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XXXII. On the differential laws for the conservation of impulse and energy in Einstein's theory of gravitation.

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Upon resuming the investigations that I presented to the Scientific Society on 25 January of this year ([2]), I have succeeded in deriving the various forms for the differential laws of the conservation of impulse and energy that were presented by various authors ([3]) for *Einstein's* theory of gravitation from a unified viewpoint, and in so doing, if I am not mistaken, I have achieved an essentially improved insight into their meaning and mutual relationships. As one will see, in the presentation that follows, I actually do not need to make any more calculations, but only to make reasonable use of the elementary formulas of the classical calculus of variations.

For the sake of brevity, I will refer to my previous Note here, and also use its notations. As the actual basis for the present advance, I can then say that I am no longer subjecting the infinitesimal transformation:

$$\delta w^\tau = p^\tau \tag{1}$$

that was used there to the restriction that it must vanish on the boundary of the integration domain in a suitable way (namely, along with its first and second derivatives with respect to the w). In that way, the integrals in question will take on boundary components whose closer examination will produce everything that follows. For the specific purpose that we

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([1]) The manuscript first took on its final form in mid-September of that year.

([2]) See the final issue of the year's volumes for 1917 of these Nachrichten: "Zu Hilbert's erster Note über die Grundlagen der Physik." [Abh. XXXI of this collection.]

([3]) The main papers by *Einstein* to consider are the combined articles of 1916: "Die Grundlagen der allgemeinen Relativitätsteorie" (Leipzig) and the communication to the Berlin Academy "Hamiltonsche Prinzip and allgemeine Relativitätsteorie" (Sitzungsbericht of 26 October 1916), by *Hilbert*, the previously-cited Note (Göttinger Nachrichten of 20 November 1915), by *Lorentz*, the four articles that he published on the basis of his lectures from March to June in 1916 at the Verslag der Amsterdamer Akademie – viz., "over Einsteins theorie der zwaartekracht." In particular, see Art. III from April (September, resp.) 1916 and Art. IV from October (May, resp.) 1917. Here, I should also cite the book *Raum-Zeit-Materie* (Berlin 1918) by *Weyl* that appeared recently, to which I shall refer later on. [*Weyl's* book is already into its third edition; in this article, it will always be the first edition that is cited.]

have in mind here, it then suffices to consider only the first of the two previously-considered integrals, namely:

$$I_1 = \iiiint K\, d\omega. \tag{2}$$

The consideration of this integral conveniently separates in such a way that I shall first regard K as any function of only $g^{\mu\nu}$, $g^{\mu\nu}_{\rho}$, $g^{\mu\nu}_{\rho\sigma}$, and then as an invariant under arbitrary transformations of the world-parameters w (which is likewise still not determined more precisely), and then, in conclusion, as an invariant with a definite structure to it.

If I apply, for example, (1) to (2) then a series of differential relations will arise that K must satisfy identically. I will now turn to physics by no longer restricting myself to the case of the free, electromagnetic field, as I did previously, but assume that one has an arbitrary "material" field. If one combines *Hilbert's* Ansatz with that of *Einstein* then, as is known, one can write the associated ten field equations of gravitation in the simple form ([4])([5]):

$$K_{\mu\nu} - \kappa T_{\mu\nu} = 0, \tag{3}$$

in which we understand $K_{\mu\nu}$ to mean the *Lagrangian* derivative with respect to the $g^{\mu\nu}$ that belongs to I_1, divided by $\sqrt{g}$, while $T_{\mu\nu}$ means the energy components of matter. *The transition to the various forms of the conservation laws is then obtained simply from the principle that one substitutes any arbitrary* $\kappa T_{\mu\nu}$ *for* $K_{\mu\nu}$ *in the identities that are derived from K.*

§ 1.

Infinitesimal transformation of the $g^{\mu\nu}$.

In order to give the reader as much help as necessary, I will first discuss the short intermediate calculations that determine the $\delta g^{\mu\nu}$ that corresponds to the infinitesimal transformation (1) of the w.

