“Über gewisse in der Liniengeometrie auftretende Differentialgleichungen,” Math. Ann. **5** (1872), and *Gesammelte Abhandlungen.*

IX. On certain differential equations that appear in line geometry.

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Translated by D. H. Delphenich

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In line geometric investigations, one is led to certain differential equations that will be formulated in what follows and then integrated in more specific cases. They essentially correspond to the following three problems:

1. Find those line complexes whose lines are the tangents to a surface.
2. Determine those congruences that belong to a given line complex whose lines are principal tangents to their focal surfaces.
3. Obtain the enveloping curve of the lines of a congruence.

In the representation that we adopt in what follows, in which the use of line coordinates will be made throughout [1]), one recognizes how these three problems can be grouped together naturally.

Problem 1 is one of the first ones in which line complexes entered into the investigation. It was also already solved by Cayley in his first communication [2]), if only approximately. In it, Cayley examined those conditions that a line complex must satisfy in order for its lines to intersect a fixed curve. He found only a first condition, which is, however, still not sufficient, and is, moreover, fulfilled when the lines of the complex envelop surface [3]). This very condition will be derived in the sequel in a completely different way, and it will be shown that, in fact, it characterizes the totality of the tangents to a surface. For complexes of second degree, in particular, for the ones that envelop a surface, the corresponding result was found by Plücker (*Neue Geometrie*, no. 341).

Problem 2 was presented by Lie, but in a different form [4]). Namely, it amounts to finding those surfaces that contact the cone of a given complex at each of their points. He then showed that the edges along which the cone of the complex contacts the desired surface are principal tangents to the surface and that this property characterizes the

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[1]) The following may likewise serve to illustrate how one can operate with line coordinates without having to revert to their connection with point and plane coordinates.

[2]) Quarterly Journal, v. 3, pp. 227. [(1860), Coll. Papers IV.]

[3]) [A complete system of conditions for the secant complex of a curve was presented by Voss in a note in the Gött. Nachr. (1875), pp. 101-123.]

[4]) Cf., the treatise of Lie in Math. Annalen, Bd. 5, and its relevant communication to the Academy in Christiania (Berichte, 1870, 71). In it, the differential equation that the problem defined was referred to briefly as “the differential equation of the given line complex.” – Darboux has also dealt with this situation; cf., a relevant remark of Lie in the aforementioned reference. [The paper of Lie that is referred to as “the aforementioned reference” here and later in the text is the treatise that was mentioned in footnote [1]) of Abh. VIII.]

surface in question. The problem then comes down to that of finding the surfaces to which a system of principal tangents belongs to the given complex. In this form, it is obviously identical with problem 2; the form that it will assume in the sequel is better adapted to the demands of the present lecture.

Finally, problem 3 arises whenever one treats line congruences. From the general theory [5]), the lines of a congruence may be summarized in a series of developables in two ways. The problem is: Determine these developables when the congruence is given.

Problem 2 shall be solved in what follows in the particular case of line complexes of second degree. Such a solution was given by Lie in another, more geometric, way in the aforementioned treatise. These results naturally agree; it will therefore be interesting to pursue how their geometric content corresponds to the analytical ones that are applied here. His result may be summarized in an obvious way in the final analytical formula that will be presented.

Likewise, by the same method adopted here, problem 3 is disposed of for those line congruences of fourth order and class that two line complexes of second degree have in common, and actually belong to the singularity surface. Among these, one finds the special type that will likewise be suggested: the general congruences of second order and class that belong to a complex of second degree and a complex of first degree [6]).

From the previously-cited treatise of Lie, one will recognize how these line geometric problems are identical to problems that relate to *sphere geometry*. Problem 1 then corresponds to the demand: Give those equations between the four coordinates of a sphere (its center coordinates and its radius), which represent a three-fold infinite system of spheres (sphere complex), whose spheres all contact a fixed surface. Problem 2 then corresponds to the problem: If a sphere complex is given, find those surfaces whose system of principal spheres belongs to the complex [7]). Finally, problem 3 leads to: If a congruence of spheres – i.e., a doubly-infinite family of spheres – is given, arrange them into a singly-infinite sequence of spheres, in which each two consecutive ones contact each other.

In other words, starting with the connection that exists between line geometry and the metric point geometry of the space of four dimensions [8]), one can present equivalent problems for this metric geometry. The corresponding problems of the metric geometry of the space of three dimensions might be stated here; their number, like the number of variables, has been diminished by one. They are the following two:

a) Give those developable surfaces that include the infinitely distant imaginary circle.

b) Determine those curves on a given surface whose tangents continually meet the infinitely distant circle (the so-called "curves without length").

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[5]) Cf., Kummer in Borchardt's Journal, Bd. 57 (1860).

[6]) Cf., theorem XXXVI of the general enumeration of ray systems of second order by Kummer. (Abhandlungen der Berl. Akad. 1866.)

[7]) As he communicated to us, Darboux had also concerned himself with this problem, which Lie treated in the aforementioned treatise, along with problem 2), more recently. He solved it in precisely the context in which the solution was given by Lie, and in which it emerges by application of the Lie transformation of line geometry into sphere geometry from solution of the line geometric problem that was given in the text. The method that Darboux employed in it was, as far as I can tell, entirely identical with the one that will be applied here.

[8]) Cf., the present collection: "Über Liniengeometrie und metrische Geometrie."

The recent French geometers especially have concerned themselves with the latter curves. Incidentally, the problem of the conformal mapping of two surfaces onto each other emerges from the search for these curves. Namely, one has only to relate the two surfaces to each other in such a way that the curves in question on the one surface correspond to those of the other. The developables a) were first examined by Darboux [9]) in regard to their distinguished metric properties. As he communicated to me, Darboux has also solved problem 2) for the surfaces of fourth degree that contain the imaginary circle. By means of the Lie map (as Lie also remarked), this corresponds to the integration of the enveloping curves of a line congruence of second order and class that I will given in the following, and which I have already communicated before on occasion [Göttinger Nachrichten, 1871, no. 1 (but not included in this collection)]. It may suffice to give us a proof of the corresponding metric problem, which naturally relates to not only the metric space of three and four dimensions, but also arbitrarily many of them [10]). I now revert to the line-geometric problems that shall constitute the actual content of this communication, and thus begin by commenting on some things about linear complexes that will be useful later.

§ 1.

Some remarks about linear complexes.

In § 1 of the foregoing reference: "Über Liniengeometrie und metrische Geometrie," I have developed the way that line geometry can be regarded as the geometry of surface of second degree in a space of five dimensions [11]). It is represented, if one understands x_1, x_2, …, x_6 to mean the homogeneous coordinates of the space of five dimensions, by:

$$\Omega(x_1, x_2, \ldots, x_6) = 0.$$

A linear complex:

$$u_1 x_1 + u_2 x_2 + \ldots + u_6 x_6 = 0$$

is, in this way of looking at things, like the planes of the space in question. From this, one concludes that *a linear complex has an invariant*, namely, the expression that expresses the fact that when it is zero the plane $u_x = 0$ contacts the surface $\Omega = 0$. This expression arises from the determinant of Ω by bordering it (Ränderung) with the coefficients u. In particular, if Ω has [12]), as will be assumed in the sequel for the sake of simplicity, the form:

$$0 = x_1^2 + x_2^2 + \cdots + x_6^2$$

then the invariant takes the form:

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[9]) Annales scientifiques de l'École Normale Supérieure, t. 2, 1865.

[10]) From the combination of the articles that are given in § 2, it is obvious how this metric problem can be treated by exactly the same formulas as the line geometric ones that are used here.

[11]) This expression shall be allowed here, since indeed no misunderstanding can arise.

[12]) Cf., "Zur Theorie der Komplexe, etc.," Math. Annalen, Bd. 2 (1870). [See Abh. II of this collection.]

$$u_1^2 + u_2^2 + \cdots + u_6^2 .$$

If the invariant vanishes then the linear complex is a so-called *special* one; i.e., it consists of the totality of all lines that meet a fixed line.

