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Mechanics in non-Euclidian space forms

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It is known that the equations of motion that are valid in three-dimensional **Euclid**ian space can be adapted to an arbitrarily-large number of dimensions, and the admissibility of that adaptation will also be recognized immediately geometrically. In the context of that, the main problem that I posed for the present article was that of deriving the equations of motion that one gets when one drops the parallel axiom, in addition to the three-dimensionality of space, from the principles of mechanics. The result that was obtained can be given the following form: One determines the equations of motion for points in an (n + 1)-fold extended **Euclid**ian space form that belong to an n-fold extended spherical structure at the onset of the motion and are constrained to remain on it during the motion. The equations that are then obtained will then be true for an n-dimensional non-**Euclid**ian space form. For a finite space, the radius and all coordinates must be real, whereas for the **Lobachevski**an space form, the radius and one coordinate must be pure imaginary, while the remaining coordinates are real. The derivation of that result is very simple and was eased substantially for me by applying the coordinates of Herrn **Weierstrass** that I had employed before in my earlier papers, which also prompted me to write this article. The applicability of those coordinates is based upon the fact that the law that was posed above for mechanics is true for the purely geometric properties. Thus, its validity for mechanics would have to be expected from the onset, from the way that Herr **Lipschitz** and others inferred the form of the equations of motion from the form of the line element. The fact that I initially derived the law for three dimensions is based upon the complexity in the expressions that a higher number of dimensions necessitates.

The general equations of motion will be applied to a series of individual problems from potential theory and the motion of solid bodies. When one considers the fact that the theory of parallels is employed in mechanical investigations quite extensively, one would not expect to find the tremendous agreement in the results that is shown by the parts that have been explored. However, that similarity is in no way restricted to the examples that were cited; it emerges perhaps even more strongly in other branches of mechanics, e.g., hydrodynamics. Only those problems in which the principle of the center of mass plays an essential role demand special treatment and are excluded here. By contrast, the **Hamilton-Jacobi** method remains unchanged, even in its external form, when one assumes that every function of the coordinates will be homogeneous of degree zero by means of the equation that exists for it. The motion of a free point that is attracted to a fixed point according to an arbitrary function of distance, or which is attracted to two fixed points according to the law that corresponds to **Newton**'s law, the infinitely-small motion of a pendulum, the shortest line on the intersection of arbitrarily-many confocal second-degree structures, and some related problems do not offer the slightest difficulty. I have added some of them in which the motion always results in a two-fold or three-fold extended plane in the first two sections.

Perhaps I did not need to go into potential theory. Once Herr **Kronecker** proved for **Euclidian**ian space forms that the laws of potentials are only entirely-special cases of very general analytical laws, and once Herr **Schering** had communicated the proposition by which the known laws could be adapted to our space forms, I could not hope to add anything substantial to the theory. However, it seemed appropriate to me to first develop the geometric foundations of that theory briefly, secondly, to add some elementary theorems, and thirdly, to give the simple form that the formulas take on when one applies **Weierstrass** coordinates.

I went into the mechanics of rigid bodies all the more precisely. Part of that realm had already found a detailed treatment for three dimensions some time ago by Herrn **Lindemann** (Math. Ann., Bd. VII), while another part of it was examined for **Euclid**ian ("planar") space forms by Herrn **Scheefer** in his dissertation (Berlin, 1880). My treatment differs from both of those by the fact that it is broader in scope and from the latter one by also the fact that it places **Riemann**ian space forms at its focus. One can state with no exaggeration that the simplest and most beautiful laws in that subject are true for the finite space forms in particular, and that rigid motion can be understood completely for the other space forms only when it is understood for the finite ones.

At one point, **Lindemann**'s results seem to contradict my own. Whereas I give only one special type of infinitely-small motion besides the rotation around a line (namely, the motion that is "self-reciprocal"), **Lindemann** enumerated four of them, of which a second one occurred in the **Lobachevski**an space form. However, on the one hand, **Lindemann** did not wish to confine himself to real motions, and on the other hand, he did not wish to pose the condition that is posed differently here that the space form must agree with the **Euclid**ian one in an infinitely-small domain. He went one step further into generality than I did and considered all projective space forms with three dimensions and six-fold mobility. The treatment of such things is most naturally connected with the analytical treatment of our space forms, although the line element cannot be postulated for several of them.

As far as the geometric theorems that are employed in this article are concerned, some of them have not been published before, in general. I hope that the brief suggestions about them that can be given in the course of the investigation will suffice for their understanding, although I point out that I have published a summary and brief derivation of them recently on a special occasion ("Über die Nicht-*Euklid*ischen Raumformen von *n* Dimensionen," Braunsberg, *Huyes* Buchhandlung). I have already pointed out before that I call any structure in which a line lies completely whenever it has two points in common with it a "principle structure" or "plane."

§ 1. – Motion of a free point in three-dimensional spaces.

The concepts of mass, density, velocity, and force are independent of the infinitude of lines and of the parallel axiom for infinite lines. Therefore, they will suffer no alterations in the non-**Euclid**ian space forms. Likewise, their persistence will still be true, and the unit of force can be reduced to the units of time and mass in the known way. The measurement of force presents no theoretical difficulties, either. In addition, since infinitely-small regions in our space forms have the same properties as in **Euclid**ian geometry, the parallelograms of motions and forces will still be valid for our infinitely-small region.

We once more base our investigations on **Weierstrass** coordinates, but replace the symbols t, u, v, w with p, x, y, z, resp., and call time t. The derivatives of the coordinates with respect to time might be denoted by p', x', y', z'; p'', …, for brevity. We will then have the relations:

(1) $$\begin{cases} k^2 p^2 + x^2 + y^2 + z^2 = k^2, \\ k^2 p\, p' + x\, x' + y\, y' + z\, z' = 0, \\ k^2 p'^2 + x'^2 + y'^2 + z'^2 + k^2 p\, p'' + x\, x'' + y\, y'' + z\, z'' = 0, \end{cases}$$

in which k^2 denotes the reciprocal value of the curvature. One immediately gets the three equations for the velocity v :

(2) $$\begin{cases} v^2 = k^2 p'^2 + x'^2 + y'^2 + z'^2, \\ v^2 + k^2 p\, p'' + x\, x'' + y\, y'' + z\, z'' = 0, \\ \dfrac{1}{2}\dfrac{d(v^2)}{dt} = k^2 p' p'' + x'\, x'' + y'\, y'' + z'\, z''. \end{cases}$$

If no forces act upon the point initially then, from what was said, the point must move with uniform velocity along a straight line. One must then have:

$$\frac{d^2 p}{dt^2} = M\,\frac{dp}{dt} + N\,p, \quad \frac{d^2 x}{dt^2} = M\,\frac{dx}{dt} + N\,x, \ldots$$

The second of equations (2) implies that $N = -\,v^2 / k^2$ and the third one implies $M = 0$. Thus, the equations of motion for a point on which no forces act will be:

(3) $$\frac{d^2 p}{dt^2} = -\frac{v^2}{k^2}\,p, \quad \frac{d^2 x}{dt^2} = -\frac{v^2}{k^2}\,x, \quad \frac{d^2 y}{dt^2} = -\frac{v^2}{k^2}\,y, \quad \frac{d^2 z}{dt^2} = -\frac{v^2}{k^2}\,z,$$

where v denotes the constant velocity.

A force R might now act upon a point (p, x, y, z) at rest. We represent that force in the known way by a straight line segment and imagine that the line is traverse with uniform velocity in a unit of time. Let the coordinates of the point to which one then arrives after a time dt be $p + (P / k^2)\,dt$, $x + X\,dt$, $y + Y\,dy$, $z + Z\,dt$. That then implies the second equation (1):

(4) $$p\,P + x\,X + y\,Y + z\,Z = 0 .$$

Furthermore, one has:

(5) $$R^2 = \frac{P^2}{k^2} + X^2 + Y^2 + Z^2.$$

Since the velocity vanishes for $t = 0$, and with it, the first derivatives of the coordinates with respect to time, a known argument will yield that for $t = 0$, one must have the following equations of motion:

$$m k^2 \frac{d^2 p}{dt^2} = P, \quad m \frac{d^2 x}{dt^2} = X, \quad m \frac{d^2 y}{dt^2} = Y, \quad m \frac{d^2 z}{dt^2} = Z; \tag{6}$$

m means the mass of the point here.

The quantities *P*, *X*, *Y*, *Z* can be defined geometrically in yet another way. One lays a plane ε through the point of application of the force that is perpendicular to the direction of the force. If it intersects the planes $x = 0$, $y = 0$, $z = 0$ with the angles α, β, γ, resp., then $X = R \cos \alpha$, $Y = R \cos \beta$, $Z = R \cos \gamma$. If one of those planes is not intersected then the distance divided by *k* will enter in place of the angle. Moreover, if *e* is the distance from the plane ε to the origin of the coordinate system then $P = k R \sin e / k$ .Therefore, *P*, *X*, *Y*, *Z* might be referred to as the *components of the force*.

When several forces act upon a point at rest, their components must be added together on the left-hand side of (6). Likewise, the parallelogram of forces teaches one that the left-hand sides of (3) and (6) must be added together when a force (*P*, *X*, *Y*, *Z*) acts upon a moving point. We then get the equations of motion in the form:

$$\left\{ \begin{aligned} m k^2 \frac{d^2 p}{dt^2} &= P - m v^2 p, \\ m \frac{d^2 x}{dt^2} &= X - \frac{mv^2}{k^2} x, \\ m \frac{d^2 y}{dt^2} &= Y - \frac{mv^2}{k^2} y, \\ m \frac{d^2 z}{dt^2} &= Z \frac{mv^2}{k^2} z. \end{aligned} \right. \tag{7}$$

Here, *v* represents the (variable) velocity. As a result of the last equation in (2), one must add the following relation to equations (4) and (5):

$$\tfrac{1}{2} m \frac{d(v^2)}{dt} = P \frac{dp}{dt} + X \frac{dx}{dt} + Y \frac{dy}{dt} + Z \frac{dz}{dt}. \tag{8}$$

If the force *R* goes through a center of attraction or repulsion then another form for the equations of motion will occasionally be quite convenient. Let p_0, x_0, y_0, z_0 be the coordinates of the attracting point, and let *e* the distance from the attracting point. One then has the equations:

(9)
$$\left\{\begin{aligned} m\frac{d^2p}{dt^2} &= \frac{R\,p_0}{k\sin\frac{e}{k}} - L\,p\,, \\ m\frac{d^2x}{dt^2} &= \frac{R\,x_0}{k\sin\frac{e}{k}} - L\,x\,, \\ m\frac{d^2y}{dt^2} &= \frac{R\,y_0}{k\sin\frac{e}{k}} - L\,y\,, \\ m\frac{d^2z}{dt^2} &= \frac{R\,z_0}{k\sin\frac{e}{k}} - L\,z\,. \end{aligned}\right.$$

The function L satisfies the equations:

(10)
$$\left\{\begin{aligned} \tfrac{1}{2}m\frac{d(v^2)}{dt} &= \frac{R}{k\sin\frac{e}{k}}\left(k^2 p_0\frac{dp}{dt} + x_0\frac{dx}{dt} + y_0\frac{dy}{dt} + z_0\frac{dz}{dt}\right), \\ m v^2 &+ \frac{R\,k\cos\frac{e}{k}}{\sin\frac{e}{k}} - L\,k^2 = 0\,. \end{aligned}\right.$$

Equations (9) and (10) can either be easily derived directly or reduced to equations (7) and (8).

Example: Planetary motion.

In order to adapt **Newton**'s law of gravitation to non-**Euclid**ian space forms, one must not start from the algebraic form of it, but one must derive the corresponding analytical expression from the geometric concepts that lie at the basis for the law. If one imagines that several spherical surfaces are described around the attracting point as their centers and that in each case equal areas are assigned equal masses then the forces that are exerted on equal masses will be inversely proportional to the areas of the spheres. Now, the area of a sphere with radius r is equal to $4\pi k^2 \sin^2 (r / k)$ .When we then assume that the Sun us immobile and consider it, as well as the planets, to be points, and ignore the mutual attraction of the planets, we can give the following expression to the problem of planetary motion:

A point moves under the influence of a force that points away from a fixed center and is inversely proportional to the square of the sine of the distance from the fixed point, divided by k.

We choose the fixed point to be the origin of the coordinate system, and since the motion obviously takes place in a plane, that plane will be $z = 0$. When we then set $x^2 + y^2 = q^2$, equations (9) will take the form:

$$p'' = \frac{\mu}{q^3} - p\,L\,,$$

$$x'' = -\,x\,L\,,$$

$$y'' = -\,y\,L\,.$$

That directly implies the integrals:

$$(\alpha)\qquad \begin{cases} x\,y' - x'\,y = c\,, \\ v^2 = 2h + \dfrac{2\mu p}{q}\,, \end{cases}$$

in which c and h mean two constants. If we then couple the second equation (1) and the first equation (2) then it will follow that:

$$(\beta)\qquad k^4\,p^2 = 2\,h\,q^2 + 2\,\mu\,p\,q - c^2.$$

If we now get the value of L from the second equation in (10) and then calculate the value of $\frac{d^2q}{dt^2} = q$ from the equation $k^2\,p^2 + q^2 = k^2$ and its first two derivatives then we will arrive at the equations:

$$q\,x'' - q''\,x = -\frac{c^2 x}{q^3} = \frac{c^2}{\mu}\,(p\,x'' - p''\,x)\,,$$

$$q\,y'' - q''\,y = -\frac{c^2 y}{q^3} = \frac{c^2}{\mu}\,(p\,y'' - p''\,y)\,.$$

Those two equations integrate to the relations:

$$\mu\,c\,q = c^3\,p - a\,y + b\,x\,,$$
$$0 = c^2\,(p\,q' - q\,p')\,q + a\,x + b\,y\,.$$

One can make $a = 0$ by a suitable rotation of the coordinate system. The last two equations will then be:

$$(\gamma)\qquad \mu\,c\,q = c^3\,p + b\,x\,,$$

$$p' = \frac{b\,y}{c^2 k^2}\,.$$

Equation (γ) teaches us that **Kepler**'s first law remains unchanged.

