INTRODUCTION TO THE THEORY OF SYSTEMS OF DIFFERENTIAL EQUATIONS

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1934
LEIPZIG AND BERLIN
PUBLISHED AND PRINTED BY B. G. TEUBNER

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Introduction.

The question of the existence of solutions to a system of arbitrarily many differential equations with arbitrarily many unknowns has already been resolved by French mathematicians for some time now. After the research and partial results of Méray and other authors, Riquier first solved the problem in general in 1893 ([1]). Later (1896), Delassus ([2]) dealt with the problem by a simpler procedure, and then he also answered the questions that had been imperfectly posed by Riquier concerning the arbitrary elements (constant or functions) that entered into the general solution. Whereas this work was done in the style of Cauchy's theory of differential equations, and can be considered to be its conclusion, E. Cartan dealt with the differential equations in the form of Pfaff equations in his examination. His far-reaching application of the theory of Pfaffian systems ([3]), which came from geometry and infinitely continuous groups, included a new solution to the problem that had been addressed by Riquier and Delassus, in a sense, and that gave a deep insight into the mechanics of differential equations. Cartan's theory rested upon the calculus of alternating differential forms, which had been founded by Grassmann, Poincaré, Cartan, and Burali-Forti, whose relationship to the other branches of mathematics (multiple integrals, topology) alone was already noteworthy, and whose invariance properties made the differential equations of geometry particularly tractable.

The present work gives a systematic introduction to the theory of systems of differential equations in which consistent use of the calculus of alternating differential forms is made. For the logical structure of the theory, we find it expedient to use, in place of the Pfaff system that was used by Cartan, an equivalent system, which originated in the annihilation of alternating differential forms of arbitrary degree, and which Goursat ([4]) has employed from the outset. Its use, as well as that of the notion of a differential ideal, seems to me to achieve a tangible simplification, but it will be essentially treated as a representation (supplemented by details) of Cartan's theory in what follows. As much as possible, the notation of Cartan will also be retained, as a result of which this work will do justice to his results, as well as being a commentary on the profound, but inaccessible, treatises of the great French mathematicians.

The aforementioned generalization of the theory of Pfaff systems is also treated in two articles of J. W. Thomas ([5]) that recently appeared, as well as a work of C. Burstin that will appear soon.

Instead of giving a detailed introduction to this book, I will content myself with the following hint for the sake of orientation: Only after one makes oneself familiar with the calculus of differential forms through a fleeting lecture in Chapter I will one sufficiently grasp the notions of "integral manifold," "integral element," "regular chain of integral

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[1]) The work of Riquier is summarized in the book: Ch. Riquier, *Les systèmes d'equations aux dérivées partielles*, Paris, 1910.

[2]) E. Delassus, "Extension du théorème de Cauchy aux systèmes les plus généraux d'equations aux dérivées partielles," Ann. Éc. Norm. (3) **13** (1896).

[3]) E. Cartan, "Sur l'intégration des systèmes d'equations aux différentielles totales," Ann. Éc. Norm. (3) **18** (1901). – "Sur la structure des groupes infinis de transformation," Ann. Éc. Norm. (3) **21** (1904), Chap. I, also cf. E Goursat, *Leçons sur le problème de Pfaff*, Chap. VIII, Paris, 1922.

[4]) E. Goursat, *Leçons sur le problème de Pfaff*, pp. 111, *et. seq*.

[5]) J.M. Thomas, "An existence theorem for generalized pfaffian systems. The condition for a pfaffian system in involution." Bulletin of the Amer. Math. Soc. **40** (1934), 309-320.

elements," that are mentioned in the titles and italicized words in order to understand the existence theorem and the computational procedures. The connection between partial differential equations and Pfaffian systems is explained on pp. 67 and 68. A proof of the fundamental theorem of Lie group theory is given in the appendix as an application of the theory.

The author would like to acknowledge the ongoing suggestions that he received from the topological-differential geometric school of Blascke as its origin. I would like to gratefully thank Herrn Blaschke for the suggestions and interest that he has shown in my work.

I have Herrn Henke to thank for his help in making corrections.

Hamburg, July 29, 1934.

I. The ring of differential forms.

1. **Algebraic calculations with differential forms**. The set of well-defined holomorphic functions on a region or at a fixed point $(x_1^0, x_2^0, \cdots, x_n^0)$ in complex (x_1, x_2, …, x_n)-space defines a domain of integrity or a *ring*, in the sense of abstract algebra: sums, differences, and products of two such functions also belong to the set. The following considerations are based on the use of one such ring: It is called the *scalar* or *function ring F*. We construct a non-commutative (for $n > 1$) ring of differential forms over F by means of certain closely-related symbols:

$$
\begin{array}{ll}
dx_i & (i = 1, 2, \ldots, n) \\
d(x_i, x_k) & (i, k = 1, 2, \ldots, n) \\
\quad\ldots\ldots\ldots\ldots & \\
d(x_{i_1}, x_{i_2}, \cdots, x_{i_p}) & (i_1, i_2, \ldots\, i_p = 1, 2, \ldots, n),
\end{array}
$$

which shall denote first, second, …, p^{th} degree *differentials*, and for which we assume from now on that:

1. *The differentials are skew-symmetric in the indices, i.e.:*

$$d(x_{i_1}, x_{i_2}, \cdots, x_{i_p}) = d(x_{k_1}, x_{k_2}, \cdots, x_{k_p})$$

or

$$d(x_{i_1}, x_{i_2}, \cdots, x_{i_p}) = -d(x_{k_1}, x_{k_2}, \cdots, x_{k_p})$$

depending on whether $k_1, k_2, \ldots, k_p$ is an even or odd permutation of $i_1, i_2, \ldots, i_p$.

2. The *multiplication of differentials* obeys the rules:

$$d(x_{i_1}, x_{i_2}, \cdots, x_{i_p})\ d(x_{k_1}, x_{k_2}, \cdots, x_{k_p}) = d(x_{i_1}, x_{i_2}, \cdots, x_{i_p}, x_{k_1}, x_{k_2}, \cdots, x_{k_p})\,.$$

The *ring of differential forms D* thus-defined is the set of all expressions (differential forms):

$$\Omega = a + \sum a_i\ dx_i + \sum a_{ik}\ d(x_i, x_k) + \ldots + \sum a_{i_1, i_2, \cdots, i_p}\ d(x_{i_1}, x_{i_2}, \cdots, x_{i_p}) + \ldots$$

in which the a, a_i, a_{ik}, … are arbitrary scalars of F.

3. The *addition* of two differential forms Ω and:

$$\Theta = b + \sum b_i\ dx_i + \sum b_{ik}\ d(x_i, x_k) + \ldots$$

is defined by:

$$\Omega + \Theta = (a + b) + \sum (a_i + b_i)\, dx_i + \sum (a_{ik} + b_{ik})\, d(x_i, x_k) + \ldots,$$

and *multiplication by a scalar c* is defined by:

$$c\Omega = \Omega c = ac + \sum a_i \, c dx_i + \sum a_{ik} \, c d(x_i, x_k) + \ldots$$

An associative multiplication of arbitrary differential forms of D is defined by assumption 2 and the use of distributivity.

The skew symmetry of differentials gives us only one differential of degree n – namely, $d(x_1, x_2, \ldots, x_n)$ – and none of higher degree. The

$$n + \binom{n}{2} + \binom{n}{3} + \ldots + \binom{n}{n}$$

various differentials, together with 1, which one may also refer to as a *differential of degree 0*, define a basis for the ring D over F.

If only differentials of degree p appear in a differential form then it is called a *homogeneous form of degree p*, or, more simply, a *(differential) form of degree p*. One also calls the forms of first degree *Pfaffian forms*. We shall assume, once and for all, that lower-case Greek symbols $\omega, \vartheta, \varpi, \varphi, \theta,$ etc., will always represent homogenous forms.

For two forms:

$$\omega = \sum a_{i_1, i_2, \cdots, i_p} \; d(x_{i_1}, x_{i_2}, \cdots, x_{i_p})$$
$$\theta = \sum b_{i_1, i_2, \cdots, i_q} \; d(x_{i_1}, x_{i_2}, \cdots, x_{i_q})$$

one has:

$$\omega\theta = \sum_{i_1, i_2, \cdots, i_p, k_1, k_2, \cdots, k_q} a_{i_1, i_2, \cdots, i_p} b_{k_1, k_2, \cdots, k_q} \; d(x_{i_1}, x_{i_2}, \cdots, x_{i_p}, x_{k_1}, \cdots, x_{k_p})$$
$$\theta\omega = \sum_{i_1, i_2, \cdots, i_p, k_1, k_2, \cdots, k_q} a_{i_1, i_2, \cdots, i_p} b_{k_1, k_2, \cdots, k_q} \; d(x_{k_1}, x_{k_2}, \cdots, x_{k_p}, x_{i_1}, x_{i_2}, \cdots, x_{i_p})$$

and, from this:

(1) $$\omega\theta = (-1)^{pq}\theta\omega.$$

Let Ω be an arbitrary differential form and let:

$$\Omega = \omega^{(0)} + \omega^{(1)} + \ldots + \omega^{(n)}$$

be its decomposition into individual homogeneous parts. For the multiplication of Ω with a homogeneous form ω of degree p, one then has:

(2) $$\omega\Omega = \Omega\,\omega \qquad \text{when } p \text{ is even}$$
$$\Omega\,\omega = \Omega^*\omega \qquad \text{when } p \text{ is odd.}$$

Here, $\Omega^* = \omega^{(0)} - \omega^{(1)} + \omega^{(2)} - \ldots + (-1)^n \omega^{(n)}$.

From (1), the product of a form of odd degree with itself is always null. *If two or more minus signs appear in the product of a form ω of odd degree with itself then the product is identically null,* since, from (2), one can write all of the individual factors of ω.

Of the *ideals* [1] that one can consider in the ring D, only one ideal is of interest to us, viz., the ideal of homogeneous forms:

$$\omega_1,\ \omega_2,\ \ldots,\ \omega_r. \tag{3}$$

Here, the difference between right and left ideals disappears; the right ideal obtained from (3), i.e., the set of elements:

$$\omega_1\Omega_1 + \omega_2\Omega_2, \ldots + \omega_r\Omega_r,$$

in which the Ω_i run through all of the forms of D, is identical with the corresponding left ideal that, from (2), one can form from Ω_i (possibly in the form of Ω_i^*) on the left-hand side of ω.

If:

$$\Theta = \theta^{(0)} + \theta^{(1)} + \ldots + \theta^{(n)}$$

belongs to the ideal that is determined by (3) then the same is true for each homogeneous part $\theta^{(i)}$ of Θ; one then collects terms of degree i in the summands $\omega_v\,\Omega_v$ on the right-hand side of:

$$\Theta = \sum_{v=1}^{r} \omega_v \Omega_v$$

in order to express the $\theta^{(i)}$ as linear combinations of the ω_v.

We have the following simple criterion for the linear dependence of Pfaff forms:

r Pfaff forms:

$$\omega_v = \sum_{k=1}^{n} a_{vk}\, dx_k \qquad (v = 1, 2, \ldots, r)$$

are linearly dependent when and only when their product vanishes:

$$\omega_1\,\omega_2 \ldots \omega_r = 0.$$

This follows immediately from:

$$\omega_1\,\omega_2 \ldots \omega_r = \sum_{i_1 < i_2 < \cdots < i_r}^{n} \begin{vmatrix} a_{1i_1} & a_{1i_2} & \cdots & a_{1i_r} \\ a_{2i_1} & a_{2i_2} & \cdots & a_{2i_r} \\ \cdots & \cdots & \cdots & \cdots \\ a_{ri_1} & a_{ri_2} & \cdots & a_{ri_r} \end{vmatrix} d(x_{i_1}, x_{i_2}, \cdots, x_{i_r}).$$

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[1]) Cf., Van der Waerden, *Moderne Algebra I*, pp. 53.

The following remarks will be useful to us in later applications:

If the Pfaffian forms $\omega_1, \omega_2, \ldots, \omega_r$ are linearly independent then all of their products $\omega_{i_1}\omega_{i_2}\cdots\omega_{i_p}$ ($i_1 < i_2 < \ldots < i_p$) are also linearly independent.

In a linear relation:

$$\sum_{i_1<i_2<\cdots<i_p}^{n} a_{i_1,i_2,\cdots,i_p}\,\omega_{i_1}\omega_{i_2}\cdots\omega_{i_p} = 0 \qquad (a \text{ belongs to } F),$$

all of the terms after the first one disappear upon multiplication by $\omega_{i_{p+1}}, \omega_{i_{p+2}}, \ldots, \omega_{i_r}$, in which $i_1, i_2, \ldots, i_r$ represents a permutation of 1, 2, …, r, since at least two equal linear factors must appear in all of the remaining terms. One also obtains:

$$\sum a_{i_1,i_2,\cdots,i_p}\,\omega_1\,\omega_2\ldots\omega_r = 0,$$

from which, since $\omega_1\,\omega_2\ldots\omega_r \neq 0$, the vanishing of the a must follow.

If $\omega_1, \omega_2, \ldots, \omega_r$ are once again arbitrary homogeneous differential forms then the expression:

$$\Omega \equiv 0 \pmod{\omega_1, \omega_2, \ldots, \omega_r}$$

means that the form Ω belongs to the ideal that is generated by the $\omega_1, \omega_2, \ldots, \omega_r$. From this, one obtains all of the rules for arithmetic with congruences ([1]).

2. Differentiation. Up till now, we have used the differential only as an algebraic symbol. In order to justify the use of the word "differential," we must first introduce a process of differentiation in D. It shall be an additive operation in D that takes a scalar $a(x_1, x_2, \ldots, x_n)$ to the Pfaffian form:

$$da = \frac{\partial a}{\partial x_1}dx_1 + \frac{\partial a}{\partial x_2}dx_2 + \cdots + \frac{\partial a}{\partial x_n}dx_n$$

and, in general, it takes a p^{th} degree "monomial":

$$\omega = a\,d(x_{i_1}, x_{i_2}, \cdots, x_{i_p})$$

to the differential form:

$$d\omega = da\,d(x_{i_1}, x_{i_2}, \cdots, x_{i_p}) = \sum_{l=1}^{n}\frac{\partial a}{\partial x_l}(x_l, x_{i_1}, x_{i_2}, \cdots, x_{i_p})$$

[1]) Cf., Van der Waerden, *Moderne Algebra I*, pp. 54.

of degree $p + 1$. Differentiation is then defined for all differential forms by requiring it to be a linear operation. A form that is produced by the differentiation of a form Ω will always be indicated by placing the letter d before it – viz., $d\Omega$ – and one calls this form the *differential* or *derivative* of Ω. [1])

The dx_i now appear as the differentials of the scalars x_i.

For two monomials:

$$\omega = a\,d(x_{i_1}, x_{i_2}, \cdots, x_{i_p}), \qquad \theta = b\,d(x_{k_1}, x_{k_2}, \cdots, x_{k_q}),$$

one has:

$$\begin{aligned} d(\omega\theta) &= d(ab\,d(x_{i_1}, \cdots, x_{i_p}, x_{k_1} \cdots, x_{k_q})) \\ &= (b\,da + a\,db)\,d(x_{i_1}, x_{i_2}, \cdots, x_{i_p}) \\ &= da\,d(x_{i_1}, \cdots, x_{i_p})\, b\,d(x_{k_1}, \cdots, x_{k_q}) \\ &\quad + (-1)^p\, a\,d(x_{i_1}, \cdots, x_{i_p})\, db\,d(x_{k_1}, \cdots, x_{k_q}) \\ &= d\omega\,\theta + (-1)^p \omega\, d\theta. \end{aligned}$$

The sign $(-1)^p$ depends only upon ω and we have the general formula for an arbitrary form Ω and a homogeneous p^{th} degree form:

$$d(\omega\Omega) = d\omega\,\Omega + (-1)^p \omega\, d\Omega. \tag{4}$$

In particular, we note the case $p = 0$:

$$d(a\Omega) = da\,\Omega + a\,d\Omega \qquad (a \text{ is a scalar}). \tag{5}$$

A differential form Ω whose derivative $d\Omega$ vanishes identically is called *integrable*.

The integrable forms define a sub-ring of D, which follows from the fact that if:

$$d\Omega = 0, \qquad d\Theta = 0$$

then, from (4):

$$d(\Omega\Theta) = 0,$$

and naturally $d(\Omega + \Theta) = 0$ is also true.

A differential form that can be represented by the derivative of another differential (in D) will be called a *total differential*.

Since we have, for a monomial $\omega = a\,d(x_{i_1}, x_{i_2}, \cdots, x_{i_p})$:

$$d(d\omega) = d\sum_{l=1}^{n} \frac{\partial a}{\partial x_l} d(x_l, x_{i_1}, x_{i_2}, \cdots x_{i_p}) = \sum_{k,l} \frac{\partial^2 a}{\partial x_k \partial x_l} d(x_k, x_l, x_{i_1}, x_{i_2}, \cdots x_{i_p}) = 0,$$

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[1]) I have taken the liberty of replacing the customary notation Ω' with $d\Omega$.

the derivative of a derivative is always null.

For any differential form Ω *we have:*

(6) $$d(d\Omega) = 0.$$

Any total differential form is therefore integrable. The converse is not true in general. If the fundamental scalar ring consists of holomorphic functions at a point $x_1^0, x_2^0, \cdots, x_n^0$ then, as we shall see later (cf. pp. 63), every integrable form is a total differential.

In a ring of differential forms in which a differentiation is defined in addition to the algebraic operations, one will need to consider, not the usual ideals, but the so-called *differential ideals,* which are ideals in the usual sense, except that the following condition is also true for them: *if* Ω *belongs to the differential ideal then so does* $d\Omega$.

We are once more interested in homogeneous ideals.

The smallest differential ideal that contains the given forms $\theta_1, \theta_2, \ldots, \theta_l$ is called the *differential ideal that is generated by* $\theta_1, \theta_2, \ldots, \theta_l$. Obviously, it must contain the forms:

(7) $$\sum_{i=1}^{l} \theta_i \Omega_i + \sum_{i=1}^{l} d\theta_i \Psi_i ,$$

in which the Ω, Ψ range over all elements of D. However, the set of these forms is already a differential ideal since, from (4) and (6), the derivative of such an expression (7) can once more be written the same form.

If θ is integrable then the differential ideal that it generates is identical with the usual ideal that it determines.

3. Substitution of variables. The importance of the calculus of differential forms that we just developed lies in its *invariance under substitution of variables.*

In the equations:

(8) $$x_i = x_i(y_1, y_2, \ldots, y_m) \qquad (i = 1, 2, \ldots, n)$$

let the right-hand sides be single-valued holomorphic functions that are defined in a particular neighborhood Y (at a point, resp.) in complex $(y_1, y_2, \ldots, y_m)$-space that take the values $x_1, x_2, \ldots, x_n$ whenever $y_1, y_2, \ldots, y_m$ vary within Y. We tacitly assume that these or corresponding regularity assumptions will be satisfied for any other variable substitutions.

Under the subtitution (8), all scalars $a(x_1, x_2, \ldots, x_n)$ go to functions:

(9) $$\bar{a}(y_1, y_2, \ldots, y_m) = a(x_1(y), x_2(y), \ldots, x_n(y))$$

in the ring $\bar{F}$ of single-valued holomorphic functions of $y_1, y_2, \ldots, y_m$ on Y.

The transformation of the differential takes place in such a manner that the processes of multiplication and differentiation are invariant operations.

Since the differentiation of a scalar is invariant we must show that when $a(x_1, x_2, \ldots, x_n)$ goes to $\bar{a}(y_1, y_2, \ldots, y_m)$ under (8) then da also goes to the differential:

$$d\bar{a} = \sum_{k=1}^{m} \frac{\partial \bar{a}}{\partial y_k} dy_k$$

in the ring $\bar{D}$ of differential forms over $\bar{F}$. For this, it suffices to observe that dx_i is transformed into:

$$d\bar{x}_i = dx_i(y) = \sum_{k=1}^{m} \frac{\partial x_i}{\partial y_k} dy_k \qquad (i = 1, 2, \ldots, n)$$

because that means that $da = \sum \frac{\partial a}{\partial x_i} dx_i$ goes to:

$$\sum \frac{\partial \bar{a}}{\partial x_i} \frac{\partial x_i}{\partial y_k} dy_k = \sum \frac{\partial \bar{a}}{\partial y_k} dy_k .$$

The transformation formulas for the higher differentials:

$$d(x_{i_1}, x_{i_2}, \cdots, x_{i_p})$$

are uniquely determined because of the invariance of multiplication; from (8), the $d(x_{i_1}, x_{i_2}, \cdots, x_{i_p})$ must go from products of $dx_{i_1}, dx_{i_2}, \cdots, dx_{i_p}$ over to:

$$\begin{aligned} d\bar{x}_{i_1} d\bar{x}_{i_2} \cdots d\bar{x}_{i_p} &= \left(\sum_{k_1} \frac{\partial x_{i_1}}{\partial y_{k_1}} dy_{k_1}\right)\left(\sum_{k_1} \frac{\partial x_{i_1}}{\partial y_{k_1}} dy_{k_1}\right)\cdots\left(\sum_{k_1} \frac{\partial x_{i_1}}{\partial y_{k_1}} dy_{k_1}\right) \\ &= \sum_{k_1<k_2<\cdots<k_p}^{n} \frac{\partial(x_{i_1}, x_{i_2}, \cdots, x_{i_p})}{\partial(y_{k_1}, y_{k_2}, \cdots, y_{k_p})} d(y_{k_1}, y_{k_2}, \cdots, y_{k_p}) . \end{aligned} \tag{10}$$

From this, it is clear what we mean when we say that we apply the substitution (8) to a differential form:

$$\Omega = a + \sum a_i \, dx_i + \sum a_{ik} \, d(x_i, x_k) + \ldots;$$

one replaces each coefficient $a_{i_1, i_2, \cdots, i_p}$ by way of (9) with the corresponding $\bar{a}_{i_1, i_2, \cdots, i_p}$ and each differential $d(x_{i_1}, x_{i_2}, \cdots, x_{i_p})$ with the expression (10).

We have the following *invariance theorem:* If $\bar{\Omega}$ ($\bar{\Theta}, \overline{\Omega\Theta}, \overline{d\Omega}$, resp.) are the differential forms that result from Ω ($\Omega, \Omega\Theta, d\Omega$, resp.) by substitution in (8) then we have:

(11) $$\overline{\Omega\Theta} = \overline{\Theta\Omega},$$

(12) $$\overline{d\Omega} = d\bar{\Omega},$$

which one can also express as:

The operations on the differential ring are commutative under variable substitution.

In order to prove (11), (12), it obviously suffices to let Ω and Θ be monomials:

$$\Omega = a\,d(x_{i_1}, x_{i_2}, \cdots, x_{i_p}) = a\,\omega,$$
$$\Theta = b\,d(x_{k_1}, x_{k_2}, \cdots, x_{k_q}) = b\,\theta.$$

We have:

$$\overline{\Omega\Theta} = \overline{ab\omega\theta} = \bar{a}\bar{b}\,\overline{\omega\theta} = \bar{a}\bar{b}\,\bar{\omega}\bar{\theta} = (\bar{a}\,\bar{\omega})(\bar{b}\,\bar{\theta}) = \overline{\Theta\Omega},$$

because under the assumption (10) on the transformation of differentials the validity of $\overline{\omega\theta} = \bar{\omega}\bar{\theta}$ is already guaranteed.

As for the derivatives, upon recalling the invariance of multiplication that we just proved, one has:

$$\overline{d\Omega} = \overline{da\omega} = \overline{da}\,\bar{\omega} = d\bar{a}\,\bar{\omega}.$$

One can also write $d\bar{a}\,\bar{\omega}$ as $d(\bar{a}\,\bar{\omega})$, since, from (5), we have:

$$d(\bar{a}\,\bar{\omega}) = d\bar{a}\,\bar{\omega} + \bar{a}\,d\,\bar{\omega},$$

and $\bar{\omega} = d\bar{x}_{i_1}\, d\bar{x}_{i_2} \cdots d\bar{x}_{i_p}$ is the product of integrable forms, so $d\,\bar{\omega} = 0$. One thus has:

$$\overline{d\Omega} = d(\bar{a}\,\bar{\omega}) = d\,\bar{\Omega}.$$

4. Differential equations. Let:

$$\Theta_1, \Theta_2, \ldots, \Theta_k$$

be certain differential forms on *D*. A substitution of variables:

(13) $$x_i = x_i(u_1, u_2, \ldots, u_m) \qquad (i = 1, 2, \ldots, n)$$

under which all of the forms Θ_i vanish is called a *solution of the system of differential equations:*

(14) $$\Theta_1 = 0, \; \Theta_2 = 0, \; \ldots, \; \Theta_k = 0.$$

Since each of the homogeneous parts of a form Θ will, from (13), obviously be annulled identically, equations (14) and the following ones:

$$\theta_1 = 0,\ \theta_2 = 0,\ \ldots,\ \theta_l = 0, \tag{15}$$

whose left-hand sides consist of the various vanishing homogeneous parts of the Θ_i, will have the same solutions. It therefore suffices to consider *systems of homogeneous differential equations.*

Among the θ, one can also find forms of degree zero whose associated equations $\theta = 0$ are scalar.

From the invariance properties of the differential ring operations, we conclude that:

Each solution of the differential equations:

$$\theta_1 = 0,\ \theta_2 = 0,\ \ldots,\ \theta_l = 0$$

also annuls each form of the differential ideal $\mathfrak{a}$ *that is generated by the left-hand sides.*

If a form Ω vanishes under the substitution (13) then, from (11), (12), $d\Omega$ and each "multiple" $\Omega\Theta$ (Θ is arbitrary in D) will also be annulled under (13). We call $\mathfrak{a}$ the *ideal that is associated with the differential equations*.

II. Theory of functions and geometrical considerations.

1. Algebraic manifolds. One understands the term *algebraic manifold* in complex $(x_1, x_2, \ldots, x_n)$-space to mean a point set that can be described by a system of equations:

$$
(1) \qquad \begin{aligned}
&\mathfrak{B}_1(x_1 - x_1^0, x_2 - x_2^0, \ldots, x_n - x_n^0) = 0 \\
&\mathfrak{B}_2(x_1 - x_1^0, x_2 - x_2^0, \ldots, x_n - x_n^0) = 0 \\
&\qquad \ldots \\
&\mathfrak{B}_s(x_1 - x_1^0, x_2 - x_2^0, \ldots, x_n - x_n^0) = 0
\end{aligned}
$$

in the neighborhood of each its points $(x_1^0, x_2^0, \cdots, x_n^0)$, in which each of the left-hand sides is a holomorphic power series in (x^0). One says that this manifold M is *regular* at (x^0), or that it is *simple* at (x^0) (it has (x^0) for a simple point), when equation (1) can be solved in a neighborhood of (x^0) in such a way that some – say, $n - r$ – of the coordinates x behave like holomorphic functions of the remaining r coordinates; the number r is the *dimension* of M at (x^0). If M is regular and r-dimensional at (x^0) then equations (1) can be expressed in terms of $n - r$ other ones:

$$
(2) \qquad \varphi_i(x_1, x_2, \ldots, x_n) = 0 \quad (\varphi \text{ is holomorphic at } (x^0)) \quad (i = 1, 2, \ldots, n - r),
$$

such that the $n - r$ differentials:

$$
d\varphi_i = \sum_{k=1}^{n} \frac{\partial \varphi_i}{\partial x_k} dx_k
$$

are linearly independent for $(x) = (x^0)$. In fact, the holomorphic solubility of (2) in terms of $n - r$ of the x follows from this, and substitution gives the equations that the remaining $n - r$ of the holomorphic x must satisfy, which is a system of equations of the form (2).

