in the textbooks on optics, it can be shown that:

$$\tan 2\psi = \tan 2\vartheta \cdot \cos\delta, \qquad \sin 2\chi = -\sin 2\vartheta \cdot \sin\delta, \tag{6}$$

when one sets:

$$\frac{B}{A} = \tan\vartheta, \qquad \sigma = \tan\chi. \tag{7}$$

One now easily recognizes the following fact: It is possible to represent any wave σ, ψ as the sum of two waves of the form:

$$\sigma_0,\ \psi_1 \qquad \text{and} \qquad -\sigma_0,\ \psi_2, \tag{8}$$

where an arbitrary value σ_0 and an arbitrary angle ψ_1 are prescribed, while ψ_2 is determined from [1]:

$$|\,\psi_1 - \psi_2\,| = \frac{\pi}{2}. \tag{9}$$

The intensities of the partial waves (8) and their relative phase are determined uniquely by σ, ψ. *The energy of the total wave σ, ψ is the sum of the energies of the components* (8).

On the basis of this fact, it seems natural for us to propose as an ideal analyzer for the polarization properties of a light quantum an apparatus that physically implements the mathematical decomposition (8) for any incident wave, so it always, for example, transmits the one component (8) and reflects the other one. Such an ideal analyzer is thus characterized in itself by the numbers σ, ψ. If we again consider a single light quantum then any analyzer σ, ψ defines a special "mechanical quantity" for us; we say that we will measure the value $+\frac{1}{2}$ or $-\frac{1}{2}$ for the quantity:

---

[1] Obviously, one can also describe the two waves (8) as follows: If the one wave is represented by:

$$x = A\sin t, \qquad y = B\sin(t+\delta)$$

then the other one has the form:

$$x = -B\sin(t+\delta'-\delta), \qquad y = A\sin(t+\delta').$$

$$(\sigma,\ \psi) \tag{10}$$

according to whether the light quantum is transmitted or reflected, respectively. The angle ψ is undetermined in the case $\sigma = \pm 1$; we denote the two quantities (10) that remain in this case alone by:

$$(\pm 1). \tag{11}$$

In the case $\sigma = 0$, we again obtain the previous quantities that were defined with the help of the Nicol, namely:

$$(0,\ \psi). \tag{12}$$

If one has measured the quantity $(\sigma,\ \psi_1)$ for a light quantum then a subsequent measurement of $(-\sigma,\ \psi_2)$, where ψ_2 satisfies equation (9), will certainly deliver *the opposite value* to the first measurement. As a generalization of (2), we will then have:

$$(\sigma,\ \psi_2) = -(-\sigma,\ \psi_1) \qquad \text{for} \qquad |\psi_1 - \psi_2| = \frac{\pi}{2}. \tag{13}$$

In particular, one has:

$$(+1) = -(-1). \tag{14}$$

If we construct the quantity:

$$C_1 + C_2(\sigma,\ \psi_1) = q \tag{15}$$

from $(\sigma,\ \psi_1)$, with two c-numbers C_1, C_2, then q has the value the value $q' = C_1 \pm C_2$ for $(\sigma,\ \psi_1) = \pm\frac{1}{2}$. We can, however, also add and multiply two different such quantities q according to the symbolic combination $q + p$, as one forms in quantum mechanics, more generally. What this addition and multiplication means in our case will be deduced from what follows.

The practicability of the aforementioned explanation might perhaps involve some excuses. In fact, for any reader that is completely familiar with Pauli's theory a brief presentation would suffice. It thus seems that it is precisely the polarization of light quantum that provides a particularly instructive example of the definition of the singular concepts in quantum mechanics. Above all, it is the elementary concept of a *physical quantity* that has experienced such an essential alteration in the development from classical to quantum mechanics, and in whose current conception the deepening of our intuitive understanding finds its expression most clearly. One is not allowed to ascribe a definite value to one *independently of the processes of observation*.

§ 2. Correspondence between the light quantum and the magnetic electron. If we pose the question of which pairs of quantities $(\sigma,\ \psi)$ and $(\bar{\sigma}, \bar{\psi})$ are to be regarded as *canonically conjugate* (for a suitable normalization) then (as for the magnetic electron) we can lean on the quantum-mechanical theorem that for given values of a quantity all values of the conjugate impulse are *equally probable*. We then see immediately that, e.g., (always for a suitable normalization) the quantity $(0,\ \psi)$ (linear polarization), with any ψ, is canonically conjugate to (+1) or (–1) (circular polarization). $(0,\ \psi)$ and $(0, \bar{\psi})$

are also conjugate when $|\ \psi - \bar{\psi}\ |$ is equal to $\pi/4$ or $3\pi/4$. Having established this, the quantum-mechanical theory of polarization can be constructed in a manner that is similar to the theory of the magnetic electron.