In place of the quantities p and q, we introduce a new variable r by the equation:

$$q = r\,p\,.$$

Equation (β) will then go to:

$$(\delta) \qquad dt = \frac{r\,dr}{\left(1+\frac{r^2}{k^2}\right)\sqrt{r^2\left(2h-\frac{c^2}{k^2}\right)+2\mu r - c^2}}\,.$$

For $y = 0$, dp / dt will be equal to zero, and therefore also dr / dt . If we denote the corresponding values of r by r_1 and r_2 then:

$$r_1 + r_2 = -\frac{2\mu}{2h-\frac{c^2}{k^2}}\,, \quad r_1\,r_2 = -\frac{c^2}{2h-\frac{c^2}{k^2}}\,,$$

so

$$\frac{\mu}{h} = -\frac{k\,(r_1+r_2)}{k^2 - r_1\,r_2} = -k\tan\frac{2a}{k}\,,$$

when the principal diameter of the conic section is set to $2a$.

One-half the orbital period of the planet will be obtained when one integrates equation (δ) between the limits r_1 and r_2 . If one performs that integration then one will get the following equation for the orbital period T :

$$T = \frac{\pi\,\mu}{(-h)^{3/2}\sqrt{\left(1+\frac{\mu^2}{h^2k^2}\right)\left(1+\sqrt{1+\frac{\mu^2}{h^2k^2}}\right)}}\,,$$

or

$$(\varepsilon) \qquad T^2 = \frac{4\pi^2 k^3 \sin^3\frac{a}{k}\cos\frac{a}{k}}{\mu}\,,$$

such that **Kepler**'s third law is altered slightly. From equation (α), one must give **Kepler**'s second law the following form: If one lengthens the radius vector beyond the planets to itself then twice the radius will describe equal areas in equal times.

That law – viz., the law of areas – is obviously true for any free central motion.

§ 2. – Motion on a surface.

If a point is constrained to move on a surface without friction then it is known that this condition can be replaced with a force that acts in the direction of the normal and whose magnitude is determined from the fact that the point remains on the surface during the motion.

If $\varphi(p, x, y, z) = 0$ is the equation of the surface then the equation of the tangent plane that is laid through the point (p, x, y, z) has the coefficients:

$$\frac{1}{k^2}\frac{\partial\varphi}{\partial p} - p\left(p\frac{\partial\varphi}{\partial p} + x\frac{\partial\varphi}{\partial x} + y\frac{\partial\varphi}{\partial y} + z\frac{\partial\varphi}{\partial z}\right),$$

$$\frac{\partial\varphi}{\partial x} - x\left(p\frac{\partial\varphi}{\partial p} + x\frac{\partial\varphi}{\partial x} + y\frac{\partial\varphi}{\partial y} + z\frac{\partial\varphi}{\partial z}\right),$$

$$\frac{\partial\varphi}{\partial y} - y\left(p\frac{\partial\varphi}{\partial p} + x\frac{\partial\varphi}{\partial x} + y\frac{\partial\varphi}{\partial y} + z\frac{\partial\varphi}{\partial z}\right),$$

$$\frac{\partial\varphi}{\partial z} - z\left(p\frac{\partial\varphi}{\partial p} + x\frac{\partial\varphi}{\partial x} + y\frac{\partial\varphi}{\partial y} + z\frac{\partial\varphi}{\partial z}\right).$$

Those four quantities are then proportional to the components of the force, which correspond to the given condition and which must be added to the given forces using the parallelogram of forces. If we denote the resultant of the given forces by (P, X, Y, Z) then we will get the equation of motion:

$$(11)\qquad \begin{cases} m k^2 \dfrac{d^2 p}{dt^2} = P - k^2 S\, p + M\dfrac{\partial\varphi}{\partial p}, \\ m\dfrac{d^2 x}{dt^2} = X - S\, x + M\dfrac{\partial\varphi}{\partial x}, \\ m\dfrac{d^2 y}{dt^2} = Y - S\, y + M\dfrac{\partial\varphi}{\partial y}, \\ m\dfrac{d^2 z}{dt^2} = Z - S\, z + M\dfrac{\partial\varphi}{\partial z}. \end{cases}$$

The velocity v will, in turn, be calculated from equation (8). Moreover, one has $k^2 S = v^2$ when φ is given in homogeneous form such that $p\frac{\partial\varphi}{\partial p} + x\frac{\partial\varphi}{\partial x} + y\frac{\partial\varphi}{\partial y} + z\frac{\partial\varphi}{\partial z}$ also vanishes along with φ.

Should the point remain at rest, then the quantities $\frac{d^2p}{dt^2}$, $\frac{d^2x}{dt^2}$, $\frac{d^2y}{dt^2}$, $\frac{d^2z}{dt^2}$, and S would have to be equal zero.

Example. The pendulum.

A point that is constrained to remain on a spherical surface is acted upon by a constant force that points away from a (real or imaginary) fixed point.

The equations will become simple when we choose the center of the sphere to be the point (1, 0, 0, 0) and lay the axis $y = z = 0$ through the attracting point, whose other coordinates might be p_0 and x_0. Let the constant force be g, let the radius of the sphere be l, let the distance from the center to the attracting point be a, and let the variable distance from the moving point to the attracting point be e. The equations of motion are then:

$$\frac{d^2 p}{dt^2} = \frac{g}{k \sin\frac{e}{k}} p_0 + L\,p + \frac{1}{k^2} M,$$

$$\frac{d^2 x}{dt^2} = \frac{g}{k \sin\frac{e}{k}} x_0 + L\,x,$$

$$\frac{d^2 y}{dt^2} = \qquad L\,y,$$

$$\frac{d^2 z}{dt^2} = \qquad L\,z.$$

Since p is constant, we can overlook the first equation entirely. It follows from the last three that:

$$v^2 = 2g\,(h - e)\,,$$

where h refers to a constant. If we then couple that with the law of areas then the quadratures that solve the problem in full generality are easy to give. Meanwhile, only the infinitely-small motions are especially significant, and for them we can assume that x and e are constant, up to infinitely-small quantities or order two. We will then have:

$$e = a - l, \quad x_0 = k \sin\frac{a}{k}, \quad x = k \sin\frac{l}{k}, \quad L = -\frac{g \sin\frac{a}{k}}{k \sin\frac{a-l}{k} \sin\frac{l}{k}},$$

and that constant value of L will give the value of $\frac{d^2y}{dt^2} : y$, as well as that of $\frac{d^2z}{dt^2} : z$. As is known, it follows from this that the motions are periodic and that the period of oscillation is equal to:

$$2\pi \cdot \sqrt{\frac{k \sin\frac{l}{k}}{g} \frac{\sin\frac{a-l}{k}}{\sin\frac{a}{k}}}\,.$$

The same result can be derived from the equation for v^2 for planar oscillations when one expresses e and h in terms of the angle of deflection and develop the expression for $h - e$ using **Tayler**'s theorem. The isochronism of small oscillations is then true for non-**Euclid**ian space forms, as well.

§ 3. – The general equations of motion in triply-extended space.

The equations of motion that were derived in the first section can be adapted to arbitrarily-many points immediately when each of then can move freely. Likewise, one can assume from the equations of the second section that the surface on which the point should remain will itself move in an arbitrarily-prescribed way. If we assume that the equation of the surface:

$$\varphi = 0$$

is homogeneous in the coordinates and that the coefficients include time then we will also have:

$$mv^2 = k^2 S$$

now, but v itself cannot be calculated in the given way. If we also assume that the conditions that exist between the moving points can be expressed by a sequence of equations in their coordinates and time and that those equations represent the effects that were given in the previous section then we will get the equations of motion that correspond to the first **Lagrangian** form. If the equations of constraint are:

(12) $$\varphi = 0, \quad \psi = 0, \ldots$$

then the equations of motion are:

(13) $$\begin{cases} m_\iota k^2 \dfrac{d^2 p_\iota}{dt^2} = P_\iota - k^2 S_\iota\, p_\iota + M \dfrac{\partial \varphi}{\partial p_\iota} + N \dfrac{\partial \psi}{\partial p_\iota} + \cdots, \\ m_\iota \dfrac{d^2 x_\iota}{dt^2} = X_\iota - S_\iota\, p_\iota + M \dfrac{\partial \varphi}{\partial x_\iota} + N \dfrac{\partial \psi}{\partial x_\iota} + \cdots, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{cases}$$

in which the equations for $\frac{d^2 y_\iota}{dt^2}$ and $\frac{d^2 z_\iota}{dt^2}$ are analogous to the one that was exhibited for x_ι . Here, P_ι, X_ι, Y_ι, Z_ι are given as components of the resultant of all of the forces that act upon the point (p_ι, …). The functions S_ι, M, N, … are calculated from the equations of constraint (12) and the magnitude of the velocity, while one must, in turn, consider equations (1).

If one now introduces virtual displacements: δp, δx, δy, δz, for which, in addition to the r equations:

(14) $$k^2\, p_\iota\, \delta p_\iota + x_\iota\, \delta x_\iota + y_\iota\, \delta y_\iota + z_\iota\, \delta z_\iota = 0 ,$$

one also has the equations:

(15) $$\sum \left(\frac{\partial \varphi}{\partial p} \delta p + \frac{\partial \varphi}{\partial x} \delta x + \frac{\partial \varphi}{\partial y} \delta y + \frac{\partial \varphi}{\partial z} \delta z \right) = 0,$$

then one will immediately derive **d’Alembert**’s principle from equations (13):

(16) $$\sum\left[\left(k^2 m\frac{d^2p}{dt^2}-P\right)\delta p+\left(m\frac{d^2x}{dt^2}-X\right)\delta x+\left(m\frac{d^2y}{dt^2}-Y\right)\delta y+\left(m\frac{d^2z}{dt^2}-Z\right)\delta z\right]=0.$$

In order to arrive at **Hamilton**'s principle one defines the work done by a force under any displacement, precisely as in **Euclid**ian geometry, to be the product of the force and the displacement with the cosines of the angle they subtend, and refers to the sum of those products for all points as the *work done* on the system. The analytical expression for the work done by a force is then:

(17) $$P\,\delta p + X\,\delta x + Y\,\delta y + Z\,\delta z\,,$$

and the total work U' will be obtained upon summing over all points. One likewise refers to the expression:

(18) $$\tfrac{1}{2}\sum m v^2 = T$$

as the *vis viva* of the system. **Hamilton**'s principle:

(19) $$\int(\delta T + U')\,dt = 0$$

is then true in the same sense and to the same extent as in **Euclid**ian space forms.

It hardly needs to be mentioned that one can also derive equations (13) and (16) from (19) here.

We do not need to go into the details of **Gauss**'s principle of least constraint. It would suffice to refer to a treatise by **Lipschitz** in this journal (Bd. 82, pp. 316).

When a potential exists, i.e., when the work done is $U' = \delta U$, one will have:

(20) $$P=\frac{\partial U}{\partial p}-k^2 E\,p,\quad X=\frac{\partial U}{\partial x}-E\,x,\quad Y=\frac{\partial U}{\partial y}-E\,y,\quad Z=\frac{\partial U}{\partial z}-E\,z$$

for every point, where E is determined by means of equation (4). If one makes U homogeneous of degree zero then E will be equal to zero.

One gets several theorems from the general equations of motion in **Euclid**ian space that are true for numerous groups of motions, such as the law of *vis viva*, the law of the center of mass, and the law of areas. The first one does not change at all for non-**Euclid**ian space forms and the law of the center of mass loses its validity, but we obtain six laws of area. I would like to go into that in some detail. One begins with the equations:

(21) $$\sum\left(p\frac{d^2x}{dt^2}-x\frac{d^2p}{dt^2}\right)=\sum\left(p\,X-\frac{xp}{k^2}\right),\quad \sum\left(y\frac{d^2z}{dt^2}-z\frac{d^2y}{dt^2}\right)=\sum(y\,Z-z\,Y),\ldots$$

One can arrive at these six equations by means of **d'Alembert**'s principle when it is possible to displace the system along each coordinate axis and rotate about it. In that regard, one has the theorem:

If a system can be rotated around two straight lines that intersect then it can be rotated around every line that goes through the point of intersection, and when it can be rotated around a line, in addition, it can be rotated around every line and displaced along it. When a system can be displaced along two lines that lie in a plane, it can be displaced along every line that lies in that plane, and if that possibility is true for yet another line then the possibility of displacement and rotation will be true in full generality

From that, one can easily predict how many areal theorems will be true for a given system. As far as the geometric meaning is concerned, one has:

$$pX - \frac{xP}{k^2} = R\cos\frac{a}{k}\cos\varphi,$$
$$yZ - zY = Rk\sin\frac{a}{k}\sin\varphi,$$

when R means the magnitude of the force, φ is the angle between the direction of the force and the $y = z = 0$ axis, and a is the distance between the two directions. That will imply the meaning of the integrals on the left-hand sides of (21).