Now let two linear complexes be given:

$$u = 0, \quad v = 0.$$

They have a linear congruence in common that likewise belongs to all complexes:

$$\lambda u + \mu v = 0.$$

Among them, one finds two special ones: the so-called *directrices.* One determines them when one defines the invariant of $\lambda u + \mu v$. Let A_{uu} be the invariant of u, A_{vv}, that of v, and to finally $A_{uu} = A_{vv}$ is the expression that comes about when one bounds (rändert) the determinant of Ω on the one side by the coefficients of u and on the other side, by those of v. I have occasionally called the expression A_{uv} the *simultaneous invariant* of the two complexes. (Its vanishing is the condition for the two complexes to be in involutory position.) By this notation, the invariant of $\lambda u + \mu v$ becomes:

$$\lambda^2 A_{uu} + 2 \lambda\mu A_{uv} + \mu^2 A_{vv} .$$

When set to zero, it yields a quadratic equation for λ / μ, and it is the one that determines the two directrices. The quadratic equation that is then obtained, when regarded as a quadratic binary form in the variables λ, μ, has an invariant:

$$A_{uu} A_{vv} - A_{uv}^2 .$$

This does not change (up to a factor) when one uses any two other complexes of the group $\lambda u + \mu v$, in place of u, v. It is then a *combination* of the two complexes u, v, and therefore an *invariant of the congruence that is determined by it.*

The vanishing of this invariant says that the quadratic equation for the determination of the directrices of the congruence has two equal roots so the directrices of the congruence coincide (cf., Plücker's *Neue Geometrie*, no. 68). The congruence shall then be called a *special* linear complex.

A further particularization enters in when not only $A_{uu} A_{vv} - A_{uv}^2 = 0$, but also A_{uu} , A_{vv}, A_{uv} vanish individually. u and v are then both special complexes (straight lines) that intersect each other. The congruence decomposes into two of them: A congruence of first order and null class that consists of the lines that go through the same intersection point, and a congruence of first class and null order that consists of the lines that run in the common plane. Such a linear congruence will be referred to in what follows as a *decomposed* one. A decomposed congruence has infinitely many directrices: The lines of the pencil that u and v belong to and that are represented by:

$$\lambda u + \mu v = 0.$$

We now consider the three linear complexes:

$$u = 0, \quad v = 0, \quad w = 0.$$

They have a ruled family in common – i.e., a generator of a surface of second degree (a one-sheeted hyperboloid). The other generators of the hyperboloids are the directrices of the congruence of any two complexes in the group:

$$\lambda u + \mu v + \nu w = 0.$$

One obtains all of the second generators when one chooses all of the values for λ, μ, ν for which the invariant of $\lambda u + \mu v$ vanishes, for which one then has:

$$0 = \lambda^2 A_{uu} + 2 \lambda \mu A_{uv} + \mu^2 A_{vv} + 2 \nu \lambda A_{uw} + 2 \mu \nu A_{vw} + \nu^2 A_{ww}.$$

If we interpret the λ, μ, ν as coordinates in the plane, this equation represents a conic section. It has an invariant under linear transformations that one can subject the λ, μ, ν to, namely, the determinant:

$$\begin{vmatrix} A_{uu} & A_{uv} & A_{uw} \\ A_{vu} & A_{vv} & A_{vw} \\ A_{wu} & A_{wv} & A_{ww} \end{vmatrix}.$$

We will refer to it as the *invariant* of the ruled surface that is common to the three complexes.

The vanishing of the invariant says: Firstly, that the conic section that is represented by the equation in λ, μ, ν decomposes. The second system of generators also decomposes then. The associated hyperboloid degenerates into two planes and two points that lie on their intersection in this case (cf., Plücker's *Neue Geometrie*, no. 144) [13]). The one generator of it consists of the lines that go through the first point in the first plane, or through the second point in the second plane; the other generator consists of the lines that go through the first point in the second plane or the second point in the first plane. The three complexes thus have two united pencils of rays in common. We will refer to such a decomposed ruled family as a *special* ruled family, in analogy with the foregoing.

A further particularization is that not only the invariant of the ruled family, but all of the sub-determinants, vanish identically. The ruled family then degenerates into two pencils of rays that cover it (cf., Plücker's *Neue Geometrie*, no. 146). The congruences of any two complexes of the family $\lambda u + \mu v + \nu w$ are then special, since their invariants are linear combinations of the vanishing sub-determinants.

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[13]) Therefore, while an F_2, considered as a point structure, yields the cone as its first particularization and, when considered as a plane structure, it yields the conic section, here, a plane-pair that is combined with a point-pair that lies in it enters in. It is interesting that one must consistently take all three particularizations into account for the general enumeration that relates to systems of surfaces of second degree. Cf., the paper of Schubert: "Zur Theorie der Characteristiken" (Borchardt's Journal, Bd. 71, 1870). The particularization of the F_2 that we spoke of here was referred to as the "limited planar section" there.

In the last case, it would be further conceivable that the second sub-determinants – i.e., A_{uu}, A_{uv}, etc. – themselves all vanish. u, v, w are then three special complexes whose axes mutually intersect, so either they have a point in common or lie in the same plane. Thus, a double infinity of lines then satisfy the three equations $u = 0$, $v = 0$, $w = 0$, namely, the ones that go through the common point (lie in the common plane, resp.). The equation of condition $\Omega = 0$ is then fulfilled identically by means of $u = 0$, $v = 0$, $w = 0$; the third equation – viz., $w = 0$ – then serves only to define one component of the decomposed congruence $u = 0$; $v = 0$, $w = 0$. This is then an essentially different case from the foregoing that will not come under consideration in what follows.

Finally, four complexes:

$$u = 0, \quad v = 0, \quad w = 0, \quad t = 0$$

can come under consideration. They have two lines in common, and for this pair of lines, one obtains the invariant:

$$\begin{vmatrix} A_{uu} & A_{uv} & A_{uw} & A_{ut} \\ A_{vu} & A_{vv} & A_{vw} & A_{vt} \\ A_{wu} & A_{wv} & A_{ww} & A_{wt} \\ A_{tu} & A_{tv} & A_{tw} & A_{tt} \end{vmatrix} .$$

If it vanishes then the two lines coincide [14]). If the first sub-determinants vanish then the two coinciding lines intersect [and thus the linear complexes have the entire pencil that is given by these two generators in common]. What the vanishing of the second and third sub-determinants means will remain unmentioned here.

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[14]) If one lets $t = 0$ mean a special complex, where A_{tt} vanishes, then the vanishing of the invariant in the text says that the line t contacts the hyperboloid of the three complexes $u = 0$, $v = 0$, $w = 0$. However, A_{tu}, A_{tv}, A_{tw}, are obviously nothing but the equations of the complexes u, v, w in which only the coordinates of the lines t are involved. However, if one briefly replaces A_{tu}, A_{tv}, A_{tw} by u, v, w then the resulting equation:

$$0 = \begin{vmatrix} A_{uu} & A_{uv} & A_{uw} & u \\ A_{vu} & A_{vv} & A_{vw} & v \\ A_{wu} & A_{wv} & A_{ww} & w \\ u & v & w & 0 \end{vmatrix}$$

represents the *equation of the hyperboloid of the complex* u, v, w, as I stated without proof in Math. Annalen, Bd. 2 (1870) [see Abh. II of this collection]. In a similar way, one finds the product of the equations of the two directrices of the congruence u, v:

$$0 = \begin{vmatrix} A_{uu} & A_{uv} & u \\ A_{vu} & A_{vv} & v \\ u & v & 0 \end{vmatrix} .$$

§ 2.

Formation of the differential equations.