The dimension of an algebraic manifold M is well-defined only at a simple point of M, and it can vary at different parts of M. If one says simply "the dimension of M" then one always means the maximum dimension that M takes at its simple points.

The set of simple points of an algebraic manifold M is always open; the non-simple, i.e., the so-called *singular* points of M are accumulations of simple points and define a lower-dimensional algebraic submanifold of M. [1])

If the set of simple points is connected then any two simple points may be connected by a path in M that consists of only simple points; one then says that M is *irreducible*. In this case, M has the same dimension everywhere.

2. Regular systems of equations. When the equations:

$$
(3) \qquad \varphi_i(x_1, x_2, \ldots, x_n) = 0 \qquad (i = 1, 2, \ldots, s)
$$

[1]) We shall content ourselves with the mere statement of this fact without going into its not-so-elementary proof.

define an r-dimensional manifold M_r through (x^0) that is simple at that point then we do not necessarily need to find $n - r$ linearly independent differentials among the $d\varphi_i$ at $(x) = (x^0)$. If that is the case then one calls the system of equations (3) *regular* at (x^0) and calls $n - r = \rho$ the *rank* of the system.

In this case, any function $\Phi(x_1, x_2, \ldots, x_n)$ that is holomorphic at (x^0) and vanishes on M_r can be expressed as a linear combination of elements on the left-hand side of (3):

$$\Phi(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{s} A_i(x_1, x_2, \ldots, x_n)\varphi_i(x_1, x_2, \ldots, x_n)$$

($A_i(x)$ is holomorphic at (x^0)).

For instance, let:

$$d\varphi_1, d\varphi_2, \ldots, d\varphi_\rho,$$

be linearly independent at (x^0). One chooses r more functions that are holomorphic at (x^0):

$$\psi_{\rho+1}, \psi_{\rho+2}, \ldots, \psi_n,$$

so that $d\varphi_1, d\varphi_2, \ldots, d\varphi_\rho, d\psi_{\rho+1}, d\psi_{\rho+2}, \ldots, d\psi_n$ are linearly independent at (x^0). The equations:

$$\varphi_i = z_i \qquad (i = 1, 2, \ldots, \rho)$$
$$\psi_i = z_i \qquad (i > \rho)$$

define a one-to-one transformation of the variables, and M_r has the following equations in terms of the z:

(4) $$z_1 = z_2 = \ldots = z_\rho = 0.$$

When the functions F are transformed into functions of z they may be developed into a power series that vanishes for (4), and can therefore be expressed in the form:

$$z_1\mathfrak{B}_1(z) + z_2\mathfrak{B}_2(z) + \ldots + z_\rho\mathfrak{B}_\rho(z) \qquad \text{(the } \mathfrak{B}\text{'s are power series).}$$

By the applying inverse transformation to the x variables one obtains the desired sort of representation for F in which only $\varphi_1, \varphi_2, \ldots, \varphi_\rho$ are involved.

A system of functions $\psi_i(x_1, \ldots, x_n)$ that are holomorphic at (x^0) and vanish on the zero manifold M_r of the functions φ_i is called a *basis* for the system of equations (3) when every function $\Phi(x_1, \ldots, x_n)$ that is holomorphic at (x^0) and vanishes on M_r can be expressed as a linear combination of the ψ_i:

$$\Phi = \sum A_i(x)\,\psi_i(x).$$

We also write:

$$\Phi \equiv 0 \qquad (\text{mod } y_1, y_2, \ldots).$$

As we have seen, the φ_i themselves define such a basis.

Choose s' ($\leq s$) of the equations (3), say:

$$\varphi_i(x_1, x_2, \ldots, x_n) = 0 \qquad (i = 1, 2, \ldots, s') \tag{5}$$

such that they define a regular system of equations of rank ρ' ($\leq \rho$). With their help, one can represent ρ' of the variables x – perhaps $x_1, x_2, \ldots, x_{\rho'}$ – as holomorphic functions of the remaining ones, and by substituting the expressions that are thus obtained:

$$x_i = f_i(x_{\rho'+1}, \ldots, x_n) \qquad (i = 1, 2, \ldots, \rho')$$

into the remaining equations (13), one obtains a new regular system of equations:

$$\bar{\varphi}_i(x_{\rho'+1}, x_{\rho'+2}, \ldots, x_n) = 0 \qquad (i = s'+1, \ldots, s), \tag{6}$$

which is indeed, as one easily sees, of rank $\rho - \rho'$, so (5) and (6) collectively define a basis for the system (3).

We observe the following properties of regular systems of equations:

1. If (3) is regular at (x^0) then it is also regular in a sufficiently small neighborhood of (x^0) in any zero locus.

2. Under a *biholomorphic transformation,* i.e., a transformation:

$$x_i = x_i(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \qquad (i = 1, 2, \ldots, n)$$

that is a one-to-one holomorphic map of a neighborhood of a point $(\bar{x}_1^0, \bar{x}_2^0, \cdots, \bar{x}_n^0)$ in $(\bar{x})$-space onto a neighborhood of $(x_1^0, x_2^0, \cdots, x_n^0)$ in (x)-space, the system of equations (3) goes to a system that is regular at $(\bar{x}^0)$:

$$\bar{\varphi}_i(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) = 0 \qquad (i = 1, 2, \ldots, s)$$

and has the same rank ρ.

3. Linear vector spaces. A p-dimensional vector subspace at a point $(x_1^0, \cdots, x_n^0)$ in the space of $(x_1, \ldots, x_n)$ is spanned by p independent vectors with the components:

$$\Delta_v x_1, \Delta_v x_2, \ldots, \Delta_v x_n, \qquad (v = 1, 2, \ldots, p).$$

The determinant:

$$z_{i_1,i_2,\cdots,i_p} = \begin{vmatrix} \Delta_1 x_{i_1} & \Delta_1 x_{i_2} & \cdots & \Delta_1 x_{i_p} \\ \Delta_2 x_{i_1} & \Delta_2 x_{i_2} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \Delta_p x_{i_1} & \Delta_p x_{i_2} & \cdots & \Delta_p x_{i_p} \end{vmatrix},$$

whose value is independent of the choice of spanning vectors, makes the direction of V_p unique. These so-called *Grassmann direction coordinates* $z_{i_1,i_2,\cdots,i_p}$ are not independent for $p > 1$, $n > 3$, because there then exist algebraic relations between them that can be interpreted as the equations for a $(n-p)p$-dimensional algebraic manifold – viz., the *Grassmann manifold* G_p^n – in the projective space with the $\binom{n}{p}$ homogeneous coordinates z. For $n = 4$, $p = 2$, one has one relation, which is known as the Plücker relation, between the six homogeneous line coordinates in R_3. Furthermore, the relations between the $z_{i_1,i_2,\cdots,i_p}$ in the general case are simple to express [1]); they follow directly from certain quadratic relations that correspond to the Plücker relations precisely. The exact form of these equations is of no interest to us; we only remark that one can always solve them in such a way that all of the inhomogeneous direction coordinates:

$$u_{i_1,i_2,\cdots,i_p} = \frac{z_{i_1,i_2,\cdots,i_p}}{z_{1,2,\cdots,p}}$$

appear as complete rational functions of the $(n-p)p$ special ones:

$$u_{k,2,3,\ldots,p},\ u_{1,k,3,\ldots,p},\ \ldots,\ u_{1,2,\ldots,p-1,k}, \qquad (k = p+1, p+2, \ldots, n).$$

Naturally, there are relationships for the other inhomogeneous coordinates, for which another z appears in the denominator instead of $u_{1,2,3,\ldots,p}$. From this, we see that G_p^n is globally simple as an algebraic manifold in the $\left(\binom{n}{p}-1\right)$-dimensional complex projective z-space: the Grassmann manifold is free of singularities.

From now on, we shall understand the notation $G_p^n(z)$ to mean a basis for the relations between the z, i.e., $G_p^n(z)$ is a finite set of polynomials (e.g., quadratic ones) $P_1(z)$, $P_2(z)$, …, $P_l(z)$ in such a way that each (naturally, homogeneous) relation that exists between the z can be expressed in terms of the vanishing of a linear combination:

$$\sum A_i(z) P_i(z) \qquad (A_i(z) \text{ are polynomials}).$$

[1]) See E. Bertini, *Geometria proiettiva degli iperspazia*, Messina 1923. pp. 45-47. In the German translation (by A. Duschek, Vienna 1924), see pp. 42-44.

$G_p^n(z) = 0$ is a regular system of equations in the neighborhood of each null point (z^0) of $G_p^n(z)$.

When one also considers the variability of these points, the totality of all p-dimensional vector subspaces can be represented in the product space $R(x, z)$ as the n-dimensional x-space and the projective z-space by way of the $(n-p)p+n$-dimensional manifold $G_p^n(z) = 0$.

It is clear what we are to understand when we say *a neighborhood of a p-dimensional vector subspace* $V_p^0 = (x^0, z^0)$: it is the set of all V_p whose points (x, z) correspond to a certain neighborhood of (x^0, z^0) in $R(x, z)$. The space $V_p = (x, z)$ is said to (arbitrarily) close to V_p^0 when (x, z) lies in an (arbitrarily small) neighborhood of (x^0, z^0).

4. **Differential equations as equations for linear vector spaces.** Let:

$$\omega = \sum a_{i_1, i_2, \cdots, i_p}(x_1, x_2, \ldots, x_n)\, d(x_{i_1}, x_{i_2}, \cdots, x_{i_p}) \tag{7}$$

be a differential form of degree p on D and let V_p^0 be a vector subspace. We say: ω *vanishes on* V_p^0, or V_p^0 *satisfies the equation* $\omega = 0$, when the coordinates (x^0, z^0) of V_p^0 make the expression:

$$\sum a_{i_1, i_2, \cdots, i_p}(x_1^0, x_2^0, \cdots, x_n^0)\, z^0_{i_1, i_2, \cdots, i_p}$$

vanish. In general, we say: *A q-dimensional vector subspace* V_q $(q \geq p)$ *satisfies the equation of degree p,* $\omega = 0$, *when each of the p-dimensional vector subspaces that are contained in it satisfies this equation.* $\omega = 0$ will imply no conditions whatsoever for subspaces V_q with $q < p$: every q-dimensional vector subspace satisfies every differential equation of degree higher than q. Finally, to say that V_q satisfies an inhomogeneous differential equation $\Omega = 0$ shall mean: V_q annuls every homogeneous part of Ω. Furthermore, these definitions shall be valid for equations $\omega = 0$ of degree 0; in that case, $\omega = 0$ is the only condition that is valid for the points V_q.

If a vector subspace satisfies the equation $\Omega = 0$ *then it also makes each multiple* $\Omega\Theta$ *of* Ω *vanish.*

In order to prove this, we can assume that Θ is a monomial $\Theta = d(x_{k_1}, x_{k_2}, \cdots, x_{k_r})$ and that Ω is homogeneous:

$$\Omega == \sum a_{i_1, i_2, \cdots, i_p}\; d(x_{i_1}, x_{i_2}, \cdots x_{i_p}).$$

In:

$$\Omega\Theta = \sum_{i_1, i_2, \cdots, i_p} a_{i_1, i_2, \cdots, i_p}\; d(x_{i_1}, x_{i_2}, \cdots x_{i_p}, x_{k_1}, \cdots, x_{k_r})$$

one imagines that the differentials of degree $p+r$ replace the determinants:

$$\Delta(x_{i_1}, \cdots, x_{k_r}) = \begin{vmatrix} \Delta_1 x_{i_1} & \Delta_1 x_{i_2} & \cdots & \Delta_1 x_{k_r} \\ \Delta_2 x_{i_1} & \Delta_2 x_{i_2} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots \\ \Delta_{p+r} x_{i_1} & \Delta_p x_{i_2} & \cdots & \Delta_{p+r} x_{k_r} \end{vmatrix}$$

that are defined by any $p+r$ vectors of V_q, and then develops that determinant into p-rowed minors of the first p columns according to the Laplace expansion theorem. One then obtains an expression:

$$\sum \Delta(x_{k_1}, x_{i_2}, \cdots, x_{k_r}) \sum_{i_1, i_2, \cdots, i_p} a_{i_1, i_2, \cdots, i_p} \, \Delta(x_{i_1}, x_{i_2}, \cdots x_{i_p}, x_{k_1}, \cdots, x_{k_r}) \, ,$$

in which the first summation sign refers to the $(p+r)$-rowed determinant. The sum:

$$\sum_{i_1, i_2, \cdots, i_p} a_{i_1, i_2, \cdots, i_p} \, \Delta(x_{i_1}, x_{i_2}, \cdots x_{i_p}, x_{k_1}, \cdots, x_{k_r})$$

vanishes everywhere that V_q satisfies the equation $\Omega = 0$.

From the theorem that we just proved, it further follows that:

If the vector subspace V_q satisfies the equations:

$$\Omega_1 = 0, \Omega_2 = 0, \ldots, \Omega_l = 0$$

then it also annuls every form of the (ordinary) ideals that are defined by the left-hand sides.

A partial converse also follows from this theorem:

If a q-vector V_q annuls every equation of degree q in an ideal $\mathfrak{a}$ with the basis Ω_1, Ω_2, …, Ω_l then the Ω_i also vanish on V_q, and with them, all of the forms of $\mathfrak{a}$.

Proof. It obviously suffices to consider the case $l = 1$ with $\Omega_1 = \Omega$ assumed to be homogeneous of degree p ($p \leq q$). Suppose $d(x_1, x_2, \ldots, x_q) \neq 0$ on V_q, for instance. One can linearly represent the components:

$$\Delta x_{q+1}, \Delta x_{q+2}, \ldots, \Delta x_n$$

of the vectors of V_q in terms of the remaining ones, which can also be expressed by saying: One can find $n - q$ Pfaffian forms of the form:

$$\omega_i = dx_i - \alpha_{i1}dx_1 - \alpha_{i2}dx_2 - \ldots - \alpha_{iq}dx_q \qquad (q < i \leq n)$$

(perhaps with constant coefficients) that vanish on V_q. Modulo these forms ω_i, Ω may be reduced to a form that involves only the differentials dx_1, dx_2, …, dx_q:

$$\Omega_0 = \sum_{i_1<i_2<\cdots<i_p}^{q} c_{i_1,i_2,\cdots,i_p}\, d(x_{i_1}, x_{i_2}, \cdots x_{i_p}),$$

i.e.:

$$\Omega \equiv \Omega_0 \qquad (\text{mod } \omega_{q+1}, \omega_{q+2}, \ldots, \omega_n).$$

Therefore, all $\Omega_0\Theta$ of degree q vanish on V_q along with all forms $\Omega\Theta$. Thus, if one picks a particular combination of indices i_1, i_2, …, i_p and chooses i_{p+1}, i_{p+2}, …, i_q such that i_1, i_2, …, i_q is a (perhaps even) permutation of 1, 2, …, q then we have that:

$$\Omega_0\, d(x_{i_1}, x_{i_2}, \cdots x_{i_p}, x_{k_1}, \cdots, x_{k_r}) = c_{i_1,i_2,\cdots,i_p}\, d(x_1, x_2, \ldots, x_q) = 0$$

on V_q . The points of V_q thus annul all of the coefficients c, i.e., $\Omega = 0$ on V_q. Q.E.D.

5. Tangent elements. Let M be an algebraic manifold, (x^0) one of its simple points, and let:

(8) $$\varphi_1 = 0, \varphi_2 = 0, \ldots, \varphi_s = 0$$

be a regular system of equations that represents M in a neigborhood of (x^0). Any vector subspace at (x^0) that satisfies the equations:

$$d\varphi_1 = 0, d\varphi_2 = 0, \ldots, d\varphi_s = 0$$

is called a *tangent element* of M at (x^0). If r is the dimension of M at (x^0) then all of these tangent elements are contained in a larger space of dimension r. This is what we always mean when we speak of *the* tangent element at (x^0).

The definition of tangent element is obviously independent of the choice of regular system of equations that represent M at (x^0).

One also sees that one can define the tangent elements of M simply by vector subspaces that satisfy the equations:

$$\varphi_1 = 0, \varphi_2 = 0, \ldots, \varphi_s = 0, d\varphi_1 = 0, d\varphi_2 = 0, \ldots, d\varphi_s = 0$$

for the points of a sufficiently small neighborhood of (x^0). Such a neighborhood must be chosen to be small enough that all of the null points of φ_1, …, φ_s still belong to M and the system of equations (8) is still regular there.

6. Direction coordinates of a vector space. *The direction of a p-dimensional vector subspace V_p may be specified in the following way:* Let:

$$(9) \qquad \omega_i = \sum_{k=1}^{n} \alpha_{ik}\ (x_1, \ldots, x_n)\ dx_k \qquad (i = 1, 2, \ldots, n)$$

be any n linearly independent Pfaff forms (say, $\omega_i = dx_i$). If $\omega_1\,\omega_2 \ldots\ \omega_p \neq 0$ on V_p then for all of the vectors of V_p the expressions:

$$\sum_{k=1}^{n} \alpha_{ik}(x_1^0, \cdots, x_n^0)\,\Delta x_k \qquad (i > p, \text{ and } (x^0) \text{ is a given point of } V_p)$$

may be linearly represented by the corresponding ones with $i \leq p$. (We will always state this fact in the form: $\omega_1, \omega_2, \ldots, \omega_p$ are linearly independent on V_p.) Thus, the constants l_{ik} in the Pfaffian forms:

$$\vartheta_i = \omega_i - \sum_{k=1}^{p} l_{ik}\,\omega_k$$

are always uniquely determined by the requirement that the ϑ_i vanish on V_p. Each system of values for x and the $(n-p)p$ *constants l corresponds to precisely one p-dimensional vector subspace*. These quantities l can be regarded as the direction coordinates of V_p, and they have the advantage over the Grassmann coordinates that they are restricted by no relations. Everything happens the same way for them on such a V_p only if $\omega_1\,\omega_2 \ldots\ \omega_p \neq 0$. Their relationship with the $z_{i_1, i_2, \cdots, i_p}$ is easy to understand.

Let:

$$dx_i = \sum_{k=1}^{n} \beta_{ik}(x)\,\omega_k \qquad (i = 1, 2, \ldots, n)$$

be the solution of (9) for the dx. If:

$$dx_i \equiv \sum_{k=1}^{n} b_{ik}(x, l)\,\omega_k \qquad (\text{mod } \vartheta_{p+1}, \vartheta_{p+2}, \ldots, \vartheta_n)$$

and:

$$d(x_{i_1}, x_{i_2}, \cdots, x_{i_p}) \equiv \varphi_{i_1, i_2, \cdots, i_p}(x, l)\,\omega_1\,\omega_2 \ldots\ \omega_p \qquad (\text{mod } \vartheta)$$

then the direction coefficients $z_{i_1, i_2, \cdots, i_p}$ of V_p are proportional to the expressions $\varphi(x, l)$. Then, for any p independent vectors in V_p:

$$\Delta_\nu x_1, \Delta_\nu x_2, \ldots, \Delta_\nu x_n \qquad (\nu = 1, 2, \ldots, p),$$

we have:

$$\Delta_\nu x_i = \sum_{k=1}^{p} b_{ik}(x, l)\, w_{k\nu} \qquad \left(\text{in which we have set } \sum_{l=1}^{n} \alpha_{kl}\,\Delta_\nu x_l = w_{k\nu}\right),$$

and from this:

(10) $$z_{i_1,i_2,\cdots,i_p} = |\,\Delta_\nu x_i\,| = \varphi_{i_1,i_2,\cdots,i_p}(x,l)\,|\,w_{k\nu}\,|.$$

From the fact that:

$$\omega_1\,\omega_2\ldots\,\omega_{k\text{-}1}\,\omega_i\,\omega_{k+1}\ldots\,\omega_p \equiv l_{ik}\,\omega_1\,\omega_2\ldots\,\omega_p \qquad (\text{mod } \vartheta) \qquad (i > \nu),$$

or, simply:

$$\sum_{k_1,\cdots,k_p} \alpha^{ik}_{k_1,k_2,\cdots,k_p}\, d(x_{k_1}, x_{k_2}, \cdots x_{k_p}) \equiv l_{ik}\,\omega_1\,\omega_2\ldots\,\omega_p\,,$$

and:

$$\sum \alpha_{k_1,k_2,\cdots,k_p}\, d(x_{k_1}, x_{k_2}, \cdots x_{k_p}) \equiv l_{ik}\,\omega_1\,\omega_2\ldots\,\omega_p$$

(the α are certain determinants of the α_{ik}), it follows conversely that:

(11) $$\begin{cases} \sum \alpha^{ik}_{k_1,k_2,\cdots,k_p}(x)\, z_{k_1,k_2,\cdots,k_p} = l_{ik}\,|\,w_{ik}\,| \\ \sum \alpha_{k_1,k_2,\cdots,k_p}(x)\, z_{k_1,k_2,\cdots,k_p} = \quad |\,w_{ik}\,|\,. \end{cases}$$

III. Integral manifolds and integral elements.

1. Integral manifolds. A system of differential equations:

(1) $$\theta_1 = 0,\ \theta_2 = 0,\ \ldots,\ \theta_l = 0$$

is called *integrable* if all of its solutions are determined.

Let:

(2) $$x_i = x_i(u_1, u_2, \ldots, u_n) \qquad (i = 1, 2, \ldots, n)$$

be a solution of (1). When there are exactly p linearly independent differentials $dx_i(u)$ the solution is called *p-dimensional*. In the neighborhood of a system of values $(u_1^0, u_2^0, \cdots, u_m^0)$, as long as the number of independent $dx_i(u)$ is not less than p (2) allows us to eliminate the arguments u such that $n - p$ of the x_i among the $x_i = x_i(u^0) = x_i^0$ can be expressed as holomorphic functions of the remaining ones, say:

$$x_i = f_i(x_1, x_2, \ldots, x_p) \qquad (i = p+1, p+2, \ldots, n).$$

From (2), (u^0) is a parametric representation of a p-dimensional algebraic manifold M_p that is regular at (x^0), at least in a neighborhood of such a system of values (u^0).

We consider any form Ω in the differential ideal $\mathfrak{a}$ such that the system (1) is satisfied. One can think of substituting from (2) into W – at least for all (u) in a sufficiently small neighborhood of (u^0) – one then obtains:

$$\begin{cases} x_i = x_i & (i = 1, 2, \cdots, p) \\ x_i = f_i(x_1, x_2, \cdots, x_p) & (i = p+1, \cdots, n) \end{cases}$$

and in the resulting form $\bar{\Omega}$, which no longer includes the variables $x_1, x_2, \ldots, x_p$, one then substitutes:

(3) $$x_i = f_i(u_1, u_2, \ldots, u_n) \qquad (i = 1, 2, \ldots, p).$$

We obviously already have $\bar{\Omega} = 0$, since otherwise the vanishing of $\bar{\Omega}$ as a result of (3) would mean the existence of a linear dependency between the products of the differentials $dx_i(u)$ $(i = 1, 2, \ldots, p)$, which is impossible, from an earlier remark (cf. sec. 6).

One now observes that the statement: W vanishes under the substitution:

$$x_i = f_i(x_1, x_2, \ldots, x_p) \qquad (i = p+1, p+2, \ldots, n)$$

is equivalent to the statement:

(4) $$\Omega \equiv 0 \qquad (\mathrm{mod}\ (x_{p+1} - f_{p+1}), \ldots, (x_n - f_n), (dx_{p+1} - df_{p+1}), \ldots, (dx_n - df_n)).$$

The fact that (4) implies the vanishing of $\bar{\Omega}$ is obvious. As for the neighoring points, one subjects W to an identity transformation in which one replaces:

$$\begin{array}{lll} x_i & \text{with} & f_i + (x_i - f_i) \\ dx_i & \text{with} & df_i + (dx_i - df_i) \end{array}$$

and develops the coefficients of Ω in powers of $x_i - f_i$. One can write the resulting expression as:

(5) $$\Omega = \bar{\Omega} + \sum_{i=p+1}^{n} (x_i - f_i)\Omega_i + \sum_{i=p+1}^{n} (dx_i - df_i)\,\Theta_i,$$

in which Ω_i, Θ_i are differential forms with coefficients that are holomorphic in (x^0). (4) then follows from $\bar{\Omega} = 0$, precisely.

Since the tangent elements of M_p annul all of the forms found in the module (4), we conclude from (4) that every tangent element of M_p satisfies the equation $\mathfrak{a} = 0$.

Definition: *An algebraic manifold (which is irreducible and possesses only simple points) whose tangent elements satisfy the equation* $\mathfrak{a} = 0$ *is called an integral manifold.*

We see that every solution (2) corresponds to an integral manifold, which one obtains when the rank of the matrix $\left(\frac{\partial x_i}{\partial u_k}\right)$ is not less than p by letting $u_1, u_2, \ldots, u_m$ range through all possible values. It is easy to see that one obtains an irreducible manifold M_p in this way, and that all of its points, which correspond to the values of u that were left out, are accumulation points of M_p; the remaining points lie in lower-dimensional algebraic manifolds.

Conversely, every integral manifold M_p corresponds to infinitely many solutions of (1).

Let:

(6) $$x_i = f_i(x_1, x_2, \ldots, x_p) \qquad (i = p+1, \ldots, n)$$

be the equation of M_p in the neighborhood of one of its points $x_1^0, x_2^0, \cdots, x_n^0$. It follows from the identity (5) that for an arbitrary differential form Ω of $\mathfrak{a}$ we have:

$$\Omega \equiv \bar{\Omega} \;(\text{mod } x_{p+1} - f_{p+1}, \ldots, x_n - f_n, dx_{p+1} - df_{p+1}, \ldots, dx_n - df_n),$$

in which $\bar{\Omega}$ is the form that Ω turns into under the substitution (6). Now, since every M_p is indeed a manifold we should have that Ω vanishes for all of the vector subspaces that satisfy the equations:

$$x_{p+1} - f_{p+1} = 0, \quad \ldots, x_n - f_n = 0, \quad dx_{p+1} - df_{p+1} = 0, \quad \ldots, dx_n - df_n = 0.$$

If $\bar{\Omega}$ were not identically null then one would certainly find an element of this vector subspace that did not annul $\bar{\Omega}$; therefore, one can always find a tangent vector to M_p for which the coordinates $x_1, x_2, \ldots, x_p$ of the point in question take on arbitrary values in the neighborhood of $x_1^0, x_2^0, \cdots, x_p^0$, as well as the components $\Delta x_1, \Delta x_2, \ldots, \Delta x_p$. One thus necessarily has that $\bar{\Omega}$ is identically zero. If one now takes an arbitrary parametric representation for M_p:

$$x_i = f_i(u_1, u_2, \ldots, u_m) \qquad (i = 1, 2, \ldots, n) \tag{7}$$

with $x_i(u_1^0, u_2^0, \cdots, u_m^0) = x_i^0$ then the substitution of (7) in Ω can be further divided into two steps: One first performs (6), and then:

$$x_i = f_i(u_1, u_2, \ldots, u_m) \qquad (i = 1, 2, \ldots, p).$$

Ω already vanishes from the first step. Every parametric representation (7) of M_p thus gives a solution of the system of differential equations.