The expressions on the right-hand sides of equations (21) will not change when any force is displaced arbitrarily along its direction. Therefore, when no equations of constraint exist, only internal forces will act upon the system, and any force will be equal to its opposite, so the right-hand sides will be equal to zero, and one will get six integral equations from (21).

One will get the conditions for equilibrium in three different forms when one sets the derivatives of the coordinates equal to zero in equations (13), (16), and (19). It is not necessary to write out the equations. Rather, I would only like to recall the facts that **Dirichlet**'s way of characterizing the stability of equilibrium is also true for non-**Euclidian** space forms, and that the conditions for the equilibrium of an inextensible string lead to very simple equations. When a string is not constrained to remain on a surface, the direction of the force at every point will lie in the osculating plane to the curve.

§ 4. – Motion in *n*-fold extended space forms. The Hamilton-Jacobi method.

Since the geometric concepts that were employed in the foregoing section were completely independent of the number of dimensions, the results obtained can be adapted immediately to an arbitrary number of dimensions. Hence, when we now assume that the motion takes place in an n-fold extended space with constant **Riemann**ian curvature $1 / k^2$, we will determine the position of every point by $n + 1$ coordinates $x_0, x_1, \ldots, x_n$, between which, one has the equation:

(22) $$k^2 x_0^2 + x_1^2 + \cdots + x_n^2 = k^2.$$

If $X_0, X_1, \ldots, X_n$ are the components of the force R that acts upon the point $(x_0, x_1, \ldots, x_n)$ then one will have:

(23)
$$X_0 x_0 + X_1 x_1 + \ldots + X_n x_n = 0\,,$$
$$R^2 = \frac{X_0^2}{k^2} + X_1^2 + \cdots + X_n^2.$$

The $X_0, X_1, \ldots, X_n$ have the corresponding meanings that were given above, namely, that X_ι is equal to the product of R with the cosine of the angle that the $(n-1)$-dimensional plane that is constructed at the point of application and perpendicular to the direction of the force makes with the plane $x_\iota = 0$. X_0 is, in turn, equal to $k\,R \sin e / k$, where e denotes the distance from point (1, 0, 0, …, 0) to the aforementioned plane.

The moving system might consist of r points with the masses m_ι and the coordinates $x_0^{(\iota)}, x_1^{(\iota)}, \ldots x_n^{(\iota)}$. The forces $X_0^{(\iota)}, X_1^{(\iota)}, \ldots X_n^{(\iota)}$ might act upon them, and the conditions $\varphi = 0$, $\psi = 0$, … might exist between them. Corresponding to equations (13), the first **Lagrange** equations will then be:

(24)
$$\left\{\begin{aligned} m_\iota k^2 \frac{d^2 x_0^{(\iota)}}{dt^2} &= X_0^{(\iota)} - k^2 S_\iota x_0^{(\iota)} + M \frac{\partial\varphi}{\partial x_0^{(\iota)}} + N \frac{\partial\psi}{\partial x_0^{(\iota)}} + \cdots, \\ m_\iota \frac{d^2 x_\kappa^{(\iota)}}{dt^2} &= X_\kappa^{(\iota)} - k^2 S_\iota x_\kappa^{(\iota)} + M \frac{\partial\varphi}{\partial x_\kappa^{(\iota)}} + N \frac{\partial\psi}{\partial x_\kappa^{(\iota)}} + \cdots, \end{aligned}\right.$$

$$(\iota = 1, \ldots, r\,;\ \kappa = 1, \ldots, n)\,.$$

The virtual displacements $\delta x_0, \ldots, \delta x_n$ must satisfy equation (22), as well as the equations $\varphi = 0$, $\psi = 0$, …, when time is considered to be constant in them. One then has **d' Alembert**'s principle:

(25)
$$\sum\left[\left(m k^2 \frac{d^2 x_0}{dt^2} - X_0\right)\delta x_0 + \left(m \frac{d^2 x_1}{dt^2} - X_1\right)\delta x_1 + \cdots + \left(m \frac{d^2 x_n}{dt^2} - X_n\right)\delta x_n\right] = 0.$$

The definition of work does not change, and its expression will become:

$$\sum (k^2 X_0\, \delta x_0 + X_1\, \delta x_1 + \ldots + X_n\, \delta x_n)\,.$$

Hamilton's principle does not change in any way. If the forces have a potential U then, with **Hamilton**, one sets:

(26)
$$T - U = H$$

and expresses all of the coordinates in terms of a sequence of independent quantities q_ι that will all satisfy the equations of constrain identically when one substitutes them in those equations. T is then a function of q_ι and $q'_\iota = dq_\iota / dt$. One sets $\partial T / \partial q'_\iota = p_\iota$ and expresses H in terms of q_ι and p_ι. **Hamilton**'s equations of motion are now:

(27) $$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}.$$

It needs no proof that the methods of **Hamilton** and **Jacobi** can also be adapted to non-**Euclid**ian space forms. Only their relationship with the **Weierstrass** coordinates requires a more detailed discussion. We will show later that those methods are also quite appropriate for the solution of problems here in some examples.

Jacobi based his method of the last multiplier upon the fact that the multiplier for a given system of differential equations can be determined before one does any integration. It is equal to an arbitrary constant and can then be set equal to unity when the point is free, as well as when one uses **Hamilton**'s equations as a basis. However, if one determines the motion in a **Euclid**ian space from the first **Lagrang**ian form and defines the quantities $(\varphi\varphi)$, $(\varphi\psi)$, … from the equations of constraint $\varphi = 0$, $\psi = 0$, … according to the rule:

$$(\varphi\psi) = \sum\left(\frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial y} + \frac{\partial\varphi}{\partial z}\frac{\partial\psi}{\partial z}\right),$$

then the multiplier of the system will be equal to the determinant:

(28) $$\begin{vmatrix} (\varphi\varphi) & (\varphi\psi) & \cdots \\ (\psi\varphi) & (\psi\psi) & \cdots \\ \cdots & \cdots & \ddots \end{vmatrix}.$$

Now, **Jacobi**'s investigations can be adapted to a **Euclid**ian space form of arbitrarily many dimensions, as he himself often emphasized. However, our general equations of motion can be summarized analytically in the following way: A **Euclid**ian space form of $n + 1$ dimensions can be based upon the rectangular coordinates $k\,x_0, x_1, \ldots, x_n$. Let the rectangular components of each force be $X_0 / k, X_1, \ldots, X_n$. In addition to the equations of constraint $\varphi = 0$, $\psi = 0$, … the condition (22) must be true for the coordinates of each point, which we would like to write briefly as:

$$\Omega_\iota = k^2.$$

Under that assumption, the equations of motion have the same form in $(n + 1)$-fold extended **Euclid**ian space form that they have in the n-fold extended non-**Euclid**ian space form whose reciprocal curvature is equal to k^2. Indeed, equation (23) says that the direction of every force belongs to the structure $\Omega = k^2$, but that condition is analytically unnecessary, since one requires only the simplest values of S_ι . It makes just as little difference in the purely-analytical treatment that k^2 is negative and every $k\,x_0$ and X_0 / k is imaginary for the **Lobachevski** space form.

It does not need to be mentioned that the multiplier of the system (27) of **Hamilton**'s equations can be set equal to unity. However, when one starts from the first **Lagrang**ian form, one will have to construct the expressions:

$$(\varphi\,\varphi),\ (\varphi\,\psi),\ \ldots, \qquad (\Omega_\iota\,\Omega_\iota),\quad (\Omega_\iota\,\Omega_\kappa),\quad (\Omega_\iota\,\varphi),\ \ldots$$

according to the rule:

$$(29)\qquad \begin{cases} (\varphi\varphi) = \sum \dfrac{1}{m}\left(\dfrac{1}{k^2}\dfrac{\partial\varphi}{\partial x_0}\dfrac{\partial\varphi}{\partial x_0} + \dfrac{\partial\varphi}{\partial x_1}\dfrac{\partial\varphi}{\partial x_1} + \cdots + \dfrac{\partial\varphi}{\partial x_n}\dfrac{\partial\varphi}{\partial x_n}\right), \\ (\varphi\psi) = (\psi\,\varphi) = \sum \dfrac{1}{m}\left(\dfrac{1}{k^2}\dfrac{\partial\varphi}{\partial x_0}\dfrac{\partial\psi}{\partial x_0} + \dfrac{\partial\varphi}{\partial x_1}\dfrac{\partial\psi}{\partial x_1} + \cdots + \dfrac{\partial\varphi}{\partial x_n}\dfrac{\partial\psi}{\partial x_n}\right), \\ \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{cases}$$

One will then have:

$$(\Omega_\iota\,\Omega_\kappa) = \begin{cases} \dfrac{k^2}{m_\iota} & \text{for} \quad \iota = \kappa, \\ 0 & \text{for} \quad \iota \neq \kappa, \end{cases}$$

and

$$(\Omega_\iota\,\varphi) = \frac{1}{m_\iota}\left(x_0^{(\iota)}\frac{\partial\varphi}{\partial x_0^{(\iota)}} + \cdots + x_n^{(\iota)}\frac{\partial\varphi}{\partial x_n^{(\iota)}}\right).$$

Thus, when the motion in the non-**Euclid**ian space form is initially free, the multiplier of the system of **Lagrange** equations will be equal to the determinant of the quantities $(\Omega_\iota\,\Omega_\kappa)$, so it will be equal to a constant that can also be chosen to be equal to unity. However, when equations of constraint exist, they shall be assumed to be homogeneous for the coordinates of any point. Each $(\Omega_\iota\,\varphi)$, $(\Omega_\iota\,\psi)$, … will also vanish then, and the determinant of the quantities $(\Omega_\iota\,\Omega_\kappa)$, $(\Omega_\iota\,\varphi)$, $(\varphi\,\psi)$, … will be equal to the determinant that is formed from the quantities $(\varphi\,\varphi)$, $(\varphi\,\psi)$, …, up to a constant factor. With the assumed form for φ, ψ, …, the multiplier of the system of **Lagrange** equations will be equal to the determinant (28), whose elements are formed according to the rule (29).

Now, as far as **Hamilton**'s method is concerned, which was perfected by **Jacobi** in his *Vorlesungen über Mechanik* (pp. 167), it is represented it in roughly the following form for the case in which the force function did not contain time explicitly:

One expresses T and U in terms of the 2μ *quantities* q_i *and* p_i *that were defined above. One then replaces* p_i *with* $\partial W / \partial q_i$ *in the equation:*

$$0 = \alpha + T - U,$$

such that this equation will become a partial differential equation for W. If one knows a complete solution of it that contains the μ *constants* $\alpha_1, \ldots, \alpha_{\mu-1}$ *, in addition to the constant that is additively coupled with W, then the integral equations for the differential equations of motion will be:*

$$\frac{\partial W}{\partial \alpha_1} = \beta_1, \quad \ldots, \frac{\partial W}{\partial \alpha_{\mu-1}} = \beta_{\mu-1}, \quad \frac{\partial W}{\partial \alpha} = \tau - t.$$

In that form, the method can also be employed for non-**Euclid**ian space forms. **Jacobi**'s general result is also valid for those spaces, namely, the general theory of perturbations. However, I believe that it is important to give the form of the differential equation when **Weierstrass** coordinates $x_0, x_1, \ldots, x_n$ are used. Rather than developing that form directly (which would not be difficult), I shall employ the equation that **Jacobi** exhibited for the case of equations of constraint and communicated in the first treatise that was printed in his lectures (pp. 376-379). That will allow us to give the result immediately.

If the function W is assumed to be homogeneous of degree zero in the coordinates of all points then it can be determined from the following differential equation for a system of free points:

(30)
$$\sum \frac{1}{m}\left\{\frac{1}{k^2}\left(\frac{\partial W}{\partial x_0}\right)^2 + \left(\frac{\partial W}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial W}{\partial x_n}\right)^2\right\} = 2U - 2\alpha .$$

When homogeneous equations of constraint exist $\varphi = 0$, $\psi = 0$, …, *W might once more be postulated to be homogeneous of degree zero. The equation will then be:*

$$\sum \frac{1}{m}\left\{\frac{1}{k^2}\left(\frac{\partial W}{\partial x_0} - \lambda\frac{\partial \varphi}{\partial x_0} - \mu\frac{\partial \psi}{\partial x_0} + \cdots\right)^2 + \left(\frac{\partial W}{\partial x_1} - \lambda\frac{\partial \varphi}{\partial x_1} - \mu\frac{\partial \psi}{\partial x_1} + \cdots\right)^2 + \cdots + \left(\frac{\partial W}{\partial x_n} - \lambda\frac{\partial \varphi}{\partial x_n} - \mu\frac{\partial \psi}{\partial x_n} + \cdots\right)^2\right\}$$
$$= 2U - 2\alpha ,$$

in which the coefficients λ, μ *are found by means of the equations:*

$$\lambda\,(\varphi\,\varphi) + \mu\,(\varphi\,\psi) + \ldots = (W\,\varphi) ,$$
$$\lambda\,(\psi\,\varphi) + \mu\,(\psi\,\psi) + \ldots = (W\,\psi) ,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots ,$$

and $(\varphi\,\varphi)$, $(\varphi\,\psi)$, … *have the meanings that they were given above.*

First example: The shortest line on an arbitrarily-extended structure.