Now, let an arbitrary complex be given:

$$\gamma = 0.$$

We consider one of its lines (x). In the vicinity of this line the complex can be regarded as a linear one; i.e., the neighboring lines are determined by a tangent linear complex, up to quantities of higher order (cf., Plücker's, *Neue Geometrie*, no. 297, et seq.). It is:

$$\left(\frac{\partial\varphi}{\partial x_1}\right) y_1 + \left(\frac{\partial\varphi}{\partial x_2}\right) y_2 + \cdots + \left(\frac{\partial\varphi}{\partial x_6}\right) y_6 = 0,$$

where the differential quotients in parentheses relate to the constant values of x. This linear tangential complex is not determined uniquely by this, but there is single infinitude of them with the same status. Namely, since the given complex:

$$\varphi = 0$$

will not change when one adds Ω to its equation with an arbitrary factor:

$$\lambda\varphi + \mu\,\Omega = 0,$$

so any linear complex that is included in the equation:

$$\sum \left(\lambda \frac{\partial\varphi}{\partial x_\alpha} + \mu \frac{\partial\varphi}{\partial x_\alpha} \right) \cdot y_\alpha = 0$$

is a linear tangential complex [15]). Therefore:

$$\frac{\partial\Omega}{\partial x_\alpha} \cdot y_\alpha = 0$$

is the equation of the special complexes whose lines all intersect the line x.

The simple infinitude of linear tangential complexes have a *special* linear congruence in common. In fact, as we will always do from now on, we take the simplified form for Ω:

$$\Omega = x_1^2 + x_2^2 + \cdots + x_6^2 ,$$

[15]) Among the linear tangential complexes there are three distinguished ones that have stationary contact. Cf., the previous treatise.

so the family of linear tangential complexes will be:

$$\sum \left(\lambda \frac{\partial \varphi}{\partial x_\alpha} + \mu x_\alpha \right) y_\alpha = 0.$$

The invariant of the individual complex is then equal to:

$$\lambda^2 \sum \left(\frac{\partial \varphi}{\partial x_\alpha} \right)^2 ,$$

since $\sum x_\alpha^2$ vanishes, on account of $\Omega = 0$, just as $\sum \frac{\partial \varphi}{\partial x_\alpha} \cdot x_\alpha$ vanishes on account of $\varphi = 0$, and when set to zero, it yields the double root $\lambda = 0$. The two directrices of the congruence then coincide, and indeed, in the given line (x).

Now, among the lines of the complex, there are, in particular, ones for which the tangent linear complexes decompose into common special congruences. The condition for this is, from the foregoing:

$$\sum \left(\frac{\partial \varphi}{\partial x_\alpha} \right)^2 = 0.$$

Then, all of the simple infinite of tangential complexes are special – i.e., straight lines – and these lines define a pencil. By means of this pencil, the given lines will be associated with a point that lies on them and a plane that goes through them. All of the lines of the complex that are infinitely close to the given one (x) and intersect it must either meet it at the associated point or run in the associated plane. The associated point is, for that reason, the common contact point for the line (x), and the complex curves that are contained in the planes that go through it; likewise, the associated planes of all cones that emanate from the points of the line (x) will contact the lines (x).

Complex lines (x) of this type are called *singular lines of the complex* by Plücker (no. 305, 306, of *Neue Geometrie*). The associated point is called the associated *singular point,* and the associated plane is called the associated *singular plane.*

If Ω has, as we assumed above, the simplified form $\sum x_\alpha^2 = 0$ then the singular lines of the complex:

$$\varphi = 0$$

will be singled out by the equation:

$$\sum \left(\frac{\partial \varphi}{\partial x_\alpha} \right)^2 = 0.$$

If φ is of degree m then this equation is of degree $2(m - 1)$; *the singular lines then define a congruence of order and class* $2m (m - 1)$.

I will, incidentally, connect this with the definition of a very important surface for the theory of complexes. Each of the two-fold infinity of singular lines is associated with a singular point and a singular plane. There is, then, a surface of singular points and a surface of singular planes. *These two surfaces are now identical and define a subset of the focal surface that is enveloped by congruence of the singular lines* [16]). In the following, I will refer to this surface as the *singularity surface* of the complex, as I have already done occasionally.

One can now give complexes of a special type – they will be called *special* complexes in what follows – for which:

$$\sum \left(\frac{\partial \varphi}{\partial x_\alpha} \right)^2 = 0,$$

vanishes identically, on account of $\sum x^2 = 0$, $\varphi = 0$, whose lines are all singular lines then. I assert that these complexes, whose lines envelop a surface, and thus, *the complexes that consist of the totality of tangents to a surface, are characterized by the differential equation* [17]):

$$\sum \left(\frac{\partial \varphi}{\partial x_\alpha} \right)^2 = 0.$$

It is now obvious that this condition is fulfilled for all complexes whose lines envelop a surface. Each line of such a complex has the character of a singularity. The complex curves – e.g., the ones that lie in the planes that go through them (the intersection curves of these planes with the enveloping surface) – contact the line at a fixed point, etc.

In order to show the converse of the theorem, we employ a lemma. Namely, let (x) be a line of the complex. Then, since:

$$\sum \left(\frac{\partial \varphi}{\partial x_\alpha} \right)^2 = 0,$$

($\partial \varphi / \partial x$) is also a straight line. Moreover, since:

$$\varphi = 0 \qquad \text{and thus} \qquad \sum x_\alpha \frac{\partial \varphi}{\partial x_\alpha} = 0,$$

it will intersect the line (x). For that reason, ($x + \lambda\, \partial \varphi / \partial x$) is a pencil of straight lines: viz., the pencil that was already discussed before of the special linear tangential complex that belongs to the singular line (x). *In the present case, this entire pencil of lines belongs to the complex* $\varphi = 0$.

The proof, which I hope to give more rigorously on another occasion, may be obtained as follows: If, as we assumed:

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[16]) It was this theorem that Pasch gave in his Habilitationschrift ("Zur Theorie der Komplexe, etc.," Giessen, 1870). In Plücker, one finds the corresponding theorem for complexes of second degree proved by a more circuitous route (no. 318-320).

[17]) [This theorem was also given for the first time in the treatise of Pasch that was cited above.]

$$\sum \left(\frac{\partial \varphi}{\partial x_\alpha} \right)^2 = 0,$$

then, by means of $\varphi = 0$, $\sum x_\alpha^2 = 0$, one can, as one can show, set [18]):

$$\sum \left(\frac{\partial \varphi}{\partial x_\alpha} \right)^2 = M\,\varphi + N \sum x_\alpha^2 .$$

One now defines:

$$\varphi\left(x_\alpha + \lambda \frac{\partial \varphi}{\partial x_\alpha} \right)$$

$$= \varphi(x_\alpha) + \lambda \sum \frac{\partial \varphi}{\partial x_\alpha} \cdot \frac{\partial \varphi}{\partial x_\alpha} + \frac{\lambda^2}{1\cdot 2} \cdot \sum \frac{\partial^2 \varphi}{\partial x_\alpha \partial x_\beta} \cdot \frac{\partial \varphi}{\partial x_\alpha} \cdot \frac{\partial \varphi}{\partial x_\beta} + \ldots$$

The terms in λ^0 and λ^1 vanish with no further restrictions when $\varphi = 0$, $\sum x_\alpha^2 = 0$. For the other terms, one can prove it by a recurrence process, in which one makes use of the given representation for $\sum \left(\frac{\partial \varphi}{\partial x_\alpha} \right)^2$.

The lines of the complex may then be arranged into the doubly infinite pencil:

$$x + \lambda \frac{\partial \varphi}{\partial x}.$$

The common intersection point of the lines of the pencil is the associated singular point for all of them, and the plane of the pencil is the singular plane that is associated with all of them. The three-fold infinitude of lines of the complex then correspond to only a two-fold infinitude of singular points and a two-fold infinitude of singular planes. There is then (as for general complexes) a surface of singular points and a surface of singular planes. Now, it is easy to see that (as with general complexes) these two surfaces are identical and the lines of the complexes are the tangents to these surfaces. In fact, any line of the complex must now contact the curve of the complex in an arbitrary plane that goes through it, as a singular line, at the associated singular points. The curve of the complex that is contained in a plane is then the intersection curve of the plane with the surface of singular points. The surface of singular points will then be enveloped by the lines of the complex. In fact, the lines of the complex thus envelop a surface: the surface of singular points. One proves the same thing for the surface of singular planes. The surface of singular points and the surface of singular planes are then identical [19]).