It is therefore simpler to present an invertible one-to-one correspondence between the observable quantities for the light quantum and magnetic electron in such a way that the additive and multiplicative combination and the probability relations between the physical quantities for light quantities are the same as they are for the corresponding quantities for the magnetic electron.

We represent the impulse component of the magnetic electron in a definite direction by a point x, y, z on the unit sphere $x^2 + y^2 + z^2 = 1$. We call its pole that lies at $z = 1$ the *positive* one and the pole at $z = -1$ the *negative* one. If we associate the positive pole with the positive circular polarization (+1) then we must, from (14), § 1, associate the negative pole with the negative circular polarization (–1). Furthermore, in hindsight of the things that we know about canonically conjugate quantities for the light quantum, on the one hand, and the magnetic electron, on the other, we must make the equatorial points of the sphere correspond to the linear polarizations in such a way that the angular difference $\Delta\psi$ between two linear polarizations is half as large as the angular distance between the two equatorial points.

Finally, we must generally associate a quantity (σ, ψ) with a positive σ with a point of the positive hemisphere and associate the quantity $(-\sigma, \psi)$ that is the mirror image of it in the equatorial plane with the negative hemisphere. We shall now determine this map more precisely.

We relate every point of the sphere to a point $x = a$, $y = b$ of the plane $z = 0$ by means of stereographic projection from the positive pole according to the formula:

$$x = \frac{2a}{1+a^2+b^2}, \qquad y = \frac{2b}{1+a^2+b^2}, \qquad z = \frac{-1+a^2+b^2}{1+a^2+b^2}. \tag{1}$$

Our problem is then to associate the points a, b (or $\eta = a + ib$) in such a way that two points of the sphere that are mirror opposites for the equatorial plane $z = 0$ correspond to the same ellipse. That reinforces the situation that two such points are represented in the complex plane by:

$$\eta_1 = \eta_2 \qquad \text{and} \qquad \eta_2 = \frac{1}{\eta^*}. \tag{2}$$

They therefore satisfy an equation:

$$\{\eta - (a + ib)\}\{(a - b)\eta - 1\} = \eta^2(a - ib) - (1 + a^2 + b^2)\,\eta + (a + ib) = 0 \tag{3}$$

that belongs to a quadratic form:

$$Q(\xi_1, \xi_2) = (a - ib)\,\xi_1^2 - (1 + a^2 + b^2)\,\xi_1\xi_2 + (a + ib)\,\xi_2^2, \tag{4}$$

which is always real for $\xi_2 = \xi_1^*$. We then define:

$$\left.\begin{aligned} -Q(x+iy, x-iy) &= -2\{a(x^2-y^2)+2bxy\}+(1+a^2+b^2)(x^2+y^2) \\ &= x^2\{(1-a)^2+b^2\}+y^2\{(1+a)^2+b^2\}-4bxy = F(x,y), \end{aligned}\right\} \tag{5}$$

and when we associate this form $F(x, y)$ with the point $\eta_1 = a + ib$ we are certain that it simultaneously makes the mirror image in the equatorial correspond to the point $\eta_1 = \frac{1}{a-ib}$. We see, in addition, that:

$$F(x, y) = \text{const.} \tag{6}$$

always represents an ellipse that only degenerates into a double line segment in the case of $a^2 + b^2 = 1$. The determinant of $F(x, y)$ is, in fact, equal to:

$$D = 4\{(1-a)^2 + b^2\}\{(1+a)^2 + b^2\} - 16b^2 = 4(a^2 + b^2 - 1)^2. \tag{7}$$

We confirm that the association possesses the following previously-obtained properties:

1. The circular polarizations (+1), (–1) correspond to the poles of the sphere. Therefore, (6) goes to a circle for these poles [$a = b = 0$, ($a = b = \infty$, resp.)].

2. The quantities (σ, ψ_1), $(-\sigma, \psi_1)$, with $|\psi_1 - \psi_2| = \pi/2$ are *opposite*. In fact, their ellipses belong to two points of the sphere that lie on a diameter, as one can see from (5).

3. The linear polarizations correspond to equatorial points of the sphere; one can deduce this from (1) and (7). We also see that the association of angles is then the correct one. From $F(x, y) = 0$ in the case of $a^2 + b^2 = 1$, it follows, in fact, that:

$$\tan\left(\psi - \frac{\pi}{2}\right) = -\frac{x}{y} = -\frac{1+a}{b}, \tag{8}$$

so:

$$\tan 2\psi = \frac{b}{a}. \tag{9}$$

By the association that we just carried out, it is now completely defined in general what we must consider to be the "sum" and "product" of two quantum-mechanical quantities in our case, and thus two polarization forms. It is also generally established which quantities are canonically conjugate to a given one. Naturally, all of these can also be easily represented in explicit formulas. We thus content ourselves with suggesting the following: Formula (9) is true in general for elliptical, as well as linear, polarizations. The ellipse $F(x, y)$ = const. that is obtained by means of a coordinate system x', y' that is rotated around its principal axis takes on the form:

$$x'^2 \cdot (1-\sqrt{a^2+b^2})^2 + y'^2 \cdot (1+\sqrt{a^2+b^2})^2 = \text{const.} \tag{10}$$