Let a spatial structure be determined by *r* equations $\varphi = 0$, $\psi = 0$, which might be homogeneous in the coordinates. A point moves in that structure without any accelerating forces acting upon it. The equations of motion will then be obtained when one sets the quantities $X_0, \ldots, X_n$ equal to zero in equations (24). The same equations are obtained from the basic theorems of the variational calculus for the shortest line, as well as the equilibrium conditions for the form that a weightless tensed string will assume on the structure. We conclude from the equations that:

If one passes an $(r + 1)$*-fold extended plane through the (r-fold extended) normal plane to an* $(n - r)$*-fold extended structure at an arbitrary point of the latter and through the tangent to a shortest line that goes through its base point then it will go through yet a third infinitely-close point of the shortest line, so the two-fold extended osculating plane will include the shortest line.*

We seek the equation of the shortest line for an arbitrary curvature structure of quadratic structures, so for an (n – r)-fold extended structure in which r confocal quadratic structures intersect. Let the equations of that structure be:

$$\frac{k^2 x_0^2}{\lambda_\iota + k^2} + \frac{x_1^2}{\lambda_\iota - \alpha_1} + \cdots + \frac{x_n^2}{\lambda_\iota - \alpha_n} = 0 \qquad (i = 1, 2, \ldots, r).$$

The equations of the shortest line are:

$$\frac{1}{x_0}\frac{d^2 x_0}{dt^2} = -\frac{c^2}{k^2} + \frac{M_1}{\lambda_1 + k^2} + \cdots + \frac{M_r}{\lambda_r + k^2},$$
$$\frac{1}{x_1}\frac{d^2 x_n}{dt^2} = -\frac{c^2}{k^2} + \frac{M_1}{\lambda_1 - \alpha_1} + \cdots + \frac{M_r}{\lambda_r - \alpha_1},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$
$$\frac{1}{x_n}\frac{d^2 x_n}{dt^2} = -\frac{c^2}{k^2} + \frac{M_1}{\lambda_1 - \alpha_n} + \cdots + \frac{M_r}{\lambda_r - \alpha_n}.$$

We integrate them using the method that **Weierstrass** gave [Berliner Monatsberichte (1861), pp. 988] and set:

$$f(\lambda) = \frac{\lambda + k^2}{k^2}(\lambda_1 - \alpha_1)\cdots(\lambda_r - \alpha_1),$$

$$\frac{\varphi(\lambda)}{f(\lambda)} = \frac{k^2 x_0^2}{\lambda + k^2} + \frac{x_1^2}{\lambda_1 - \alpha_1} + \cdots + \frac{x_n^2}{\lambda_1 - \alpha_n},$$

$$\frac{\varphi_1(\lambda)}{f(\lambda)} = \frac{k^2 x_0'^2}{\lambda + k^2} + \frac{x_1'^2}{\lambda_1 - \alpha_1} + \cdots + \frac{x_n'^2}{\lambda_1 - \alpha_n}.$$

If we next consider λ to be independent of t then we will prove by differentiation that:

$$\left(\frac{1}{2}\frac{d\varphi(\lambda)}{dt}\right)^2 - \varphi(\lambda)\,\varphi_1(\lambda)$$

is independent of t, so it is just a function of λ. As such, it had degree $2n$ and vanishes for $-k^2$, $\alpha_1, \ldots, \alpha_n$, $\lambda_1, \ldots, \lambda_r$, so it will also vanish for the $n - r - 1$ values $\beta_1, \ldots, \beta_{n-r-1}$, in addition. We then set:

$$R(\lambda) = -\frac{\lambda + k^2}{k^2}(\lambda - \alpha_1)\ldots(\lambda - \alpha_n)(\lambda - \lambda_1)\ldots(\lambda - \lambda_r)(\lambda - \beta_1)\ldots(\lambda - \beta_{n-r-1}),$$

and can now also consider λ to depend upon t in the equation:

$$\left(\frac{1}{2}\frac{d\varphi(\lambda)}{dt}\right)^2 - \varphi(\lambda)\,\varphi_1(\lambda) = c^2\,R(\lambda),$$

when we set:

$$\frac{1}{2}\frac{d\varphi(\lambda)}{dt} = \frac{k^2 x_0\,x_0'}{\lambda + k^2} + \frac{x_0\,x_0'}{\lambda - \alpha_a} + \cdots + \frac{x_n\,x_n'}{\lambda - \alpha_n},$$

and assign it the same values $\lambda_{r+1}, \ldots, \lambda_n$ for which $\varphi(\lambda)$ vanishes, in addition to $\lambda_1, \ldots, \lambda_r$. If we introduce the abbreviation:

$$g(\lambda) = (\lambda - \lambda_1)\ \ldots\ (\lambda - \lambda_r)$$

then the following equations will emerge from the considerations that were posed:

$$\frac{g(\lambda_{r+1})\,d\lambda_{r+1}}{\sqrt{R(\lambda_{r+1})}} + \cdots + \frac{(\lambda_n)\,d\lambda_n}{\sqrt{R(\lambda_n)}} = 0,$$

$$\frac{\lambda_{r+1}\,g(\lambda_{r+1})\,d\lambda_{r+1}}{\sqrt{R(\lambda_{r+1})}} + \cdots + \frac{\lambda_n\,g(\lambda_n)\,d\lambda_n}{\sqrt{R(\lambda_n)}} = 0,$$

……………………………………………………..

$$\frac{\lambda_{r+1}^{n-r-2}\,g(\lambda_{r+1})\,d\lambda_{r+1}}{\sqrt{R(\lambda_{r+1})}} + \cdots + \frac{\lambda_n^{n-r-2}\,g(\lambda_n)\,d\lambda_n}{\sqrt{R(\lambda_n)}} = 0,$$

$$\frac{\lambda_{r+1}^{n-r-1}\,g(\lambda_{r+1})\,d\lambda_{r+1}}{\sqrt{R(\lambda_{r+1})}} + \cdots + \frac{\lambda_n^{n-r-1}\,g(\lambda_n)\,d\lambda_n}{\sqrt{R(\lambda_n)}} = 2c\,dt.$$

In particular, when $r = 1$, so one seeks the shortest line on a quadratic structure, one can express the elliptic coordinates, and then also the coordinates $x_0, \ldots, x_n$ of the shortest line, most simply in terms of hyperelliptic functions of the $n - 1$ quantities $u, u', \ldots, u^{(n-2)}$, the first of which is variable, while the remaining ones are constant.

The **Jacobi** method leads to that goal quite simply. If λ_κ mean the n roots of the equation:

$$\frac{k^2 x_0^2}{\lambda + k^2} - \frac{x_1^2}{\alpha_1 - \lambda_1} + \cdots + \frac{x_n^2}{\alpha_n - \lambda_n} = 0$$

then

$$x_0^2 = \frac{(k^2 + \lambda_1)\cdots(k^2 + \lambda_n)}{(k^2 + \alpha_1)\cdots(k^2 + \alpha_n)},$$

$$x_\kappa^2 = \frac{k^2(\alpha_\kappa - \lambda_1)\cdots(\alpha_\kappa - \lambda_n)}{(\alpha_\kappa + k^2)(\alpha_\kappa - \alpha_1)\cdots(\alpha_\kappa - \alpha_{\kappa-1})(\alpha_\kappa - \alpha_{\kappa+1})\cdots(\alpha_\kappa - \alpha_n)},$$

and

$$4\,ds^2 = \sum N_\kappa\,d\lambda_\kappa^2,$$

where:

$$N_\kappa = \frac{k^2(\lambda_1 - \lambda_\kappa)\cdots(\lambda_n - \lambda_\kappa)}{(k^2+\lambda_\kappa)(\alpha_1 - \lambda_\kappa)\cdots(\alpha_n - \lambda_\kappa)}.$$

In our case, $\lambda_1, \ldots, \lambda_r$ are constant. In the formula:

$$2T = \tfrac{1}{4} N_{r+1} \lambda_{r+1}'^2 + \cdots + \tfrac{1}{4} N_n \lambda_n'^2 = c^2,$$

we must then set:

$$\frac{\partial T}{\partial \lambda_\nu'} = \tfrac{1}{4} N_\nu \lambda_\nu' = \frac{\partial W}{\partial \lambda_\nu},$$

and obtain the differential equation:

$$\tfrac{1}{2} c^2 = \frac{1}{N_{r+1}}\left(\frac{\partial W}{\partial \lambda_{r+1}}\right)^2 + \cdots + \frac{1}{N_n}\left(\frac{\partial W}{\partial \lambda_n}\right)^2,$$

which will decompose into the following ones:

$$\frac{f(\lambda_\mu)}{g(\lambda_\mu)}\left(\frac{\partial W}{\partial \lambda_\nu}\right)^2 = \tfrac{1}{2} c^2 (\lambda_\nu - \beta_1)\cdots+(\lambda_\nu - \beta_{n-r-1}), \qquad \text{for } \nu = r+1, \ldots, n,$$

when one employs the notation above.

The differentiation of W with respect to the β and c^2 will lead to the equations above.

Permit us to add the solution to the following problem here: Define a two-fold extended curvature structure of quadratic structures on the **Euclid**ian plane. Once that problem has been solved, the mapping of that space form to the plane will present no difficulties.

Since all $d\lambda_\iota$ in the expression for the line element vanish, except for $d\lambda_{n-1}$ and $d\lambda_n$, from the known method, one can set:

$$du \pm i\,dv = \frac{a \pm b\,i}{2}\left\{\sqrt{\frac{k^2(\lambda_{n-1}-\lambda_1)\cdots(\lambda_{n-1}-\lambda_n)}{(\lambda_{n-1}+k^2)(\lambda_{n-1}-\alpha_1)\cdots(\lambda_{n-1}-\alpha_n)}}\,d\lambda_{n-1} + i\sqrt{\frac{k^2(\lambda_{n-1}-\lambda_1)\cdots(\lambda_n-\lambda_{n-1})}{(\lambda_n+k^2)(\lambda_n-\alpha_1)\cdots(\lambda_n-\alpha_n)}}\,d\lambda_n\right\}.$$

Here, u and v are rectangular coordinates in the mapping plane, and either the upper or the lower sign in $\pm$ is chosen everywhere. Now, if $\lambda_{n-1} > \lambda_n$ then one chooses $a = \frac{2}{\sqrt{\lambda_{n-1}-\lambda_n}}$, $b = 0$. In that way, u will become a function of just λ_{n-1} and v, a function of just λ_n, and their representation will be obvious.

Second example: A free point is attracted to a fixed point according to an arbitrary function of the distance between them.

Although the solution is very simple, one might be permitted to say a few words about it. If one would like to employ the usual methods then one would perhaps start from the fact that the motion takes place in a two-fold extended plane, as one learns from the theory of surfaces. If one would like to integrate the partial differential equation then one would introduce polar coordinates (cf., **Jacobi**, *loc. cit.*, pp. 185 and 344). The equation then decomposes into a sequence of other ones that can be integrated immediately. When the law of attraction is the one that was posed in the example in § **1**, one can very easily determine the position of the point in terms of its distance r from the attracting point and in terms of the distance ρ from a second arbitrarily-chosen point. If one chooses the latter to be along the path of the moving point at a distance r_0 from the center then one will have:

$$W = \int_{r-\rho}^{r+\rho} ds \sqrt{\frac{\mu}{2k}\cot\frac{s+r_0}{2k} - \tfrac{1}{2}\alpha}\,,$$

which is an equation that can be obtained by differentiating the equations of motion.

Third example: A point moves in an n-dimensional space form under the influence of two forces that point from fixed centers and are inversely-proportional to the square of the sine of the distance divided by k.

We can start from the fact that the moving point remains the triply-extended principal structure, which is determined by the two attracting points and the initial direction of the moving point. Meanwhile, the calculation will not simplify in that way. We then pay no attention to the initial direction of the point and lay the planes $x_2 = 0$, $x_3 = 0$, …, $x_n = 0$ of a **Weierstrass** coordinate system through the line that connects the attracting points and give the structure $x_1 = 0$ a symmetric position with respect to those two points. We set:

$$x_2 = s \cdot \cos \varphi_1\,,$$
$$x_3 = s \cdot \sin \varphi_1 \cos \varphi_2\,,$$
$$\cdots\cdots\cdots\cdots$$
$$x_n = s \cdot \sin \varphi_1 \sin \varphi_2 \ldots \sin \varphi_{n-2}\,,$$

such that we will have:

$$x_2^2 + x_3^2 + \cdots + x_n^2 = s^2, \quad k^2 x_0^2 + x_1^2 + s^2 = k^2.$$

In that way, the differential equation (30) will go to:

$$\frac{1}{k^2}\left(\frac{\partial W}{\partial x_0}\right)^2 + \left(\frac{\partial W}{\partial x_1}\right)^2 + \left(\frac{\partial W}{\partial s}\right)^2 + \frac{1}{s^2}\left(\frac{\partial W}{\partial \varphi_1}\right)^2 + \frac{1}{s^2 \sin^2 \varphi_1}\left(\frac{\partial W}{\partial \varphi_2}\right)^2 + \cdots + \frac{1}{s^2 \sin^2 \varphi_1 \cdots \sin^2 \varphi_{n-2}}\left(\frac{\partial W}{\partial \varphi_{n-2}}\right)^2$$

$$= 2U + 2h\,,$$

which immediately decomposes into the two equations:

$$\frac{1}{k^2}\left(\frac{\partial W}{\partial x_0}\right)^2 + \left(\frac{\partial W}{\partial x_1}\right)^2 + \left(\frac{\partial W}{\partial s}\right)^2 = 2U + 2h - \frac{2\rho}{s^2}\,,$$

$$\left(\frac{\partial W}{\partial \varphi_1}\right)^2 + \frac{1}{s^2 \sin^2 \varphi_1}\left(\frac{\partial W}{\partial \varphi_2}\right)^2 + \cdots = 2\rho,$$

where r denotes an arbitrary constant. I express the first equation in terms of elliptic coordinates, for which the focal points coincide with the attracting points. The first example shows one how to convert the left-hand side. On the right-hand side, one sets:

$$U = \frac{m}{k}\cot\frac{r}{k} + \frac{m_1}{k}\cot\frac{r_1}{k},$$

where r and r_1 mean the distances from the foci. An easy conversion will give:

$$U = \frac{(m+m_1)\sqrt{\frac{k^2+\lambda_1}{k^2}(\alpha_1-\lambda_1)} + (m-m_1)\sqrt{\frac{k^2+\lambda_2}{k^2}(\alpha_1-\lambda_2)}}{\lambda_1-\lambda_2},$$

and one will find that:

$$\frac{1}{s^2} = \frac{(k^2+\alpha_1)(\alpha_1-\alpha_2)}{k^2(\lambda_1-\lambda_2)}\left\{\frac{1}{\lambda_1-\alpha_2} + \frac{1}{\alpha_2-\lambda_2}\right\}.$$