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[18]) [The allowability of the Ansatz for algebraic complexes is obtained the argument that was developed in the note: "Über einen lineiengeometrischen Satz (Gött. Nachr., 1872, Math. Ann., Bd. 22, and Abh. X of this collection).]

[19]) One can, as one might do for the moment here, define the singularity surface of a complex as those special complexes that circumscribe the complex, and the focal surface of a congruence as those special

We now consider the congruence that is common to two complexes:

$$\varphi = 0, \quad \psi = 0.$$

A line of it has the coordinates x. In the vicinity of it, one can replace the two complexes with a linear tangential complex to it:

$$\sum \left(\frac{\partial \varphi}{\partial x_\alpha} + \lambda x_\alpha \right) y_\alpha = 0,$$
$$\sum \left(\frac{\partial \psi}{\partial x_\alpha} + \lambda x_\alpha \right) y_\alpha = 0.$$

One can replace the given congruence in the vicinity of (x) by the linear congruence that is common to any two of these complexes. *There is then a two-fold infinitude of linear congruences that contact a given congruence in one of its lines* (x).

These linear congruences all have a ruled family in common:

$$\sum \frac{\partial \varphi}{\partial x_\alpha} \cdot y_\alpha = 0, \qquad \sum \frac{\partial \psi}{\partial x_\alpha} \cdot y_\alpha = 0, \qquad \sum x_\alpha y_\alpha = 0.$$

However, this divides into two pencils, since its invariant:

$$\begin{vmatrix} \sum \left(\frac{\partial \varphi}{\partial x_\alpha} \right)^2 & \sum \frac{\partial \varphi}{\partial x_\alpha} \cdot \frac{\partial \psi}{\partial x_\alpha} & \varphi \\ \sum \frac{\partial \varphi}{\partial x_\alpha} \cdot \frac{\partial \psi}{\partial x_\alpha} & \sum \left(\frac{\partial \psi}{\partial x_\alpha} \right)^2 & \psi \\ \varphi & \psi & \sum x_\alpha^2 \end{vmatrix}$$

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complexes that the congruence belong to. [The intrinsic connection between singularity surfaces and the other structure that is present here with the corresponding differential equations will perhaps become more obvious when I mention the original concept that I was led to by this line of reasoning. One starts with the line coordinates that satisfy the identity $x_1^2 + x_2^2 + x_3^2 + x_4^2 + pq = 0$, sets $q = 1$, and then eliminates p from the equation of the complex. Then, e.g., the partial differential equation of the special complex will become:

$$\left(\frac{\partial \Phi}{\partial x_1} \right)^2 + \left(\frac{\partial \Phi}{\partial x_2} \right)^2 + \left(\frac{\partial \Phi}{\partial x_3} \right)^2 + \left(\frac{\partial \Phi}{\partial x_4} \right)^2 = 0.$$

One sees the analogy with the developables that are circumscribed by the spherical circle in ordinary R_3. The "pencil" of complexes corresponds to the generators of the developables (and thus the characteristics of the partial differential equation), the union of the positions of consecutive pencils of the intersection of successive generators (the union of the positions of consecutive characteristic strips). K.]

vanishes, by means of $\varphi = 0$, $\psi = 0$, $\sum x_\alpha^2 = 0$. The directrices of the two-fold infinitude of tangent congruences then consist of two pencils that have the lie (x) in common. Any two lines that are taken from the two pencils are directrices of a tangent congruence. In these two pencils, one recognizes the tangent pencil [20]) of the focal surface to the congruence in whose contact points with the lines (x). (?)

Among the lines of a congruence there will, in particular, be ones for which the two pencils – i.e., the two contact points with the focal surface – coincide. The line of the congruence, which was previously a double tangent to the focal surface, generally becomes a tangent with four-point contact. These lines of the congruence are represented by the condition that for them the sub-determinants of the foregoing invariants vanish, which can, by means of the simplified form for the latter, be reduced to one condition:

$$\sum\left(\frac{\partial\varphi}{\partial x_\alpha}\right)^2\cdot\sum\left(\frac{\partial\psi}{\partial x_\alpha}\right)^2-\left(\sum\frac{\partial\varphi}{\partial x_\alpha}\cdot\frac{\partial\psi}{\partial x_\alpha}\right)^2=0.$$

This equation, together with:

$$\varphi = 0,\quad \psi = 0,\quad \sum x_\alpha^2 = 0,$$

represents a line surface of the congruence that has four-point contact with the focal surface. If φ is of degree m and ψ is of degree n then this surface will be of degree $4mn(m + n - 2)$.

Those lines of a congruence $[m, n]$ *that have four-point contact with the focal surface generally define a line surface of degree:*

$$4m\,n\,(m + n - 2).$$

There will now be certain congruences – which shall be called *special* congruences – for which the present equation:

$$\sum\left(\frac{\partial\varphi}{\partial x_\alpha}\right)^2\cdot\sum\left(\frac{\partial\psi}{\partial x_\alpha}\right)^2-\left(\sum\frac{\partial\varphi}{\partial x_\alpha}\cdot\frac{\partial\psi}{\partial x_\alpha}\right)^2=0$$

is fulfilled identically, by means of:

$$\varphi = 0,\quad \psi = 0,\quad \sum x_\alpha^2 = 0.$$

These have the peculiarity that all of their lines contact the focal surface in coincident points. They are the congruences whose lines are principal tangents to the focal surface

[20]) In the ordinary representation, one distinguishes the lines of the two pencils that are perpendicular to (x) and refers to them as the *focal lines* of the infinitely thin bundle of rays that run in the vicinity of (x). Any other line pair that is taken from the two pencils is equivalent in the projective sense.

[21]), in contrast to the general congruences whose lines are double tangents to the focal surface.

With this, problem (2) *is then also formulated* [22]). If a line complex $\varphi = 0$ is given then one seeks those congruences $\varphi = 0$, $\psi = 0$ such that for the lines of the congruence:

$$\sum\left(\frac{\partial\varphi}{\partial x_\alpha}\right)^2 \cdot \sum\left(\frac{\partial\psi}{\partial x_\alpha}\right)^2 - \left(\sum\frac{\partial\varphi}{\partial x_\alpha}\cdot\frac{\partial\psi}{\partial x_\alpha}\right)^2 = 0.$$

In particular, if $\varphi = 0$ is a special complex – i.e., a surface – then this equation reduced to:

$$\sum\frac{\partial\varphi}{\partial x_\alpha}\cdot\frac{\partial\psi}{\partial x_\alpha} = 0.$$

Finally one might be given three complexes:

$$\varphi = 0, \quad \psi = 0, \quad \chi = 0.$$

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[21]) Cf., Kummer, "Allgemeine Theorie der Strahlensysteme," § 8 (Borchardt's Journal, Bd. 57).

[22]) Lie already endowed this problem with a similar form. Namely, he found that under a transformation that arises from introducing the lines of the complex as space elements, the differential equation in question goes to an equation of second degree. In particular, he gave the equation:

$$p\cdot\frac{\partial H}{\partial x} + q\frac{\partial H}{\partial y} - \frac{\partial H}{\partial z} = \sqrt{1+p^2+q^2}\cdot\sqrt{\left(\frac{\partial H}{\partial x}\right)^2+\left(\frac{\partial H}{\partial y}\right)^2+\left(\frac{\partial H}{\partial z}\right)^2-1}$$

(Bericht der Akademie zu Christiana, 1870, December). If one would like to apply point coordinates instead of line coordinates then one would be led to a partial differential equation that is, on first glance, very different. If one has, by the use of the p_{ik} coordinates, the equation of the complex:

$$\varphi(p_{ik}) = 0$$

then, when one confers fixed values to x, the equation:

$$\varphi(x_i\, y_k - y_i\, x_k) = 0$$

represents the cone of the complex that emanates from the point (x). Problem (2) now consists of finding those surfaces $y(x) = 0$ that will contact the cone of the complex in question at each of their points. If one then expresses the condition that the plane:

$$\frac{\partial\psi}{\partial x_1}\cdot y_1 + \frac{\partial\psi}{\partial x_2}\cdot y_2 + \frac{\partial\psi}{\partial x_3}\cdot y_3 + \frac{\partial\psi}{\partial x_4}\cdot y_4 = 0$$

contacts the cone:

$$\psi(x_i\, y_k - y_i\, x_k) = 0$$

then one has the differential equation of the problem. If φ is of degree m then the differential quotients ∂y / ∂x will generally be of degree $m\,(m-1)$.