We will now see how the *problems of integration theory* appear in the production of complete integral manifolds. One can formulate these problems analytically in the following way:

Determine all regular systems of equations:

$$\varphi_1 = 0, \varphi_2 = 0, \ldots, \varphi_s = 0, \tag{8}$$

in such a way that:

$$\mathfrak{a} \equiv 0 \quad (\bmod\ \varphi_1, \varphi_2, \ldots, \varphi_s, d\varphi_1, d\varphi_2, \ldots, d\varphi_s),$$

i.e., such that one obtains the ideal $\mathfrak{a}$ that includes the differential ideal that is generated by the functions $\varphi_1, \varphi_2, \ldots, \varphi_s$.

2. Integral elements. A vector subspace that satisfies the equations $\mathfrak{a} = 0$ is called an *integral element.*

Since the vanishing of all of the forms of $\mathfrak{a}$ follows from the vanishing of all of the forms of degree p in $\mathfrak{a}$ for a p-dimensional vector subspace, one can characterize the p-dimensional integral elements E_p by the equations:

$$\mathfrak{a}_p = 0,$$

which immediately refer to the Grassmann direction coordinates of E_p. We shall use the notation $\mathfrak{a}_p$ to mean the totality of all forms of degree p in $\mathfrak{a}$. We would also like to consider the case $p = 0$. The points that satisfy the scalar equations $\mathfrak{a}_0 = 0$ shall be regarded as integral points or 0-dimensional integral elements.

Every lower-dimensional vector subspace that is contained in a p-dimensional integral element is also an integral element, since it also satisfies the equations $\mathfrak{a} = 0$.

If one understands $\mathfrak{a}_p(x, z)$ to mean the replacement of the $d(x_{i_1}, x_{i_2}, \cdots, x_{i_p})$ in $\mathfrak{a}_p$ with expressions in terms of $z_{i_1, i_2, \cdots, i_p}$ then one can represent the totality of all p-dimensional integral elements E_p in the space $R(x, z)$ by the algebraic manifolds:

(9) $$\mathfrak{a}_p(x, z) = 0, \qquad G_p^n(z) = 0,$$

which represent a continuous band of planes in the Grassmann manifold (Concerning $G_p^n(z)$, cf. pp. 15).

From this, we obtain a (p+1)-dimensional integral element E_{p+1} that $E_p = (x, z)$ goes through (i.e., one that contains E_p).

The $\binom{n}{p+1}$ homogeneous direction coordinates $w_{i_1, i_2, \cdots, i_{p+1}}$ of E_{p+1} must be represented in the form:

(10) $$w_{i_1, i_2, \cdots, i_{p+1}} = \Delta x_{i_1} z_{i_2, \cdots, i_{p+1}} - \Delta x_{i_2} z_{i_2, i_3, \cdots, i_{p+1}} + (-1)^p \Delta x_{i_{p+1}} z_{i_2, i_2, \cdots, i_p}$$

since there will be a vector:

$$\Delta x_1, \Delta x_2, \ldots, \Delta x_n$$

that, together with E_p, spans E_{p+1}. Since E_{p+1} is an integral element, one must have the following equations, which are analogous to (9):

$$\mathfrak{a}_{p+1}(x, w) = 0, \qquad G_{p+1}^n(w) = 0.$$

By introducing the expressions (10), the equations $G_{p+1}^n(w) = 0$, and therefore the equations $G_p^n(z) = 0$, will be satisfied identically (because the w can be written as a (p+1)-rowed determinant, on the basis of the fact that $G_p^n(z) = 0$), although $\mathfrak{a}_{p+1}(x, w) = 0$ might go over to:

$$\mathfrak{a}_{p+1}(x, z, \Delta x) = 0.$$

These are homogeneous linear equations in the Δx; every vector of E_p satisfies them. If $r_{p+1} + 1$ ($r_{p+1} \geq -1$) is the number of solutions Δx in this trivially independent system then one can say: $\infty^{r_{p+1}}$ *(p+1)-dimensional integral elements go through E_p* ($\infty^r = 1$ for r = 0 and = 0 for $r = -1$).

This number r_{p+1} can vary with E_p .

The notion of regular integral elements plays a fundamental role in integration theory.

Definition: *A p-dimensional integral element* $E_p^0 = (x^0, z^0)$ *is called regular when:*

1. *The system of equations:*

$$\mathfrak{a}_p(x, z) = 0, \qquad G_p^n(z) = 0$$

is regular in the neighborhood of (x^0, z^0);

2. *No* $(p+1)$*-dimensional integral elements pass through* E_p^0 *that do not also pass through the integral-*E_p *in the neighborhood of* E_p^0.

If N is the dimension of the manifold (9) in the neighborhood of (x^0, z^0) then (due to the homogeneity of the z) $N+1$ is the dimension of the system of solutions (x, z) of (9), and condition 1 persists, viz., that among the differentials:

$$d\mathfrak{a}_p(x, z), \qquad dG_p^n(z)$$

in the left-hand of (9) one can find precisely $n + \binom{n}{p} - N - 1$ of them that remain linearly independent at (x^0, z^0).

As far as the second condition is concerned, it may be expressed analytically: the linear system of equations $\mathfrak{a}_{p+1}(x, z, \Delta x) = 0$ has the same rank $n - r_{p+1} - p - 1$ for all values of the parameters x, z that are sufficiently close to (x^0, z^0) and satisfy equations (9). When condition 1 is already satisfied then the rank of this linear system of equations at (x^0, z^0) can be at most less than it is for the general (x, z) that satisfies the relations (9).

From the definition it immediately follows that any integral-E_p that is sufficiently close to E_p^0 is also regular when E_p^0 is, as well.

We will call a p-dimensional integral manifold *regular* when one finds at least one integral-E_p in its tangent elements.

3. Invariance under biholomorphic transformations. The fact that the partitioning of the integral elements into regular and singular (i.e., non-regular) elements is meaningful follows from their *invariance under biholomorphic transformations.*

Let:

$$x_i = x_i(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \qquad (i = 1, 2, \ldots, n) \tag{11}$$

be a biholomorphic transformation in a neighborhood of $(x_1^0, x_2^0, \cdots, x_n^0)$, $(\bar{x}_1^0, \bar{x}_2^0, \cdots, \bar{x}_n^0)$, resp., and let:

$$\bar{\theta}_1 = 0,\ \bar{\theta}_2 = 0,\ \ldots, \bar{\theta}_l = 0$$

be the new system of differential equations that results from using (11) in the left-hand side of the system (1). Due to the invariance of the differential ring operations and (11), the differential ideal $\mathfrak{a}$, which is generated by $\theta_1, \ldots, \theta_l$ over the ring f of functions that are holomorphic at (x^0), goes to the differential ideal $\bar{\mathfrak{a}}$ that is generated by $\bar{\theta}_1, \bar{\theta}_2, \cdots, \bar{\theta}_l$ over the ring of functions that are holomorphic at $(\bar{x}^0)$. The equations:

(12) $$\bar{\mathfrak{a}}_p(\bar{x}, \bar{z}) = 0, \qquad G_p^n(\bar{z}) = 0$$

that the p-dimensional integral elements $(\bar{x}, \bar{z})$ of the new systems define [1]) can thus be obtained directly from:

(13) $$\mathfrak{a}_p(x, z) = 0, \qquad G_p^n(z) = 0,$$

in which one has made the substitution:

(14) $$x_i = x_i(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n),$$
$$z_{i_1, i_2, \cdots, i_p} = \sum_{k_1 < k_2 < \cdots < k_p} \frac{\partial(x_{i_1}, x_{i_2}, \cdots, x_{i_p})}{\partial(\bar{x}_{k_1}, \bar{x}_{k_2}, \cdots, \bar{x}_{k_p})} \bar{z}_{k_1, k_2, \cdots, k_p}.$$

(14) represents a biholomorphic transformation in the variables (x, z) and $(\bar{x}, \bar{z})$ for all (x, z) [$(\bar{x}, \bar{z})$, resp.] for which (x) [$(\bar{x})$, resp.] lies in the neighborhood of (x^0) [$(\bar{x}^0)$, resp.]. If (13) is regular at (x^0, z^0) then the system of equations (12) is also regular at the corresponding point $(\bar{x}^0, \bar{z}^0)$. Furthermore, the equations:

$$\bar{\mathfrak{a}}_{p+1}(\bar{x}, \bar{z}, \Delta\bar{x}) = 0 \qquad \text{go to} \qquad \mathfrak{a}_{p+1}(x, z, \Delta x) = 0$$

through the use of (14) along with:

$$\Delta x_i = \sum_k \frac{\partial x_i}{\partial \bar{x}_k} \Delta \bar{x}_k \,,$$

from which it follows that the number r_{p+1} is the same for the corresponding elements at (x, z) and $(\bar{x}, \bar{z})$. From these remarks, it is clear that the regular integral elements of one system correspond to the regular elements of the other.

4. On the first regularity condition for integral elements. *In order to specify whether a given integral element* $E_p^0 = (\mathrm{x}^0, z^0)$ *satisfies the first regularity condition*, one can proceed as follows:

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[1]) As far as the definition and regularity of the integral element E_p, whose point in question (x) lies sufficiently close to (x^0), is concerned, it is obviously unimportant whether one considers $\mathfrak{a}$ to be defined over the original ring F or over f.

Suppose that $z^0_{1,2,\cdots,p} \neq 0$, for instance, so that one can use the inhomogeneous coordinates:

$$u_{i_1,i_2,\cdots i_p} = \frac{z_{i_1,i_2,\cdots i_p}}{z_{1,2,\cdots,p}}.$$

From this, the system:

(15) $$\mathfrak{a}_p(x, z) = 0, \qquad G^n_p(z) = 0$$

is regular for (x^0, z^0) if and only if the corresponding system of equations in u under the transformation:

$$(x) = (x^0), \qquad u_{i_1,i_2,\cdots i_p} = u^0_{i_1,i_2,\cdots i_p} = \frac{z^0_{i_1,i_2,\cdots i_p}}{z^0_{1,2,\cdots,p}}$$

is also regular.

Since the system of equations $G^n_p(z) = 0$ is already regular, one can replace it in the neighborhood of (u^0) with the equations:

(16) $$u_{i_1,i_2,\cdots i_p} - \varphi_{i_1,i_2,\cdots i_p}(u) = 0,$$

in which $\binom{n}{p} - 1 - p(n - p)$ of the u are expressed in terms of the remaining ones, perhaps:

(17) $$u_{k,2,3,\ldots,p}, \qquad u_{1,k,3,\ldots,p}, \ \ldots, \ u_{1,2,\ldots,p-1,k}, \qquad (k = p+1, \ldots, n).$$

From an earlier general remark concerning regular systems of equations (cf. pp. 14), the system:

(18) $$\mathfrak{a}_p(x, u) = 0$$

that is obtained from the system $\mathfrak{a}_p(x, z) = 0$ by replacing the u with their expressions in the $(n - p)p$ variables (17) must also be regular, and conversely the regularity of (15) follows from that of (18).

For practical applications, one is accustomed to seeing p-dimensional vector subspaces represented by equations in the style of pp. 19:

(19) $$\omega_i - \sum_{k=1}^{p} l^0_{ik}\,\omega_k = 0 \qquad (i = p+1, \ldots, n).$$

Because of this, we will consider *how one knows whether a vector subspace that is defined by the point* (x^0) *and the equations:*

$$\omega_i - \sum_{k=1}^{p} l^0_{ik}\,\omega_k = 0$$

is an integral element, and, in particular, whether it is a regular integral element.

Every differential form of degree p can be represented – modulo the left-hand side of (19) – in the form:

$$a(x, l)\,\omega_1\,\omega_2\,\ldots\,\omega_p\,,$$

in which $a(x, l)$ is a polynomial in l. From this, if $\mathfrak{a}_p(x, l)$ means the totality of functions $a(x, l)$ that are obtained from the forms of $\mathfrak{a}_p$ in this way then:

$$\mathfrak{a}_p(x^0, l^0) = 0$$

is the necessary and sufficient condition for (x^0, l^0) to be an integral element.

If we further assume that $z^0_{1,2,\cdots,p} \neq 0$ is the vector subspace that is defined at (x^0, l^0) and let $u^0_{i_1,i_2,\cdots i_p}$ denote the value of the inhomogeneous direction coordinates for (x^0, l^0) then equations (10) and (11) that were presented in sec. 18 teach us that the variables:

$$x_1, \ldots, x_n, \qquad u_{1,2,\ldots,\,i-1,\,k,\,i+1,\ldots,\,p} \qquad (i = 1, 2, \ldots, p,\; k = p+1, \ldots, n)$$

and the (x, l) in the neighborhood of (x^0, u^0) [(x^0, l^0), resp.] correspond to each other biholomorphically, and from this, equations (18) and:

$$\mathfrak{a}_p(x, l) = 0 \tag{20}$$

are always both regular. The integral element (x^0, l^0) thus satisfies the regularity condition 1 when and only when the system (20) is regular for (x^0, l^0).

As for the second regularity condition, we will not go into its interpretation here, since we shall have more to say about that later in the context of the construction of regular chains of integral elements.

IV. Existence Theorems for Integral Manifolds

1. Proof of the first existence theorem. The equations $\mathfrak{a}_p(x^0, z^0, \Delta x) = 0$, which are satisfied by all of the vectors Δx and, together with $E_p^0 = (x^0, z^0)$, span an integral element, determine an $(r_{p+1} + p + 1)$-dimensional vector subspace at (x^0), viz., the so-called *polar element* of E_p^0; we denote it by $H(E_p^0)$. It contains all of the integral elements that go through E_p^0. One can specify the single $(p+1)$-dimensional integral element E_{p+1}^0 that goes through E_p^0 by using it, since one intersects $H(E_p^0)$ with an $(n - r_{p+1})$-dimensional vector subspace $V_{n-r_{p+1}}^0$ that lies in E_p^0; this intersecting plane then has dimension $p+1$ for a general point in $V_{n-r_{p+1}}^0$.

First existence theorem. *Let M_p be a regular p-dimensional integral manifold, let $E_p^0 = (x^0, z^0)$ be its regular elements, and let $F_{n-r_{p+1}}$ be an $(n - r_{p+1})$-dimensional manifold through M_p whose tangent element at (x^0) has only one E_{p+1}^0 in common with $H(E_p^0)$. There then exists exactly one $(p+1)$-dimensional integral manifold M_{p+1} in the neighborhood of (x^0) that goes through M_p and is contained in $F_{n-r_{p+1}}$.*

We prove this in three steps.

A. Lemma. Since the statement of theorem is biholomorphically invariant, we can assume that:

$$x_{n-r+1} = x_{n-r+2} = \ldots = x_n = 0 \qquad (r = r_{p+1})$$

are the equations for $F_{n-r_{p+1}}$ and:

$$(1) \qquad \begin{aligned} x_{p+1} &= 0, && \\ x_i &= \varphi_i(x_1, x_2, \ldots, x_p) && (p+1 < i \leq n-r), \\ x_i &= 0 && (n-r < i < \leq n), \end{aligned}$$

$$(\varphi_i \text{ is holomorphic for } x_1^0, x_2^0, \cdots, x_p^0)$$

are the equations for M_p, and also assume that the direction coordinate $w_{1,2,\ldots,p+1}$ is not zero on E_{p+1}^0.

The equations of the desired manifold M_{p+1} can be expressed in the form:

$$(2) \qquad \begin{aligned} x_i &= f_i(x_1, x_2, \ldots, x_p) && (p+1 < i \leq n-r) \\ x_i &= 0 && (n-r < i). \end{aligned}$$

Since (x^0, z^0) shall be a regular integral element, the system of equations:

$$\mathfrak{a}_p(x, z) = 0, \qquad G_p^n = 0,$$

is regular in (x^0, z^0), and one can therefore choose one of them in the left-hand side of these equations, which we denote by:

$$\psi_1(x, z),\ \psi_2(x, z),\ \ldots,\ \psi_\rho(x, z),$$

in such a way that every function $F(x, z)$ that is holomorphic at (x^0, z^0) and vanishes for all integral elements E_p in the neighborhood of (x^0, z^0) can be represented in the form:

$$F(x, z) = \sum_{i=1}^{\rho} A_i(x, z)\, \psi_i(x, z) \qquad (A_i \text{ holomorphic at } x^0, z^0).$$

In this case, we simply write:

$$F(x, z) \equiv 0 \qquad (\text{mod } \psi_1, \psi_2, \ldots, \psi_\rho).$$

One finds exactly $t = n - r_{p+1} - p - 1$ linearly independent equations among those of $\mathfrak{a}_p(x^0, z^0, \Delta x) = 0$. Let:

$$\Phi_h = \sum a^{(h)}_{i_1, i_2, \cdots, i_{p+1}}\, d(x_{i_1}, x_{i_2}, \cdots, x_{i_{p+1}}) \qquad (h = 1, 2, \ldots, t)$$

be t differential forms of $\mathfrak{a}_{p+1}$ that correspond to t such independent equations, and let:

$$\Phi = \sum a_{i_1, i_2, \cdots, i_{p+1}}\, d(x_{i_1}, x_{i_2}, \cdots, x_{i_{p+1}})$$

mean a completely arbitrary form of $\mathfrak{a}_{p+1}$.

We consider the $t+1$ expressions:

$$\begin{aligned}
\Phi_h(x, z, \Delta x) &= \sum a^{(h)}_{i_1, i_2, \cdots, i_{p+1}} (\Delta x_{i_1}, z_{i_2, \cdots, i_{p+1}} - \Delta x_{i_2}, z_{i_2, i_3, \cdots, i_{p+1}} \cdots + \cdots) \qquad (h = 1, 2, \ldots, t), \\
\Phi(x, z, \Delta x) &= \sum a_{i_1, i_2, \cdots, i_{p+1}} (\Delta x_{i_1}, z_{i_2, \cdots, i_{p+1}} - \Delta x_{i_2}, z_{i_2, i_3, \cdots, i_{p+1}} \cdots + \cdots).
\end{aligned}$$

The last linear form is linearly independent of the higher t for (x^0, z^0) and also for every integral element (x, z) close to it since (x^0, z^0) is a regular element. All $(t+1)$-rowed determinants in the coefficient matrix of this linear form are therefore $\equiv 0$ (mod $\psi_1, \psi_2, \ldots, \psi_\rho$). If one then multiplies the Φ_h and Φ by suitable sub-determinants $U_h(x, z)$ (a matrix, $U(x, z)$, resp.) then, by addition, one obtains an expression:

$$\sum_{h=1}^{t} U_h(x, z)\, \Phi_h(x, z, \Delta x) + U(x, z)\, \Phi(x, z, \Delta z),$$

which can be written in the form:

$$\sum_{i=1}^{n} f_i(x,z)\,\Delta x_i$$

with $f_i(x, z) \equiv 0 \pmod{\psi_1, \psi_2, \ldots, \psi_\rho}$. Since $U(x, z)$ can be constructed from the minors of any of the first t rows of any matrix, one can deduce that $U(x^0, z^0) \neq 0$, and one is led to the conclusion that:

For an arbitrary differential form F in $\mathfrak{a}_{p+1}$, *we have:*

$$\Phi(x, z, \Delta x) \equiv \sum_{h=1}^{t} A_h(x,z)\,\Phi_h(x, z, \Delta x) \qquad \pmod{\psi_1, \psi_2, \ldots, \psi_\rho} \tag{3}$$

($A_h(x, z)$ is holomorphic at x^0, z^0).

B. Construction of a Cauchy-Kowalewskian system of differential equations.

In order to simplify the formalism, we would like to write assumption (2) for the desired integral manifold M_{p+1} as follows:

$$x_i = x_i(x_1, x_2, \ldots, x_{p+1}) \qquad (i = 1, 2, \ldots, n), \tag{4}$$

in which we naturally have:

$$x_i(x_1, x_2, \ldots, x_{p+1}) = x_i \qquad \text{for } i \leq p + 1,$$
$$x_i(x_1, x_2, \ldots, x_{p+1}) = 0 \qquad \text{for } i > n - r.$$

We must determine these functions $x_i(x)$ so that every Φ of $\mathfrak{a}_{p+1}$ will be annulled when substituted in (4):

$$\left(\sum a_{i_1, i_2, \cdots, i_{p+1}} \frac{\partial(x_{i_1}, x_{i_2}, \cdots, x_{i_{p+1}})}{\partial(x_1, x_2, \cdots, x_{p+1})}\right) d(x_1, x_1, \ldots, x_{p+1}) = 0$$

and $x_i(x_1, x_2, \ldots, x_p, 0)$ represents the manifold M_p. To this end, we next consider the equations:

$$\sum a_{i_1, i_2, \cdots, i_{p+1}} \frac{\partial(x_{i_1}, x_{i_2}, \cdots, x_{i_{p+1}})}{\partial(x_1, x_2, \cdots, x_{p+1})} = 0 \qquad (h = 1, 2, \ldots, t). \tag{5}$$

If one thinks of the determinants that appear here as being developed in terms of the elements $\dfrac{\partial x_i}{\partial x_{p+1}}$ then, up to sign, the left-hand sides are equal to the expressions:

$$\Phi_h(x, z, \Delta x),$$

when one introduces the expressions:

$$z_{i_1, i_2, \cdots, i_p} = \frac{\partial(x_{i_1}, x_{i_2}, \cdots, x_{i_p})}{\partial(x_1, x_2, \cdots, x_p)}, \quad \Delta x_i = \frac{\partial x_i}{\partial x_{p+1}}$$

into them.

From the assumptions that we made about E_p^0 and F_{n-r}, it now follows that the equations:

(6) $$\Phi_h(x^0, z^0, \Delta x) = 0 \qquad (h = 1, 2, \ldots, t)$$

can be solved in terms of the t quantities Δx_{p+2}, Δx_{p+3}, …, Δx_{n-r}. Otherwise, one would have a linear relation:

$$\sum_{i=1}^{p+1} \alpha_i \Delta x_i + \sum_{i=n-r+1}^{n} \beta_i \Delta x_i = 0,$$

which cannot be the case. For the E_{p+1}^0 that are determined by:

(7) $$\Delta x_{n-r+1} = \Delta x_{n-r+2} = \ldots = \Delta x_n = 0,$$

we shall indeed have that $w_{1,2,\ldots,p+1} \neq 0$, from which the vanishing of all α follows. One must then have $\beta = 0$, because otherwise, from (7), the integral E_{p+1} that goes through E_p^0 would no longer be uniquely defined. Thus, the intersection of $V_{n-r_{p+1}}^0$ and $H(E_p^0)$ will be of dimension higher than $p+1$.

Equations (5) can thus be solved for the derivatives:

$$\frac{\partial x_{p+1+k}}{\partial x_{p+1}} \qquad (k = 1, 2, \ldots, t),$$

and upon setting x_{n-r+1}, …, x_n to zero, the functions:

$$\frac{\partial x_{p+2}}{\partial x_{p+1}} = H_1\left(x_1, x_2, \cdots, x_{n-r}, \frac{\partial(x_{i_1}, x_{i_2}, \cdots, x_{i_p})}{\partial(x_1, x_2, \cdots, x_p)}\right)$$

$$\frac{\partial x_{p+3}}{\partial x_{p+1}} = H_2\left(x_1, x_2, \cdots, x_{n-r}, \frac{\partial(x_{i_1}, x_{i_2}, \cdots, x_{i_p})}{\partial(x_1, x_2, \cdots, x_p)}\right)$$

$$\cdots$$

$$\frac{\partial x_{n-r}}{\partial x_{p+1}} = H_t\left(x_1, x_2, \cdots, x_{n-r}, \frac{\partial(x_{i_1}, x_{i_2}, \cdots, x_{i_p})}{\partial(x_1, x_2, \cdots, x_p)}\right)$$

appear on the right-hand side of the solution formulas, which are holomorphic in all $n - r + \binom{n}{p}$ arguments in the neighborhood of:

$$x_1 = x_1^0, \quad x_{n-r} = x_{n-r}^0, \quad \frac{\partial(x_{i_1}, x_{i_2}, \cdots, x_{i_p})}{\partial(x_1, x_2, \cdots, x_p)} = z^0_{i_1, i_2, \cdots, i_p}.$$

From the classical theorem of *Cauchy* and *Sonja Kowalewski* [1]), there is precisely one solution:

$$x_i = f_i(x_1, x_2, \ldots, x_{p+1}) \qquad (i = p+2, \ldots, n-r)$$

of this system of partial differential equations that reduces to:

$$f_i(x_1, x_2, \ldots, x_p, 0) = \varphi_i(x_1, x_2, \ldots, x_p)$$

for $x_{p+1} = 0$. (φ_i are functions that enter into the equations (1) for M_p.)

A $(p+1)$-dimensional manifold M_{p+1} in F_{n-r} that goes through M_p is defined by:

$$x_i = x_i(x_1, x_2, \ldots, x_{p+1}) = \begin{cases} x_i & (i = 1, 2, \cdots, p+1) \\ f_i(x_1, x_2, \cdots, x_{p+1}) & (i = p+2, \cdots, n-r) \\ 0 & (i = n-r+1, \cdots, n), \end{cases} \tag{8}$$

which is an integral manifold of the system:

$$\Phi_h = 0 \qquad (h = 1, 2, \ldots, t).$$

We will now show that M_{p+1} makes all other forms Φ in $\mathfrak{a}_{p+1}$ vanish automatically, so it is the desired integral manifold.

C. Proof that M_{p+1} satisfies the equations $\mathfrak{a}_{p+1} = 0$. Among the functions $\psi_i(x, z)$ that were just introduced, perhaps the first σ of them can go in the left-hand side of:

$$\mathfrak{a}_{p+1}(x, z) = 0,$$

whereas the rest of them are found among the $G_p^n(z)$.

We now consider the differential forms that correspond to $\psi_k(x, z)$:

$$\psi_k = \sum b^{(k)}_{i_1, i_2, \cdots, i_p}\, d(x_{i_1}, x_{i_2}, \cdots, x_{i_p}) \qquad (k = 1, 2, \ldots, \sigma).$$

By substituting in (8), they may become:

$$\bar{\psi}_k = \sum_{l=1}^{p+1} (-1)^{l-1}\, V_{kl}(x_1, x_2, \ldots, x_{p+1})\, d(x_1, \ldots, x_{l-1}, x_{l+1}, \ldots, x_{p+1}). \tag{9}$$

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[1]) See, e.g., E. GOURSAT, *Leçons sur l'intégration des équations aux dérivées partielles du premier ordre*, Paris 1921, pp. 2.