We then find that:

$$W = \int d\lambda_1 \sqrt{\frac{\frac{1}{2}h\lambda_1 - \frac{1}{2}(m+m_1)\sqrt{\frac{k^2+\lambda_1}{k^2}(\alpha_1-\lambda_1)} - \frac{1}{2}\rho\frac{(k^2+\alpha_2)(\alpha_1-\alpha_2)}{k^2(\lambda_1-\alpha_2)} + \mu}{\frac{k^2+\lambda_1}{k^2}(\alpha_1-\lambda_1)(\alpha_2-\lambda_1)}}$$

$$+\int d\lambda_2 \sqrt{\frac{\frac{1}{2}h\lambda_2 - \frac{1}{2}(m-m_1)\sqrt{\frac{k^2+\lambda_2}{k^2}(\alpha_1-\lambda_2)} + \frac{1}{2}\rho\frac{(k^2+\alpha_2)(\alpha_1-\alpha_2)}{k^2(\alpha_2-\lambda_2)} + \mu}{\frac{k^2+\lambda_2}{k^2}(\alpha_1-\lambda_2)(\alpha_2-\lambda_2)}}$$

$$+ F(\varphi_1, \ldots, \varphi_{n-2}),$$

in which μ denotes a constant, and F is just a function of $\varphi_1, \ldots, \varphi_{n-2}$ that is obtained from the second differential equation by further decomposition.

§ 5. – Extension of Newton's law.

In an n-dimensional space, we imagine that an $(n - 1)$-fold extended spherical surface has been described about an attracting mass-point at its center. If mass has been distributed on parts of the surface then according to the ideas that are at the basis for **Newton**'s law, the attraction that is exerted towards the center is directly proportional to the mass and inversely proportional to the area. Now, if the surface is a structure for which the radius r is equal to $\varpi k^{n-1} \sin^{n-1} r / k$, where

ϖ is expressed in terms of n and π in a known way. As an extension of **Newton**'s law of attraction for an n-dimensional space form, we then consider an attraction that is directly proportional to the mass and inversely proportional to the $(n - 1)^{\text{th}}$ power of the since of the distance divided by k.

For the sake of brevity, we might be permitted in this section to refer to a part of space that is bounded by an $(n - 1)$-fold extended spherical surface as a "sphere," the boundary surface as a "spherical surface," and the space that is bounded by two concentric $(n - 1)$-fold extended spherical surfaces as a "spherical shell." The word "ellipsoid" might have a corresponding definition.

Theorem IV in Band v, pp. 9, of **Gauss**'s *Werke* corresponds to the following one:

If dS denotes an element of an $(n - 1)$*-fold extended structure S that is simply included in a finite space, while P denotes a point of that element, M denotes a fixed point in space, r denotes the distance MP, and u denotes the angle between MP and the interior normal to dS then the following integral, which extends over the entire structure S:*

$$\int \frac{dS \cdot \cos u}{k^{n-1} \sin^{n-1} \frac{r}{k}}$$

will be equal to 0 *or* ϖ *or* $\frac{1}{2}\varpi$ *according to whether M lies outside of S, inside of it, or on it, respectively.*

In order to prove that, one describes a spherical surface around M with an infinitely-small radius ν and draws straight lines from M to all of the points on the boundary of dS. If the conical structure that arises in that way cuts out the region $\nu^{n-1}\, d\sigma$ from the spherical surface then:

$$\frac{dS \cdot \cos u}{d\sigma} = k^{n-1} \sin^{n-1} \frac{r}{k},$$

from which, the theorem follows immediately.

That theorem leads to an extension of another theorem that **Gauss** proved (*Werke*, V, pp. 225). The extension was presented by Herrn **Schering** in the treatise: "Die Schwerkraft in mehrfach ausgedehnten *Gauss*ischen und *Riemann*schen Räumen" [Göttinger Nachrichten (1873), pp. 154] as Lehrsatz III.

The attraction that an infinitely-thin homogeneous spherical shell exerts is zero for interior points, and for exterior points, it is so large that it is as if the mass were concentrated at the center. That immediately implies the attraction that an arbitrary homogeneous spherical shell or solid ball exerts upon any point.

If φ is a homogeneous function of degree two in the coordinates, and if Ω has the meaning that it was given above then the structures that are represented by the equations:

$$\varphi = 0 \quad \text{and} \quad \varphi + \frac{\lambda}{k^2}\Omega = 0$$

shall be referred to as *similar concentric structures*. The space that is contained between two infinitely-close similar ellipsoids:

$$\frac{x_1^2}{\alpha_1}+\cdots+\frac{x_n^2}{\alpha_n}-x_0^2=0 \quad \text{and} \quad \frac{x_1^2}{\alpha_1}+\cdots+\frac{x_n^2}{\alpha_n}-x_0^2=2\,d\tau$$

might be uniformly filled with mass. The mass thus-distributed exerts no attraction on any interior point. We shall add a proof that will allow us to communicate several important formulas for the ellipsoid.

The density at the point x is proportional to $\psi\,d\tau$, where ψ means the sine of the distance from the center to the tangent plane and is obtained from the formula:

$$\frac{1}{\psi^2}=\frac{x_1^2}{\alpha_1^2}+\cdots+\frac{x_n^2}{\alpha_n^2}+\frac{x_0^2}{k^2}\,.$$

The attracting point ξ has the distance r from the point x, and the line r defines an angle φ with normal at x; one then has:

$$\cos\varphi=\frac{x_0\,\xi_0-\dfrac{x_1\,\xi_1}{\alpha_1}-\cdots-\dfrac{x_n\,\xi_n}{\alpha_n}}{\psi k\sin\dfrac{r}{k}}\,.$$

If one describes a sphere around ξ with the infinitely-small radius ν and if an element dS of the attracting body that contains the point x determines the element $\nu^{n-1}\,d\sigma$ on the sphere then the attraction that is exerted upon dS will be proportional to:

$$\frac{d\sigma\cdot d\tau}{\psi\cdot\cos\varphi}=\frac{k\sin\dfrac{r}{k}\,d\sigma\cdot d\tau}{x_0\,\xi_0-\dfrac{x_1\,\xi_1}{\alpha_1}-\cdots-\dfrac{x_n\,\xi_n}{\alpha_n}}\,.$$

As the polar properties of the ellipsoid would imply, that expression will remain unchanged when one takes the element whose boundary, in conjunction with ξ, leads to the same conical structure instead of dS.

When one associates those points on two confocal ellipsoids that coincide in n – 1 elliptic coordinates, one will arrive at the following theorem in the known way:

Any confocal ellipsoid is a level surface for an infinitely-thin layer that is bounded by similar ellipsoids.

The form that the potential assumes for the given law and the most important laws for it were given before in the aforementioned paper by **Schering** [Gött. Nachr. (1873), pp. 159 to 159]. The function $w_n(r)$ of the distance r with which the mass is multiplied is different for even and odd n, and indeed, for an odd n, it is:

$$w_n(r) = \frac{\cot\frac{r}{k}}{(n-2)\,k^{n-2}} \sum_{\nu=0}^{(n-3)/2} \frac{2\nu+2}{2\nu+1} \cdot \frac{2\nu+4}{2\nu+3} \cdots \frac{n-5}{n-6}\,\frac{n-3}{n-4}\,\frac{1}{\sin^{2\nu}\frac{r}{k}}\,, \tag{31}$$

whereas for an even $n > 2$:

$$w_n(r) = \frac{3}{2}\cdot\frac{5}{4}\cdots\frac{n-3}{n-4}\,\frac{\log\left(\frac{1}{2k}\cot\frac{r}{2k}\right)}{(n-2)\,k^{n-2}} + \frac{\cos\frac{r}{k}}{(n-2)\,k^{n-2}\sin^2\frac{r}{k}} \sum_{\nu=0}^{(n-4)/2} \frac{2\nu+3}{2\nu+2}\,\frac{2\nu+5}{2\nu+4}\cdots\frac{n-3}{n-4}\,\frac{1}{\sin^{2\nu}\frac{r}{k}}\,. \tag{32}$$

In order for the potential function:

$$V = \int dm\, w(r) \tag{33}$$

to also preserve the known exterior properties, we represent it as a homogeneous function of degree zero in the coordinates $x_0, x_1, \ldots, x_n$ and also let the first differential quotients that are used for defining the second partial differential quotients consist of functions of degree – 1 in the coordinates. The components of the force are then:

$$X_0 = -\frac{\partial V}{\partial x_0}\,, \quad X_1 = -\frac{\partial V}{\partial x_1}\,, \quad \ldots, \quad X_n = -\frac{\partial V}{\partial x_n}\,.$$

If only an infinitely-small mass is contained in any infinitely-small region of space then the potential and the first derivatives will change continuously in all of space. By contrast, the differential expression that is defined by the rule above will be:

$$\Delta V = \frac{1}{k^2}\frac{\partial^2 V}{\partial x_0^2} + \frac{\partial^2 V}{\partial x_1^2} + \cdots + \frac{\partial^2 V}{\partial x_n^2} = -\varpi\rho\,, \tag{34}$$

in which ϖ denotes the coefficient that was given above, and ρ is the density at the point x. When the mass is condensed into $(n – 1)$-fold extended structures, the potential itself will be continuous, but the derivative in the direction of the normal will change by $\varpi\rho$ when one passes through the mass structure. When the mass is condensed even further, the potential will become infinite at the mass, and indeed it will become logarithmically infinite for an $(n – 2)$-fold extended structure. (**Schering** expressed those theorems in Lehrsatzen V, VI, VII.)

As far as the proof is concerned, one can perhaps initially prove the formula $\Delta V = 0$ for a point that lies outside the mass, which can be accomplished by differentiating under the integral sign. For the general proof of the formula $\Delta V = -\varpi\rho$ and for the proof of the other theorems, one can restrict oneself to an infinitely-small region, and the laws of **Euclid**ian space forms will be valid for it.

The further properties of the potential are based upon **Green**'s theorem (Lehrsatz VIII in **Schering**), which assumes the following form with our notation:

$$\int\left(\frac{1}{k^2}\frac{\partial U}{\partial x_0}\frac{\partial V}{\partial x_0}+\frac{\partial U}{\partial x_1}\frac{\partial V}{\partial x_1}+\cdots+\frac{\partial U}{\partial x_n}\frac{\partial V}{\partial x_n}\right)dR_n+\int U\cdot\Delta V\cdot dR_n=\int U\,\frac{\partial V}{\partial N}\,dR_{n-1}\,. \tag{35}$$

Here, the spatial region R_n is singly-enclosed by the structure R_{n-1}, the differentiation $\partial V / \partial N$ is along the outward-pointing normal, and ΔV has the meaning that it was given above. The integration on the left is over the entire region R_n, while it extends over the entire boundary R_{n-1} on the right.

We believe that we do not need to add a proof of that theorem. On the other hand, we would like to briefly make note of the fact that the idea that **Somoff** connected with that theorem in the second volume of his *Mechanik* (German edition, pp. 147) also remains true here. If U means the density of a mass that is distributed in R_n, and V means the potential of the velocity with which each particle moves then we can easily show that we have:

$$\frac{\delta\, dR_n}{dR_n}=\Delta V\cdot\delta t\,.$$

Hence, the left-hand side of the equation above, as well as the right, represents the change in mass, and indeed the former expression will be obtained by considering every element, while the latter by determining the entrances and exits that occur in the boundary.

Therefore, the properties of the potential remain true. Since the conversion is quite simple, it will suffice to give those points at which the potential differs for **Euclid**ian and non-**Euclid**ian space forms.

In the **Euclid**ian space forms, the potential and the force are equal to zero at infinity, and the development in $1 / r$ contains only positive powers for which the coefficient of the lowest-order one can be given immediately. That breaks down in non-**Euclid**ian space forms completely. In the **Lobachevski**an space forms of odd dimensions, the potential will be equal to the mass multiplied by a certain constant.

When an infinite ν-fold extended plane in an n-dimensional **Euclid**ian space form is uniformly covered with mass, the law of attraction to a point will be the one that is true for the attraction by a point in an $(n - \nu)$-fold extended **Euclid**ian space form. Completely different laws of attraction will be true for non-**Euclid**ian space forms under the assumptions that were made.

§ 6. – Infinitely-small motion of a rigid body.