They have a common line surface. Let (x) be a line on it. It has the following tangential complex relative to φ, ψ, χ:

$$\sum\left(\frac{\partial\varphi}{\partial x_\alpha}+\lambda x_\alpha\right)y_\alpha=0,$$

$$\sum\left(\frac{\partial\lambda}{\partial x_\alpha}+\mu x_\alpha\right)y_\alpha=0,$$

$$\sum\left(\frac{\partial\chi}{\partial x_\alpha}+\nu x_\alpha\right)y_\alpha=0.$$

Any three complexes from these three families have a common hyperboloid that contacts the rectilinear surface that is common to the three given complexes at (x). *There is a three-fold infinitude of such contacting hyperboloids.* All have two coinciding lines in common, namely, (x) and the neighboring generators, which are the common lines of the four complexes:

$$\sum\frac{\partial\varphi}{\partial x_\alpha}\cdot y_\alpha=0,\qquad \sum\frac{\partial\psi}{\partial x_\alpha}\cdot y_\alpha=0,\qquad \sum\frac{\partial\chi}{\partial x_\alpha}\cdot y_\alpha=0,\qquad \sum x_\alpha y_\alpha=0.$$

The invariant of the line-pairs that are common to these complexes:

$$\begin{vmatrix}
\sum\left(\frac{\partial\varphi}{\partial x_\alpha}\right)^2 & \sum\frac{\partial\varphi}{\partial x_\alpha}\cdot\frac{\partial\psi}{\partial x_\alpha} & \sum\frac{\partial\varphi}{\partial x_\alpha}\cdot\frac{\partial\chi}{\partial x_\alpha} & \varphi \\
\sum\frac{\partial\psi}{\partial x_\alpha}\cdot\frac{\partial\varphi}{\partial x_\alpha} & \sum\left(\frac{\partial\psi}{\partial x_\alpha}\right)^2 & \sum\frac{\partial\psi}{\partial x_\alpha}\cdot\frac{\partial\chi}{\partial x_\alpha} & \psi \\
\sum\frac{\partial\chi}{\partial x_\alpha}\cdot\frac{\partial\varphi}{\partial x_\alpha} & \sum\frac{\partial\chi}{\partial x_\alpha}\cdot\frac{\partial\psi}{\partial x_\alpha} & \sum\left(\frac{\partial\chi}{\partial x_\alpha}\right)^2 & \chi \\
\varphi & \psi & \chi & \sum x_\alpha^2
\end{vmatrix}$$

then vanishes.

For particular lines of the line surface all of the sub-determinants of this invariant will also vanish. They are the so-called *singular* generators of the line surface that intersect it consecutively. For their determination, one obtains:

$$0 = \begin{vmatrix} \sum\left(\frac{\partial\varphi}{\partial x_\alpha}\right)^2 & \sum\frac{\partial\varphi}{\partial x_\alpha}\cdot\frac{\partial\psi}{\partial x_\alpha} & \sum\frac{\partial\varphi}{\partial x_\alpha}\cdot\frac{\partial\chi}{\partial x_\alpha} \\ \sum\frac{\partial\psi}{\partial x_\alpha}\cdot\frac{\partial\varphi}{\partial x_\alpha} & \sum\left(\frac{\partial\psi}{\partial x_\alpha}\right)^2 & \sum\frac{\partial\psi}{\partial x_\alpha}\cdot\frac{\partial\chi}{\partial x_\alpha} \\ \sum\frac{\partial\chi}{\partial x_\alpha}\cdot\frac{\partial\varphi}{\partial x_\alpha} & \sum\frac{\partial\chi}{\partial x_\alpha}\cdot\frac{\partial\psi}{\partial x_\alpha} & \sum\left(\frac{\partial\chi}{\partial x_\alpha}\right)^2 \end{vmatrix}.$$

When φ, ψ, χ are of degree m, n, p, resp., this equation is of degree $2(m + n + p - 3)$. One then obtains the theorem: *The line surface that is common to three complexes of degree m, n, p, resp., has:*

$$4m\,n\,p(m + n + p - 3)$$

singular generators, in general [23]).

There is now a *special* line surface whose lines are all singular generators – i.e., they intersect consecutively: These are the *developables.* They are characterized by the fact that for them, by means of $\varphi = 0$, $\psi = 0$, $\chi = 0$, the present equation is fulfilled identically. Thus if φ and ψ are given and one regards χ as unknown then *this equation represents the differential equation for the developables of the congruence* $\varphi = 0$, $\psi = 0$, $\chi = 0$, which was problem (3). It will be linear, in particular, when $\varphi = 0$, $\psi = 0$, $\chi = 0$ is a special congruence.

We will determine the enveloping curves of the congruence of two complexes that are associated with the same singularity surface in a somewhat different way, namely, we express the lines of the congruence by two parameters and then present the condition for two neighboring lines of the congruence to intersect. To that end, the condition may be given here under which the two neighboring lines (x) and ($x + dx$) intersect, at all. In order for two lines (x) and (y) to intersect, one must have:

$$\sum x_\alpha y_\alpha = 0.$$

However, if $y_\alpha = x_\alpha + dx_\alpha$ then this equation is satisfied identically, because the y_α, as line coordinates, are linked with the equation $\sum y_\alpha^2 = 0$, which, since $\sum x_\alpha^2 = 0$, leads to $\sum x_\alpha dx_\alpha = 0$. If we now set $y_\alpha = x_\alpha + dx_\alpha + d^2x_\alpha$ then we have, since $\sum y_\alpha^2 = 0$:

$$\sum x_\alpha dx_\alpha = 0, \qquad \sum (2x_\alpha d^2x_\alpha + dx_\alpha^2) = 0.$$

On the other hand, the condition for the intersection becomes:

$$\sum x_\alpha d^2x_\alpha = 0$$

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[23]) The same number was derived in a somewhat different way by Lüroth: "Zur Theorie der windschiefen Flächen" (Borchardt's Journal, Bd. 67, 1867).

and by means of the last equation this reduces to:

$$\sum dx_\alpha^2 = 0.$$

This is the condition for the intersection of two consecutive lines that will be applied in what follows [24]).

§ 3.

Elliptical coordinates for the determination of straight lines [25]). Determination of various enveloping curves.

I will now introduce, in place of the homogeneous line coordinates $x_1, \ldots, x_6$ that we have been using up to now, and which are coupled by the equation of condition:

$$\sum x_\alpha^2 = 0,$$

four mutually independent, inhomogeneous coordinates $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. They will be defined in the following way:

In an earlier paper [Math. Ann., Bd. 2 (see Abh II of this collection)], I showed that the complexes of second degree with a common singularity surface can be represented by a parameter t in the following way:

$$0 = \frac{x_1^2}{k_1-\lambda} + \frac{x_2^2}{k_2-\lambda} + \cdots + \frac{x_6^2}{k_6-\lambda},$$

where $x_1, x_2, \ldots, x_6$ are coordinates of the type that was just considered. In the general case in which this canonical form applies, the common singularity surface is a Kummer surface of fourth degree with 16 nodes.