On the other hand, we consider the forms that belong to $\mathfrak{a}_{p+1}$:

$$\Phi_{kl} = dx_l \, \psi_k \qquad \begin{pmatrix} k = 1, 2, \cdots, \sigma \\ l = 1, 2, \cdots, p+1 \end{pmatrix}.$$

Let $\Phi_{kl}(x, z, \Delta x)$ be the expression that one obtains by replacing $d(x_{i_1}, x_{i_2}, \cdots, x_{i_p})$ with:

$$\Delta x_{i_1}, z_{i_2, i_3, \cdots, i_{p+1}} - \Delta x_{i_2}, z_{i_2, i_3, \cdots, i_{p+1}} + \cdots$$

in Φ_{kl}. Under the substitution:

$$(10) \qquad \begin{cases} x_i = x_i(x_1, x_2, \cdots, x_{p+1}), \\ z_{i_1, i_2, \cdots, i_p} = \dfrac{\partial(x_{i_1}, x_{i_2}, \cdots, x_{i_p})}{\partial(x_1, x_2, \cdots, x_{p+1})}, \\ \Delta x_i = \dfrac{\partial x_i}{\partial x_{p+1}}, \end{cases}$$

$\Phi_{kl}(x, z, \Delta x)$ goes to $V_{kl}(x_1, x_2, \ldots, x_{p+1})$. We apply the result (3) to the functions $\Phi_{kl}(x, z, \Delta x)$ and use the expressions (10) for $x, z, \Delta x$. We then observe that with this substitution:

1. All $\Phi_h(x, z, \Delta x)$ vanish identically, because M_{p+1} satisfies the differential equation $\Phi_h = 0$;

2. All $\psi_k(x, z)$ with $k > \sigma$ vanish, because the equations for Grassmann manifold G_p^n are satisfied identically when one substitutes the p-rowed determinant of any matrix with p rows and n columns for z;

3. The $\psi_k(x, z)$ with $k \leq \sigma$ reduce to:

$$(-1)^p V_{k,p+1}(x_1, x_2, \ldots, x_{p+1}) = V_k(x_1, x_2, \ldots, x_{p+1}),$$

from which we conclude that we can set:

$$(11) \qquad V_{kl}(x_1, x_2, \ldots, x_{p+1}) = A_{kl1} V_1 + A_{kl2} V_2 + \ldots + A_{kl\sigma} V_\sigma.$$

(A is holomorphic at (x^0))

The same considerations may be applied to $d\psi_k$. It is also a form in $\mathfrak{a}_{p+1}$, and one shows, with the same conclusions as above, that the form $\overline{d\psi}_k = d\psi_k$ that is obtained from $d\psi_k$ by substitution in (8) may be written in the form:

$$\overline{d\psi}_k = (B_{k1}V_1 + B_{k2}V_2 + \ldots + B_{k\sigma}V_\sigma)\, d(x_1, x_2, \ldots, x_{p+1}).$$

One can also obtain $\overline{d\psi}_k$ directly by differentiating (9):

$$\overline{d\psi}_k = \left(\frac{\partial \mathrm{V}_{k1}}{\partial x_1} + \frac{\partial \mathrm{V}_{k2}}{\partial x_2} + \cdots + \frac{\partial \mathrm{V}_{k,p+1}}{\partial x_{p+1}}\right) d(x_1, x_2, \ldots, x_{p+1}),$$

and one then has:

$$\frac{\partial V_{k1}}{\partial x_1} + \frac{\partial V_{k2}}{\partial x_2} + \cdots + \frac{\partial V_{k,p+1}}{\partial x_{p+1}} = \sum_{\nu=1}^{\sigma} B_{k\nu} V_\nu \,.$$

If one uses the expressions (11) for the V_{kn} then one obtains a system of linear partial differential equations for the functions $V_1, V_2, \ldots, V_\sigma$ that can be solved for holomorphic

$$\frac{\partial V_1}{\partial x_{p+1}}, \frac{\partial V_2}{\partial x_{p+1}}, \ldots, \frac{\partial V_\sigma}{\partial x_{p+1}},$$

and are homogeneous in the V and their derivatives.

For the functions $V_k(x_1, x_2, \ldots, x_{p+1})$, one has:

(12) $$V_k(x_1, x_2, \ldots, x_p, 0) = 0 \qquad (k = 1, 2, \ldots, \sigma);$$

this makes:

$$(-1)^p\, V_k(x_1, x_2, \ldots, x_p, 0)\, d(x_1, x_2, \ldots, x_p)$$

the form that is obtained from ψ_k by the substitution:

(13) $$x_i = x_i(x_1, x_2, \ldots, x_p, 0) \qquad (i = 1, 2, \ldots, n),$$

and (13) represents the manifold M_p that all of the forms of $\mathfrak{a}_p$ annul (as a p-dimensional integral manifold).

From the form of the functions V_k that are obtained from the system of partial differential equations, it follows that:

(14) $$V_1 = V_2 = \ldots = V_\sigma = 0$$

is the only solution that satisfies the condition (12).

One now proves, as above for Φ_{kl} and $d\Phi$, that for any form Φ in $\mathfrak{a}_{p+1}$, substitution of (8) produces an expression:

$$\left(\sum_{k=1}^{\sigma} A_k V_k\right) d(x_1, x_2, \ldots, x_{p+1}),$$

which vanishes identically on account of (14).

Thus, since M_{p+1} satisfies the differential equations $\mathfrak{a}_{p+1} = 0$, it already follows that it satisfies all of the equations $\mathfrak{a} = 0$, because $\mathfrak{a}_{p+1} = 0$ implies that every $(p+1)$-dimensional – and therefore every lower-dimensional tangent element of M_{p+1} – is an integral element.

M_{p+1} is the desired integral manifold, and it is also the only one that satisfies the required conditions since M_{p+1} is already uniquely determined by $\Phi_h = 0$ ($h = 1, 2, \ldots, t$).

The existence theorem that is thus proved can also be formulated in a slightly more precise way:

When $r_{p+1} \geq 0$ ($r_{p+1} = 0$, resp.), $(p+1)$-*dimensional integral manifolds go through a regular integral-*M_p *, and they depend on* r_{p+1} *arbitrary functions of* $p+1$ *variables* (*are uniquely determined, resp.*).

Any of the integral-M_p near M_{p+1} that were constructed above can be represented in the form:

$$x_i = g_i(x_1, x_2, \ldots, x_{p+1}) \qquad (i = 1, 2, \ldots, n),$$

and from this, the functions:

(15) $$g_i(x_1, x_2, \ldots, x_{p+1}) \qquad (n - r < i \leq n)$$

may be prescribed arbitrarily, up to the supplementary condition that:

(16) $$g_i(x_1, x_2, \ldots, x_p, 0) = \varphi_i(x_1, x_2, \ldots, x_p) \qquad (n - r < i \leq n).$$

By assuming (15), one can infer that M_{p+1} shall lie in the manifold F_{n-r}:

$$x_i - g_i(x_1, x_2, \ldots, x_{p+1}) = 0 \qquad (n - r < i \leq n)$$

and (16) expresses the idea that these F_{n-r} go through the given integral-M_p.

One easily proves an extension of this existence theorem that is needed for many purposes:

If the manifolds M_p *and* F_{n-r} *depend upon certain parameters* $\alpha_1, \alpha_2, \ldots, \alpha_\tau$ *in the neighborhood of a holomorphic system of values* $\alpha_1^0, \alpha_2^0, \cdots, \alpha_\tau^0$ *then the integral manifold* M_{p+1} *also varies holomorphically with the* α [1]).

---

[1]) A q-dimensional manifold $M(\alpha)$ [which is defined in the neighborhood of a point (x^0)] "depends holomorphically on the parameters $\alpha_1, \alpha_2, \ldots, \alpha_\tau$ in the neighborhood of a system of values $\alpha_1^0, \alpha_2^0, \cdots, \alpha_\tau^0$" when $n - q$ of the coordinates x of any arbitrary point neighboring (x^0) on $M(\alpha)$ can be represented as holomorphic functions of the others.

Concerning the aforementioned existence proof, we can still say the following: The bihilomorphic transformation that takes $F_{n-r} = F_{n-r}(\alpha)$ to:

$$x_{n-r+1} = x_{n-r+2} = \ldots = x_n = 0$$

now depends on α, and the parameter α therefore enters the differential equation $\mathfrak{a} = 0$ holomorphically. One further deduces that the functions φ_i that appear in the equations of $M_p = M_p(\alpha)$ now depend on the α variables:

$$\varphi_i = \varphi_i(x_1, x_2, \ldots, x_n, \alpha_1, \ldots, \alpha_\tau)$$

$$(\varphi_i(x, \alpha) \text{ is holomorphic for } (x) = (x^0), (\alpha) = (\alpha^0)).$$

The Cauchy-Kowalewski system of differential equations that are obtained by construction on $M_{p+1} = M_{p+1}(\alpha)$, like the initial conditions:

$$f_i(x_1, x_2, \ldots, x_p, 0) = \varphi_i(x_1, x_2, \ldots, x_n, \alpha_1, \ldots, \alpha_\tau) \qquad (p+1 < i \leq n - r)$$

now depends holomorphically on the α, and a simple extension of the Cauchy-Kowalewski theorem shows that the solutions $x_i = f_i(x_1, x_2, \ldots, x_{p+1}) = \varphi_i(x_1, x_2, \ldots, x_{p+1}, \alpha_1, \ldots, \alpha_\tau)$ are holomorphic functions of the α in the neighborhood of $(\alpha) = (\alpha^0)$.

2. Regular chains of integral elements. If one has a sequence of regular integral elements:

$$E_0^0, E_1^0, E_2^0, \cdots, E_{p-1}^0$$

that relates to the integral element E_p^0, such that each E_i^0 is contained in the following E_{i+1}^0 then one can speak of a regular chain:

$$E_0^0 \subset E_1^0 \subset E_2^0 \subset \cdots \subset E_{p\text{-}1}^0 \subset E_p^0$$

that ends with E_p^0. [1]) The element E_p^0 itself does not need to be regular. Such a chain is indexed by a sequence of whole numbers (we call them the *characteristic numbers* of the sequence):

$$r_1, r_2, \ldots, r_{p-1}, r_p,$$

in which r_i means that ∞^{r_i} integral-E_i go through E_{i-1}^0, or – what amounts to the same thing – that the polar element $H(E_{i-1}^0)$ has the dimension $r_i + i$.

---

[1]) Here, as in all other cases, the notation $A \subset B$ shall mean that of the two vector spaces (or manifolds) A, B the first one is contained in the second one.

Since all of the vectors that span an integral element, together with E_{i+1}^0, also define an integral element, together with E_i^0, that is contained in E_{i+1}^0, one has that $H(E_i^0)$ is contained in $H(E_{i+1}^0)$ and that:

$$r_i + i + 1 \leq r_i + i,$$

or:

$$s_i = r_i - r_{i+1} - 1 \geq 0 \qquad (i = 1, 2, \ldots, p-1).$$

If one lets r_0 denote the dimension of the algebraic manifold $\mathfrak{a}_0 = 0$ (the manifold of integral points) in the neighborhood of $E_0^0 = (x_1^0, x_2^0, \cdots, x_n^0)$ then the number s_0, which is defined by:

$$s_0 = r_0 - r_1 - 1,$$

is also positive or zero, because the r_0-dimensional tangent element of $\mathfrak{a}_0 = 0$ in (x^0) contains the (r_1+1)-dimensional polar space $H(E_0^0)$:

$$\mathfrak{a}_1(x^0, \Delta x) = 0.$$

All of the forms $d\mathfrak{a}_0$ belong to $\mathfrak{a}_1$.

One now chooses – as is always possible – any sequence:

$$V_{n-r_1}^0 \subset V_{n-r_2}^0 \subset \cdots \subset V_{n-r_p}^0$$

of $(n - r_i)$-dimensional vector spaces at (x^0) that are contained in each other in the specified way and are such that the intersection of $V_{n-r_i}^0$ with $H(E_{i-1}^0)$ is precisely E_i^0. If one has any sequence:

(17) $$V_{n-r_1} \subset V_{n-r_2} \subset \cdots \subset V_{n-r_p}$$

of analogously-ordered vector spaces then they uniquely determine a regular chain:

(18) $$E_0 \subset E_1 \subset E_2 \subset \ldots \subset E_p$$

of integral elements that are close to the E_i^0, assuming that the V_{n-r_i} are arbitrarily close to the $V_{n-r_i}^0$ [thus, among other things, the point (x) of V is arbitrarily close to (x^0).] One takes E_0 to be the point (x); thus, if one has already obtained:

$$E_0 \subset E_1 \subset E_2 \subset \ldots \subset E_{i-1}$$

then one obtains E_i as the intersection of $H(E_{i-1})$ and V_{n-r_i}. Due to the fact that:

$$E_{i-1} \subset H(E_{i-1}), \qquad E_{i-1} \subset V_{n-r_i-1} \subset V_{n-r_i},$$

this intersection space indeed goes through E_{i-1}, and it is also exactly i-dimensional and close to E_i^0 when one already assumes that $E_0, E_1, \ldots, E_n$ are arbitrarily close to E_0^0, E_1^0, …, E_{i-1}^0, just as one assumes that the V_{n-r_ν} are arbitrarily close to the $V_{n-r_\nu}^0$.

Since the intersection of $V_{n-r_1}^0$ and $H(E_0^0)$ is non-degenerate the same is true for the intersection of the tangent elements of $\mathfrak{a}_0 = 0$ at (x^0) with $V_{n-r_1}^0$, which therefore has the dimension $r_0 - r_1 - 1$. On the basis of this fact and the fact that $n - r_0 < n - r_1$ one can determine an $(n - r_0)$-dimensional vector subspace of $V_{n-r_1}^0$ that contains no tangents to $\mathfrak{a}_0 = 0$, and in a number of ways.

3. Corollaries to the first existence theorem. The second existence theorem. Now let:

$$F_{n-r_0} \subset F_{n-r_1} \subset F_{n-r_2} \subset \cdots \subset F_{n-r_p} \tag{19}$$

be $(n - r_i)$-dimensional algebraic manifolds that go through (x^0) and are regular (simple) at (x^0) and have the $V_{n-r_i}^0$ for tangent elements there.

Due to the assumptions on $V_{n-r_0}^0$, F_{n-r_0} and $\mathfrak{a}_0 = 0$ have only one point $M_0 = (x^0)$ in common in the neighborhood of (x^0). From the existence theorem that was proved above, there is precisely one one-dimensional manifold M_1 that that goes through M_0 and is contained in F_{n-r_1}, and there is precisely one two-dimensional integral manifold M_2 through M_1 that is in F_{n-r_2}, etc. By induction, one concludes the existence and uniqueness of a sequence of integral manifolds:

$$M_0 \subset M_1 \subset M_2 \subset \ldots \subset M_p \tag{20}$$

that have the relationship to the sequence (19) that the sequence (18) has to the chain (17).

One can also express the *second existence theorem* that is thus posed as:

In a neighborhood of (x^0), *the sequence* (19) *of manifolds* F_{n-r_i} *leaves exactly one p-dimensional manifold* M_p *fixed, under the requirement that* M_p *and* F_{n-r_i} *have an i-dimensional intersection* ($i = 0, 1, 2, \ldots, p$).

In fact, the intersection of M_p and F_{n-r_i} must be an integral manifold since every lower-dimensional manifold that lies in an integral manifold is itself an integral manifold. If one then seeks to construct the M_p of the sequence $M_0, M_1, M_2, \ldots$ then one will be unavoidably led to the above construction.

The actual meaning of the previous existence theorem first becomes clear in the analytical formulation.

Let a coordinate system:

$$z_i = z_i(x_1, x_2, \ldots, x_n) \qquad (i = 1, 2, \ldots, n)$$

$$[dz_1\, dz_2 \ldots dz_p \neq 0 \text{ at } (x^0) \qquad \text{and} \qquad z_i(x_1^0, x_2^0, \cdots, x_n^0) = 0]$$

be introduced in the neighborhood of (x^0) such that E_ν^0 can be determined from E_p^0 by means of the equations:

$$dz_{\nu+1} = dz_{\nu+2} = \ldots = dz_p = 0 \qquad (n = 1, 2, \ldots, p-1).$$

$dz_1, dz_2, \ldots, dz_p$ are then linearly independent on E_p^0, i.e.:

$$d(z_1, z_2, \ldots, z_p) \neq 0$$

on E_p^0. Since $V_{n-r_p}^0$ contains the element E_p^0, we also have that $dz_1\, dz_2 \ldots dz_p \neq 0$ on $V_{n-r_p}^0$, and it is therefore possible to find r_p Pfaffian forms that vanish on V_{n-r_p} and are independent of $dz_1, dz_2, \ldots, dz_p$ and each other. One can put them into the form:

$$dz_i - \sum_{k=1}^{p} l_{ik}^0 dz_k \qquad (p < i \leq r_p + p)$$

by indexing the z_i $(i > p)$ in an appropriate manner. The $V_{n-r_p-1}^0$ that lies in $V_{n-r_p}^0$ then annuls not only these forms, but also dz_p and $r_{p-1} - r_p - 1 = s_{p-1}$ more Pfaffian forms, which can be written, if necessary by re-ordering the z_i with $i > r_p + p$, in the form:

$$dz_i - \sum_{k=1}^{p-1} l_{ik}^0 dz_k \qquad (r_p + p < i \leq r_{p-1} + p - 1).$$

One sees as one continues that the ordering of the z_i $(i > p)$ may be arranged so that in general $V_{n-r_\nu}^0$ can be described by means of the equations for $V_{n-r_\nu+1}^0$, along with $s_\nu + 1$ more equations:

(21) $$dz_{\nu+1} = 0, \qquad dz_i - \sum_{k=1}^{\nu} l_{ik}^0 dz_k = 0 \qquad (r_{\nu+1} + \nu + 1 < i \leq r_\nu + \nu).$$

Finally, let $V_{n-r_0}^0$ be represented by the equations for $V_{n-r_1}^0$ and:

$$dz_i = 0 \qquad (r_1 + 1 < i \leq r_0).$$

If one is given the point (x_0) and the:

$$pr_p + (p-1)s_{p-1} + (p-2)s_{p-2} + \ldots + 2s_2 + s_1$$

constants l_{ik}^0 then the sequence:

(22) $$V_{n-r_1}^0 \subset V_{n-r_2}^0 \subset \cdots \subset V_{n-r_p}^0 ,$$

and therefore also the chain:

(23) $$(x^0) = E_0^0 \subset E_1^0 \subset E_2^0 \subset \cdots \subset E_p^0 ,$$

is uniquely determined. If one varies the point $(x_1, x_2, \ldots, x_n)$ on the manifold $\mathfrak{a}_0 = 0$ in a neighborhood of (x^0) as well as the constants l_{ik} in the neighborhood of the system of values l_{ik}^0 in the equations:

$$V_{n-r_\nu} : \begin{cases} dz_i - \sum_{k=1}^{p} l_{ik} dz_k = 0 & (p < i \leq r_p + p), \\ dz_i - \sum_{k=1}^{p-1} l_{ik} dz_k = 0 & (r_p + p < i \leq r_{p-1} + p - 1), \\ \cdots & \cdots \\ dz_i - \sum_{k=1}^{\nu} l_{ik} dz_k = 0 & (r_{\nu+1} + \nu + 1 < i \leq r_\nu + \nu), \\ dz_{\nu+1} = dz_{\nu+2} = \cdots = dz_p = 0 \end{cases}$$

$$(\nu = p, p - 1, \ldots, 1)$$

then one obtains all of the sequences that are close to (22) [(23), resp.]:

$$V_{n-r_1} \subset V_{n-r_2} \subset \cdots \subset V_{n-r_p}$$

$$(x) = E_0 \subset E_1 \subset E_2 \subset \ldots \subset E_p,$$

and one also obtains every integral element that is close to E_p^0 exactly once. We also remark that the manifold of p-dimensional integral elements in the neighborhood of E_p^0 has dimension:

(24) $$r_0 + s_1 + 2s_2 + 3s_3 + \ldots + (p - 1)s_{p-1} + pr_p .$$

One now considers a sequence of algebraic manifolds:

$$F_{n-r_0} \subset F_{n-r_1} \subset F_{n-r_2} \subset \cdots \subset F_{n-r_p}$$

whose elements are defined in the following way:

F_{n-r_p} is given by:

$$z_i - \varphi_i(z_1, z_2, \ldots, z_p) = 0 \qquad (p < i \leq r_p + p - 1);$$

F_{n-r_p-1} is the intersection of F_{n-r_p} with:

$$z_p = 0, \qquad z_i - \varphi_i(z_1, z_2, \ldots, z_{p-1}) = 0 \qquad (r_p + p < i \leq r_{p-1} + p - 1).$$

In general: F_{n-r_ν} is the intersection of $F_{n-r_\nu+1}$ with:

$$z_{\nu+1} = 0, \qquad z_i - \varphi_i(z_1, z_2, \ldots, z_\nu) = 0 \qquad (r_{\nu+1} + \nu + 1 < i \leq r_\nu + \nu)$$

and finally F_{n-r_0} is the intersection of F_{n-r_1} with:

$$z_i - \varphi_i = 0 \qquad \begin{pmatrix} r_1 + 1 < i \leq r_0 \\ \varphi = \text{const.} \end{pmatrix}.$$

When the functions (constants, resp.) φ, which are assumed to be holomorphic at $(z) = 0$, satisfy only the condition that for:

$$z_1 = z_2 = \ldots = z_p = 0$$

the Pfaffian forms:

$$dz_i - d\varphi_i(z_1, z_2, \ldots, z_\nu) \qquad \begin{pmatrix} r_{\nu+1} + \nu + 1 < i \leq r_\nu + \nu \\ \nu = 1, 2, \cdots, p \\ r_{p+1} = -1, \text{ by assumption} \end{pmatrix}$$

are arbitrarily close to the forms that were considered above:

$$dz_i - \sum_{k=1}^{\nu} l_{ik}^0 dz_k \qquad \begin{pmatrix} r_{\nu+1} + \nu + 1 < i \leq r_\nu + \nu \\ \nu = 1, 2, \cdots, p \end{pmatrix},$$

and the values of φ_i are arbitrarily small, then the manifolds F that are so defined satisfy the assumptions that were demanded on pp. 39. They thus determine a p-dimensional integral manifold M_p. Since M_p possesses a tangent element that neighbors on E_p^0 and $d(z_1, z_2, \ldots, z_p) \neq 0$ for E_p^0, it may be represented in the form:

$$z_i = f_i(z_1, z_2, \ldots, z_p) \qquad (i = p+1, \ldots, n).$$

The fact that M_p has F_{n-r_ν} in common with M_ν can be deduced from:

$$f_i(z_1, z_2, \ldots, z_\nu, 0, \ldots, 0) = \varphi_i(z_1, z_2, \ldots, z_p).$$

One sees from this that a choice of function (or constant) φ will determine a choice of integral-M_p. By varying the φ, one obtains *all* p-dimensional integral manifolds that possess a tangent element that neighbors on E_p^0. The totality of these integral manifolds therefore depends on:

s_0	arbitrary constants,			
s_1	"	functions of	1 variable	
s_2	"	"	2	"
	…			
s_{p-1}	"	"	$p-1$	"
r_p	"	"	p	"

If we do without any mention of V_{n-r_ν} and the condition that the $V^0_{n-r_0}$ that is determined by:

$$dz_1 = dz_2 = \ldots = dz_{r_0} = 0$$

contains no tangent to $\mathfrak{a}_0 = 0$, which replaces the equivalent condition that:

$$dz_1 dz_2 \ldots dz_{r_0} \neq 0 \qquad \text{on } \mathfrak{a}_0 = 0 \text{ at } (x^0),$$

then we can summarize the result so obtained in the following way:

Second existence theorem. *Let:*

$$(x^0) = E_0^0 \subset E_1^0 \subset E_2^0 \subset \cdots \subset E_p^0$$

be a regular chain of integral elements, and let:

$$r_0, r_1, r_2, \ldots, r_p$$

be the associated characteristic numbers. Let the coordinate system:

$$z_i = z_i(x_1, x_2, \ldots, x_n) \qquad \begin{pmatrix} i = 1, 2, \cdots, n \\ z_i(x_1^0, \cdots, x_n^0) = 0 \end{pmatrix}$$

and the Pfaffian forms:

$$\vartheta_i = dz_i - \sum_{k=1}^{\nu} l_{ik}^0 dz_k \qquad \begin{pmatrix} r_{\nu+1} + \nu + 1 < i \leq r_\nu + \nu \\ \nu = 1, 2, \cdots, p \\ r_{p+1} = -1, \text{ by assumption} \end{pmatrix}$$

be selected in such a way that E_ν^0 *is determined on* E_p^0 *by:*

$$dz_{n+1} = dz_{\nu+2} = \ldots = dz_p = 0$$

and on the polar space $H(E_{\nu-1}^0)$ *by:*

$$\vartheta_i = 0 \qquad (p < i \leq r_\nu + \nu)$$

$$dz_{n+1} = dz_{\nu+2} = \ldots = dz_p = 0,$$

and that:

$$dz_1 dz_2 \ldots dz_{r_0} \neq 0$$

for the tangent element to $\mathfrak{a}_0 = 0$ *at* (x^0). *The p-dimensional integral manifold* M_p*, which possesses a tangent element close to* E_p^0*, may be represented in the form:*

$$z_i = f_i(z_1, z_2, \ldots, z_p) \qquad (p < i \leq n),$$

in which:

$$(23) \qquad \begin{cases} f_i(z_1, z_2, \cdots, z_{p-1}, z_p) & (p < i \leq r_p + p), \\ f_i(z_1, z_2, \cdots, z_{p-1}, 0) & (r_p + p < i \leq r_{p-1} + p - 1), \\ \cdots & \cdots \\ f_i(z_1, 0, \cdots, 0, 0) & (r_2 + 2 < i \leq r_1 + 1), \\ f_i(0, 0, \cdots, 0, 0) & (r_1 + 1 < i \leq r_0) \end{cases}$$

can be arbitrarily assigned, assuming that the values of these functions are sufficiently small for $(z) = 0$ *and that for* $(z) = 0$ *the forms:*

$$df_i(z_1, z_2, \ldots, z_\nu, 0, \ldots, 0) \qquad (r_{\nu+1} + \nu + 1 \leq i < r_\nu + \nu)$$

are arbitrarily close to the corresponding Pfaffian forms:

$$\sum_{k=1}^{\nu} l_{ik}^0 dz_k .$$

M_p *is then uniquely determined by the data* (23).

On the basis of the remarks that were made at the end of the proof of the existence theorem (pp. 36) one can further add that if the initial data (23) are holomorphic functions of definite parameters $\alpha_1, \alpha_2, \ldots, \alpha_\tau$ in the neighborhood of a system of values $(\alpha) = (\alpha_0)$ then the solution $z_i = f_i(z_1, z_2, \ldots, z_p)$ also depends holomorphically on the α.