In order for the distance between all points to remain unchanged under the infinitely-small motion:

(36) $$k^2\,\frac{dx_0}{dt} = \sum_\kappa \mu_{0\kappa}\,x_\kappa\,, \quad \frac{dx_\iota}{dt} = \sum_\kappa \mu_{\iota\kappa}\,x_\kappa\,,$$

the following conditions must be fulfilled:

(37) $$\mu_{\mu} = 0\,, \qquad \mu_{\iota\kappa} + \mu_{\kappa\iota} = 0\,.$$

The square of the velocity of the point x is then:

(38) $$\left\{\begin{array}{l} \rho^2 = x_0^2\,(\mu_{10}^2 + \cdots + \mu_{n0}^2) + x_1^2\left(\dfrac{\mu_{10}^2}{k^2} + \mu_{21}^2 + \cdots + \mu_{n1}^2\right) + \cdots \\ \cdots + 2\,x_0\,x_1(\mu_{20}\,\mu_{21} + \cdots + \mu_{n0}\,\mu_{n1}) + \cdots + 2\,x_1\,x_2\left(\dfrac{\mu_{01}\,\mu_{02}}{k^2} + \mu_{31}\,\mu_{32} + \cdots\right) \\ + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right.$$

All points that possess that velocity lie on a second-order structure. That structure will be displaced into itself by the motion. All structures of that type have the same midpoint and are similar to each other. The problem of determining the nature of that structure is identical to the problem of seeking the maxima and minima of the velocity. Both of them lead to the determination of the elementary divisors of the determinant:

(39) $$D = \begin{vmatrix} \mu_{10}^2 + \cdots + \mu_{n0}^2 - \rho^2 & \mu_{20}\,\mu_{21} + \cdots + \mu_{n0}\,\mu_{n1} & \cdots & \mu_{10}\,\mu_{1n} + \mu_{20}\,\mu_{2n} + \cdots \\ \mu_{20}\,\mu_{21} + \cdots + \mu_{n0}\,\mu_{n1} & \dfrac{\mu_{01}^2}{k^2} + \mu_{21}^2 + \cdots + \mu_{n1}^2 - \dfrac{\rho^2}{k^2} & \cdots & \dfrac{\mu_{01}\,\mu_{0n}}{k^2} + \mu_{21}\,\mu_{2n} + \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \mu_{1n}\,\mu_{10} + \mu_{2n}\,\mu_{20} + \cdots & \dfrac{\mu_{n0}\,\mu_{01}}{k^2} + \mu_{2n}\,\mu_{21} + \cdots & \cdots & \dfrac{\mu_{0n}^2}{k^2} + \mu_{1n}^2 + \cdots + \mu_{n1}^2 - \dfrac{\rho^2}{k^2} \end{vmatrix}.$$

It is the square of the determinant:

(40) $$\Delta = \begin{vmatrix} \rho\,i & \mu_{01} & \cdots & \mu_{0n} \\ \dfrac{\mu_{01}}{k} & \dfrac{\rho\,i}{k} & \cdots & \mu_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\mu_{n0}}{k} & \mu_{n1} & \cdots & \dfrac{\rho\,i}{k} \end{vmatrix}.$$

When the last n rows in this are multiplied by k, we will get a skew determinant for $\rho = 0$. The known properties of such things imply a theorem that was probably first expressed for arbitrary

Euclidian space forms by **Jordan** (C. R. Acad. Sci. Paris, LXXV, pp. 1614) and first proved for our space forms by **Beltrami** (*):

In a space form with an even number of dimensions, any instantaneous motion consists of a rotation about a point. For an odd number of dimensions, no point remains at rest, in general, but whenever a point is kept fixed, a rotation around a line will take place.

That theorem admits a significant generalization. One proves it by means of either the theorems that **Frobenius** (this journal, Bd. 82, pp. 244 and 245) presented in regard to the vanishing of the sub-determinants of a skew determinant, or also geometrically by repeated application of the foregoing theorem. We give that extension the form:

A structure that remains at rest under the motion of an even-dimensional space form always has an odd number of dimensions. Therefore, whenever a second point remains at rest, in addition to a point at rest (and its opposite point in a ***Riemann****ian space), at least all points of a two-fold extended plane must keep their positions. Whenever a point remains at rest, in addition to such a plane, the same thing will be true for a four-fold extended plane, etc.*

In order to go into more detail about the motion that is characterized by equations (36) and (37), we next consider the case in which n is odd, k^2 is positive, and the equation $\Delta = 0$ is not satisfied by the value $\rho = 0$. The roots of $\Delta = 0$ are then pair-wise equal and opposite, and $D = 0$ has only paired equal roots. By converting the coordinate system, one can give equation (38) the form:

$$\rho^2 = a_0\,(k^2\,x_0^2 + x_1^2) + a_1\,(x_2^2 + x_1^2) + \cdots + a_{(n-1)/2}\,(x_{n-1}^2 + x_n^2).$$

Here, a_0, a_1, … are the roots of the equation $D = 0$. If they are mutually distinct then the velocity $\rho = \sqrt{a_\iota}$ will define a structure that can be represented homogeneously by $n - 1$ coordinates. I shall call such a thing a *conical structure* with singular lines. The conical structure that corresponds to the largest and smallest velocity is imaginary, except for the singular lines, and real for the other roots. Those considerations lead to the following theorem:

For the most general infinitely-small motion of a finite space form with an odd number n of dimensions, $(n + 1) / 2$ *straight lines will be displaced into themselves. Each of those lines defines a right angle with the other ones and has a distance from them of* $\frac{1}{2}k\pi$*, so it belongs to the absolute polar structure of the remaining ones. Once any one of them has been chosen, a second one can be chosen arbitrarily on an* $(n - 2)$*-fold extended plane, and then a third one can be chosen arbitrarily on an* $(n - 4)$*-fold extended plane, etc. One of those lines has the property that its points have greatest velocity, and the points that lie on a second of those lines move the slowest. For all other points, the velocity lies between those limits. The velocity that one of the remaining lines that*

(*) **Darboux** and **Hoüel**, Bull. (1), t. IX, pp. 239. Unfortunately, I am not familiar with the paper itself, but only the reference to it "Fortschritten der Mathematik."

is displaced into itself has is also possessed by the points of a quadratic conical structure for which that line is a double line. If one associates every point with all of the points that have the same velocity that it has then space will be decomposed into similar concentric second-order structures. All of those structures have the peculiarity that their midpoints fill up $(n + 1) / 2$ *lines, namely, the straight lines that are displaced into themselves.*

Any infinitely-small motion can be composed of $(n + 1) / 2$ *displacements along a line* [$(n + 1) / 2$ *rotations around an* $(n – 2)$*-fold extended plane, resp.*]. *Those lines can be chosen such that each of them belongs to the absolute polar plane of the others.*

Any infinitely-small motion can be composed of a displacement along a line and a rotation around that line, and indeed, the line can be chosen in $(n + 1) / 2$ *different ways if that motion is to arise.*

One now assumes that the equation $\Delta = 0$ has nothing but equal roots. One might believe that this requirement leads to $(n – 1) / 2$ equations of constraint. However, since $\Delta = 0$ is itself the condition for a maximum, those conditions must split into a larger number. I would not like to go into the search for those conditions here, whose number amounts to $\left(\frac{n+1}{2}\right)^2 - 1$, and how one expresses the $\mu_{\iota\kappa}$ in terms of $(n^2 + 3) / 4$ arbitrarily-chosen quantities. Namely, since the equation $D = 0$ might possess only one root, and all elements must vanish for it, we will get a sequence of conditions that indeed no longer have to be mutually-independent, but which will allow one to easily derive the geometric properties of that motion. (If one would like to have the formulas in their simplest form then one let all $\mu_{\iota\kappa}$ vanish, except for μ_{01} / k, μ_{23}, μ_{45}, …, $\mu_{n-1,n}$, for a special choice of coordinates, and set the latter equal to each other.) The line that connects the point x to the infinitely-close position of that point contains all points whose coordinates are $\alpha x_\iota + \beta \sum_\kappa \mu_{\iota\kappa} x_\kappa$ when α and β satisfy the condition that: $k^2\alpha^2 + \beta^2\rho^2 = k^2$. If one replaces x with any point on that line then equations that exist between the quantities $\mu_{\iota\kappa}$ will show that the new line coincides with the previous one. Thus, a line that moves into itself will go through any point in space. The set of all those lines has an important property in common with the parallels in **Euclid**ian space: Whereas, in general, for any two lines in a finite space, two straight lines can be found that intersect both of them perpendicularly, and both of them at are unequal distances, the known method for determining those distances will yield two equal roots for two straight lines in that system. However, at the same time, the base-points of the common perpendiculars will be undetermined, and all lines that go from a point of the one straight line to the other one will be perpendicular to the first one and equal to each other. That property can be verified directly very easily. If one would like to adapt the concepts of angle and the distance between two lines to the finite space forms then one would have to refer to the smaller of the two common perpendiculars that were found above as the “distance between them” and the larger one, divided by k, as the “angle between them.” We can also say then that for any two lines in the system, the distance between them is equal to the angle multiplied by k. Hence, we can characterize that special motion, which I would like to refer to as *self-reciprocal*, in the following way:

If the equation $\Delta = 0$ *possesses nothing but equal roots in* ρ^2 *then all point in space will move in straight lines with equal velocity. A straight line that is displaced into itself goes through each point in space. The* $(n - 1)$*-fold extended system of those lines differs from the system of parallels in a* ***Euclid****ian space by the fact that no two of them can lie on the same two-fold extended plane, but it agrees with that system in that any two lines have the same distance between them everywhere and that a line goes through every point of the one that intersects both of them at right angles. If one lays a two-fold extended plane through any line of the system then the angle through which that plane rotates around the line is equal to the displacement along the line divided by k. A threefold-extended plane goes through two lines of the system that contains a two-fold infinitude of lines that move into themselves. Every four-fold plane that goes through that plane makes the same rotation around the three-fold extended plane that moves into itself as the two-fold extended one does around the line that moves into itself. m lines of the system determine a* $(2m - 1)$*-fold extended plane, in general, and each of those* $2m$*-dimensional planes that go through it rotate with the point velocity divided by k.*

Now, the $(n + 1) / 2$ roots of the equation $\Delta = 0$ might be split into groups of α, β, … equal ones, but all of the remaining roots are different, and none of those roots can be equal to zero. The α-fold root corresponds to a velocity that all points of a $(2\alpha - 1)$-fold extended principal structure possess. A straight line that is displaced into itself goes through each point of the principal structure, and any two of those lines have the same distance between them everywhere. Similarly, a $(2\beta - 1)$-fold extended plane that belongs to the absolute polar plane of the first, $(2\alpha - 1)$-fold extended principal structure will possess a velocity that corresponds to the β-fold root, and in that way, which we have just characterized as a self-reciprocal motion, will displace into itself. The same thing is true of the remaining roots.

When a principal structure is at rest, the motion of an even-dimensional (*paarig ausgedehnten*) and an odd-dimensional (*unpaarig ausgedehnten*) space form differ by only the number of dimensions of the structure at rest. If the structure at rest has $n - 2m$ dimensions then the $(2m)$-fold extended plane that is perpendicular to it at one of its points will rotate around the fixed point of intersection. Each $(2m - 1)$-fold extended spherical structure that is contained in such a plane has the properties of a **Riemann**ian space form, so it contains at least m principal circles that are displaced into themselves. Therefore, at least m two-dimensional planes will go through each point of the structure at rest that rotate about that point. However, if the equation $\Delta = 0$ has several non-vanishing roots, in addition to the vanishing ones, then there will be several extended planes with an even number of dimensions that go through each point of the structure at rest, and for them, the velocity of each point will depend upon only its distance from the structure at rest, and a two-fold extended plane will go through each point that rotates into itself.

Thus, the most general motion an even-dimensional space form consists of a fixed point and $n / 2$ two-fold extended planes going through it that are displaced into themselves; those planes are mutually-perpendicular. If r is the distance from a moving point to the fixed point, and if the velocity is equal to $k\,\lambda \sin r / k$ then there will be $n / 2$ singular values for λ ; the largest and smallest values of λ each belong to one of those planes. Whereas the general value of λ belongs to a conical structure that has the fixed point as its vertex, for the stationary values of λ, it will contain a two-dimensional double plane. Therefore, the most general motion of such a space form can be composed of n / 2 special motions along n / 2 mutually-perpendicular, two-fold extended planes that go through that point. Each of those special motions not only rotates the two-fold extended plane into itself about the fixed point, but also displaces every three-fold extended plane that is laid through the two-fold extended plane into itself, and every point of the (n – 2)-fold extended plane that is perpendicular to the first plane at the point of rotation will remain at rest.

When the equation $\Delta = 0$ has several non-zero roots in addition to the zero root, there will be even-dimensional planes with a point that remains fixed and in which a two-fold extended plane that rotates into itself will go through each point. That system of planes deserves attention. Whereas, in general, two two-dimensional planes that have a point in common will define two angles with each other, one will get only one angle for two planes of the aforementioned system, and the angle that a line that is drawn in one of the planes through the common point makes with its projection onto the other plane will be constant. If one describes a circle in one of the planes about the common point then all of its points will have the same distance from the other plane, and the foot of the perpendicular that is dropped from one of them to the other plane will again lie on a circle.