When one sets the x equal to the coordinates of a straight line, one can now consider the present equation as an equation for λ. It is of fourth degree, since the power λ^5 that appears by multiplication has the vanishing factor $\sum x^2$. The four roots of the equation shall be called $\lambda_1, \lambda_2, \lambda_3, \lambda_4$; these are the ones that will be employed as coordinates of the line from now on. *These four coordinates then give the value of the parameter λ for those four complexes of the system that are associated with the line in question.*

As one sees this coordinate determination is analogous to the general Jacobi method for elliptical coordinates. In the Jacobi method, one has only one equation, and indeed an inhomogeneous one, of the form [26]):

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[24]) Lie employed the following condition for the intersection of two consecutive lines (or the contact of two consecutive spheres):

$$dx^2 + dy^2 + dz^2 + (i\, dH)^2 = 0.$$

[25]) Cf., a note in the Göttinger Nachrichten, 1871, no. 1. [Not included in the present collection, because it included the developments of the text.]

[26]) For Jacobi, the parameter λ is given another sign, which does not, however, seem advantageous.

$$\sum_{\alpha=1}^{n} \frac{x_\alpha^2}{k_\alpha - \lambda} = 1,$$

while two homogeneous equations are given here:

$$\sum_{\alpha=1}^{n+1} \frac{x_\alpha^2}{k_\alpha - \lambda} = 0, \qquad \sum_{\alpha=1}^{n+1} x_\alpha^2 = 0.$$

This general [27]) type of elliptical coordinates was first mentioned by Darboux [28]) and were developed by him in a recent treatise [29]). He referred to them as the first derivation of the ordinary elliptical coordinates, insofar as he was led to the first derivation by a single application of a process by which he could derive a new orthogonal system from any orthogonal system. When applied to the system of confocal surfaces of second degree, this process yields the Darboux-Moutard orthogonal system of the surface of fourth degree that includes the imaginary circle, and the new coordinates refer to this system. (Cf., on this, the aforementioned treatise, § 2, as well.)

We will next express the previous coordinates x_α by the new ones λ_α. To that end, $f(\lambda)$ might refer to the expression:

$$f(\lambda) = (k_1 - \lambda)(k_2 - \lambda) \ldots (k_6 - \lambda).$$

One then has, as is well-known, the relations:

$$\sum \frac{1}{f'(k_\alpha)} = 0, \ \sum \frac{k_\alpha}{f'(k_\alpha)} = 0, \ \sum \frac{k_\alpha^2}{f'(k_\alpha)} = 0, \ \sum \frac{k_\alpha^3}{f'(k_\alpha)} = 0, \ \sum \frac{k_\alpha^4}{f'(k_\alpha)} \ \sum \frac{k_\alpha^3}{f'(k_\alpha)} = 0.$$

As a result, the x_a are given by the following equation:

$$\rho x_\alpha^2 = \frac{(k_\alpha - \lambda_1)(k_\alpha - \lambda_2)(k_\alpha - \lambda_3)(k_\alpha - \lambda_4)}{f'(k_\alpha)}.$$

In fact, one convinces oneself that, as a result of the equations that exist between $f'(k)$ and nothing more, these values of x_α^2 satisfy the equation $\sum x_\alpha^2 = 0$, as well as the four complex equations that correspond to the values $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ of λ.

[27]) One can refer to these coordinates are *more general*, since they yield the ordinary elliptical coordinates when two x_a coincide. This would correspond to a degeneracy of the Kummer surface into a surface with double lines; i.e., into a surface of a Plücker complex.

[28]) Comptes rendus, t. 69, 1869, 2. "Sur une nouvelle séries de systèmes orthogonaux algébriques."

[29]) Comptes rendus, t. 73, 1871, 2. "Des courbes tracées sur une surface et dont la sphère osculatrice est tangente en chaque point à la surface."

We might remark incidentally how the main elements of the given system of complexes will be represented by the parameter λ and the Kummer surface that is linked with it [30]).

If one sets two of the λ parameters – say, λ_3 and λ_4 – equal to each other then one has a tangent to the Kummer surface. If one considers λ_1 and λ_2 to be constant, while $\lambda_3 = \lambda_4$ runs through all values then one obtains the pencil of all tangents that contact the surface at a point. λ_1 and λ_2 then characterize the contact point; one can regard them as coordinates of the points on the surface [31]). The two pencils that are composed of the tangents at two points (λ_1, λ_2) and (λ_1', λ_2') are then, corresponding to $\lambda_3 = \lambda_4$, uniquely – and therefore, projectively – related to each other.

If one sets the three parameters equal to each other then one obtains the principal tangents of the surface.

If one takes the four parameters to be pair-wise equal then one has the lines that fill up the 16 double planes of the surface and the ones that go through the 16 double points.

Finally, the assumption that all of the parameters are equal to each other yields the tangents of the contacting conic sections that lie in the 16 double planes, as well as the generators of the cone that contacts the 16 nodes.

One obtains the lines that belong to a certain complex of the system when one sets one of the parameters – say, λ_4 – equal to the λ in question. If one takes two parameters – say, λ_3 and λ_4 – to be constant then one gets the lines of the congruence that are common to the two complexes $\lambda = \lambda_3$ and $\lambda = \lambda_4$. Thus, if one likewise has $\lambda_3 = \lambda_4$ then one gets the *singular lines* of the complexes $\lambda = \lambda_3 = \lambda_4$. Therefore, the tangents in any pencil of tangents to the Kummer surface that are associated with the complex $\lambda = \lambda_3 = \lambda_4$ as singular lines will be determined through the values of $\lambda_3 = \lambda_4$. If $\lambda_3 = \lambda_4 = k_\alpha$ then the singular lines are *double tangents* to the Kummer surface, namely, the ones that belong to the linear (fundamental) complex $x_\alpha = 0$ that is found among the complexes of the system. – If three parameters are constant – say, λ_2, λ_3, λ_4 – then one obtains the generators of the line surfaces that are common to the three complexes $\lambda = \lambda_2$, $\lambda = \lambda_3$, $\lambda = \lambda_4$. Thus, if $\lambda_2 = \lambda_3 = \lambda_4$ then on gets the singular lines of the complex $\lambda = \lambda_2 = \lambda_3 = \lambda_4$ that osculate the Kummer surface. When the common value of $\lambda_2 = \lambda_3 = \lambda_4$ is equal to k_α, they are then lines with four-point contact. Indeed, when one endows α with the values 1, …, 6, one obtains all of the lines of the Kummer surface that have four-point contact, except for the ones that are tangent to the conic section of contact in the 16 double planes [32]). – Finally, the assumption that all parameters are constant yields the 32 straight lines that are common to the complexes $\lambda = \lambda_1$, $\lambda = \lambda_2$, $\lambda = \lambda_3$, $\lambda = \lambda_4$. Therefore, if $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$ then one has the 32 distinguished singular lines of the complex $\lambda = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$ that are tangents to the conic section of contact in the double planes (generators of the contact cone in the double points, resp.).

We would now like to substitute the new coordinates λ_1, λ_2, λ_3, λ_4 in the equation:

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[30]) For the proof, see the paper: "Zur Theorie der Komplex, etc.," Math. Annalen, Bd. 2 (1870). [See Abh. II of this collection.]

[31]) As shall be shown, the curves $\lambda_1 = \rho$, $\lambda_2 = \sigma$ are the principal tangent curves to the Kummer surface.

[32]) Cf., the paper of Lie and myself: "Über die Hauptangentenkurven der Kummerschen Fläche." Monasberichte der Berliner Akademie, 1870, December. [See Abh. VI of this collection.]

$$\sum dx_\alpha^2 = 0,$$

which expresses the fact that two consecutive lines intersect. One finds:

$$\begin{aligned}
0 = \; & d\lambda_1^2 \cdot \frac{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}{f(\lambda_1)} \\
& + d\lambda_2^2 \cdot \frac{(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)(\lambda_2-\lambda_1)}{f(\lambda_2)} \\
& + d\lambda_3^2 \cdot \frac{(\lambda_3-\lambda_2)(\lambda_3-\lambda_1)(\lambda_3-\lambda_4)}{f(\lambda_3)} \\
& + d\lambda_4^2 \cdot \frac{(\lambda_4-\lambda_1)(\lambda_4-\lambda_2)(\lambda_4-\lambda_3)}{f(\lambda_4)}.
\end{aligned}$$

This differential equation will now be integrated with no further assumptions.