V. Remarks on the computational aspects of the results obtained.

1. Determination of chains of integral elements. For practical applications of the existence theorems that were just proved, it becomes necessary to develop a procedure for determining the regular chains and characteristic numbers. We would like to treat this problem in the following form:

Problem: *Let $\omega_1, \omega_2, \ldots, \omega_p$ be p linearly independent Pfaffian forms. Determine all p-dimensional integral elements on which $\omega_1, \omega_2, \ldots, \omega_p$ are linearly independent and intersected by a sequence of regular integral-E_ν by way of:*

$$\omega_{\nu+1} = \omega_{\nu+2} = \ldots = \omega_p = 0$$

for $\nu = 0, 1, 2, \ldots, p-1$.

Along with the ω one chooses $n-p$ more Pfaffian forms $\omega_{p+1}, \ldots, \omega_n$ in such a way that $\omega_1\omega_2 \ldots \omega_n \neq 0$. For the p-dimensional integral element E_p on which $\omega_1, \omega_2, \ldots, \omega_p$ are linearly independent, one can make the Ansatz:

$$\varpi_i = \omega_i - \sum_{k=1}^{p} l_{ik}\,\omega_k \qquad (p < i \leq n), \tag{1}$$

and E_ν is then determined by appending the equations:

$$\omega_i = 0 \qquad (n < i \leq p). \tag{2}$$

Modulo the left-hand side of (1) and (2), every form in $\mathfrak{a}_\nu$ may be uniquely reduced to the form:

$$a(x, l_1, l_2, \ldots, l_\nu)\ \omega_1\omega_2 \ldots \omega_\nu ,$$

in which $a(x, l_1, l_2, \ldots, l_\nu)$ is linear in each of the variables l_k:

$$l_{p+1,k}\,,\ l_{p+2,k}\,,\ \ldots,\ l_{nk} \qquad (k = 1, 2, \ldots, \nu).$$

The functions $a(x, l_1, l_2, \ldots, l_\nu)$ that thus appear in the forms $\mathfrak{a}_\nu$ may be completed with $\mathfrak{a}_\nu(x, l_1, l_2, \ldots, l_\nu)$, and one has:

$$\mathfrak{a}_\nu \equiv \mathfrak{a}_\nu(x, l_1, l_2, \ldots, l_\nu)\ \omega_1\omega_2 \ldots \omega_\nu \qquad (\text{mod } \omega_{\nu+1}, \ldots \omega_p, \varpi_{p+1}, \ldots, \varpi_n).$$

Modulo the same thing, one obviously has:

$$\begin{aligned}\mathfrak{a}_{\nu-1} \equiv \mathfrak{a}_{\nu-1}&(x, l_1, l_2, \ldots, l_{\nu-1})\ \omega_1\omega_2 \ldots \omega_{\nu-1} \\ &+ \mathfrak{a}_{\nu-1}(x, l_1, l_2, \ldots, l_{\nu-2}, l_\nu)\ \omega_1\omega_2 \ldots \omega_{\nu-2}\,\omega_\nu + \ldots \\ &+ \mathfrak{a}_{\nu-1}(x, l_\nu, l_2, \ldots, l_{\nu-1})\ \omega_\nu\omega_2 \ldots \omega_{\nu-1},\end{aligned}$$

and from this, we have:

$$\mathfrak{a}_{\nu-1}\,\omega_\nu \equiv \mathfrak{a}_{\nu-1}(x, l_1, l_2, \ldots, l_{\nu-1})\ \omega_1\,\omega_2 \ldots \omega_\nu\,,$$
$$\mathfrak{a}_{\nu-1}\,\omega_{\nu-1} \equiv \mathfrak{a}_{\nu-1}(x, l_1, \ldots, l_{\nu-2}, l_\nu)\ \omega_1\,\omega_2 \ldots \omega_\nu\,.$$

Since $\mathfrak{a}_{\nu-1}\,\omega_\nu$ and $\mathfrak{a}_{\nu-1}\,\omega_{\nu-1}$ are contained in $\mathfrak{a}_\nu$, we conclude that the $\mathfrak{a}_{\nu-1}(x, l_1, l_2, \ldots, l_{\nu-1})$ and the $\mathfrak{a}_{\nu-1}(x, l_1, \ldots, l_{\nu-2}, l_\nu)$ occur in the expressions $\mathfrak{a}_\nu \equiv \mathfrak{a}_\nu(x, l_1, l_2, \ldots, l_\nu)$ in the same way.

Because of this, in order for the p-dimensional vector space at (x) whose direction is determined by (1) to be an integral element it is necessary and sufficient that:

$$\mathfrak{a}_p(x, l_1, l_2, \ldots, l_p) = 0.$$

Let $(x^0, l_1^0, l_2^0, \cdots, l_p^0)$ be a solution of these equations and let E_p^0 be the associated integral element. How is one to know whether the chain that is determined by E_p^0 using (2), viz.:

$$(x^0) = E_0^0 \subset E_1^0 \subset E_2^0 \subset \cdots \subset E_p^0$$

is regular?

2. Search for regular chains. We first pursue the consequences of the given assumption that $(x^0, l_1^0, l_2^0, \cdots, l_p^0)$ determines a regular chain.

Since (x^0) is a simple point of the manifold $\mathfrak{a}_0 = 0$, the equations $\mathfrak{a}_0(x) = 0$ can be holomorphically solved for $n - r_0$ of the x – say, $x_{r_0+1}, \ldots, x_n$ – in the neighborhood of (x^0). In general, one has:

For an appropriate ordering of the ω_i $(i > p)$ *the equations:*

$$\mathfrak{a}_\nu(x, l_1, l_2, \ldots, l_\nu) = 0$$

may be solved in a neighborhood of (x^0, l^0) *in such a way that the:*

$$l_{ik} \qquad (i > r_k + k,\ k = 1, 2, \ldots, \nu)$$

are then represented as holomorphic functions of $x_1, x_2, \ldots, x_{r_0}$ *and the remaining* $\sum_{k=1}^{\nu}(r_k + k - p)$ *quantities* l_{ik}.

Proof: The case $\nu = 0$ has already been settled. We assume that the assertion has already been proved for the case of the equation:

$$\mathfrak{a}_{\nu-1}(x, l_1, l_2, \ldots, l_{\nu-1}) = 0. \tag{3}$$

In order to render the given indices no longer necessary we consider the variables:

$$x_i \qquad (i > r_0), \qquad\qquad l_{ik} \qquad (i > r_k + k,\ k = 1, 2, \ldots, n-1)$$

to be *principal* and the remaining x, l to be *parametric.*

Every system of values for the parametric variables that lies in a neighborhood of (x^0, l^0) leaves precisely one $(\nu-1)$-dimensional integral element $E_{\nu-1} = (x, l_1, l_2, \ldots, l_{\nu-1}, l_\nu)$ in the neighborhood of $E_{\nu-1}^0$ fixed, and the condition for this is:

(4) $$\mathfrak{a}_\nu(x, l_1, l_2, \ldots, l_{\nu-1}, l_\nu) = 0,$$

since the ν-dimensional vector space that goes through $E_{\nu-1}$ and satisfies the equations:

$$\omega_i - \sum_{k=1}^{\nu} l_{ik}\,\omega_k = 0 \qquad (p < i \leq n)$$

$$\omega_i = 0 \qquad (\nu < i \leq p)$$

is an integral-E_ν. If one then lets the l_ν vary then one obtains all of the integral-E_ν that go through $E_{\nu-1}$ and satisfy the equations:

$$\omega_i = 0 \qquad (\nu < i \leq p).$$

The vector space $V(x)$ that is defined by the point (x) and the last equations has $E_{\nu-1}$ in common with $H(E_{\nu-1})$ in any case (because $\omega_\nu = \omega_{\nu+1} = \ldots = \omega_p = 0$ on it). The intersection of $V(x)$ and $H(E_{\nu-1})$ thus remains on the original integral-E_ν. When $E_{\nu-1}$ is sufficiently close to $E_{\nu-1}^0$, this intersection cannot be degenerate. Due to the regularity of $E_{\nu-1}^0$, $H(E_{\nu-1})$ varies continuously when $E_{\nu-1}$ varies continuously in the neighborhood of $E_{\nu-1}^0$ (regularity condition 2), and the intersection of $H(E_{\nu-1}^0)$ and $V(x^0)$ is certainly non-degenerate. Otherwise, the forms $\omega_{\nu+1}, \ldots, \omega_p$ are linearly independent on $H(E_{\nu-1}^0)$, which cannot be true, since E_p^0 lies in $H(E_{\nu-1}^0)$, and all of the ω_i $(i = 1, 2, \ldots, p)$ on E_p^0 are linearly independent.

The intersection of $H(E_{\nu-1})$ and $V(x)$ therefore has the dimension $r_\nu + \nu - (p - \nu)$; i.e., it consists of $\infty^{r_\nu+\nu-p}$ integral-E_ν. If one therefore introduces the expressions for the x, l_1, $l_2, \ldots, l_{\nu-1}$ into equations (4) through the parametric variables then for all values of the parametric variables that are sufficiently close to (x^0, l^0) one reduces them to $n - p - (r_\nu + \nu - p) = n - r_\nu - \nu$ independent equations, which can be solved for $n - r_\nu - \nu$ of the variables l. These are then holomorphic in the parameters x, l_1, l_2, …, $l_{\nu-1}$ and represented in terms of the remaining $r_\nu + \nu - p$ quantities l_ν (they are, in fact, linear in them).

One finds the equations:

$$\mathfrak{a}_{\nu-1}(x, l_1, \ldots, l_{\nu-2}, l_\nu) = 0$$

amongst (4), and, due to their similarity with (3), the $l_{i\nu}$ ($i > r_{\nu-1} + \nu - 1$) can be deduced as holomorphic functions of the parameters x, l_1, …, $l_{\nu-2}$ and the remaining $l_{i\nu}$, so equations (4) render $l_{i\nu}$ ($i > r_{\nu-1} + \nu - 1$), as well as $(r_{\nu-1} + \nu - 1 - p) - (r_{\nu-1} + \nu - p)$ more $= s_{\nu-1}$ quantities $l_{i\nu}$ soluble. For an appropriate ordering of the ω_i ($r_{\nu-1} + \nu - 1 \geq i > p$) one can assume that they are, in fact, the $l_{i\nu}$ with $r_{\nu-1} + \nu - 1 \geq i > r_{\nu-1} + \nu$. The theorem is thus proved.

Under the assumption that there is a system of solutions $(x^0, l_1^0, \cdots, l_p^0)$ *that defines a regular chain* $E_0^0 \subset E_1^0 \subset \cdots \subset E_p^0$, *the equations:*

$$\mathfrak{a}_p(x, l_1, l_2, \ldots, l_\nu) = 0$$

can be completely solved in the neighborhood of $(x^0, l_1^0, \cdots, l_p^0)$ *by means of the successive solution of the partial system:*

$$(5)\qquad \begin{cases} \mathfrak{a}_0(x) & = 0, \\ \mathfrak{a}_1(x, l_1) & = 0, \\ \mathfrak{a}_2(x, l_1, l_2) & = 0, \\ \cdots & \cdots \\ \mathfrak{a}_p(x, l_1, l_2, \cdots, l_p) & = 0. \end{cases}$$

Under back-substitution, the foregoing equations reduce to:

$$\mathfrak{a}_\nu(x, l_1, l_2, \ldots, l_\nu) = 0$$

in the $n - r_\nu - n$ *linearly independent equations in terms of the* l_ν. *In light of the fact that* $\mathfrak{a}_0(x) = 0$, *one therefore has only linear equations to solve at every step. The numbering of the x and the* ω_i ($i > p$) *and the solution of* (5) *can be so arranged that the* (*so-called principal*) *quantities:*

$$x_i\ (i > r_0), \qquad l_{ik} \qquad (i > r_k + k,\ k = 1, 2, \ldots, p)$$

appear as holomorphic functions of the remaining (*so-called parametric*) *variables x, l.*

3. Criterion for regular chains. We will now show that, *conversely, if the equations:*

$$\mathfrak{a}_p(x, l_1, l_2, \ldots, l_p) = 0$$

can be solved in the manner that was just described in the neighborhood of the solution $(x^0, l_1^0, \cdots, l_p^0)$ *then a regular chain of integral elements can be determined from* $(x^0, l_1^0, \cdots, l_p^0)$.

Proof. We shall let $n - \rho_\nu - \nu$ ($\nu = 1, 2, \ldots, p$) denote the number of independent l_ν in the equations:

(6) $$\mathfrak{a}_\nu (x^0, l_1^0, l_2^0, \cdots, l_{\nu-1}^0, l_\nu) = 0,$$

although we do not know how they are connected with the characteristic numbers r of the (possibly non-regular) chain $E_0^0 \subset E_1^0 \subset \cdots \subset E_p^0$ that is determined by $(x^0, l_1^0, \cdots, l_p^0)$.

Since the equations:

$$\mathfrak{a}_{\nu-1} (x^0, l_1^0, \cdots, l_{\nu-2}^0, l_\nu) = 0$$

are generally contained in:

$$\mathfrak{a}_\nu (x^0, l_1^0, \cdots, l_{\nu-1}^0, l_\nu) = 0,$$

we can choose the numbering of the ω_i ($i > p$) in such a way that one can find $n - \rho_\nu - \nu$ independent variables $l_{i\nu}$ ($i > \rho_\nu + \nu$) among the equations (6).

Among the equations in $\mathfrak{a}_0(x)$, we select $n - \rho_0$ of them, in which we have denoted the dimension of $\mathfrak{a}_0(x) = 0$ at (x^0) by ρ_0, such that at (x^0) $n - \rho_0$ of the x – say x_i ($i > \rho_0$) – can be solved holomorphically. Likewise, we choose $n - \rho_\mu - \mu$ corresponding equations from:

$$\mathfrak{a}_\mu (x, l_1, l_2, \ldots, l_{\mu-1}) = 0$$

that admit a solution from the $l_{i\mu}$ ($i > \rho_\mu + \mu$) for $(x) = (x^0)$, $(l_1) = (l_1^0)$, …, $(l_{\mu-1}) = (l_{\mu-1}^0)$. The totality of the equations thus obtained for $m = 0, 1, 2, \ldots, n$ will be denoted by:

(7) $$\mathfrak{a}'_\nu (x, l_1, l_2, \ldots, l_\nu) = 0.$$

Among the:

(8) $$\mathfrak{a}'_p (x, l_1, l_2, \ldots, l_p) = 0$$

we then have:

$$\sum_{k=0}^{p} (n - \rho_k - k)$$

equations that can be solved in the neighborhood of (x^0, l^0) in terms of just as many variables, namely, the "principal" x_i ($i > \rho_0$), l_{ik} ($i > \rho_k + k$, $k = 1, 2, \ldots, p$). *The statement that we just proved can now be rephrased as:*

By substituting the expressions for the principal x, l *that are obtained from* (8) *into* $\mathfrak{a}_p(x, l_1, l_2, \ldots, l_p) = 0$, *all of these equations are already satisfied.*

In order to show that E_ν^0 is regular, we make the following Ansatz for the integral-E_ν that neighbor E_ν^0:

$$\omega_i - \sum_{k=1}^{\nu} l_{ik}\,\omega_k = 0 \qquad (p < i \leq n),$$

$$\omega_i - \sum_{k=1}^{\nu} l_{ik}\,\omega_k = 0 \qquad (\nu < i \leq p).$$

Modulo the left-hand side of these equations, one may set:

$$\mathfrak{a}_\nu \equiv \mathfrak{a}_\nu(x, l_1, l_2, \ldots, l_\nu, \lambda)\ \omega_1\,\omega_2 \ldots \omega_\nu ,$$

such that:

(9) $$\mathfrak{a}_\nu(x, l_1, l_2, \ldots, l_\nu, \lambda) = 0$$

is the condition for the integral-E_ν.

The first regularity condition for E_ν^0 demands that the last system of equations is regular for:

(10) $$(x) = (x^0), \qquad (l) = (l^0), \quad (\lambda) = (0)$$

(cf. pp. 28).

Among the equations (9), let $\sum_{k=0}^{\nu}(n - \rho_k - k)$ equations:

(11) $$\mathfrak{a}'_\nu(x, l_1, l_2, \ldots, l_\nu, \lambda) = 0$$

be so selected that for $(\lambda) = (0)$ they go to (7). By construction, this system is regular for the system of values (10). We shall show that the expressions that are obtained from them for the principal x, l as holomorphic functions of the remaining x, l and the λ by substitution in (9) already satisfy these equations identically!

We now assume that the aforementioned substitution does not make all of the expressions $\mathfrak{a}_\nu(x, l, \lambda)$ vanish. There will then be one or more relations between the "parametric" variables x_i $(i \leq r_0)$, l_{ik} $(p < i \leq \rho_k + k,\ k = 1, 2, \ldots, \nu)$ and the λ that are satisfied for $(\lambda) = (0)$, but not generally all values of λ. This means that the manifold of integral-E_ν, which satisfies the equations:

(12) $$\omega_i - \sum_{k=1}^{\nu} \lambda_{ik}\,\omega_k = 0 \qquad (\nu < i \leq p),$$

has a lower dimension for general values of λ than it does for $(\lambda) = (0)$, where it has the value:

$$N_\nu = \rho_0 + \sum_{k=1}^{\nu} (\rho_k + k - p).$$

However, this is not the case, since, by (12), there will be at least ∞^{N_ν} integral-E_ν in the neighborhood of E_p^0 that intersect the integral-E_p. We shall make this conclusion more precise:

Let $x_i = x_i(x)$, $l_{ik} = l_{ik}(x, l)$ be the expressions that are determined by $\mathfrak{a}_p(x, l) = 0$ in the manner just described, with all of the x, l in the parametric variables. The formulas:

$$x_i = x_i(x),$$

$$\omega_i - \sum_{k=1}^{p} L_{ik}(x,l)\omega_k = 0 \qquad (p < i \leq n) \tag{13}$$

then deliver all of the integral-E_p that neighbor E_p^0. Along with (12) these formulas deliver a collection of integral-E_ν whose dimension $N'_\nu(\lambda)$ is immediately obvious when one uses the ω_k ($k > n$), which are expressed in terms of $\omega_1, \omega_2, \ldots, \omega_\nu$ by (12), in (13), from which it might arise that:

$$\omega_i - \sum_{k=1}^{p} L_{ik}(x,l)\omega_k = 0 \qquad (p < i \leq n).$$

$N'_\nu(\lambda) = \rho_0$ is then the rank of the matrix of derivatives of the L_{ik} with respect to the parameters x, l. In any case, this dimension cannot be greater in special cases of λ – e.g., $(\lambda) = (0)$ – than in the general case of λ that lie in the neighborhood of $(\lambda) = (0)$. For $(\lambda) = (0)$, however, from (12) and (13) one obtains all of the E_ν that are determined from:

$$\mathfrak{a}_\nu(x, l_1, l_2, \ldots, l_\nu) = 0;$$

i.e., $N'_\nu(0) = N_\nu$, and therefore $N'_\nu(\lambda) \geq N_\nu$.

What the second regularity condition for E_ν^0 entails, as we shall observe next, is that for any integral-E_ν on which $\omega_{\nu+1} = 0$ the equations:

$$\omega_i - \sum_{k=1}^{\nu+1} l_{ik}\omega_k = 0 \qquad (p < i \leq n),$$

$$\omega_i - \sum_{k=1}^{\nu+1} \lambda_{ik}\omega_k = 0 \qquad (\nu + 1 < i \leq p),$$

along with the following conditions, which are analogous to (9):

$$\mathfrak{a}_{\nu+1}(x, l_1, l_2, \ldots, l_{\nu+1}, l) = 0, \tag{14}$$

collectively deliver $\infty^{\rho_\nu+1}$ integral-$E_{\nu+1}$ that go through them; then, as we have just seen, equations (14) can be solved in such a way that the $\rho_{\nu+1}$ quantities:

$$
\begin{array}{ll}
l_{i\nu+1} & (p < i \leq \rho_{\nu+1} + \nu + 1), \\
\lambda_{i\nu+1} & (\nu + 1 < i \leq p)
\end{array}
$$

stay arbitrary.

Now the regularity of the system of equations (14) remains valid when one transforms it to other direction coordinates $\bar{l}, \bar{\lambda}$ that enter into the new Pfaffian forms:

$$\bar{\omega}_i = \sum_{k=1}^{n} \rho_{ik}\omega_k \qquad (i = 1, 2, \ldots, n) \qquad (|\rho_{ik}| \neq 0)$$

in the same manner as the l, λ do in ω. If one is careful that the matrix (ρ_{ik}) is sufficiently small compared to the identity matrix then the new equations:

$$\mathfrak{a}_{\nu+1}(x, \bar{l}_1, \bar{l}_2, \cdots, \bar{l}_{\nu+1}, \bar{\lambda}) = 0$$

can be solved in the same way as (14). In particular, they therefore leave:

$$
\begin{array}{ll}
\bar{l}_{i\nu+1} & (p < i \leq \rho_{\nu+1} + \nu + 1), \\
\bar{\lambda}_{i\nu+1} & (\nu + 1 < i \leq p)
\end{array}
$$

arbitrary, and one concludes that $\infty^{\rho_\nu+1}$ integral-$E_{\nu+1}$ go through any E_ν that satisfies the equation $\bar{\omega}_{\nu+1} = 0$. Due to the fact that the arbitrariness in the choice of $\bar{\omega}_{\nu+1}$ is restricted only by inequalities, it follows that $\infty^{\rho_\nu+1}$ integral-$E_{\nu+1}$ go through the general integral-E_ν in the neighborhood of E_ν^0, hence, just as many as go through E_ν^0 itself.

The regularity of the chain that is determined by $(x^0, l_1^0, l_2^0, \cdots, l_p^0)$, viz.:

$$E_0^0 \subset E_1^0 \subset E_2^0 \subset \cdots \subset E_p^0,$$

is thus proved.

The rest follows from $\rho_\nu = r_\nu$ ($\nu = 1, 2, \ldots, p$); $\rho_0 = r_0$ was clear to begin with.

4. Another formulation of the criterion. The question of whether a solution $(x^0, l_1^0, l_2^0, \cdots, l_p^0)$ of:

$$\mathfrak{a}_\nu(x^0, l_1^0, l_2^0, \cdots, l_p^0) = 0 \tag{16}$$

corresponds to a regular chain can sometimes be addressed as follows:

One first sees whether (x^0) is a simple point of $\mathfrak{a}_0 = 0$. If this is the case and r_0 is the dimension of $\mathfrak{a}_0 = 0$ at (x^0) then *one further notes the number of equations in the left-hand side of:*

(16) $$\mathfrak{a}_\nu(x^0, l_1^0, l_2^0, \cdots, l_{\nu-1}^0, l_\nu) = 0$$

that are independent of l_ν and sets them equal to $n - \rho_\nu - \nu$. Furthermore, let N be the dimension of the zero manifold of $\mathfrak{a}_p(x, l_1, l_2, \ldots, l_p)$ in the neighborhood of $(x^0, l_1^0, l_2^0, \cdots, l_p^0)$. *One then always has:*

(17) $$r_0 + \rho_1 + \rho_2 + \ldots - \frac{p(p-1)}{2} \geq N,$$

and $(x^0, l_1^0, l_2^0, \cdots, l_p^0)$ therefore determines one and only one regular chain when one demands equality in the latter expression. In this case, $\rho_\nu = r_\nu$.

If one chooses $n - \rho_\nu - \nu$ equations from $\mathfrak{a}_\nu(x, l_1, l_2, \ldots, l_\nu) = 0$ whose left-hand sides are independent of l_ν for:

$$(x, l_1, \ldots, l_{\nu-1}) = (x^0, l_1^0, \cdots, l_{\nu-1}^0),$$

and solves the equations that are thus obtained for $\nu = 1, 2, \ldots, p$ then one can express $\sum_{\nu=1}^{p}(n - \rho_\nu - \nu)$ of the l in terms of r_0, x, and the remaining:

$$\sum_{\nu=1}^{p}(\rho_\nu + \nu - p) = + \rho_1 + \rho_2 + \ldots + \rho_p - \frac{p(p-1)}{2}$$

l holomorphically. It might be that all of the equations $\mathfrak{a}_\nu(x, l_1, l_2, \ldots, l_\nu) = 0$ are already satisfied; from the criterion that was just established, $(x^0, l_1^0, \cdots, l_p^0)$ determines a regular chain. Otherwise, (17) is a strict inequality, and any chain that belongs to $(x^0, l_1^0, \cdots, l_p^0)$ is certainly not regular.

5. Another formulation of the second existence theorem. Now let $\omega_i = dx_i$, in particular. The solution $(x^0, l_1^0, \cdots, l_p^0)$ of $\mathfrak{a}_p(x, l_1, l_2, \ldots, l_p) = 0$ may determine a regular chain, and the quantities:

$$x_i \quad (i > r_0) \qquad \text{and} \qquad l_{ik} \quad (i > r_k + k)$$

become principal, as above.

The forms:

$$\vartheta_i = dx_i - \sum_{k=1}^{\nu} l_{ik}^0 dx_k \qquad (r_\nu + \nu + 1 < i \leq r_\nu + \nu)$$

then have the properties that were required by the existence theorem on pp. 43.

The equations:

(18) $$dx_i - \sum_{k=1}^{\nu-1} l_{ik}^0 dx_k \quad - l_{i\nu} dx_\nu = 0 \qquad (p < i \leq \nu),$$

$$dx_i \qquad\qquad - \lambda_{i\nu} dx_\nu = 0 \qquad (\nu < i \leq p),$$

together with [1]):

(19) $$\mathfrak{a}_\nu(x^0, l_1^0, l_2^0, \cdots, l_{\nu-1}^0, l_\nu^0, \lambda_\nu) = 0,$$

determine all of the integral-E_ν that go through $E_{\nu-1}^0$, and thus also the polar space $H(E_{\nu-1}^0)$. $E_{\nu-1}^0$ is uniquely determined on $H(E_{\nu-1}^0)$, as required, by:

(20) $$dx_i = 0 \qquad (\nu < i \leq p),$$

$$\vartheta_i = 0 \qquad (p < i \leq r_\nu + \nu),$$

since (18) and (20) have:

$$\lambda_{i\nu} = 0,$$

$$l_{i\nu} = l_{i\nu}^0 \qquad (p < i \leq r_\nu + \nu),$$

as consequences, and the remaining $l_{i\nu}$ are uniquely determined by (19).

On the basis of the general existence theorems, one thus has:

The integral manifolds M_p that possess tangent elements that neighbor on $E_p^0 = (x^0, l_1^0, \cdots, l_p^0)$ may be represented in the form:

$$x_i = f_i(x_1, x_2, \ldots, x_p) \qquad (i = p + 1, \ldots, n).$$

For those x_i that have exactly ν parametric "derivatives:"

$$l_{ik} = \frac{\partial x_i}{\partial x_k} \qquad (k = 1, 2, \ldots, p)$$

the:

$$f_i(x_1, x_2, \ldots, x_\nu, x_{\nu+1}^0, \cdots, x_p^0),$$

and, for the parametric x_i that have no parametric derivatives, the:

$$f_i(x_1^0, x_2^0, \cdots, x_p^0),$$

can be described arbitrarily. Therefore, it is assumed that the differences between:

$$f_i(x_1^0, x_2^0, \cdots, x_p^0) \text{ and } x_1^0 \qquad (i > p)$$

[1]) These equations go to (9) when one sets all of the $l_{i\mu}$ ($\mu < \nu$) to zero.

$$\left(\frac{\partial f_i}{\partial x_k}\right)_0 \textit{ and } l_{ik}^0$$

are sufficiently small.