If one would like to foresee the various possibilities that are possible for the motion of an n-fold extended space form then one can proceed in the following way:

If no structure is to remain at rest then one decomposes the number $(n + 1) / 2$ *into a sum of whole numbers all different ways. If* $\alpha, \beta, \ldots$ *is such a decomposition then one associates each of them with a certain positive quantity* ρ_α, ρ_β, …; *the quantities* ρ_α, ρ_β, … *shall be distinct. The velocity* ρ_α *of all points is then associated with a* $(2\alpha - 1)$*-fold extended principal structure that moves within itself; likewise, all points of a* $(2\beta - 1)$*-fold extended principal structure will move with the velocity* ρ_β*, etc. The motion of each such structure is of the type that was described above.*

Should an r-dimensional principal structure remain at rest, then one could decompose the number $(n - r) / 2$ *into numbers* $\gamma, \delta, \ldots$ *in all different ways and associate each of them with a particular velocity* ρ_γ, ρ_δ, … *A* (2γ)*-fold extended plane will then go through each point of the structure at rest that is not only displaced into itself, but in which a two-fold extended plane that is displaced into itself will also go through each point. The other numbers determine corresponding planes of dimension* 2δ, … *that possess that property.*

The following theorem deserves to be pointed out:

An even-dimensional plane that moves within itself cannot occur in isolation within a space form with an odd number of dimensions but must belong to a family of such planes in an odd-dimensional principal structure that possesses that property. Similarly, one can combine all principal structures that continue to cover their initial position under an infinitely-small motion of an even-dimensional space form into a finite number of such planes, each of the latter, in turn, has an even number of dimensions.

I do not need to go further into the **Lobachevski**an space forms. If one considers that every displacement along a n-fold extended plane comes from a rotation about its absolute polar structure, so about an $(n - \nu - 1)$-fold extended plane then one will easily see how the foregoing result must be altered. Indeed, in its proof, we made use of the fact that for a positive k^2, every bundle of similar quadratic structures can be represented by $n + 1$ squares. Instead of them, we must now also look at the bundle that can be defined for negative k^2, according to **Weierstrass** (Berliner Monatsberichte, May 1868).

If one would like to characterize the infinitely-small motions in more detail then one could also study their close relationship to the straight lines in space here, which is a relationship that has proved to be so fruitful for $n = 3$ in recent papers. If one considers the x to be given and the ξ to be variable in the equation:

$$\sum \mu_{\iota\kappa}\, x_\iota\, \xi_\kappa = 0$$

then that equation will represent the plane that is perpendicular to the direction of motion of the point x at that point. However, if one considers the $x_\iota\, \xi_\kappa - x_\kappa\, \xi_\iota$ to be the coordinates of a straight line then every line that satisfies the equation will have the property that each of its points is perpendicular to the direction of motion. The "line complex" characterizes the motion considered completely. It has a close relationship with the study of the composition of motions, degrees of mobility, constraints, and the like. Namely, the polar properties also prove to be especially important in that context. However, I believe that I shall not go further into that theory at this point.

The theory of force that act upon a fixed body is not essentially different from the theory of infinitely-small motions. If the forces $X^{(\alpha)}$ act on the points $x^{(\alpha)}$, and we derive the equations of motion from, say, **d'Alembert**'s principle then we will see that in addition to the initial state, we will be dealing with only the $n\,(n + 1) / 2$ quantities:

$$M_{\kappa 0} = \sum \left(X_\kappa\, x_0 - \frac{X_0\, x_\kappa}{k^2} \right), \quad M_{\iota\kappa} = \sum (X_\iota\, x_\kappa - X_\kappa\, x_\iota).$$

Those quantities have the properties that were assumed for the $\mu_{\iota\kappa}$ in equation (37). Thus, the same laws are true for the classification of force-systems and their composition that are true for infinitely-small motions. The work that the force system $M_{\iota\kappa}$ does under an infinitely-small motion is:

$$\sum \mu_{\alpha\beta}\, M_{\alpha\beta}\, dt.$$

That gives the meaning of the $M_{\iota\kappa}$: M_{01} is the work that the given system of forces does when the body is displaced with unit velocity along the axis $x_1 = x_2 = \ldots = x_n = 0$.

§ 7. – The theory of moments of inertia.

If shall preface the further examination of the motion of a rigid body with the theory of moments of inertia. I define the moment of a mass-point relative to any principal structure to be k^2 times the product of the mass with the square of the sine of the distance divided by k. I further assume that the moment of a system shall be equal to the sum of the moments of the individual points.

If $a_0, a_1, \ldots, a_n$ are the coordinates of an $(n-1)$-extended plane, which satisfy the equation:

$$\frac{a_0^2}{k^2} + a_1^2 + \cdots + a_n^2 = 1,$$

and if ρ is the distance from the point x to that plane then:

$$k \sin \frac{\rho}{k} = a_0\, x_0 + a_1\, x_1 + \ldots + a_n\, x_n\,.$$

If r is the distance from the point to an $(n-l)$-fold extended plane, moreover, then one lays l mutually-perpendicular $(n-1)$-fold extended planes. If the point has the distances $\rho_1, \ldots, \rho_l$ from them then one will have:

$$\sin^2 \frac{r}{k} = \sin^2 \frac{\rho_1}{k} + \cdots + \sin^2 \frac{\rho_l}{k}.$$

If we multiply both sides of that equation by $k^2\, m$ and add over all mass-points then we will get the following theorem:

The moment of a mass-system for any $(n-l)$*-fold extended plane is equal to the sum of the moments of the system for* l $(n-1)$*-dimensional planes that all go through the* $(n-l)$*-fold extended plane and are perpendicular to each other. In particular, the moment at a point is equal to the sum of the moments relative to* n $(n-1)$*-fold extended planes that go through the point and are perpendicular to each other but are otherwise chosen arbitrarily.*

That theorem makes it seem appropriate to next investigate the moment of inertia for $(n-1)$-fold extended planes. If the points $x^{(\alpha)}$ have the masses m_α then the moment of inertia λ for the plane α will be:

$$\lambda = \sum m_\alpha\, [a_0\, x_0^{(\alpha)} + a_1\, x_1^{(\alpha)} + \cdots + a_n\, x_n^{(\alpha)}]^2\,. \tag{41}$$

Some theorems will follow from that form that are closely connected with the theorem that was just cited and which we would like to express in the following way:

If each of $n+1$ $(n-1)$*-dimensional planes is perpendicular to the other* n *then the sum of the moments of a mass system relative to that plane will be equal to the mass times* k^2*.*

The moment of a mass relative to an arbitrary plane the moment of that mass relative to the absolute polar plane of the former differ from each other by the mass times k^2*.*

If one lets the quantities α in equation (41) vary in such a way that λ remains constant then the plane will continue to be the tangent plane to a structure of class two. The structure for which $\lambda = 0$ corresponds to the imaginary **Hess**ian structure of the body. For varying λ, equation (41) represents a family of confocal structures. That gives the following theorem:

All (n – 1)*-fold extended planes for which a given mass-system possesses the same moment will contact the same* (n – 1)*-fold extended surface of degree two. All of the structures that possess that property define a family of confocal surfaces.*

Since the moment is independent of the choice of coordinate system, one can refer the structure (41) to the principal coordinate system and represent it in the form:

(42) $$\lambda = A_0\, a_0^2 + A_1\, a_1^2 + \cdots + A_n\, a_n^2 \,.$$

That yields the following theorems:

The moment of inertia will assume its stationary value for the n + 1 *symmetry planes of the aforementioned quadratic structure. It attains its largest and smallest values only for those planes, but every other stationary value will be attained for the tangent planes to an* (n – 2)*-fold extended structure.*

One can distribute the mass at the n + 1 *midpoints of the structure* (42) *in such a way that the moment of those* n + 1 *mass-points relative to each* (n – 1)*-fold extended plane and is therefore equal to the moment of the given body relative to every principal structure.*

The last theorem can be extended in conjunction with a theorem by **Reye** (**Schlömilch**’s Zeitschrift, Bd. X and this journal Bd. 72).

Among the (n – 1)-fold extended principal structures that go through the same point, there are n of them for which the moment will be stationary. They are perpendicular to each other and define the symmetry planes for all tangent conical structures that can be laid from the point to the structure (42). Two of those n stationary moments can be equal to each other only when the point lies on a focal structure of the family. Naturally, the moments are also equal for all planes of a bundle in that case. However, they cannot be equal to more of them when the quantities $k^2 A_0, A_1, \ldots, A_n$ are distinct. However, if $k^2 A_0 = A_1 = \ldots = A_n = k^2 M / (n + 1)$, in particular, where M denotes the total mass, then that value will give the moment relative to each plane.

The theory of the moments of inertia relative to the points in space is obtained from the foregoing by means of the reciprocity theorem. According to it, all points for which a mass-system possesses equal moment lie on a structure of degree two. All such structures are similar and concentric. The moment assumes its greatest value for one of their midpoints and its least value for a second one. The moments that are obtained at all the other points then defines a conical structure, as well. Those quadratic structures have the same midpoint as the aforementioned one; it occurs at perhaps the center of mass of the **Euclid**ian space. We refer to it as the *center of inertia* and the symmetry planes of the quadratic structures as the *principal planes of inertia* of space. The moment of inertia assumes its stationary values for n points on every (n – 1)-fold extended plane when that plane does not contact a conical structure that belongs to the bundle of similar structures. Analogous statements are true for every plane of lower dimension.

The **Poinsot** moment ellipsoid can be adapted in several ways. If an r-fold extended plane is given, and if λ is the moment of inertial for an (r + 1)-fold extended plane that goes through the given plane then one raises a perpendicular to the first plane at one of its points that lies in the $(r + 1)$-fold extended plane and carries a line segment a from its foot for which $k \sin a / k$ is inversely proportional to the square root of λ. One makes that construction for all $(r + 1)$-fold extended planes that go through the given plane when one leaves the foot of the perpendicular unchanged. The end points of the perpendiculars then lie on a structure of degree two.

Since the proof for different values of r does not change very much, we choose $r = 0$ and then determine the moment of inertia for all lines that go through the same point. We choose that point to the initial point (1, 0, …, 0) of a **Weierstrass** coordinate system and choose the planes $x_1, \ldots, x_n$ to be the planes for which the moment of inertia assumes its stationary values. If a_ι denotes the cosine of the angle that an $(n - 1)$-fold extended plane that goes through the point will define the plane $x_\iota = 0$ then the moment of inertia for that plane will be equal to:

$$A_1 a_1^2 + \cdots + A_n a_n^2 .$$

For a line that goes through the point of intersection and is perpendicular to that plane, the moment will be equal to the difference between the moments that are true for the point and the plane, so it will equal:

$$(A_1 + \cdots + A_n) - (A_1 a_1^2 + \cdots + A_n a_n^2) = a_1^2 (A_2 + \cdots + A_n) + a_2^2 (A_1 + A_3 + \cdots + A_n) + \cdots + a_n^2 (A_1 + \cdots + A_{n-1}).$$

If one then draws a line segment r along that line from the origin then one will have:

$$x_\iota = a_\iota \, k \sin \frac{r}{k}$$

for its endpoint, which implies the theorem immediately. If one makes the line segment infinitely small then one will have to make the square root inversely proportional to the moment.

The theory of the moment of inertial finds its true meaning in its relationship to the *vis viva*. Up to a certain factor, it depends upon the position of the principal structure that moves within itself and the ratios of the velocities with which it moves. Therefore, if any motion is given then we can leave the ratios of the $\mu_{\alpha\beta}$ unchanged, but assume that the sum of the stationary values of r^2 is equal to k^2, so we can set:

$$\sum \mu_{0\kappa}^2 + k^2 \sum \mu_{\iota\kappa}^2 = k^2 .$$

Every motion that is defined in that way (every force-system of that kind, resp.) can be considered to be an element of an $\{\frac{1}{2} n (n + 1) - 1\}$-fold extended space, and the work done by a motion under the influence of a force-system implies the concept of distance. The *vis viva* will then lead to similar concentric second-order structures whose midpoints will represent certain motions, as the moment of inertia did just now. The importance of that consideration lies in the fact that a restricted mobility is represented in full generality in an especially simple way. I hope that these suggestions will suffice for everything that is known in the relevant investigations for $n = 3$.

§ 8. – On the finite motion of a rigid body.

Obviously, the previous formulas would simplify if I had taken $k\,x_0$ to be the coordinate and imposed the demand that it should be imaginary for **Lobachevski**an space forms. However, I feel that it would be appropriate to operate with real quantities, and $\cos r / k$, $k \sin r / k$, etc, are as well for negative k^2. Furthermore, I desire that it should emerge as clearly as possible that our formulas go to the known formulas of **Euclid**ian geometry for $k = \infty$. Thus, I shall also preserve the factor k^2 in what follows, although I could otherwise employ the known formulas of orthogonal substitution.