Instead of (1), I next write:

$$\bar{w}^{\mu} = w^{\mu} + p^{\mu}$$

([4]) See, e.g., *Herglotz* in the Sächsischen Berichten 1916, pp. 402, formula (16). For the sake of precision, I then remark that the constant κ (which I called $-\alpha$ in my previous Note in connection with *Hilbert*) will have the value that I employed there, namely:

$$\kappa = 1.87 \cdot 10^{-27} \text{ cm g}^{-1},$$

only when the basic ds^2 agrees in dimensions and signs with the $d\tau^2$ of the special theory of relativity:

$$d\tau^2 = dt^2 - \frac{dx^2+dy^2+dz^2}{c^2} \sim \text{s}^2.$$

([5]) [Here and in what follows, the signs of $T_{\mu\nu}$, $\mathfrak{T}_{\mu\nu}$, T^{σ}_{τ}, $\mathfrak{T}^{\sigma}_{\tau}$, t^{σ}_{τ}, $\mathfrak{t}^{\sigma}_{\tau}$ were inverted during reprinting in order to come into agreement with the usual notations in physics. For example, T_{44} will then be positive. K.]

(in which the "auxiliary vector" p and its differential quotients p_ρ, $p_{\rho'}$ are to be computed, moreover, in such a way that one can neglect all terms of order higher than linear). We will then have:

$$d\overline{w}^\mu = dw^\mu + \sum p_\tau^\mu dw^\tau .$$

Now, by definition, the $g^{\mu\nu}$ are cogredient to the products $dw^\mu\, dw^\nu$. One will then get:

$$\overline{g}^{\mu\nu}(\overline{w}) = g^{\mu\nu}(w) + \sum g^{\mu\tau}(w)\, p_\tau^\nu + \sum g^{\nu\tau}(w)\, p_\tau^\mu .$$

However:

$$g^{\mu\nu}(\overline{w}) = \overline{g}^{\mu\nu}(w) + \sum g_\tau^{\mu\nu}(w) \cdot p^\tau .$$

Now, it was the difference $g^{\mu\nu}(\overline{w}) - \overline{g}^{\mu\nu}(\overline{w})$ *that I called* $\delta g^{\mu\nu}$ *in my previous Note and which I will now follow Hilbert's Note and denote by* $p^{\mu\nu}$. One will then have:

(4)
$$\delta g^{\mu\nu} = p^{\mu\nu} = \sum_\tau (g_\tau^{\mu\nu}\, p^\tau - g^{\mu\tau}\, p_\tau^\nu - g^{\nu\tau}\, p_\tau^\mu) .$$

Furthermore, the differential quotients of the $p^{\mu\nu}$ with respect to the w will be denoted by $p_\rho^{\mu\nu}$, $p_{\rho\sigma}^{\mu\nu}$ as in *Hilbert*.

I shall also notate the values that the $p^{\mu\nu}$ take on in the case of constant p^τ, which will be considered especially later on [where I will write $\underset{0}{p}^{\mu\nu}$ ($\underset{0}{p}^{\tau}$, resp.)]:

(5)
$$\underset{0}{p}^{\mu\nu} = \sum_\tau g_\tau^{\mu\nu}\, \underset{0}{p}^{\tau} .$$

In this case, it is then as if the $g^{\mu\nu}$ are previously-given fixed functions of the w [a scalar] (that do not undergo a substitution that is induced by the respective transformation of the w).

§ 2.

Calculation of the δI_1 under the sole assumption that K is a function of the $g^{\mu\nu}$, $g_\rho^{\mu\nu}$, $g_{\rho\sigma}^{\mu\nu}$. – A main theorem.