In order to simplify the application in practice, we have made no reference to the ordering of the x_i ($i > p$) in these formulas. We also point out that the solution of the equation:

$$\mathfrak{a}_\nu(x, l_1, l_2, \ldots, l_\nu) = 0$$

must always be determined in such a way that *if* $l_{ik} = \frac{\partial x_i}{\partial x_k}$ *is parametric then all of the "preceding" derivatives:*

$$\frac{\partial x_i}{\partial x_1}, \frac{\partial x_i}{\partial x_2}, \ldots, \frac{\partial x_i}{\partial x_{k-1}}$$

are also parametric.

6. Setting up the equations $\mathfrak{a}_\nu(x, l_1, l_2, \ldots, l_\nu) = 0$. Let:

$$\Psi_1, \Psi_2, \ldots, \Psi_h$$

be homogeneous forms of $\mathfrak{a}$ that define a basis for this ideal, so that any form Φ of $\mathfrak{a}$ can be represented in the form:

(21) $$\Phi = \Psi_1\Phi_1 + \Psi_2\Phi_2 + \ldots + \Psi_h\Phi_h .$$

For example:

$$\theta_1, \theta_2, \ldots, \theta_l, \theta_1, \theta_2, \ldots, \theta_l$$

define such a basis.

One reduces the Ψ modulo the forms:

(22) $$\omega_i - \sum_{k=1}^{p} l_{ik}\,\omega_k \qquad (i = p + 1, \ldots, n)$$

so that they are expressed only in terms of the $\omega_1, \omega_2, \ldots, \omega_p$:

$$\Psi \equiv \sum_{k_1<k_2<\cdots<k_q}^{p} b_{k_1,k_2,\cdots,k_q}\,\omega_{k_1}\omega_{k_2}\cdots\omega_{k_q} .$$

By setting all of the coefficients $b_{k_1,k_2,\cdots,k_q}$ *that have* $k_1, k_2, \ldots, k_q \le \nu$ *equal to zero, one obtains a basis for the system of equations* $\mathfrak{a}_\nu(x, l_1, l_2, \ldots, l_\nu) = 0$. Obviously, the equations:

$$b_{k_1,k_2,\cdots,k_q} = 0 \qquad (k_1, k_2, \ldots, k_q \leq \nu)$$

are found among the $\mathfrak{a}_\nu(x, l) = 0$, and, from (21), all of the expressions $\mathfrak{a}_\nu(x, l)$ will be linear combinations of these b.

Since the variables l_k enter into the forms (22) symmetrically, all of the coefficients $b_{k_1,k_2,\cdots,k_q}$ that belong to the same Ψ can be obtained from one of them – say, $b_{1,2,\ldots,q} = b(x, l_1, l_2, \ldots, l_q)$ – by permutation of the variables:

$$b_{k_1,k_2,\cdots,k_q} = b(x, l_{k_1}, l_{k_2}, \cdots, l_{k_q}).$$

7. Prolongation of a system of differential equations. If one assumes that no regular chains intersect:

$$\omega_i = 0 \qquad (n < i \leq p)$$

on an integral-E_p on which $\omega_1 \omega_2 \ldots \omega_p \neq 0$ for $n = 1, 2, \ldots, p$ then one next seeks to represent E_p as the last link of a regular chain by setting the $\omega_1, \omega_2, \ldots, \omega_p$ equal to suitable linear combinations:

$$\bar{\omega}_i = \sum_{k=1}^{p} \rho_{ik}\omega_k \qquad (i = 1, 2, \ldots, p).$$

It can therefore happen that some E_p cannot be reached by any regular chain, at all. The integral manifolds M_p, whose tangent-E_p are all singular, in this sense, will not be immediately obtained from the existence theorems that were stated above. In this case, which is certainly not rare, one proceeds as follows:

As above, let:

$$(23) \qquad \begin{cases} \omega_i - \sum_{k=1}^{p} l_{ik}\omega_k = 0 & (i = p+1, \cdots, n), \\ \mathfrak{a}_\nu(x, l_1, l_2, \cdots, l_p) = 0 \end{cases}$$

be the equations that determine the integral-E_p on which $\omega_1 \omega_2 \ldots \omega_p \neq 0$. If the situation that was just described is the case then one regards (23) as a system of differential equations in the variables x, l, and deals with them (the so-called *prolonged system* of the original one) in the same way as with the old system. If the desired integral manifolds are still not attainable then one defines the next prolongation, etc.

It remains to be shown that after finitely many steps one will either come to the conclusion that there are absolutely no integral-M_p of the desired sort, or obtain a prolonged system in which every integral-M_p can be obtained by means of our existence theorems [1])

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[1]) See E. Cartan, "Sur la structure des groupes infinis de transformations," no. 10-12, Ann. de l'Ecole Normale (3) **21** (1904).

We incidentally remark that if one takes into account the appearance of new variables l then any system of differential equations can be transformed into a system (23) of scalar and Pfaff equations.

8. Rules for calculation in an important special case. With regard to the applications to the theory of systems of partial differential equations (cf. pp. 67, 68) we would like to take a closer look at the case for which the basis for $\mathfrak{a}$ is chosen in such a way that except for the scalar equations $\mathfrak{a}_0 = 0$ it contains only certain Pfaffian forms:

$$\theta_1, \theta_2, \ldots, \theta_h$$

and certain forms of degree two of the form:

$$\psi_i = \sum_{k=2}^{p} \varpi_{ik}\omega_k \qquad (i = 1, 2, \ldots, m) \qquad (\varpi \text{ are Pfaffian forms}).$$

We therefore assume that $(x) = (x^0)$ is a simple point of the manifold $\mathfrak{a}_0(x) = 0$, and that the forms $\theta_1, \theta_2, \ldots, \theta_h$ are linearly independent for $(x) = (x^0)$; the Pfaffian forms $\omega_1, \omega_2, \ldots, \omega_p$ have the same meaning as above.

If there exists a linear relation:

$$(24) \qquad \sum a_i\, \omega_i + \sum b_i\, q_i \equiv 0 \qquad (\text{mod } \mathfrak{a}_0)$$

between the forms θ, ω then it is clear that for any of the desired integral manifolds (on which $\omega_1, \omega_2, \ldots, \omega_p$ shall still be linearly independent) we must have:

$$a_1 = 0, a_2 = 0, \ldots, a_p = 0.$$

One then must add the last equation to $\mathfrak{a}_0 = 0$ and $da_1, da_2, \ldots, da_p$ to the θ. If (x^0) is still a simple point of the manifold:

$$a_1 = 0, a_2 = 0, \ldots, a_p = 0$$

then one recalls the procedure by which one either establishes that there are no integral manifolds of the desired type that go through (x^0) or a prolonged system for which $(x) = (x^0)$ is no longer a simple point of the scalar equations, or one ultimately obtains a system for which (x^0) is a simple point of $\mathfrak{a}_0 = 0$ and the equations θ, together with the ω, are linearly independent, mod $\mathfrak{a}_0$. We would like to pursue this last possibility further. We therefore assume that there are no relations of the form (24), and make the restriction that the θ, ω also remain linearly independent for $(x) = (x^0)$.

Along with these $p + h$ forms, let there be $q = n - p - h$ more Pfaffian forms:

$$\varpi_1, \varpi_2, \ldots, \varpi_q,$$

which are chosen in such a way that the ω, θ, ϖ are collectively n linearly independent forms at $(x) = (x^0)$.

According to the general procedure, we make the Ansatz:

$$\theta_i = \sum_{k=1}^{p} l'_{ik}\omega_k \qquad (i = 1, 2, \ldots, h)$$

$$\varpi_i = \sum_{k=1}^{p} l_{ik}\omega_k \qquad (i = 1, 2, \ldots, q),$$

in which we can set $l'_{ik} = 0$ from the outset. Only when we want to simplify the comparison with earlier considerations will we introduce the l'. The l and l' collectively play the role that was previously played by the variables l alone.

The equations $\mathfrak{a}_1(x, l) = 0$ now read:

$$l'_{i1} = 0 \qquad (i = 1, 2, \ldots, h),$$

and when we set:

$$\varpi_{ik} \equiv \sum_{\rho=1}^{q} a_{ik\rho}\varpi_\rho \qquad (\text{mod } \theta_1, \theta_2, \ldots, \theta_h, \omega_1, \omega_2, \ldots, \omega_p),$$

we get:

$$l'_{i\nu} = 0 \qquad (i = 1, 2, \ldots, h),$$

$$\sum_{\rho=1}^{q} a_{ik\rho} l_{\rho\nu} + \ldots = 0 \qquad \begin{pmatrix} i = 1, 2, \cdots, m \\ j = 1, 2, \cdots, \nu - 1 \end{pmatrix}$$

for those $\nu > 1$ equations (partial equations, resp.) for which the solution in terms of l_ν, l'_ν is at issue.

With our previous notation, we thus have:

$$(25) \qquad \begin{aligned} n - \rho_1 - 1 &= h \\ n - \rho_\nu - \nu &= h + \sigma_1 + \sigma_2 + \ldots + \sigma_{\nu-1}, \end{aligned}$$

when $\sigma_1 + \sigma_2 + \ldots + \sigma_{\nu-1}$ denotes the rank of the system of homogeneous equations:

$$a_{ij1}\, l_{1\nu} + a_{ij2}\, l_{2\nu} + \ldots + a_{ijq}\, l_{q\nu} = 0 \qquad \begin{pmatrix} i = 1, 2, \cdots, m \\ j = 1, 2, \cdots, \nu - 1 \end{pmatrix}.$$

One can also say: $\sigma_1 + \sigma_2 + \ldots + \sigma_{\nu-1}$ is the number of forms:

$$\sum_{\rho=1}^{q} a_{ik\rho}\varpi_\rho \qquad \begin{pmatrix} i = 1, 2, \cdots, m \\ j = 1, 2, \cdots, \nu - 1 \end{pmatrix}$$

that are linearly independent for $(x) = (x^0)$.

From (25), one computes:

$$\sum_{\nu=1}^{p} (\rho_\nu + \nu - p) = p(\rho_1 + 1 - p) - (p-1)\sigma_1 - (p-2)\sigma_1 - \ldots - \sigma_{p-1} ,$$

or, since $q = n - h - p = \rho_1 + 1 - p$:

$$\sum_{\nu=1}^{p} (\rho_\nu + \nu - p) = pq - (p-1)\sigma_1 - (p-2)\sigma_1 - \ldots - \sigma_{p-1} .$$

On the other hand, let:

$$\psi_i \equiv \sum_{\rho,k} a_{ik\rho}\, \varpi_\rho\, \omega_k + \tfrac{1}{2}\sum_{j,k} c_{ijk}\; \omega_j \omega_k \qquad (\mathrm{mod}\ \theta_1, \ldots, \theta_h) \qquad (c_{ijk} = -c_{ikj}).$$

We then have that:

$$(26) \qquad \begin{cases} \mathfrak{a}_0(x) = 0, & \begin{pmatrix} i = 1, 2, \cdots, h \\ k = 1, 2, \cdots, p \end{pmatrix} \\ l'_{ik} = 0, & \\ \displaystyle\sum_{\rho=1}^{q} a_{ik\vartheta} l_{\rho j} - a_{ij\rho} l_{\rho k} + \tfrac{1}{2} c_{ijk} = 0 & \begin{pmatrix} i = 1, 2, \cdots, m \\ j < k \le p \end{pmatrix} \end{cases}$$

are the equations that were previously notated by $\mathfrak{a}_p(x, l) = 0$. Therefore, if $N = r_0 + M$ is the dimension of the manifold in the space of variables x, l that is described by (26) for $(x) = (x^0)$ then, from the remarks on pp. 51 one always has the inequality:

$$(27) \qquad M \le pq - (p-1)\sigma_1 - (p-2)\sigma_2 - \ldots - \sigma_{p-1},$$

in which the validity of the equality sign is characteristic of the existence of a regular chain of the desired type.

If the inequality sign is valid then it can just as well happen (cf. pp 53) that by replacing the $\omega_1, \omega_2, \ldots, \omega_p$ with other forms $\bar{\omega}_1, \bar{\omega}_2, \cdots, \bar{\omega}_p$ that are related by:

$$\omega_i = \sum_{k=1}^{p} u_i^{(k)} \bar{\omega}_k \qquad (i = 1, 2, \ldots, p)$$

then for the new numbers $\sigma_1, \sigma_2, \ldots, \sigma_{p-1}, M$, we have:

$$M = pq - (p-1)\sigma_1 - (p-2)\sigma_2 - \ldots - \sigma_{p-1}.$$

Obviously, $\sigma_1 + \sigma_2 + \ldots + \sigma_\nu$ is equal to the number of linearly independent forms amongst the:

$$\sum_{k,\rho} a_{ik\rho} u_k^{(j)} \varpi_\rho \qquad \begin{pmatrix} i = 1, 2, \cdots, m \\ j = 1, 2, \cdots, \nu \end{pmatrix}.$$

We summarize the results obtained:

The basis for the ideal $\mathfrak{a}$ *consists of:*

1. *The scalar equations* $\mathfrak{a}_0(x) = 0$, *which define a simple* r_0*-dimensional manifold at* $(x) = (x^0)$;

2. *The Pfaffian forms:*

$$\theta_1, \theta_2, \ldots, \theta_h,$$

which are linearly independent of each other and $\omega_1, \omega_2, \ldots, \omega_p$ *at* $(x) = (x^0)$;

3. *Certain forms of degree two of the form:*

$$\psi_i = \sum_{k=2}^{p} \varpi_{ik} \omega_k \qquad (i = 1, 2, \ldots, m).$$

By the use of $q = n - p - h$ *of the Pfaffian forms* $\varpi_1, \varpi_2, \ldots, \varpi_q$, *which are independent of* ω, θ, *one sets:*

$$\varpi_{ik} \equiv \sum_{\rho=1}^{q} a_{ik\rho} \varpi_\rho \qquad (\mathrm{mod}\ \omega, \theta)$$

and computes σ_1 *as the number of linearly independent Pfaffian forms:*

$$\sum_{\rho=1}^{q} \left(\sum_{k=1}^{p} a_{ik\rho} u_k^{(1)} \right) \varpi_\rho \qquad (i = 1, 2, \ldots, m),$$

$\sigma_1 + \sigma_2$ *as the number of linearly independent forms:*

$$\sum_{\rho=1}^{q} \left(\sum_{k=1}^{p} a_{ik\rho} u_k^{(1)} \right) \varpi_\rho \qquad (i = 1, 2, \ldots, m),$$

$$\sum_{\rho=1}^{q} \left(\sum_{k=1}^{p} a_{ik\rho} u_k^{(2)} \right) \varpi_\rho$$

etc., and finally $\sigma_1 + \sigma_2 + \ldots + \sigma_{p-1}$ *is the number of linearly independent forms that appear in:*

$$\sum_{\rho=1}^{q}\left(\sum_{k=1}^{p} a_{ik\rho}u_k^{(j)}\right)\varpi_\rho \qquad \left(\begin{array}{l} i = 1, 2, \cdots, m \\ j = 1, 2, \cdots, p-1 \end{array}\right)$$

for $(x) = (x^0)$. *The* $u_k^{(j)}$ *are therefore to be considered to be unknowns.*

One further prolongs the resulting system of differential equations by the Ansatz:

$$\varpi_\rho = \sum_{k=1}^{p} l_{\rho k}\omega_k \qquad (\rho = 1, 2, \ldots, q)$$

and lets $r_0 + M$ *denote the dimension of the manifold that is defined by the scalar equations of the new system in the neighborhood of* $(x) = (x^0)$, *with l arbitrary.*

One then always has:

$$M \leq pq - (p-1)\sigma_1 - (p-2)\sigma_2 - \ldots - \sigma_{p-1},$$

and the p-dimensional integral element at $(x) = (x^0)$ *is representable as the final link in a regular chain when and only when one has strict equality.*

In the last case, the general integral manifold that passes through the neighborhood of $(x) = (x^0)$ depends on:

s_0	$= r_0 - p - q$	arbitrary constants,		
s_1	$= \sigma_1$	"	functions of	1 variable,
s_2	$= \sigma_2$	"	"	2 variables,
		…		
s_{p-1}	$= \sigma_{p-1}$	"	"	$p-1$ ",
r_p	$= q - \sigma_1 - \sigma_2 - \ldots - \sigma_{p-1}$	"	"	p ".

In order to define these arbitrarily determined pieces of M_p more precisely, one needs only to specify that the $u_i^{(k)}$ be constants, in such a way that among the aforementioned Pfaffian forms just as many of them are linearly independent as for the undetermined u. If one then computes $\bar{\omega}_1, \bar{\omega}_2, \cdots, \bar{\omega}_p$ from:

$$\omega_i = \sum_{k=1}^{p} u_i^{(k)}\bar{\omega}_k \qquad (i = 1, 2, \ldots, p)$$

then regular integral-E_ν will intersect:

$$\bar{\omega}_{\nu+1} = \bar{\omega}_{\nu+2} = \cdots = \bar{\omega}_p = 0$$

on the integral-E_p with $\omega_1\,\omega_2 \dots \omega_p \neq 0$ that goes through (x^0). Therefore, everything has been prepared for the application of the existence theorem.

One observes that something was not said in the foregoing cases for *all* of the integral-E_p, independently of the direction of the elements, as long as $\omega_1\,\omega_2 \dots \omega_p \neq 0$. It consists of the statement that everything is linear in the determination of the l standard equations.

The procedure that was given then proves to be suitable when one is compelled to go to more prolongations of a system (cf. the remarks on pp. 56). Under prolongation the system retains the assumed normal form, and indeed even when one prolongs an arbitrary system one arrives at a system for which the considerations that we just presented are applicable.

With the previous notation (cf. section 7), let:

$$(23)\qquad \begin{aligned} \omega_i - \sum_{k=1}^{p} l_{ik}\,\omega_k &= 0 \qquad (i = p+1, \cdots, n), \\ \mathfrak{a}_p(x, l_1, l_2, \cdots, l_p) &= 0 \end{aligned}$$

be the preceding Pfaffian system that is obtained from a system with the ideal $\mathfrak{a}$ by prolongation. The ideal $\mathfrak{A}$ that belongs to (23) has a basis that consists of:

the scalars: $\mathfrak{a}_p(l_1, l_2, \dots, l_p)$,

the Pfaffian forms:

$$d\mathfrak{a}_p(l_1, l_2, \dots, l_p),$$

$$\theta_i = \omega_i - \sum_{k=1}^{p} l_{ik}\,\omega_k \qquad (i = p + 1, \dots, n),$$

the forms of degree two: $d\theta_i$.

Since the forms ω_i ($i = 1, 2, \dots, n$) involve only the variables x, one can represent the $d\omega_i$ as linear combinations of the products $\omega_k\,\omega_l$, and by reduction mod $\theta_{p+1}, \theta_{p+2}, \dots, \theta_n$ they may even be expressed in terms of the $\omega_k\,\omega_l$ ($k < l \leq p$) alone. After these transformations, the forms:

$$d\theta_i = d\omega_i - \sum_{k=1}^{p} dl_{ik}\,\omega_k - \sum_{k=1}^{p} l_{ik}\,d\omega_k$$

take the desired form:

$$\psi_i = \sum_{k=2}^{p} \varpi_{ik}\,\omega_k\ ,$$

and, together with $\mathfrak{a}_p(x, l_1, \dots, l_p)$, $d\mathfrak{a}_p(x, l)$, and the θ, they still define a basis for $\mathfrak{A}$.

VI. Applications and Examples

1. Theorem on total differentials. *Any differential form ϖ over the ring of holomorphic functions at a point* $(x_1^0, \cdots . x_n^0)$ *whose derivative vanishes identically is a total differential.*

Proof: By normalization, we set:

$$\varpi = \frac{1}{q!} \sum_{i_1, i_2, \cdots, i_q}^{n} a_{i_1, i_2, \cdots, i_q} (x_1, x_2, \cdots, x_n) d(x_{i_1}, x_{i_2}, \cdots, x_{i_q}),$$

in which the $a_{i_1, i_2, \cdots, i_q}$ are skew-symmetric in the indices. We must determine a form:

$$\theta = \frac{1}{(q-1)!} \sum_{i_1, i_2, \cdots, i_{q-1}}^{n} v_{i_1, i_2, \cdots, i_{q-1}} d(x_{i_1}, x_{i_2}, \cdots, x_{i_{q-1}})$$

($v_{i_1, i_2, \cdots, i_{q-1}}$ skew-symmetric)

such that:

$$d\theta - \varpi = \frac{1}{(q-1)!} \sum_{i_1, i_2, \cdots, i_{q-1}} dv_{i_1, i_2, \cdots, i_{q-1}} d(x_{i_1}, x_{i_2}, \cdots, x_{i_{q-1}})$$

$$- \frac{1}{q!} \sum_{i_1, i_2, \cdots, i_q} a_{i_1, i_2, \cdots, i_q} d(x_{i_1}, x_{i_2}, \cdots, x_{i_q}) = 0 . \tag{1}$$

We regard this as a differential equation in the $n + \binom{n}{q+1}$ variables x, v and we can formulate our problem in the following way: Determine an n-dimensional integral manifold of (1) that is representable in the form:

$$v_{i_1, i_2, \cdots, i_{q-1}} = f_{i_1, i_2, \cdots, i_{q-1}} (x_1, x_2, \cdots, x_n) .$$

To this end, we seek an n-dimensional integral element on which $d(x_1, x_2, \ldots, x_n) \neq 0$ and is such that a regular integral chain intersects:

$$dx_i = 0 \qquad (v < i \leq n)$$

for $v = 0, 1, 2, \ldots, n - 1$.

By substitution in (1) for the coefficients $d(x_{i_1}, x_{i_2}, \cdots, x_{i_q})$ (up to sign), the Ansatz:

$$dv_{i_1, i_2, \cdots, i_{q-1}} = \sum_{i_q = 1}^{n} l_{i_1, i_2, \cdots, i_{q-1}, i_q} dx_{i_q}$$

gives the result:

$$L_{i_1,i_2,\cdots,i_q} = l_{i_1,i_2,\cdots,i_q} - l_{i_1,i_2,\cdots,i_q,i_{q-1}} - l_{i_1,i_2,\cdots,i_q,i_{q-1},i_{q-2}} - \cdots - l_{i_q,i_2,\cdots,i_{q-1},i_1} + (-1)^q a_{i_1,i_2,\cdots,i_q,} ,$$

in which the indices are skew-symmetric.

Since $d(d\theta - \varpi) = 0$, by assumption, $d\theta - \varpi$ defines a basis for the ideal $\mathfrak{a}$, and the equations $\mathfrak{a}_\nu(x, l_1, \ldots, l_\nu) = 0$ are equivalent to:

(2) $$L_{i_1,i_2,\cdots,i_q} = 0 \qquad (i_1 < i_2, \ldots, < i_q \leq \nu).$$

One must regard all of the $l_{i_1,i_2,\cdots,i_{q-1},\nu}$ as the variables l_ν. One sees that equations (2) differ from the given $\mathfrak{a}_{\nu-1}(x, l_1, \ldots, l_{\nu-1}) = 0$ by the appearance of:

$$L_{i_1,i_2,\cdots,i_q,\nu} = 0 \qquad (i_1 < i_2, \ldots, < i_{q-1} < \nu),$$

and that the latter succeeds in determining the corresponding $l_{i_1,i_2,\cdots,i_q,\nu}$. The regularity of the desired chain follows from this and therefore also the existence of an integral manifold with the desired properties.

It is also easy determine the arbitrary functions that enter into the general solution of the problems; however, we would therefore not wish to dwell on this fact, especially since the indeterminacy of the solution can be established directly in this case.

One has two forms θ_1, θ_2 for which:

$$d\theta_1 = \varpi, \qquad d\theta_1 = \varpi,$$

so we have $d(\theta_1 - \theta_2) = 0$, from which, by an application of the theorem that we just proved it follows that θ_1 and θ_2 differ by a total differential. Conversely, it then follows from $\theta_1 = \theta_2 + d\vartheta$ that $d\theta_1 = d\theta_2$.

2. Completely integrable Pfaff systems. A system of differential equations:

$$\theta_1 = 0,\ \theta_2 = 0,\ \ldots,\ \theta_l = 0,$$

in which the Pfaffian forms in the left-hand side are:

$$\theta_i = \sum_{k=1}^{n} a_{ik}\, dx_k ,$$

is called a *Pfaffian system*, and, in particular, it is called *completely integrable* when θ_1, θ_2, …, θ_l already define a basis for the differential ideal that they generate, hence, when:

$$d\theta_i \equiv 0 \pmod{\theta_1, \theta_2, \ldots, \theta_l} \qquad (i = 1, 2, \ldots, l).$$

With no loss of generality, we can assume that the q are linearly independent and perhaps choose them so that they are linearly independent in the dx_i with $i > n - l$ in a neighborhood of $(x_1^0, \cdots, x_n^0)$. Consequently, there can be no integral manifolds of dimension higher than $n - l$, but there is always one of dimension $n - l$.

By substitution in the θ_i as coefficients of dx_ν, the Ansatz:

$$dx_i = \sum_{k=1}^{n-l} l_{ik}\, dx_k \qquad (i > n - l)$$

then gives the expressions:

$$L_{i\nu} = \sum_{k>n-l} a_{ik}\, l_{k\nu} + a_{i\nu},$$

and the new equations that enter $\mathfrak{a}_\nu(x, l_1, \ldots, l_\nu) = 0$, when compared to $\mathfrak{a}_{\nu-1}(x, l_1, \ldots, l_{\nu-1}) = 0$, are:

(3) $$L_{i\nu} = 0 \qquad (i = 1, 2, \ldots, l).$$

All $l_{k\nu}$ are uniquely determined from (3); there are no parametric l_{ik}, and all x_i are parametric (because we have no equation $\mathfrak{a}_0 = 0$). From the remarks on pp. 54, we conclude:

In the neighborhood of $x_1^0, x_2^0, \cdots, x_n^0$ the integral-M_{n-l} can be represented in the form:

(4) $$x_i = f_i(x_1, x_2, \ldots, x_{n-l}) \qquad (i > n - l)$$

and they depend on exactly l constants, since the values:

$$f_{n-l+i}(x_1^0, x_2^0, \cdots, x_{n-l}^0) = c_i \qquad (i = 1, 2, \ldots, l)$$

can be assigned arbitrarily. There is thus exactly one $(n - l)$-dimensional integral manifold through each point in the neighborhood of (x^0).