We imagine that one coordinate system is coupled with the moving body, while a second one is fixed in space. We denote the coordinates of the former by ξ, while the coordinates of the former will be denoted by x. We will then have the equations:

$$(43)\qquad \begin{cases} k^2\,\xi_0 = k^2 a_{00}\,x_0 + a_{01}\,x_1 + \cdots + a_{0n}\,x_n\,, \\ \xi_\kappa = \quad a_{\kappa 0}\,x_0 + a_{\kappa 1}\,x_1 + \cdots + a_{\kappa n}\,x_n\,. \end{cases}$$

Despite their similarity, we shall combine the equations of motion with the known equations:

$$(44)\qquad \begin{cases} k^2 a_{00}^2 + a_{10}^2 + \cdots + a_{n0}^2 = k^2 a_{00}^2 + a_{01}^2 + \cdots + a_{0n}^2 = k^2, \\ a_{00}\,a_{0\kappa} + a_{10}\,a_{1\kappa} + \cdots + a_{n0}\,a_{n\kappa} = a_{00}\,a_{\kappa 0} + a_{01}\,a_{\kappa 1} + \cdots + a_{0n}\,a_{\kappa n} = 0\,, \\ \dfrac{a_{0\iota}\,a_{0\kappa}}{k^2} + a_{1\iota}\,a_{1\kappa} + \cdots + a_{n\iota}\,a_{n\kappa} = \dfrac{a_{\iota 0}\,a_{\kappa 0}}{k^2} + a_{\iota 1}\,a_{\kappa 1} + \cdots + a_{\iota n}\,a_{\kappa n} = \delta_{\iota\kappa}\ (=1 \text{ or } 0). \end{cases}$$

We know that the final position that a solid body will attain under any motion can always be obtained by a uniform motion. Indeed, the proof that **Scheefer** gave for **Euclid**ian space forms in his dissertation (Berlin, 1880) is also true for the other spaces. One can also derive a proof from the treatise by **Rosanes** in volume 80 of this journal. However, I would like to add two other proofs.

The first one, which I would like to suggest only briefly, is based upon the problem: Find the points whose coordinates have the same ratios in the second position of the body that they had in the first. If we then set $\xi_\iota = \omega\,x_\iota$ then ω will be determined from the equation:

$$(45)\qquad E(\omega) = \begin{vmatrix} a_{00} - \omega & \dfrac{a_{01}}{k^2} & \cdots & \dfrac{a_{0n}}{k^2} \\ a_{10} & a_{11} - \omega & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n0} & a_{n1} & \cdots & a_{nn} - \omega \end{vmatrix} = 0.$$

That equation is reciprocal. If we next remove the roots ± 1 and determine those points that belong to two non-reciprocal roots then their distance will be $k\,\pi/2$. However, if x and x' belong to reciprocal roots then any point whose coordinates are $\alpha\,x + \alpha'\,x'$ and which belongs to the real domain under the condition that $2\alpha\,\alpha'\,(k^2\,x_0\,x'_0 + x_1\,x'_1 + \ldots + x_n\,x'_n) = k^2$ will take on the coordinates $\alpha\,\omega\,x + (\alpha'/\omega)\,x'$ under the motion, so it will belong to the same line. When we likewise examine the multiple roots, we will easily confirm that certain principal structures overlap in both positions, and that will support the further proof.

The following proof is even simpler:

Obviously, one has:

$$k^2 x_0\,\xi_0 + x_1\,\xi_1 + \cdots + x_n\,\xi_n = k^2\,a_{00}\,x_0^2 + a_{11}\,x_1^2 + \cdots + a_{nn}\,x_n^2 + (a_{01} + a_{10})\,x_0\,x_1 + \cdots$$
$$= k^2\,a_{00}\,\xi_0^2 + a_{11}\,\xi_1^2 + \cdots + a_{nn}\,\xi_n^2 + (a_{01} + a_{10})\,\xi_0\,\xi_1 + \cdots$$

The determinant of the form:

$$k^2\,a_{00}\,x_0^2 + a_{11}\,x_1^2 + \cdots + a_{nn}\,x_n^2 + (a_{01} + a_{10})\,x_0\,x_1 + \cdots - \sigma\,\Omega$$

will be the square of the determinant $E(\omega)$ when one sets $(1 + \omega^2)/\omega = 2\sigma$. (In order to obtain that product, one multiplies $E(\omega)$ with a determinant that differs from it by the fact that the element $a_{\iota 0}$ has the divisor k^2, instead of $a_{0\iota}$.)

That implies the following theorem:

All points that have a prescribed distance from their first position to the second one lie on a quadratic structure. Any such structure coincides with its initial position in its second position. All such structures belong to a similarity bundle. That bundle has the same type as the ones that have the same property under an infinitely-small motion.

One now determines those infinitely-small motions whose characteristic bundle is identical to the one that was just found, and the lines that move into themselves can also coincide for several roots of $E(\omega) = 0$. One gives the conical structures of the bundle the velocity ρ that corresponds to the value of σ for the conical structure that is obtained from the equation $\sigma = \cos \rho / k$. The continuation of that motion will take the body to the desired final position.

In order to obtain the differential equations of the motion of a solid body upon which arbitrary forces act, one forms the components of the motion along axes that are fixed in the body by setting:

$$(46)\qquad \left\{\begin{aligned} a_{00}\,da_{0\iota} + a_{10}\,da_{1\iota} + \cdots + a_{n0}\,da_{n\iota} &= \mu_{0\iota}\,dt, \\ \frac{a_{0\iota}\,da_{0\kappa}}{k^2} + a_{1\iota}\,da_{1\kappa} + \cdots + a_{n\iota}\,da_{n\kappa} &= \mu_{\iota\kappa}\,dt; \end{aligned}\right.$$

one then has:

$$\mu_{\alpha\beta} + \mu_{\beta\alpha} = 0 .$$

The moment of inertia is a homogeneous function of degree two in the quantities $\mu_{\alpha\beta}$, and one will obtain it when one multiplies both sides of equation (38) by *m* / 2 and sums over all points of the body. If the quantities $M_{\alpha\beta}$ have the meanings that they were given at the end of § **6** then one can write the desired equations in the form:

$$(47)\qquad \begin{cases} \dfrac{d}{dt}\dfrac{\partial T}{\partial \mu_{\kappa 0}} = \sum_{\iota}\left\{\dfrac{\partial T}{\partial \mu_{\iota 0}}\mu_{\iota\kappa} - \dfrac{1}{k^2}\mu_{\iota 0}\dfrac{\partial T}{\partial \mu_{\iota\kappa}}\right\} + M_{\kappa 0}, \\ \dfrac{d}{dt}\dfrac{\partial T}{\partial \mu_{\iota\kappa}} = \sum_{\alpha}\left\{\mu_{\alpha\iota}\dfrac{\partial T}{\partial \mu_{\alpha\kappa}} - \mu_{\alpha\kappa}\dfrac{\partial T}{\partial \mu_{\alpha\iota}}\right\} + M_{\iota\kappa}, \end{cases}$$

in which the summation over α also includes the index zero.

Whether the body is free or certain equations restrict its mobility, one can always find mutually-independent linear functions of μ for which *T* can be represents as a sum of squares. We first consider the case in which no constraint equations exist. We then choose the principal planes of inertia to be the coordinate planes that are coupled with the body. Since the *vis viva T* will then take the form:

$$2T = \mu_{01}^2\left(A_0 + \frac{A_1}{k^2}\right) + \cdots + \mu_{0n}^2\left(A_0 + \frac{A_n}{k^2}\right) + \mu_{12}^2\left(A_1 + A_2\right) + \cdots + \mu_{n-1,n}^2\left(A_{n-1} + A_n\right),$$

the equations of motion will now take the simple form:

$$(48)\qquad \begin{cases} \dfrac{d\mu_{\kappa 0}}{dt} = \dfrac{k^2 A_0 - A_1}{k^2 A_0 + A_1}(\mu_{01}\,\mu_{\kappa 1} + \mu_{02}\,\mu_{\kappa 2} + \cdots + \mu_{0n}\,\mu_{\kappa n}) + \dfrac{k^2 M_{\kappa 0}}{k^2 A_0 + A_1}, \\ \dfrac{d\mu_{\iota\kappa}}{dt} = \dfrac{A_\kappa - A_\iota}{A_\kappa + A_\iota}\left(\dfrac{\mu_{\iota 0}\,\mu_{\kappa 0}}{k^2} + \mu_{\iota 1}\,\mu_{\kappa 1} + \cdots + \mu_{\iota n}\,\mu_{\kappa n}\right) + \dfrac{M_{\iota\kappa}}{A_\kappa + A_\iota}. \end{cases}$$

Up to now, I have not succeeded in integrating those equations, even for the case in which all of the quantities $M_{\alpha\beta}$ vanish. A sequence of integrals can be obtained with somewhat greater ease, and for special values of the individual integration constants, the complete solution is not difficult either. However, due to the nature of this article, I believe that I do not need to go into the details of those analytical investigations, and at this point, I would then like to infer only the consequences of our equations that can be known from them without any calculation. We first ask: When will the motion be uniform? That is, how must the initial motion of a solid body be arranged in order for it to leave the body to itself?

That will happen when the first derivatives of the quantities $\mu_{\alpha\beta}$ vanish for $t = 0$, since the higher derivatives, which are homogeneous linear functions of the first derivatives, will likewise all vanish. When we compare the right-hand sides of equations (48) with equation (38) for the case in which the principal moments of inertia are all different from each other, we will arrive at the theorem:

For a given initial motion of a rigid body whose principal moments of inertia are all different from each other, one seeks the lines that are displaced into themselves. If a line that is displaced *into itself then goes through the all of the centers of inertia of the body then as long as it does not remain at rest, the body will continue its initial motion until external forces produce a different motion.*

One can also give that theorem the following form:

One determines the bundle of similar structures that are determined by the initial position. If the center of inertia is, at the same time, the midpoint for that structure then the body will preserve its initial motion.

It seems appropriate to me to apply that theorem to the various types of infinitely-small motions that were enumerated in the penultimate section. In that way, we will get the following theorem:

In a space form of an odd number of dimensions, in the absence of external forces, the general motion of a rigid body whose principal moments of inertia are unequal can be uniformly continued only when each of the $(n + 1) / 2$ *lines that are displaced into themselves contain two centers of inertia.*

If the equation $\Delta = 0$ has no vanishing root, but the roots split into groups of α, β, … equal ones (say, $\rho_\alpha^2, \rho_\beta^2, \ldots$) then the plane that moves with velocity ρ_α must contain 2α centers of inertia, and likewise the principal structure whose points all have the velocity ρ_β must include 2β of those points, etc.

If the motion is reciprocal then it will continue uniformly in the absence of external forces.

Should a space form of an even number of dimensions continue its motion uniformly, then a center of inertia would have to remain at rest. That condition is also sufficient only when the motion with which the absolute polar plane of the fixed point is displaced into itself is reciprocal to itself, or in other words, when a two-fold extended plane that moves within itself goes through every point of the body.

In order for an even-dimensional space form to continue the most general motion, two of the n moving centers of inertia must lie on a line that is displaced into itself.

If the structure at rest has r dimensions under a motion, and if the equation $\Delta = 0$ in ρ^2 has groups of γ, δ, … equal roots, in addition, then $r + 1$ centers of inertia must remain at rest, 2γ must move rectilinearly with velocity ρ_γ, 2δ must move likewise with velocity ρ_δ, etc.

It deserves to be pointed out that although self-reciprocal motion is more general than the parallel displacement in **Euclid**ian space, it still possesses all of its characteristic properties. I shall also draw attention to the fact that in a **Lobachevski**an space form, and likewise in an even-dimensional finite space (except for a an exception that will be mentioned later), there is no motion that continues uniformly for a completely-arbitrary position of the fixed body, as would be the case for parallel displacement in any **Euclid**ian space and the self-reciprocal motion in an odd-dimensional finite space form.

When the principal moments of inertia split into groups of ε, ζ, … equal ones, there will be an $(\varepsilon - 1)$-fold extended plane such that every point is a center of inertia, etc. Should every $d\mu_{\iota\kappa} / dt$

in equation (48) vanish for $t = 0$, then the coefficients of the $x_\alpha x_\beta$ in equation (38) for which A_α and A_β do not belong to the same group for unequal α and β do not have to vanish. The right-hand side of equation (38) will then split into groups, one of which contains only ε variables, a second contains only ζ, etc. The condition that a motion continues will now consist of saying that in any $(\varepsilon - 1)$-fold extended principal structure that contains nothing but centers of inertia, there are ε points that move in straight lines.

Finally, if all moments of inertia are equal to each other then any motion of the body will continue until external forces produce another motion.

Once that question has been treated exhaustively, we would like to mention a few conditions under which all continuations of a motion belong to the same (linear) group. In that regard, we have the general theorem:

If the initial motion of a body displaces a plane (straight line, resp.) into itself, with respect to which the imaginary image of the body (and therefore every structure that is confocal to it) is symmetric, then the continuation of that motion in the absence of external forces will preserve the same property.

The following remark is closely connected with the last theorem: When the constraint that acts upon a body consists of keeping one of the planes that were referred to in its initial position, in the absence of external forces, that motion will be precisely the same as when the body is free and only the initial motion satisfies the condition that was imposed.

Among the various types of motions that are not completely free, the rotation about a point is most important. We choose the fixed point to be the origin $(1, 0, \ldots, 0)$ of a **Weierstrass** coordinate system and the stationary planes of inertia of the point to be the planes $x_1, \ldots, x_n$. The motions will then assume the form (48) when one sets each $\mu_{\kappa 0}$ and $M_{\kappa 0}$ in them equal to zero and no longer understands the A_κ to mean principal moments of inertia, but the stationary values that the moment assumes for the planes that are laid through the fixed point. The equations of motion will also keep that form for any other constraint, except that the meanings of the $\mu_{\alpha\beta}$ and $M_{\alpha\beta}$ will change.

Braunsberg, in January 1884.

Remark. – It was only belatedly that it was possible for me to confer the paper by **Clifford**: "On the free motion under no forces of a rigid system in an *n*-fold Homaloid," Proc. London Math. Soc. VIII, pp. 67. In it, equations (48) were already developed for a body on which no forces acted that moved in a finite space form. However, when **Clifford** asserted that those equations could be integrated by simple ϑ-quotients, he forgot to point out that the solution, which is possible under only certain initial conditions, lacks the character of generality.

Braunsberg, in October 1884.