That means that K does not depend upon the w explicitly. We will then have for our infinitesimal transformation (which affects the dependent variables in the integral I_1, as well as the independent ones):

(6)
$$\delta I_1 = -\iiiint \sum_{\mu\nu} \left(\frac{\partial \sqrt{g}\, K}{\partial g^{\mu\nu}}\, p^{\mu\nu} + \sum_\rho \frac{\partial \sqrt{g}\, K}{\partial g_\rho^{\mu\nu}}\, p_\rho^{\mu\nu} + \sum_\rho \frac{\partial \sqrt{g}\, K}{\partial g_{\rho\sigma}^{\mu\nu}}\, p_{\rho\sigma}^{\mu\nu} \right) dS$$

$$+ \iiint \sqrt{g}\, K(p^I dw^{II} dw^{III} dw^{IV} + \cdots + p^{IV} dw^I dw^{II} dw^{III});$$

dS is written in place of $dw^I\, dw^{II}\, dw^{III}\, dw^{IV}$, while the triple integral is extended over the boundary of the integration domain of I_1 in a well-known vectorial way ([6]).

Here, we will now address the differential quotients $p_\rho^{\mu\nu}$ ($p_{\rho\sigma}^{\mu\nu}$, resp.) that appear under the four-fold integral using the older Lagrange procedure by partially integrating once, (twice, resp.), and then replacing $p^{\mu\nu}$ with its value (4) and eliminating the differential quotients p_τ^ν and p_τ^μ that then arise by a fourth partial integration. *We thus find that:*

$$\delta I_1 = - \iiiint \sum_\tau \left(\sqrt{g} \sum_{\mu\nu} K_{\mu\nu} g_\tau^{\mu\nu} + 2 \sum_\sigma \frac{\partial \sqrt{g}\, K_\tau^\sigma}{\partial w^\sigma} \right) p^\tau \cdot dS \tag{7}$$
$$+ \iiint \sqrt{g} \left(\varepsilon^I dw^{II} dw^{III} dw^{IV} + \cdots + \varepsilon^{IV} dw^I dw^{II} dw^{III} \right),$$

into which the following abbreviations have been introduced:

1. $K_{\mu\nu}$ is the *Lagrangian* derivative that was already used in (3), divided by $\sqrt{g}$:

$$K_{\mu\nu} = - \left(\frac{\partial \sqrt{g}\, K}{\partial g^{\mu\nu}} - \sum_\rho \frac{\partial \left(\dfrac{\partial \sqrt{g}\, K}{\partial g_\rho^{\mu\nu}} \right)}{\partial w^\rho} + \sum_\rho \frac{\partial \left(\dfrac{\partial \sqrt{g}\, K}{\partial g_{\rho\sigma}^{\mu\nu}} \right)}{\partial w^\rho\, \partial w^\sigma} \right) : \sqrt{g}\,. \tag{8}$$

2. K_τ^σ are the following linear combinations of the $K_{\mu\nu}$:

$$K_\tau^\sigma = \sum_\mu K_{\mu\tau} g^{\mu\sigma}\,. \tag{9}$$

3. For $\sigma = 1, 2, 3, 4$, ε^σ are the five-term expressions that I write, from the outset (when I especially emphasize the terms that originate with the fourth partial integration):

$$\varepsilon^\sigma = \eta^\sigma + 2 \sum_\tau K_\tau^\sigma p^\tau\,. \tag{10}$$

Here, one will then have:

([6]) [It is an essential difference, and at the same time, an advance over my previous Note, that here, I am assuming nothing about the behavior of the p^τ, $p^{\mu\nu}$, $p_\rho^{\mu\nu}$, $p_{\rho\sigma}^{\mu\nu}$ on the boundary of the domain of integration. K.]

$$\eta^\sigma = K\,p^\sigma - \sum_{\mu\nu} \frac{\partial K}{\partial g^{\mu\nu}_\sigma}\,p^{\mu\nu} - \sum_{\mu\nu} \frac{\partial K}{\partial g^{\mu\nu}_{\rho\sigma}}\,p^{\mu\nu}_\rho + \frac{1}{\sqrt{g}} \frac{\partial\left(\dfrac{\partial \sqrt{g}\,K}{\partial g^{\mu\nu}_{\rho\sigma}}\right)}{\partial w^\rho}\,p^{\mu\nu}\,. \tag{11}$$

From now on, the four-fold integral that appears in the new expression (7) for δI_1, after dropping the minus sign, shall be called *the integral A*.