As long as the c remain in the neighborhood of the value $c_i = x_{n-l+1}^0$, the M_{n-l} depend holomorphically on the c; i.e., the right-hand sides of (4), $f_i(x_1, x_2, \ldots, x_{n-l}) = f_i(x, c)$ are holomorphic functions of c. Due to the fact that:

$$f_{n-l+i}(x^0, c) = c_i ,$$

its functional determinant in the c is different from zero. By solving the equation:

(5) $$x_i - f_i(x_1, x_2, \ldots, x_{n-l}, c_1, c_2, \ldots, c_l) = 0 \qquad (i > n - l)$$

for c, one obtains the equations:

$$F_i(x_1, x_2, \ldots, x_n) = c_i \qquad (i = 1, 2, \ldots, l),$$

in which the left-hand sides represent so-called *integrals.* By the "integral" of a completely integral Pfaffian system, one generally understands that term to mean a function that takes a constant value for any integral manifold.

The functions $F_i(x_1, x_2, \ldots, x_n)$ define a *complete system of integrals,* in the sense that any other integral $\Phi(x_1, x_2, \ldots, x_n)$ that is holomorphic at (x^0) can be expressed as a function of the F. Namely, let $\varphi(c_1, c_2, \ldots, c_l)$ be the holomorphic functions in $(c_1, c_2, \ldots, c_l) = (x^0_{n-l+1}, \cdots, x^0_n)$ that reduce to Φ when one considers (5). Then:

$$\Phi(x_1, x_2, \ldots, x_n) - \varphi(F_1, F_2, \ldots, F_l)$$

is an integral that vanishes on all integral-M_{n-l}. However, an integral-M_{n-l} goes through each point in the neighborhood of (x^0), from which it follows that:

$$\Phi(x_1, x_2, \ldots, x_n) = \varphi(F_1, F_2, \ldots, F_l).$$

For every integral Φ one obviously has:

$$d\Phi \equiv 0 \qquad (\mathrm{mod}\ \theta_1, \theta_2, \ldots, \theta_l).$$

In particular, one has:

(6) $$dF_i \equiv 0 \qquad (\mathrm{mod}\ \theta_1, \theta_2, \ldots, \theta_l) \quad (i = 1, 2, \ldots, l),$$

and since the dF_i are linearly independent, one can use these instead of the θ as a basis for the ideal $\mathfrak{a}$.

Conversely, if one has a system of l linearly independent Pfaffian equations [at (x^0)]:

(7) $$\theta_1 = 0,\ \theta_2 = 0,\ \ldots,\ \theta_l = 0,$$

and l functions $F_1, F_2, \ldots, F_l$ whose differentials at (x^0) are linearly independent and satisfy the congruence (6) then (7) is completely integrable. One then has:

$$\theta_i \equiv 0 \qquad (\mathrm{mod}\ dF_1, dF_2, \ldots, dF_l),$$

from which it follows that:

$$d\theta_i \equiv 0 (\mathrm{mod}\ \theta_1, \theta_2, \ldots, \theta_l).$$

One must ultimately remember that systems of ordinary differential equations are also obtained from completely integrable systems (for $l = n - 1$).

3. Differential equations for the characteristic surfaces in spaces of two complex variables. In four-dimensional space R_4 , when the two complex variables:

$$x = x_1 + ix_2, \qquad y = x_3 + ix_4$$

are two-dimensional integral manifolds of the system:

(8) $$d(x, y) = 0, \qquad d(\bar{x}, \bar{y}) = 0 \qquad \begin{pmatrix} x = x_1 - ix_2 \\ \bar{y} = x_3 - ix_4 \end{pmatrix}$$

or:

$$d(x_1, x_3) - d(x_2, x_4) = 0, \qquad d(x_1, x_4) - d(x_2, x_3) = 0$$

they are precisely the *characteristic surfaces*, i.e., surfaces that are representable as the null hypersurface of a holomorphic function $F(x, y) = 0$.

From (8), it follows that one cannot also have $d(x, \bar{x}) = 0$ for a two-dimensional integral element E_2; therefore, let $dx = d\bar{x} = 0$. Otherwise, all three differentials $d\bar{x}, d\bar{y}, d\bar{z}$ of dx will be linearly dependent on E_2. As long as one is not dealing with the integral manifold:

$$x = \text{const.}, \qquad \bar{x} = \text{const.},$$

one can therefore make the Ansatz:

$$y = f(x, \bar{x}),$$
$$\bar{y} = g(x, \bar{x})$$

for the integral-M_2, and one obtains from (2):

$$\frac{\partial f}{\partial \bar{x}} = 0, \qquad \frac{\partial g}{\partial x} = 0,$$

and therefore:

$$y = f(x), \qquad \bar{y} = \bar{f}(\bar{x}) \qquad \text{Q.E.D.}$$

Every vector E_1 in R_4 is an integral element, and exactly one integral-E_2 goes through every E_1. If z_1, z_2, z_3, z_4 are the direction components of E_1 then:

$$z_1 \Delta x_3 - z_3 \Delta x_1 - z_2 \Delta x_4 + z_4 \Delta x_2 = 0,$$
$$z_1 \Delta x_4 - z_4 \Delta x_1 - z_2 \Delta x_3 + z_3 \Delta x_2 = 0$$

are the equations $\mathfrak{a}_2(x, z, \Delta z) = 0$ for the polar element $H(E_1)$. Only for $z_1 = z_2 = z_3 = z_4 = 0$ are they linearly dependent. One therefore always has $r_2 + 2 = 2$; i.e., $r_2 = 0$.

With the help of the first existence theorem (pp. 29) we then state the following theorem, which is due to Levi-Civita: *exactly one characteristic surface goes through any regular analytical curve segment.*

4. Partial differential equations. The true significance of the theory that we just developed is clear from the remark that any system of partial differential equations in arbitrarily many unknowns and arbitrarily many equations can be handled by the calculus of differential forms.

We then consider, perhaps, the case of a system of h partial differential equations of second order:

$$F_i\left(x_1, x_2, \cdots, x_n, z, \frac{\partial z}{\partial x_1}, \cdots, \frac{\partial z}{\partial x_n}, \frac{\partial^2 z}{\partial x_1^2}, \frac{\partial^2 z}{\partial x_1 \partial x_2}, \cdots, \frac{\partial^2 z}{\partial x_n^2},\right) = 0 \qquad (i = 1, 2, \ldots, h)$$

in the unknowns $z(x_1, \ldots, x_n)$.

The integration of these systems is equivalent to the problem of determining all of the n-dimensional integral manifolds that are representable in the form:

$$\begin{aligned} z &= \varphi(x_1, \ldots, x_n), \\ p_i &= \varphi_i(x_1, \ldots, x_n), \\ r_{ik} &= \varphi_{ik}(x_1, \ldots, x_n) \end{aligned}$$

for the systems of differential equations that are composed of the h equations of degree 0:

$$F_i(x_1, x_2, \ldots, x_n, z, p_1, \ldots, p_n, r_{11}, r_{12}, \ldots, r_{nn}) = 0 \qquad (i = 1, 2, \ldots, h)$$

and the $n + 1$ Pfaff equations:

$$\text{(9)} \qquad \begin{aligned} dz - \sum_{k=1}^{n} p_k dx_k &= 0, \\ dp_i - \sum_{k=1}^{n} r_{ik} dx_k &= 0 \end{aligned} \qquad (i = 1, 2, \ldots, n).$$

In this way, any system of partial differential equations turns into to a system of scalar and Pfaff equations.

The previously-remarked situation that it is sometimes impossible to represent the integral-E_p in the form (9) as the final link of a regular chain corresponds precisely to the well-known fact that a system of partial differential equations can lead to the given independent equations by restating the integrability conditions.

Appendix.

The Main Theorems of Lie Group Theory.

1. Definition of a Lie group. An r-parameter Lie group germ is a collection of elements that can be represented by symbols:

$$S_{a_1, a_2, \cdots, a_r}$$

with r real or complex numbers $a_1, a_2, \ldots, a_r$ as indices, and between which relations of the following type exist:

1. An element, perhaps:

$$S_0 = S_{0,0,\ldots,0}\,,$$

is distinguished as the so-called *identity element.*

2. One assumes that when the systems of values $(a_1, a_2, \ldots, a_r)$, $(b_1, b_2, \ldots, b_r)$ lie in a certain neighborhood U of the system of values $(0, 0, \ldots, 0)$ there is a multiplication of both elements:

$$S_a = S_{a_1, a_2, \cdots, a_r}\,, \qquad\qquad S_b = S_{b_1, b_2, \cdots, b_r}$$

that is defined, and which leads to a uniquely determined element:

$$S_{c_1, c_2, \cdots, c_r} = S_a\, S_b\,.$$

The *multiplication law* will be described by:

$$c_\nu = \varphi_\nu(a_1, a_2, \ldots, a_r; b_1, b_2, \ldots, b_r) \qquad (\nu = 1, 2, \ldots, r)$$

in which the $\varphi_\nu(a, b)$ are analytical functions that are holomorphic for all (a), (b) in U.

3. We have:

$$S_a\, S_0 = S_0\, S_a = S_a\,,$$

i.e.:

$$\varphi_\nu(a_1, a_2, \ldots, a_r; 0, 0, \ldots, 0) = \varphi_\nu(0, \ldots, 0, a_1, a_2, \ldots, a_r) = a_\nu \qquad (\nu = 1, 2, \ldots, r),$$

and:

$$\frac{\partial(\varphi_1(a,b), \cdots, \varphi_r(a,b))}{\partial(a_1, a_2, \cdots, a_r)} \neq 0$$

for $(a) = (b) = (0)$.

4. The associativity law:

$$S_a(S_b\, S_c) = (S_a\, S_b)\, S_c$$

holds when all of the elements being multiplied satisfy condition 2.

According to 3, the equation:

$$S_x S_b = S_c$$

can be solved for S_x only when (b) and (c) are sufficiently close to (0), and the solution is uniquely determined when one further assumes that (x) lies in a fixed, sufficiently small neighborhood of the system of values (0). In particular, we can solve:

$$S_x S_a = S_0 ,$$

and by equating:

$$S_a (S_x S_a) = (S_a S_x) S_a = S_a$$

with:

$$S_0 S_a = S_a ,$$

it then follows for sufficiently small a that:

$$S_x S_a = S_0 ,$$

due to the uniqueness of the solution S_y of:

$$S_y S_a = S_a .$$

There is therefore an *inverse* $S_x = S_a^{-1}$ with the property:

$$S_a^{-1} S_a = S_a S_a^{-1} = S_0 .$$

The equation:

$$S_a S_x = S_c$$

can then be uniquely solved for:

$$S_x = S_a^{-1} S_c ,$$

which one can easily conclude from the fact that:

$$\frac{\partial(\varphi_1, \varphi_2, \cdots, \varphi_r)}{\partial(b_1, b_2, \cdots, b_r)} \neq 0$$

for $(a) = (b) = (0)$.

In the sequel, we will assume for all of the elements with which we will be concerned that the *parameters* (= indices) lie in a sufficiently small neighborhood of $(a) = (0)$. Furthermore, we will simplify the phrase "group germ" to "group."

2. The invariant Pfaffian forms ω. Structure constants. Let S_{a+da} be an element that corresponds to the parameter values:

$$a_1 + da_1, a_2 + da_2, \ldots, a_r + da_r,$$

in which we understand $da_1, da_2, \ldots, da_r$ to mean that all of the computations will be regarded as infinitesimal quantities in which all of the products that they define of second and higher order will be ignored.

For the element:

$$S_c = S_a^{-1}\, S_{a+da}\,,$$

one then computes:

$$\varphi_\nu(a, c) = a_\nu + da_\nu \qquad (\nu = 1, 2, \ldots, r),$$

from which it follows tht:

$$c_\nu = \omega_\nu(a, da) \qquad (\nu = 1, 2, \ldots, r),$$

when one understands $\omega_\nu(a, da)$ to mean the linear form in the da that is defined by:

(1) $$da_\nu = \sum_{\mu=1}^{r} \left(\frac{\partial \varphi_\nu(a,c)}{\partial c_\mu}\right)_{c=0} \omega_\mu(a, da).$$

The expressions $\omega_\nu(a, da)$ are regarded as Pfaffian forms over the ring R of holomorphic functions of $a_1, a_2, \ldots, a_r$ at $(a) = (0)$ and play an important role in the theory of Lie groups.

Let $S_k = S_{k_1, k_2, \cdots, k_r}$ be an arbitrary element of the group. Due to the fact that:

$$S_a^{-1} S_{a+da} = (S_k\, S_a)^{-1} S_k\, S_{a+da}$$

and:

$$S_k\, S_a = S_{\varphi(k,\, a)}\,,$$

$$S_k\, S_{a+da} = S_{\varphi(k,\, a) + d\varphi(k,\, a)} \quad \left(d\varphi_\nu(k,a) = \sum_\mu \left(\frac{\partial \varphi_\nu(k,a)}{\partial a_\mu}\right) da_\mu\right),$$

we have:

$$\omega_\nu(a, da) = \omega_\nu(\varphi(k, a), d\varphi(k, a));$$

i.e., the Pfaffian forms:

$$\omega_1(a, da),\ \omega_2(a, da),\ \ldots,\ \omega_r(a, da)$$

remain invariant under the substitutions:

(2) $$a_\nu \to \varphi_\nu(k_1, \ldots, k_r; a_1, \ldots, a_r) \qquad (\nu = 1, 2, \ldots, r).$$

We thus determine all of the forms in the ring of differential forms over R that remain invariant under the latter substitution! Let:

$$\Phi = \sum_{i_1, i_2, \cdots, i_p}^{r} g_{i_1, i_2, \cdots, i_r}\, da_{i_1}\, da_{i_2} \cdots da_{i_p}$$

be such a differential form of degree p. From (1), one can also express it in the form:

(3) $$\Phi = \sum_{i_1<i_2<\cdots<i_p}^{r} h_{i_1,i_2,\cdots,i_p}\,\omega_{i_1}\omega_{i_2}\cdots\omega_{i_p}\,.$$

Due to the invariance and the linear independence of the product of the ω's, the coefficients $h_{i_1,i_2,\cdots,i_p}$ must remain separately invariant when such an expression undergoes the substitution (2). For each of these coefficients h, one thus has:

$$h(a_1, a_2, \ldots, a_r) = h(\varphi_1(k, a), \varphi_2(k, a), \ldots, \varphi_r(k, a)),$$

from which it follows for $(a) = (0)$ that:

$$h(0, 0, \ldots, 0) = h(k_1, k_2, \ldots, k_r)$$

when one regards $\varphi_\nu(k_\nu, 0)$; i.e., h is a constant.

With this, in order for a differential form Φ *to remain invariant under the substitution* (2) *it is necessary and sufficient that when it is expressed in the form* (3) *it has nothing but constant coefficients.*

In particular, the constant linear combinations of the ω:

$$h_1\omega_1 + h_2\omega_2 + \ldots + h_r\omega_r$$

are characterized as the Pfaffian forms that are invariant under (2).

Due to their invariance under the substitutions (2), the differential forms:

$$d\omega_1,\ d\omega_2,\ \ldots,\ d\omega_r$$

go to products $\omega_i\omega_k$ that are linear combinations with constant coefficients. We write:

(4) $$d\omega_\nu = \tfrac{1}{2}\sum_{\rho,\sigma} c_\nu^{\rho\sigma}\omega_\rho\omega_\sigma \qquad (c_\nu^{\rho\sigma} = -c_\nu^{\sigma\rho}).$$

For reasons that we shall explain, the constants $c_\nu^{\rho\sigma}$ that appear in this expression are called the *structure constants* of the group. They satisfy certain quadratic relations that one obtains by differentiating (4):

$$d\,d\omega_\nu = \tfrac{1}{2}\sum_{\rho,\sigma} c_\nu^{\rho\sigma}\,d\omega_\rho\,\omega_\sigma - \tfrac{1}{2}\sum_{\rho,\sigma} c_\nu^{\rho\sigma}\,\omega_\rho\,d\omega_\sigma$$

or:

$$\sum_{\rho} c_\nu^{\rho\sigma}\,d\omega_\rho\,\omega_\sigma = 0.$$

By taking (4) into account, one then has:

$$\tfrac{1}{2}\sum_{\rho,\alpha,\beta,\gamma=1}^{r} c_{\nu}^{\rho\sigma} c_{\rho}^{\alpha\beta} \omega_{\alpha}\omega_{\beta}\omega_{\gamma} = 0$$

or:

(5) $$\sum_{\rho=1}^{r}(c_{\rho}^{\alpha\beta}c_{\nu}^{\rho\gamma} + c_{\rho}^{\beta\gamma}c_{\nu}^{\rho\alpha} + c_{\rho}^{\gamma\alpha}c_{\nu}^{\rho\beta}) = 0 \qquad (\nu, \alpha, \beta, \gamma = 1, 2, \ldots, r).$$

3. Determination of multiplication formulas from the forms ω. *The substitutions* (2) *are characterized by the property that they leave the forms $\omega_1, \omega_2, \ldots, \omega_r$ invariant.*

Proof: Let $\bar{\omega}_1, \bar{\omega}_2, \cdots, \bar{\omega}_r$ be the same forms as $\omega_1, \omega_2, \ldots, \omega_r$, only with $\bar{a}_1, \cdots, \bar{a}_r$ ($d\bar{a}_1, \cdots, d\bar{a}_r$, resp.) written in place of $a_1, \ldots, a_r$ ($da_1, \ldots, da_r$, resp.).

The Pfaffian system:

(6) $$\bar{\omega}_{\nu} - \omega_{\nu} = 0 \qquad (\nu = 1, 2, \ldots, r)$$

in the $2r$ variables $a, \bar{a}$ is completely integrable, since:

$$d\bar{\omega}_{\nu} - d\omega_{\nu} = \tfrac{1}{2}\sum c_{\nu}^{\rho\sigma}(\bar{\omega}_{\rho}\bar{\omega}_{\sigma} - \omega_{\rho}\omega_{\sigma}) \equiv 0 \qquad (\text{mod } \bar{\omega}_1 - \omega_1, \ldots, \bar{\omega}_r - \omega_r).$$

The left-hand sides are linearly independent in the $d\bar{a}_1, \ldots, d\bar{a}_r$, and thus there is a solution of the form:

$$\bar{a}_{\nu} = f_{\nu}(a_1, \ldots, a_r) \qquad (\nu = 1, 2, \ldots, r)$$

that is holomorphic at the point $(a) = (0)$ and takes arbitrary (sufficiently small) values $k_1, \ldots, k_r$ for $(a) = (0)$. Since this solution is uniquely determined by the initial data, and, on the other hand, since the functions:

(7) $$\bar{a}_{\nu} = \varphi_{\nu}(k_1, k_2, \ldots, k_r; a_1, \ldots, a_r) \qquad (\nu = 1, 2, \ldots, r)$$

satisfy the differential equations (6) and the same initial conditions, they give the general solution of (6). Q.E.D.

4. Isomorphisms. Two r-parameter Lie groups G and $\bar{G}$ are called *isomorphic* when their elements:

$$S_a = S_{a_1, a_1, \cdots, \bar{a}_r} \qquad (T_{\bar{a}} = T_{\bar{a}_1, \bar{a}_1, \cdots, \bar{a}_r}, \text{ resp.})$$

can be related to each other with the help of a transformation that is biholomorphic in the neighborhood of $(a) = (0)$, $(\bar{a}) = (0)$:

$$a_{\nu} = g_{\nu}(\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_r) \qquad \leftrightarrow \qquad \bar{a}_{\nu} = \bar{g}_{\nu}(a_1, a_2, \ldots, a_r)$$

$$(g_\nu(0, 0, \ldots, 0) = 0, \ \nu = 1, 2, \ldots, r)$$

in such a way that from:

$$S_a S_b = S_c$$

one infers the relations:

$$T_{\bar{g}(a)} T_{\bar{g}(b)} = T_{\bar{g}(c)},$$

and conversely, from:

$$T_{\bar{a}} T_{\bar{b}} = T_{\bar{c}}$$

one infers that:

$$S_{g(\bar{a})} S_{g(\bar{b})} = S_{g(\bar{c})}.$$

If we let $\bar{\omega}_\nu(\bar{a}, d\bar{a})$ denote the Pfaffian forms that are defined by:

$$T_{\bar{a}}^{-1} T_{\bar{a}+d\bar{a}} = T_{\bar{\omega}(\bar{a}, d\bar{a})}$$

then from the fact that:

$$S_{g(\bar{a})}^{-1} S_{g(\bar{a})+dg(\bar{a})} = S_{g(\bar{\omega}(\bar{a}, d\bar{a}))}$$

we conclude that:

(8) $$\omega_\nu(g(\bar{a}), dg(\bar{a})) = \sum_{\mu=1}^{r} h_{\nu\mu} \bar{\omega}_\mu(\bar{a}, d\bar{a}) \qquad (\nu = 1, 2, \ldots, r).$$

The determinant that is formed from the constants $h_{\nu\mu}$ is non-zero, since it is equal to:

$$\left(\frac{\partial(g_1, g_2, \cdots, g_r)}{\partial(\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_r)} \right)_{\bar{a}=0}.$$

Due to the invariance of the differential ring operations, the Pfaffian forms on the left-hand side of (8) have the same derivatives:

$$d\omega_\nu = \tfrac{1}{2} \sum_{\rho,\sigma} c_\nu^{\rho\sigma} \omega_\rho \omega_\sigma$$

as the original ω's. One thus has:

$$d\left(\sum_\mu h_{\nu\mu} \bar{\omega}_\mu \right) = \tfrac{1}{2} \sum_{\rho,\sigma,\alpha,\beta} c_\nu^{\rho\sigma} h_{\rho\alpha} h_{\sigma\beta} \bar{\omega}_\alpha \bar{\omega}_\beta,$$

and from this it follows that:

$$d\bar{\omega}_\lambda = \tfrac{1}{2} \sum \bar{c}_\nu^{\alpha\beta} \bar{\omega}_\alpha \bar{\omega}_\beta,$$

with:

(9) $$\bar{c}_\lambda^{\alpha\beta} = \sum_{\nu,\rho,\sigma} c_\nu^{\rho\sigma} h_\lambda^\nu h_{\rho\alpha} h_{\sigma\beta},$$

in which (h_i^k) denotes the inverse matrix to (h_{ik}):

$$h_\lambda^\nu h_{\nu\mu} = \delta_{\lambda\mu}.$$

For isomorphic groups G and $\bar{G}$ *the structure constants* $c, \bar{c}$ *have the relation to each other that is expressed by* (9).

Conversely, the isomorphism of the groups G and $\bar{G}$ *follows from the existence of relation* (9).

Proof: From earlier remarks, one can obtain the multiplication formula in G:

$$S_a S_b = S_{\varphi(a,b)}$$

in such a way that the substitutions:

(10) $$a_\nu \to \varphi_\nu(k_1, \ldots, k_r, a_1, \ldots, a_r) \qquad (n = 1, 2, \ldots, r)$$

leave the $\omega_1, \omega_2, \ldots, \omega_r$ invariant and the system of values $(0, 0, \ldots, 0)$ goes to $(k_1, \ldots, k_r)$. Correspondingly, if $\bar{\varphi}$ are the functions that appear in:

$$T_{\bar{a}} T_{\bar{b}} = T_{\bar{\varphi}(\bar{a}, \bar{b})}$$

under the substitution:

$$\bar{a}_\nu \to \bar{\varphi}_\nu(\bar{k}_1, \cdots, \bar{k}_r, \bar{a}_1, \cdots, \bar{a}_r)$$

then the $\bar{\omega}_1, \bar{\omega}_2, \cdots, \bar{\omega}_r$, or – what amounts to the same thing – the forms:

$$\theta_\nu(\bar{a}, d\bar{a}) = \sum_{\mu=1}^{r} h_{\nu\mu} \bar{\omega}_\mu(\bar{a}, d\bar{a}) \qquad (\nu = 1, 2, \ldots, r)$$

are left unchanged, and the condition $\bar{\varphi}_\nu(\bar{k}, 0) = \bar{k}_\nu$ is satisfied.

Since one has, from (9), that:

$$d\theta_\nu = \tfrac{1}{2} \sum_{\rho,\sigma} c_\nu^{\rho\sigma} \theta_\rho \theta_\sigma,$$

the Pfaffian system:

$$\theta_\nu(\bar{a}, d\bar{a}) - \omega_\nu(a, da) = 0 \qquad (\nu = 1, 2, \ldots, r)$$

is completely integrable:

$$d(\theta_\nu - \omega_\nu) = \tfrac{1}{2} \sum_{\rho,\sigma} c_\nu^{\rho\sigma} (\theta_\rho \theta_\sigma - \omega_\rho \omega_\sigma) \equiv 0 \qquad (\text{mod } \theta_1 - \omega_1, \ldots, \theta_r - \omega_r).$$

There is therefore a solution:

(11) $$\bar{a}_\nu = \bar{g}_\nu(a_1, a_2, \ldots, a_r) \qquad (\nu = 1, 2, \ldots, r)$$

with:

$$\bar{g}_\nu(0, 0, \ldots, 0) = 0,$$

and this comes about by means of a biholomorphic transformation that takes the systems of forms:

$$\omega_1,\ \omega_2,\ \ldots,\ \omega_r; \qquad\qquad \theta_1,\ \theta_2,\ \ldots,\ \theta_r$$

into each other.

The substitutions that are defined by:

$$\bar{g}_\nu(a) \to \bar{\varphi}_\nu(\bar{g}(k), \bar{g}(a))$$

therefore leave the $\omega_1,\ \omega_2,\ \ldots,\ \omega_r$ invariant and take $\bar{g}_\nu(0)$ to:

$$\bar{\varphi}_\nu(\bar{g}(k), \bar{g}(0)) = \bar{\varphi}_\nu(\bar{g}(k), 0) = \bar{g}_\nu(k),$$

and therefore take (0, 0, …, 0) to (k_1, k_2, …, k_r). They thus represent the same substitution as (10), and it follows that:

$$\bar{\varphi}_\nu(\bar{g}(k), \bar{g}(a)) = \bar{g}_\nu(\varphi(k, a)).$$

In other words; from:

$$S_a\, S_b = S_{\varphi(a,b)}$$

it follows that:

$$T_{\bar{g}(k)} T_{\bar{g}(a)} = T_{\bar{g}(\varphi(k,a))}.$$

Due to the invertibility of the transformation (11), we also have the invertibility of this relation, and the isomorphism of the groups G and $\bar{G}$ is proved.

5. Determination of a group with given structure constants. Not every system of $\frac{1}{2} r^2 (r-1)$ constants $c_\nu^{\rho\sigma}$ can be occur as the system of structure constants of an r-parameter Lie group; in any case, relation (5) must be satisfied. However, the existence of this relation is, as we shall see, already sufficient for the existence of an associated Lie group.