It especially arises for the congruences of any two complexes of the given system. Namely, if one sets λ_3 and λ_4 equal to constants then $d\lambda_3$, $d\lambda_4$ will be equal to zero, the factor $(\lambda_1 - \lambda_2)$ will drop away, and *one will obtain the differential equation of the enveloping curve of the congruence in the form:*

$$d\lambda_1 \sqrt{\frac{(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}{f(\lambda_1)}} = \pm\, d\lambda_2 \sqrt{\frac{(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)}{f(\lambda_2)}}\,.$$

If one sets $\lambda_3 = \lambda_4$ in this equation then one has *the enveloping curves of those double tangents to the Kummer surface that belong to the complex* $x_\alpha = 0$. As is known, these double tangents define a general congruence of second order and second class; Problem 3 is then solved for these congruences.

If one sets $\lambda = \lambda_2$, $\lambda_3 = \lambda_4$ in the present differential equation then it will be satisfied identically. The congruence $\lambda_1 = \lambda_2$, $\lambda_3 = \lambda_4$ is then one in which all lines intersect all of their neighbors. In fact, as we already remarked, the equations $\lambda_1 = \lambda_2$, $\lambda_3 = \lambda_4$ represent the lines that either lie in a double plane of the Kummer surface or go through a double point. Their totality defines a congruence of order and class 16 that generally has the required property.

Finally, let $\lambda_2 = \lambda_3 = \lambda_4$. The present differential equation will then be:

$$d\lambda_1 = 0, \qquad \text{then} \qquad \lambda_1 = \text{const.}$$

These are the principal tangent curves of the Kummer surfaces. In words: The principal tangent curves of the Kummer surface will, in any case, be defined by the points of the surface at which the second principal tangent belongs to a certain complex of the system as a singular line. They are then the curves of order 16 that were considered in no. 18 of the earlier paper: "Zur Theorie, etc." (Math. Annalen, Bd. 2 [see Abh. II of this collection]). The fact that the principal tangent curves of the Kummer surface are algebraic curves of first order was first discovered by Lie when he was studying his map from line geometry to sphere geometry that takes principal tangent curves to curvature curves. Thus, I remarked that there is an identity between the principal tangent curves and the curve systems that I had previously examined and determined in connection with the singularities themselves [33]). I then found the analytical proof [34]) that is presented here, and finally realized [35]) that it subsumed the entire process of determining the principal tangent curves by means of the associated complex under a general, line-geometric theorem that corresponded to Dupin's theorem of metric geometry. The latter is thoroughly presented in the previous paper, in which it is also shown how it encompasses the determination of the principal tangent curves of the Kummer surface. – Let it be remarked that the six principal tangent curves $\lambda_1 = k_a$ are the curves of four-point contact with the Kummer surface.

§ 4.

Determination of the integral surfaces for the general complex of second degree.

The introduction of new variables λ_1, λ_2, λ_3, λ_4 into the differential equation:

$$\sum dx_\alpha^2 = 0$$

yields:

$$0 = d\lambda_1^2 \cdot \frac{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)}{f(\lambda_1)} + \dots$$

The partial differential equation that characterizes the special complexes:

$$\sum \left(\frac{\partial\varphi}{\partial x_\alpha}\right)^2 = 0$$

will then be converted, by known methods, into:

$$0 = \left(\frac{\partial\varphi}{\partial\lambda_1}\right)^2 \cdot \frac{f(\lambda_1)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} + \left(\frac{\partial\varphi}{\partial\lambda_2}\right)^2 \cdot \frac{f(\lambda_2)}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_4)(\lambda_2-\lambda_1)} + \dots$$

Now, it is, however, known [36]) that a differential equation like the present one admits a complete solution (with three arbitrary constants) with no further assumptions, namely:

$$\varphi = \int d\lambda_1 \frac{\sqrt{(\lambda_1-a)(\lambda_1-b)}}{\sqrt{f(\lambda_1)}} + \int d\lambda_2 \frac{\sqrt{(\lambda_2-a)(\lambda_2-b)}}{\sqrt{f(\lambda_2)}} + \int d\lambda_3 \frac{\sqrt{(\lambda_3-a)(\lambda_3-b)}}{\sqrt{f(\lambda_3)}}$$

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[33]) Monatsberichte der Berl. Akademie, 1870, December. [See Abh. VI of this collection.]
[34]) Göttinger Nachrichten, 1871, no. 1. [Not included in this collection.]
[35]) Göttinger Nachrichten, 1871, no. 3. [See Abh. VII of this collection.]
[36]) Cf., Jacobi's *Vorlesungen über Dynamik.*

$$+\int d\lambda_4 \frac{\sqrt{(\lambda_4 - a)(\lambda_4 - b)}}{\sqrt{f(\lambda_4)}} + C.$$

If we let a, b, C take on the sequence of all possible values then for us the equation:

$$\varphi = 0$$

represents a three-fold infinitude of special complexes – viz., a three-fold infinitude of surfaces. Any complex that is included in the general solution – i.e., any complex whose lines envelop a surface – will then, as the enveloping structure, include a two-fold infinitude of these surfaces.

I now assert that *the surface* $\varphi = 0$ *with the constants a, b, C is the common integral of the two complexes* $\lambda = a$ *and* $\lambda = b$ [37]); i.e., that the one system of principal tangent curves of the surface belongs to the complex $\lambda = a$, while the other one belongs to the complex $\lambda = b$, or, what amounts to the same thing, that the surface has a special congruence in common with the complex $\lambda = a$, as well as the complex $\lambda = b$.

In order to prove this, one only has to show that the differential equation of the special congruences:

$$\left(\sum \frac{\partial\varphi}{\partial x_\alpha}\cdot\frac{\partial\psi}{\partial x_\alpha}\right)^2 - \sum\left(\frac{\partial\varphi}{\partial x_\alpha}\right)^2 \cdot \sum\left(\frac{\partial\psi}{\partial x_\alpha}\right)^2 = 0.$$

Sufficiency comes about when one takes one of the surfaces that were found here in place of φ and perhaps $(\lambda_4 - a)$ or $(\lambda_4 - b)$, in place of ψ. $\sum\left(\frac{\partial\varphi}{\partial x_\alpha}\right)^2$ vanishes, however, since φ is a special complex. All that remains is:

$$\sum \frac{\partial\varphi}{\partial x_\alpha}\cdot\frac{\partial\psi}{\partial x_\alpha} = 0,$$

or, upon introduction of the λ coordinates:

$$0 = \frac{\partial\varphi}{\partial\lambda_1}\cdot\frac{\partial\psi}{\partial\lambda_1}\cdot\frac{f(\lambda_1)}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)(\lambda_1-\lambda_4)} + \ldots$$

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[37]) The fact that two complexes of the system have a simple infinitude of common integral surfaces defines the starting point for the corresponding argument of Lie.

However, $\frac{\partial\psi}{\partial\lambda_1}$, $\frac{\partial\psi}{\partial\lambda_2}$, $\frac{\partial\psi}{\partial\lambda_3}$ vanish, since either $\psi = \lambda_4 - a$ or $\psi = \lambda_4 - b$ depends only upon λ_4. On the other hand, $\frac{\partial\varphi}{\partial\lambda_4} = \frac{\sqrt{(\lambda_4 - a)(\lambda_4 - b)}}{\sqrt{f(\lambda_4)}}$, which vanishes when one sets $\lambda_4 = a$ or $\lambda_4 = b$. The differential equation of the special congruences will thus be sufficiently resolved, in general.