Moreover, we will convert the triple integral that appears in (7) into a *second* four-fold integral by an elementary divergence map:

$$\iiiint \left(\frac{\partial \sqrt{g}\,\varepsilon^I}{\partial w^I} + \cdots + \frac{\partial \sqrt{g}\,\varepsilon^{IV}}{\partial w^{IV}}\right) dS\,, \tag{12}$$

which will be called *the integral B*. Thus:

$$\delta I_1 = -A + B. \tag{13}$$

Here, the following important remark suggests itself, that $A \equiv B$ when the p^τ are chosen to be constant, and thus, equal to $\underset{0}{p}^{\mu\nu}$.

In fact, the original value (6) of δI_1 will then vanish identically, *since K does not contain the w explicitly*, when one employs the values of the $\underset{0}{p}^{\mu\nu}$ that were given in (5).

However, due to the completely arbitrary nature of the choice of integration domain, we conclude from the fact that $A \equiv B$ that the integrands of A and B must also agree. We will then have:

$$\begin{aligned} \sum_\tau \left(\sqrt{g} \sum_{\mu\nu} K_{\mu\nu} g^{\mu\nu}_\tau + 2\sum_\sigma \frac{\partial \sqrt{g}\,K^\sigma_\tau}{\partial w^\sigma}\right) \underset{0}{p}^\tau &\equiv \sum_\sigma \frac{\partial \sqrt{g}\,\underset{0}{\varepsilon}^\sigma}{\partial w^\sigma} \\ &\equiv \sum_\sigma \frac{\partial \sqrt{g}\,\underset{0}{\eta}^\sigma}{\partial w^\sigma} + 2\sum_\sigma \frac{\partial \sqrt{g}\,K^\sigma_\tau}{\partial w^\sigma}\,\underset{0}{p}^\tau\,. \end{aligned} \tag{14}$$

This identity shall be called the main theorem, moreover.

Naturally, we can drop the terms in K^σ_τ on both sides. We would then like to write down the $\underset{0}{\eta}^\sigma$ as functions of the $\underset{0}{p}^\tau$ as follows:

$$\underset{0}{\eta}^\sigma = 2\sum U^\sigma_\tau\,\underset{0}{p}^\tau\,. \tag{15}$$

(We have inserted the 2 into the right-hand side of this, because later on, it will become necessary to divide by 2.) Therefore (with the use of the usual notation that δ^σ_τ is equal to 1 or 0, according to whether $\sigma = \tau$ or $\sigma \neq \tau$, resp.), one will have:

(16) $$2U_\tau^\sigma = K\,\delta_\tau^\sigma - \sum_{\mu\nu} \frac{\partial K}{\partial g_\sigma^{\mu\nu}} g_\tau^{\mu\nu} - \sum_{\mu\nu\rho} \frac{\partial K}{\partial g_{\sigma\sigma}^{\mu\nu}} g_{\rho\tau}^{\mu\nu} + \frac{1}{\sqrt{g}} \frac{\partial\left(\dfrac{\partial \sqrt{g}\,K}{\partial g_{\sigma\sigma}^{\mu\nu}}\right)}{\partial w^\rho} g_\tau^{\mu\nu}\,.$$

However, the main theorem assumes the following form:

(17) $$\sum_{\mu\nu} \sqrt{g}\,K_{\mu\nu}\,g_\tau^{\mu\nu} \equiv 2\sum_\sigma \frac{\partial \sqrt{g}\,U_\tau^\sigma}{\partial w^\sigma}, \qquad \text{for } \tau = 1, 2, 3, 4.$$

The expressions on the left-hand side are then transformed into elementary divergences.