If the $\frac{1}{2} r^2 (r-1)$ constants:

$$c_\nu^{\rho\sigma} \qquad\qquad (\nu, \rho, \sigma = 1, 2, \ldots, r,\ c_\nu^{\rho\sigma} = -c_\nu^{\sigma\rho})$$

satisfy the relations:

(12) $$\sum_{\rho=1}^{r} (c_\rho^{\alpha\beta} c_\nu^{\rho\gamma} + c_\rho^{\beta\gamma} c_\nu^{\rho\alpha} + c_\rho^{\gamma\alpha} c_\nu^{\rho\beta}) = 0 \qquad (\alpha, \beta, \gamma, \nu = 1, 2, \ldots, r)$$

then there are r linearly independent Pfaffian forms:

$$\omega_\nu = \sum_{\mu=1}^{r} b_{\nu\mu}(a_1, a_2, \dots, a_r)\, da_\mu$$

whose derivatives are:

(13) $$d\omega_\nu = \tfrac{1}{2}\sum_{\rho,\sigma} c_\nu^{\rho\sigma}\omega_\rho\omega_\sigma .$$

Proof: If one starts with:

$$\omega_\nu = \sum_{\mu=1}^{r} b_{\nu\mu}\, da_\mu$$

in (13) then one obtains:

$$\theta_\lambda = \sum_\beta db_{\lambda\beta} da_\beta - \tfrac{1}{2}\sum_{\rho,\sigma,\alpha,\beta} c_\lambda^{\rho\sigma} b_{\rho\alpha} b_{\sigma\beta} da_\alpha da_\beta = 0 \qquad (\lambda = 1, 2, \dots, r).$$

We regard this as a system of r differential equations of degree two in the $r + r^2$ variables a, b. We must then concern ourselves with finding r-dimensional integral manifolds for this system that can be represented in the form:

$$b_{\alpha\beta} = f_{\alpha\beta}(a_1, a_2, \dots, a_r).$$

If one takes relations (12) into account then one easily computes that:

$$d\theta_\alpha \equiv 0 \qquad (\text{mod } \theta_1, \theta_2, \dots, \theta_r).$$

The ideal $\mathfrak{a}$ that is associated with the system:

$$\theta_1 = 0,\ \theta_2 = 0,\ \dots,\ \theta_r = 0$$

thus has $\theta_1, \theta_2, \dots, \theta_r$ as a basis already.

From the general procedures (cf. pp. 45-55) we make the Ansatz:

$$db_{\lambda\beta} = \sum_\alpha l_{\lambda\beta\alpha}\, da_\alpha \qquad (\lambda, \beta = 1, 2, \dots, r)$$

and substitute this in the forms θ_λ. We then get the following expressions for the coefficients of $d(a_\alpha, a_\beta)$ (up to sign):

$$L_{\lambda\alpha\beta} = l_{\lambda\alpha\beta} - l_{\lambda\beta\alpha} - \tfrac{1}{2}\sum_{\rho,\sigma} c_\lambda^{\rho\sigma}(b_{\rho\alpha}b_{\alpha\beta} - b_{\rho\beta}b_{\sigma\alpha})$$
$$(L_{\lambda\alpha\beta} = -L_{\lambda\beta\alpha}).$$

If one therefore seeks the r-dimensional integral elements, on which $da_1, da_2, \dots, da_r$ are linearly independent and whose intersections with:

$$da_{\nu+1} = da_{\nu+2} = \dots = da_r = 0$$

will be regular integral-E_ν, then the equations:

$$L_{\lambda\alpha\beta} = 0 \qquad \begin{pmatrix} \lambda = 1, 2, \cdots, r \\ \alpha < \beta \leq \nu \end{pmatrix} \tag{14}$$

define a basis for the equations that we previously denoted by:

$$\mathfrak{a}_\nu(x, l_1, l_2, \ldots, l_\nu) = 0.$$

The quantities $l_{\lambda\alpha\beta}$ with $\beta = \nu$ take over the role of l_ν.
Equations (14) are soluble in terms of the quantities:

$$l_{\lambda\alpha\beta} \quad \text{with } \alpha < \nu,$$

and the totality of equations (14), when taken for $\nu = 2, 3, \ldots, r$, succeed in expressing all of these $l_{\lambda\alpha\beta}$ ($\alpha < \nu$) in terms of the remaining ones $l_{\lambda\alpha\beta}$ ($\alpha \geq \nu$).

From this, one speaks of the quantities $l_{\lambda\alpha\beta}$ with $\alpha < \nu$ as *principal*, and the other ones as *parametric* derivatives. All of the assumptions in the existence theorems are satisfied, as well as the assumption that along with:

$$l_{\lambda\alpha\beta} = \frac{\partial b_{\lambda\alpha}}{\partial a_\beta} \qquad (\alpha \geq \beta)$$

all of the preceding derivatives $l_{\lambda\alpha 1}, l_{\lambda\alpha 2}, \ldots, l_{\lambda\alpha,\beta-1}$ shall also be parametric, and we conclude the existence of solutions:

$$b_{\lambda\alpha} = f_{\lambda\alpha}(a_1, a_2, \ldots, a_r) \qquad (\lambda, \alpha = 1, 2, \ldots, r),$$

in which the section:

$$f_{\lambda\alpha}(a_1, a_2, \ldots, a_\alpha, 0, 0, \ldots, 0)$$

can be arbitrarily prescribed. In any case, one can therefore reach the conclusion that the Pfaffian forms:

$$\omega_\lambda = \sum_{\alpha=1}^{r} f_{\lambda\alpha}(a_1, a_2, \ldots, a_r)\, da_\alpha$$

that one constructs in the neighborhood of $(a) = (0)$ are linearly independent. Q.E.D.

Once the Pfaffian forms have been determined, by integrating a completely integrable system (cf. sec. 3), one then computes the substitutions:

$$a_\nu \to \varphi_\nu(k, a)$$

that leave $\omega_1, \omega_2, \ldots, \omega_r$ invariant and take $(a) = (0)$ to $(a) = (k)$. Then, by way of:

$$S_a\, S_b = S_{\varphi(a,\, b)}$$

one will state the multiplication laws of a Lie group whose structure constants will be precisely the given $c_\nu^{\rho\sigma}$. For two arbitrary systems of values:

$$(k_1, k_2, \ldots, k_r),\ (l_1, l_2, \ldots, l_r)$$

that are sufficiently close to (0, 0, …, 0) one likewise has:

$$\varphi(k, \varphi(l, a)) = \varphi(\varphi(k, l), a),$$

because:

$$a_\nu \to \varphi_\nu(k, \varphi(l, a))$$

is the substitution that leaves $\omega_1, \omega_2, \ldots, \omega_r$ fixed and takes $(a) = (0)$ to $a_\nu = \varphi_\nu(k, \varphi(l, 0)) = \varphi_\nu(k, l)$. We therefore have associativity, and one immediately sees that the remaining characteristic properties of a group germ will also satisfy:

The $\frac{1}{2} r^2 (r-1)$ *constants:*

$$c_\nu^{\rho\sigma} \qquad (\nu, \rho, \sigma = 1, 2, \ldots, r,\ c_\nu^{\rho\sigma} = -c_\nu^{\sigma\rho})$$

are structure constants of an r-parameter Lie group when and only when the satisfy the relations:

$$\sum_{\rho=1}^{r} (c_\rho^{\alpha\beta} c_\nu^{\rho\gamma} + c_\rho^{\beta\gamma} c_\nu^{\rho\alpha} + c_\rho^{\gamma\alpha} c_\nu^{\rho\beta}) = 0 \qquad (\alpha, \beta, \gamma, \nu = 1, 2, \ldots, r).$$

6. Representation of a Lie group by substitutions or transformations. Let G be a Lie group with the parameters $a_1, a_2, \ldots, a_r$. Furthermore, let:

$$f_i(x, a) = f_i(x_1, x_2, \ldots, x_n, a_1, a_2, \ldots, a_r) \qquad (i = 1, 2, \ldots, n)$$

be analytical functions of x, a that are holomorphic in the neighborhood of, say:

$$(x) = (c), \qquad (a) = (0)$$

and reduce to:

$$f_i(x, 0) = x_i$$

for $(a) = (0)$.

When these functions satisfy the functional equations:

$$f_i(f_1(x, b), \ldots, f_n(x, b), a_1, a_2, \ldots, a_r) = f_i(x_1, \ldots, x_n, \varphi_1(a, b), \ldots, \varphi_r(a, b)) \tag{15}$$
$$(i = 1, 2, \ldots, n)$$

one says that they provide a *representation of G as a substitution group.* On the other hand, if they satisfy the relations:

(16) $$f_i(f_1(x, a), \ldots, f_n(x, a), b_1, b_2, \ldots, b_r) = f_i(x_1, \ldots, x_n, \varphi_1(a, b), \ldots, \varphi_r(a, b)) \qquad (i = 1, 2, \ldots, n)$$

then they provide a *representation of G as a transformation group.* In fact, the replacement:

(17) $$x_i \to f_i(x, a) \qquad (i = 1, 2, \ldots, n)$$

can be regarded as a substitution in the first case and, in the second case, a transformation to a group Γ that is homomorphic to G. Naturally, the existence of (16) and (17) is required only for sufficiently small (a), (b), and for a neighborhood of $(x) = (c)$. Strictly speaking, we are therefore dealing with only a germ of a substitution (transformation, resp.).

Due to the fact that:

$$\varphi(k, \varphi(l, a)) = \varphi(\varphi(k, l), a),$$

the substitution:

$$a_\nu \to \varphi_\nu(k, a)$$

provides a representation of G as a substitution group if the transformed variables are a_1, a_2, …, a_r. Likewise, by means of:

$$a_\nu \to \varphi_\nu(a, k)$$

we can define a representation of G as a transformation group. One calls the group thus defined the *parameter group* of G.

In the sequel, we will always regard the replacement:

(18) $$x_i \to f_i(x, a) \qquad (i = 1, 2, \ldots, n)$$

as a substitution, and briefly denote it by S_a, in which we observe that various elements of G: S_a, S_b, …, can be indexed quite well by the same substitution.

We denote the function that results from the holomorphic function $F(x_1, x_2, \ldots, x_n)$ at $(x) = (x_0)$ by an application of (18), namely:

$$F(f_1(x, a), f_2(x, a), \ldots, f_n(x, a))$$

by:

(19) $$F(x_1, x_2, \ldots, x_n)^{S_a}.$$

Let:

$$x_i + \sum_{\nu=1}^{r} a_\nu\, \xi_{\nu i}(x_1, x_2, \ldots, x_n) + \ldots$$

be the leading terms in the development of $f_i(x, a)$ as a power series in a. The development of (19) then leads to the following:

$$F(x_1, x_2, \ldots, x_n)^{S_a} = F(x_1, x_2, \ldots, x_n) + \sum_{\nu=1}^{r} a_\nu\, X_\nu F + \ldots$$

when one understands X_ν to mean the operator:

$$X_\nu = \sum_{i=1}^{n} \xi_{\nu i}(x_1, x_2, \ldots, x_n)\frac{\partial}{\partial x_i}.$$

These operators and their linear combinations:

$$\sum \lambda_\nu X_\nu$$

are called the *infinitesimal operators* of the group. One also speaks – if imprecisely – of infinitesimal substitutions (transformations, resp.).

7. Determination of a substitution group from its infinitesimal operators. From:

$$f_i(x, a) = x_i^{S_a} = x_i^{S_{a+da}(S_{a+da}^{-1}S_a)} = f_i(x, a + da)^{(S_a^{-1}S_{a+da})^{-1}}$$

and the readily apparent relation:

$$(S_a^{-1}S_{a+da})^{-1} = S_{-\omega(a,\,da)},$$

it follows that:

$$df_i(x, a) = \sum_k \frac{\partial f_i(x,a)}{\partial x_k}dx_k + \sum_k \frac{\partial f_i(x,a)}{\partial a_\nu}da_\nu = 0$$

when the Pfaffian forms:

$$\theta_i = dx_i + \sum_{\nu=1}^{r} \omega_\nu\,(a, da)\;\xi_{\nu i}(x_1, x_2, \ldots, x_n) \qquad (i = 1, 2, \ldots, n)$$

vanish. Otherwise, one deduces that:

$$df_i(x, a) \equiv 0 \qquad (\text{mod } \theta_1, \theta_2, \ldots, \theta_n).$$

Due to the fact that $f_i(x, 0) = x_i$, the differentials $df_i(x, a)$ are linearly independent, so one also has, conversely, that:

$$\theta_i \equiv 0 \qquad (\text{mod } df_1, df_2, \ldots, df_n),$$

from which it then follows that:

(20) $$\theta_i \equiv 0 \qquad (\text{mod } \theta_1, \theta_2, \ldots, \theta_n) \qquad (i = 1, 2, \ldots, n).$$

The Pfaffian system:

(21) $$dx_i + \sum_{\nu=1}^{r} \omega_\nu\,(a, da)\;\xi_{\nu i}(x_1, x_2, \ldots, x_n) = 0 \qquad (i = 1, 2, \ldots, n)$$

is thus completely integrable, and the functions:

$$f_i(x, a) = f_i(x_1, x_2, \ldots, x_n, a_1, a_2, \ldots, a_r) \qquad (i = 1, 2, \ldots, n)$$

are integrals that take the values x_i^0 (i =1, 2, …, n) *on the integral manifolds that go through the point* [1]) $x_1 = x_i^0, \ldots, x_n = x_n^0$, $a_1 = a_2 = \ldots = a_\nu = 0$.

This theorem makes it possible to determine the substitutions $f_i(x, a)$ when the infinitesimal operators and the $\omega_\nu(a, da)$ are given.

The problem of *finding all representations of a Lie group that is described by the Pfaffian forms* $\omega_\nu(a, da)$ (ν = 1, 2, …, r) *as a substitution group* is equivalent to the problem of determining all systems of operators:

$$X_\nu = \sum_{i=1}^{n} \xi_{\nu i}(x_1, x_2, \ldots, x_n)\frac{\partial}{\partial x_i} \qquad (i = 1, 2, \ldots, r)$$

in such a way that the Pfaff system that is constructed from these $\xi_{\nu i}$ is completely integrable. There are then n independent integrals:

$$f_i(x, a) = f_i(x_1, x_2, \ldots, x_n, a_1, a_2, \ldots, a_r) \qquad (i = 1, 2, \ldots, n)$$

that reduce to x_1^0 ($x_2^0, \ldots, x_n^0$, resp.) on the integral manifolds M that go through $(x) = (x^0)$, $(a) = (0)$. Since the Pfaff forms ω_ν remain invariant under the substitutions:

$$a_\nu \to \varphi_\nu(k, a) \qquad (i = 1, 2, \ldots, r),$$

the functions:

$$f_i(x_1, \ldots, x_n, \varphi_1(k, a), \ldots, \varphi_r(k, a)) \qquad (i = 1, 2, \ldots, n)$$

are also integrals of (21), and due to the fact that $\varphi_\nu(k, 0) = 0$ they reduce to:

(22) $$f_i(x_1^0, \ldots, x_n^0, k_1, k_2, \ldots, k_r) \qquad (i = 1, 2, \ldots, n)$$

on the integral manifold M. On the other hand, the:

$$f_i(f_1(k, a), \ldots, f_n(k, a), k_1, k_2, \ldots, k_r) \qquad (i = 1, 2, \ldots, n)$$

are integrals of (21) that take the same values (22) on M. One therefore has (cf. pp. 65):

[1]) We assume that $(x_1^0, \cdots, x_n^0)$ is a system of values that lies in a neighborhood of ($c_1, c_2, \ldots, c_n$).

$$f_i(f_1(k, a), \ldots, f_n(k, a), k_1, k_2, \ldots, k_r) = f_i(x_1, \ldots, x_n, \varphi_1(k, a), \ldots, \varphi_r(k, a))$$

$$(i = 1, 2, \ldots, n)$$

for an arbitrary system of values for the variables x, a, k in the neighborhood of $(x) = (c)$ ($(a) = (0)$, $(k) = (0)$, resp.), and these functional equations express the fact that a representation of G as a substitution group is given by:

$$x_i \to f_i(x, a).$$

In order to obtain the conditions for complete integrability of the system (21), we summarize the forms θ with the help of an arbitrary (holomorphic at $(x) = (c)$) function $F(x_1, \ldots, x_n)$ by the expressions:

$$\sum_{i=1}^{n} \theta_i \frac{\partial F}{\partial x_i} = dF + \sum_{\nu=1}^{r} \omega_\nu X_\nu F \, .$$

The complete integrability of (21) is then obviously equivalent to the condition:

$$d(dF + \sum_{\nu=1}^{r} \omega_\nu X_\nu F \,) \equiv 0 \qquad (\mathrm{mod}\ \theta_1, \theta_2, \ldots, \theta_n)$$

for arbitrary F.

From the identity:

$$dF + \sum \omega_\nu X_\nu F \equiv 0 \qquad (\mathrm{mod}\ \theta_1, \theta_2, \ldots, \theta_n) \tag{23}$$

and the observation that:

$$d\theta_i \equiv 0 \qquad (\mathrm{mod}\ \theta_1, \theta_2, \ldots, \theta_n),$$

differentiation gives the relation:

$$\sum d\omega_\nu X_\nu F - \sum \omega_\nu dX_\nu F \equiv 0 \quad (\mathrm{mod}\ \theta). \tag{24}$$

By applying (23) to $X_\nu F$ instead of F, one obtains:

$$dX_\nu F \equiv -\sum_{\mu} \omega_\mu X_\mu X_\nu F \qquad (\mathrm{mod}\ \theta),$$

such that (24) can also be written as:

$$\sum d\omega_\nu X_\nu F - \tfrac{1}{2} \sum_{\rho,\sigma} \omega_\rho \omega_\sigma \,(X_\rho X_\sigma F - X_\sigma X_\rho F) \equiv 0 \qquad (\mathrm{mod}\ \theta).$$

Only the differentials *da* appear in the left-hand side of this congruence, although a dx_i must necessarily appear in any differential form of the form:

$$\Omega_1 \theta_1 + \Omega_2 \theta_2 + \ldots + \Omega_n \theta_n \qquad \text{(the } \Omega \text{ are Pfaffian forms)}$$

that does not vanish identically. One thus has:

$$\sum d\omega_\nu X_\nu - \tfrac{1}{2} \sum_{\rho,\sigma} \omega_\rho \omega_\sigma \, (X_\rho \, X_\sigma) = 0, \tag{25}$$

in which $(X_\rho \, X_\sigma)$ means the well-known bracket operation:

$$(X_\rho \, X_\sigma) = X_\rho \, X_\sigma - X_\sigma X_\rho \, .$$

Due to the linear independence of the ω, by substituting:

$$d\omega_\nu = \tfrac{1}{2} \sum c_\nu^{\rho\sigma} \omega_\rho \omega_\sigma$$

in (25) one obtains the so-called composition formulas:

$$(X_\rho \, X_\sigma) = \sum_{\nu=1}^{r} c_\nu^{\rho\sigma} X_\nu \qquad (\rho, \sigma = 1, 2, \ldots, r).$$

This shows that:

The r operators:

$$X_\nu = \sum_{i=1}^{n} \xi_{\nu i}(x_1, x_2, \ldots, x_n) \frac{\partial}{\partial x_i} \qquad (i = 1, 2, \ldots, r)$$

correspond to a representation of G as a substitution group when and only when the composition formulas:

$$(X_\rho X_\sigma) = \sum_{\nu=1}^{r} c_\nu^{\upsilon\sigma} X_\nu \qquad (\rho, \sigma = 1, 2, \ldots, r) \tag{26}$$

are satisfied.

From this, the $c_\nu^{\rho\sigma}$ are the structure constants of the group G.

8. Linear dependence of infinitesimal operators. Let $s < r$ of the operators X_ν be linearly independent. Choose any s independent linear combinations:

$$Y_\mu = \sum_{\nu=1}^{r} h_{\mu\nu} X_\nu \qquad (\mu = 1, 2, \ldots, s)$$

(with constant h), and, conversely, let:

$$X_\nu = \sum_{\mu=1}^{s} g_{\nu\mu} Y_\nu \qquad (\nu = 1, 2, \ldots, r).$$

Correspondingly, if one sets:

$$(Y_\rho Y_\sigma) = \sum_{\mu=1}^{s} \gamma_\lambda^{\rho\sigma} Y_\lambda$$

$$\sum_{\nu=1}^{r} g_{\nu\mu}\ \omega_\nu(a, da) = \tau_\mu(a, da) \qquad (\mu = 1, 2, \ldots, s).$$

Then, from:

$$\sum d\omega_\nu \mathrm{X}_\nu - \tfrac{1}{2} \sum_{\rho,\sigma} \omega_\rho \omega_\sigma\ (X_\rho X_\sigma) = 0$$

it follows that:

$$\sum d\omega_\nu X_\nu - \tfrac{1}{2} \sum_{\rho,\sigma} \omega_\rho \omega_\sigma\ (X_\rho X_\sigma) = 0,$$

or:

$$\sum_{\lambda=1}^{S} (d\tau_\lambda - \tfrac{1}{2} \sum_{\rho,\sigma} \gamma_\lambda^{\rho\sigma} \tau_\rho \tau_\sigma)\, X_\lambda = 0,$$

which has the relations:

(27) $$d\tau_\lambda = \tfrac{1}{2} \sum \gamma_\lambda^{\rho\sigma} \tau_\rho \tau_\sigma \qquad (\lambda = 1, 2, ..., s)$$

as a consequence, due to the linear independence of the Y.

Thus, the system of s linearly independent Pfaff equations at $(a) = (0)$:

$$\tau_\lambda(a, da) = 0 \qquad (\lambda = 1, 2, \ldots, s)$$

is completely integrable. In addition to the s independent integrals of this system that are holomorphic at $(a) = (0)$:

$$b_\nu(a_1, a_2, \ldots, a_r) \qquad (\nu = 1, 2, \ldots, s),$$

which might possibly vanish for $(a) = (0)$, one chooses, when $s < r$, any $r - s$ more holomorphic functions that vanish at $(a) = (0)$:

$$b_\nu(a_1, a_2, \ldots, a_r) \qquad (\nu = s+1, \ldots, r),$$

in such a way that:

$$b_\nu = b_\nu(a_1, a_2, \ldots, a_r) \qquad (\nu = 1, 2, \ldots, r)$$

defines a biholomorphic transformation in the neighborhood of $(a) = (0)$. By introducing the variables b instead of the a, the forms:

$$\tau_\nu(a, da) = \pi_\nu(b, db)$$

depend upon only the differentials db_ν with $\nu \leq s$:

(28) $$\pi_\nu = \sum_{\mu=1}^{s} \psi_{\lambda\mu}\,(b_1, b_2, \ldots, b_r)\, db_\mu.$$

The equation:

(29) $$d\pi_\nu = \tfrac{1}{2}\sum \gamma_\lambda^{\rho\sigma}\pi_\rho\pi_\sigma \qquad (\lambda = 1, 2, ..., s),$$

which follows from (27), then shows, when one introduces (28) here, that one must have:

$$\frac{\partial \psi_{\lambda\mu}}{\partial b_\nu} = 0 \qquad (\nu = s+1, \ldots, r),$$

i.e., the forms $\pi_1, \pi_2, \ldots, \pi_s$ depend upon only the variables $b_1, b_2, \ldots b_s$.

By using the b instead of the a, the differential equations (21) take the simple form:

$$dx_i + \sum_{\nu=1}^{s} \pi_\nu\,(b, db)\,\eta_{\nu i}(x_1, x_2, \ldots, x_n)\,\frac{\partial}{\partial x_i} \qquad (i = 1, 2, \ldots, n),$$

in which we have set:

$$Y_\nu = \sum_{i=1}^{n} \eta_{\nu i}\,(x_1, x_2, \ldots, x_n)\,\frac{\partial}{\partial x_i}\,.$$

We see that the integrals:

$$f_i(x_1, x_2, \ldots, x_n, a_1, a_2, \ldots a_r)$$

can also be written in the form:

$$g_i(x_1, x_2, \ldots, x_n, b_1(a), b_2(a), \ldots b_r(a)),$$

such that when $s < r$ only s of the parameters in the substitution group:

(30) $$x_i \to f_i(x, a)$$

are essential. They can also be regarded as a representation of the s-parameter Lie group g with structure constants $\gamma_\lambda^{\rho\sigma}$ ($\nu, \rho, \sigma = 1, 2, \ldots, s$) that is determined by:

$$\pi_1, \pi_2, \ldots, \pi_s\,.$$

If the operators $X_1, X_2, \ldots, X_r$ are linearly independent then the substitution (30) is a one-to-one correspondence with the elements of G; one has a so-called *faithful representation* of G. Both parameter groups give faithful representations of G as substitution (transformation, resp.) groups.

9. The invariant Pfaffian form ϖ. Up till now, our considerations were based on the Pfaffian forms $\omega_1, \omega_2, \ldots, \omega_r$ that were introduced as the parameters of the infinitesimal substitution $S_a^{-1}S_{a+da}$. Instead of these forms, one can reach the same conclusions by starting with the forms:

(31) $$\varpi_1(a, da),\ \varpi_2(a, da),\ \dots,\ \varpi_r(a, da),$$

which are defined by:

$$S_{a+da}S_a^{-1} = S_{\varpi_1,\varpi_2,\cdots,\varpi_r}\,.$$

One then easily establishes that the forms (31) are invariant under the substitutions:

$$a_\nu \to \varphi_\nu(a_1, a_2, \dots a_r, k_1, k_2, \dots k_r) \qquad (\nu = 1, 2, \dots, r).$$

Likewise, their derivatives naturally have constant coefficients; however, in order to comprehend their relationship with the $\gamma_\lambda^{\rho\sigma}$, it is expedient to first interpret the considerations of sec. 7 in the foregoing case. One immediately infers that the functions:

$$g_i(x_1, x_2, \dots, x_n, a_1, a_2, \dots a_r) \qquad (i = 1, 2, \dots, n)$$

that correspond to the substitutions S_a^{-1}:

$$x_i \to g_i(x, a)$$

are integrals of the Pfaffian system:

(32) $$\vartheta_i = dx_i - \sum \varpi_\nu(a, da)\xi_{\nu i}(x_1, x_2, \dots, x_n) = 0 \qquad (i = 1, 2, \dots, n).$$

One obtains the functions $f_i(x, a)$ by determining those solutions:

$$x_i = f_i(x_1^0, x_2^0, \dots, x_n^0, a_1, a_2, \dots a_r) \qquad (i = 1, 2, \dots, n)$$

that reduce to $(x_1^0, x_2^0, \dots, x_n^0)$ for $(a) = (0)$. From the complete integrability of the system (32), one infers (just as one did for the derivatives of equations (25)) the relations:

(33) $$\sum d\varpi_\nu\, X_\nu + \tfrac{1}{2}\sum_{\rho,\sigma}\omega_\rho\omega_\sigma\,(X_\rho\, X_\sigma) = 0,$$

from which, by (26), one has:

$$\sum_{\nu=1}^{r}(d\varpi_\nu + \tfrac{1}{2}\sum_{\rho,\sigma}c_\nu^{\rho\sigma}\varpi_\rho\varpi_\sigma)\ X_\nu = 0.$$

These formulas are true for any representation of G; in particular, they are true for a faithful representation, for which the operators X_ν are certainly independent. One thus has:

(34) $$d\varpi_\nu = -\tfrac{1}{2}\sum_{\rho,\sigma}c_\nu^{\rho\sigma}\varpi_\rho\varpi_\sigma \qquad (\nu = 1, 2, \dots, r).$$

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