The value of the constant C in the equation for φ does not come into consideration at all then, so it remains arbitrary. We then have the theorem: *Any two complexes $\lambda = a$, $\lambda = b$ of the family that is associated with the Kummer surface have a simple infinitude of common integral surfaces.*

If one lets b vary, in addition to C, then one obtains a two-fold infinitude of integral surfaces of the complex $\lambda = a$, and therefore a complete solution of the partial differential equation that is linked with the complex. The general solution encompasses all surfaces that are the enveloping structure to a simple infinitude of surfaces from the doubly infinite family thus determined. – *With that, problem* 2) *is disposed of for the general complex of second degree.*

The integral surfaces that are found, which are common to the complexes $\lambda = a$ and $\lambda = b$, have a remarkable relationship with the enveloping lines of the congruence $\lambda = a$, $\lambda = b$ that was determined in the previous paragraphs.

[In fact, the equation of the individual integral surfaces when we take λ_3 to be a constant equal to a and $\lambda_4 = b$ reduces to the differential equation of the aforementioned enveloping curve. The relation that then exists between the integral surface and the focal surface of the congruence $\lambda_3 = a$, $\lambda_4 = b$ remains to be developed more rigorously [38]).]

From the meaning of the singularity surface it follows, moreover, that the integral surface must contact it everywhere it meets the singularity surface. The cone of the complex a or b that emanates from a point of the singularity surface then degenerates to a point-pair whose intersection – the associated singular line – contacts the singularity surface. The integral surface can contact the degenerate cone nowhere else, except where it contacts the singular line. The integral surface then contacts the two associated singular lines of the complexes a and b at each of its points in which it meets the singularity surface; i.e., it contacts the singularity surface itself.

We obtain the singular lines of the complexes a that belong with the points of the contact curve when we set $\lambda_3 = \lambda_4 = a$ in the equation for the integral surface. What remains is:

$$\int d\lambda_1 \frac{\sqrt{(\lambda_1 - a)(\lambda_1 - b)}}{\sqrt{f(\lambda_1)}} + \int d\lambda_2 \frac{\sqrt{(\lambda_2 - a)(\lambda_2 - b)}}{\sqrt{f(\lambda_2)}} + C = 0.$$

This equation, together with $\lambda_3 = \lambda_4 = a$, determines the singular lines in question. On the other hand, since, from the previous paragraphs, λ_1 and λ_2 can be regarded as the

38) [The more precise details on this, which were originally discussed in Math. Ann., Bd. 5, no longer apply here, due to the objection that was raised by A. Voss in Math. Ann., Bd. 9, pp. 134-135. This state of affairs warrant further investigation. K]

coordinates of a point on the singularity surface, we can even use it as the equation of the contact curve. We then have the theorem:

The integral surface contacts the singularity surface along a curve segment whose equation is the aforementioned one.

The singularity surface then corresponds to a *singular solution* of the differential equation that is linked with the complex, in that sense that it will contact all of the integral surfaces of the complex along a curve.

Finally, we might make the following remark: If we set $\lambda_4 = a$ in the equation for an integral surface of the complex a then one comes to the representation of the associated special congruence that relates to the complex a:

$$\int d\lambda_1 \frac{\sqrt{(\lambda_1 - a)(\lambda_1 - b)}}{\sqrt{f(\lambda_1)}} + \int d\lambda_2 \frac{\sqrt{(\lambda_2 - a)(\lambda_2 - b)}}{\sqrt{f(\lambda_2)}} + \int d\lambda_3 \frac{\sqrt{(\lambda_3 - a)(\lambda_3 - b)}}{\sqrt{f(\lambda_3)}} + C = 0.$$

Now, the integrals that arise here are hyperelliptic ones that correspond to $p = 3$. This indicates an inverse problem to the aforementioned equation. To that end, we write it in the form:

$$\int d\lambda_1 \frac{\sqrt{(\lambda_1 - a)(\lambda_1 - b)}}{\sqrt{f(\lambda_1)}} + \ldots = u$$

and link it with two similarly constructed equations, whose integrals are obtained from the foregoing by differentiation with respect to the parameters a, b:

$$\int d\lambda_1 \frac{\sqrt{\lambda_1 - b}}{\sqrt{\lambda_1 - a \cdot \sqrt{f(\lambda_1)}}} + \ldots = v$$

$$\int d\lambda_1 \frac{\sqrt{\lambda_1 - a}}{\sqrt{\lambda_1 - b \cdot \sqrt{f(\lambda_1)}}} + \ldots = w.$$

These equations then serve to express the λ_1, λ_2, λ_3, and furthermore, the $x_1, \ldots, x_6$ of the lines of the complex, in terms of the u, v, w, and since x_α is connected with the λ by an equation of the symmetric form:

$$\rho x_\alpha^2 = \frac{(k_\alpha - \lambda_1)(k_\alpha - \lambda_2)(k_\alpha - \lambda_3)(k_\alpha - \lambda_4)}{f(x_\alpha)},$$

they essentially express the x_α as hyperelliptic functions of the u, v, w.

The coordinates x_α of the lines of a line complex of second degree are then represented by six-fold periodic hyperelliptic functions of three parameters u, v, w, on the basis of a second complex.

It was already stressed repeatedly that the orthogonal system that is defined for surfaces of fourth order with imaginary double circles corresponds to the system of line complexes of second degree with a common singularity surface. One disregards how one has a theorem for these surfaces, and similarly for these complexes, that says that *the coordinates of the points of a surface of orthogonal system may be represented as four-fold periodic hyperelliptic functions of two parameters.* Darboux gave this theorem without proof in the Comptes rendus (t. 68, 1869, 1: "Mémoire sur une classe de courbes et de surfaces"); he especially emphasized that it might find applications to the general surfaces of third order, since three of the surfaces of the orthogonal system are general surfaces of third degree. A similar theorem is obviously true for the corresponding structure in arbitrarily many dimensions.

One will then be led to a second inverse problem by the equation for the enveloping curve of the singular lines:

$$\int \frac{d\lambda_1(\lambda_1 - a)}{\sqrt{f(\lambda_1)}} + \int \frac{d\lambda_2(\lambda_2 - a)}{\sqrt{f(\lambda_2)}} + C = 0,$$

since for them, as well, the number of summed integrals coincides with the p of the hyperelliptic functions that appear. To that end, we set:

$$\int \frac{d\lambda_1(\lambda_1 - a)}{\sqrt{f(\lambda_1)}} + \int \frac{d\lambda_2(\lambda_2 - a)}{\sqrt{f(\lambda_2)}} = u,$$

and add a similar equation:

$$\int \frac{d\lambda_1(\lambda_1 - b)}{\sqrt{f(\lambda_1)}} + \int \frac{d\lambda_2(\lambda_2 - b)}{\sqrt{f(\lambda_2)}} = v.$$

These define two families of curves that run in the singularity surface that is common to all of the complexes: the enveloping curves of the singular lines of the complexes $\lambda = a$ and $\lambda = b$. When we replace u and v with linear combinations of those parameters, we can take a and b to be equal to two of the six quantities k_α, in particular. The aforementioned equations then define two families of enveloping curves that the six double tangent systems of the surface possess. *Relative to two such systems of curves, the coordinates of the points of the Kummer surface are then represented by four-fold periodic hyperelliptic functions.*

For special line complexes of second degree, the hyperelliptic functions that appear in the aforementioned inverse problem naturally simplify. For instance, if the k_α are pair-wise equal then they will be logarithms. The complex is then converted into the known complex whose lines intersect a fixed tetrahedron with constant double ratios. The singularity surface is degenerate in this tetrahedron. In fact, the common integral surfaces of two complexes that belong to the stated tetrahedron are represented by an equation in the logarithms of the coordinates, namely, by a linear equation in them.

These are the same surfaces that Lie and myself examined [39]) in the form of "*W* surfaces", and whose analogues in the plane we recently considered in a common treatise in these Annalen [40]).

Göttingen, in November 1871.

[39]) Comptes rendus. 1870, 1. "Sur une certaine classe de courbes et de surfaces." [See Abh. XXV of this collection.] The conception of the *W* surfaces as the common integral surfaces of two of the complexes associated with the stated tetrahedron is due to Lie.

[40]) "Über diejenigen ebenen Kurven, etc." Math. Ann., Bd. 4 (1871). [See Abh. XXVI of this collection.]