In summary, we thus obtain the representation of the numbers σ, ψ in terms of a, b by means of:

$$\left.\begin{aligned} \tan 2\psi &= \frac{b}{a}, \\ -\sigma = \frac{1-\rho}{1+\rho}, &\quad \rho = \sqrt{a^2+b^2}. \end{aligned}\right\} \tag{11}$$

Conversely [sic]:

$$\left.\begin{aligned} \frac{b}{a} &= \arctan 2\psi, \\ \rho &= \frac{1+\sigma}{1-\sigma}. \end{aligned}\right\} \tag{12}$$

§ 3. Probabilities and intensities. We now ask what the probability is for a quantum-mechanically determined quantity for a light quantum to have the value $+\frac{1}{2}$ or $-\frac{1}{2}$, under the assumption that the value of another quantity is known. More precisely, we then pose the following question: Let a light quantum be examined by our analyzer σ, ψ that yields the value $+\frac{1}{2}$ or $-\frac{1}{2}$ (transmission or reflection, resp.). This light quantum is again examined by another analyzer $\bar{\sigma}$, $\bar{\psi}$. How big is the probability that it then undergoes transmission (reflection, resp.)?

We now have two methods for answering this question: The first one is the classical one: We get our answer by the mathematical decomposition of the wave σ, ψ into two components $\bar{\sigma}$, $\bar{\psi}$ and $-\bar{\sigma}$, $\bar{\psi}_2$, where $|\bar{\psi} - \bar{\psi}_2| = \pi/2$. Secondly, the explanation that is given by *quantum mechanics* (*Pauli's theory of the magnetic electron*, resp.) provides an answer: We determine two points on the unit sphere that correspond to those quantum-mechanical quantities, the first of which would give $+\frac{1}{2}$ for a first measurement, and the second of which would be determined by the second measurement with the desired probability of $+\frac{1}{2}$. The probability will be equal to:

$$\cos^2 \frac{\Theta}{2}, \tag{1}$$

where Θ is the angular distance between the two points of the sphere.

The quantum-mechanical answer obviously agrees with the classical one in the case where the two analyzers that used are Nicol prisms. The two spherical points then lie on the equator, and their distance Θ is equal to twice the angle between the polarization directions of the two Nicols. This situation gets more involved in the general case of two elliptical polarizers. From (1), § 2, the angle Θ between two points a_1, b_1 and a_2, b_2 is given by:

$$\cos\Theta = \frac{4a_1a_2 + 4b_1b_2 + (1-a_1^2-b_1^2)(1-a_2^2-b_2^2)}{(1+a_1^2+b_1^2)(1+a_2^2+b_2^2)}, \tag{2}$$

so:

$$\left.\begin{aligned} \cos\Theta &= \frac{4a_1a_2 + 4b_1b_2 + (1-\rho_1^2)(1-\rho_2^2)}{(1+\rho_1^2)(1+\rho_2^2)} \\ &= \frac{4\rho_1\rho_2\cos(\psi_1-\psi_2) + (1-\rho_1^2)(1-\rho_2^2)}{(1+\rho_1^2)(1+\rho_2^2)}. \end{aligned}\right\} \tag{3}$$

This then yields the probability:

$$\cos^2\frac{\Theta}{2} = \frac{1+2\rho_1\rho_2\cos(\psi_1-\psi_2)+\rho_1^2\rho_2^2}{(1+\rho_1^2)(1+\rho_2^2)}, \qquad \rho_{1,2} = \frac{1+\sigma_{1,2}}{1-\sigma_{1,2}}, \tag{4}$$

or also:

$$\cos^2\frac{\Theta}{2} = \frac{(1+\sigma_1\sigma_2)^2 + (\sigma_1+\sigma_2)^2 + (1-\sigma_1^2)(1-\sigma_2^2)\cos(\psi_1-\psi_2)}{2(1+\sigma_1^2)(1+\sigma_2^2)}. \tag{5}$$

Finally, we must compare this formula with the statement of the classical theory. A somewhat lengthy calculation – for which, one can employ formulas (6), (7), § 1 and the formulas of the footnote on pp. 3, where nothing special needs to be said about the units – also leads classically, in fact, to a result that means the same thing as (4), (5).

In conclusion, I would like to express my deepest thanks for the fact that I got the impetus to carry out this argument from a conversation with Herrn Prof. C. G. Darwin. Darwin expressed the belief that a representation with the aid of polarized Schrödinger waves could also be achieved for Pauli's theory of magnetic electrons. It then seemed appealing to me to examine the converse.

I am indebted to Prof. N. Bohr for many stimulating conversations and to the International Education Board for making my sojourn in Copenhagen possible.

____