§ 3.

Simplified notation for the formulas. – An extension of the main theorem.

In the previous paragraph, I chose the notation in such a way that it would seem suitable for the later invariant-theoretic evaluations, and corresponded to the older custom, moreover. In the meantime, much can be abbreviated by using *Einstein*'s suggestions:

1. One can already save some work, when one replaces the product of $\sqrt{g}$ with any quantity that is denoted by upper-case Latin symbols with the corresponding German notation. Thus, one replaces $\sqrt{g}\,K$ with $\mathfrak{K}$, $\sqrt{g}\,K_{\mu\nu}$ with $\mathfrak{K}_{\mu\nu}$, $\sqrt{g}\,K_\tau^\sigma$ with $\mathfrak{K}_\tau^\sigma$, $\sqrt{g}\,U_\tau^\sigma$ with $\mathfrak{U}_\tau^\sigma$. (In the sense of this abbreviation, we will write an *elementary divergence* as follows:

(18) $$\frac{\partial \mathfrak{B}^I}{\partial w^I} + \frac{\partial \mathfrak{B}^{II}}{\partial w^{II}} + \frac{\partial \mathfrak{B}^{III}}{\partial w^{III}} + \frac{\partial \mathfrak{B}^{IV}}{\partial w^{IV}} = \mathfrak{Div}.$$

From now on, the $\mathfrak{B}^I, \ldots, \mathfrak{B}^{IV}$ shall depend upon only the $g^{\mu\nu}$, $g_\rho^{\mu\nu}$ in such a way that our $\mathfrak{Div}$ will become a special case of the functions $\mathfrak{K}$ that we have considered up to now.)

2. Furthermore, the summation symbols in the summation can be omitted when one remarks that one always sums over those indices that appear twice (once above and once below).

3. Finally, on the same grounds, the summation sign itself can also be dropped.

We shall make use of the abbreviations thus introduced more or less as it seems appropriate. For example, formulas (17) can then be written as:

(19) $$\mathfrak{K}_{\mu\nu}\, g_\tau^{\mu\nu} \equiv 2\, \frac{\partial \mathfrak{U}_\tau^\sigma}{\partial w^\sigma}\,.$$

In connection with this, I can think of a noteworthy generalization of the formulas (17) [(19), resp.].

The *Lagrangian* derivatives will vanish identically for the divergences ($\mathfrak{Div}$) that were just introduced in a well-known way:

(20) $$\mathfrak{Div}_{\mu\nu} \equiv 0.$$

If we then replace the $\mathfrak{K}_{\mu\nu}$ in (19) with the *Lagrangian* derivatives of a function $\mathfrak{K}^*$ that is connected to $\mathfrak{K}$ by an equation:

(21) $$\mathfrak{K}^* = \mathfrak{K} + \mathfrak{Div}$$

then the left-hand side of (19) will remain unchanged, while a new function $\mathfrak{U}_\tau^{*\sigma}$ will appear in place of $\mathfrak{U}_\tau^\sigma$ on the right-hand side. We will then have:

(22) $$\mathfrak{K}_{\mu\nu}\, g_\tau^{\mu\nu} \equiv 2\, \frac{\partial \mathfrak{U}_\tau^{*\sigma}}{\partial w^\sigma}\,.$$

Formula (19) *is then generalized in a remarkable way.* (Naturally, the $\mathfrak{U}_\tau^{*\sigma}$ differ from the $\mathfrak{U}_\tau^\sigma$ for a fixed τ by only a term whose elementary divergence $\sum \frac{\partial}{\partial w^\sigma}$ vanishes identically.)

§ 4.

Invariant-theoretic viewpoint.

In the spirit of the general theory of relativity, we will now assume that K is invariant under the group of all transformations of the w (which we must naturally think of as having been "extended" by adding the corresponding conversions of the $g^{\mu\nu}$).

When $d\omega$ is an invariant to begin with, the same thing will be true of the integral I_1.

$K_{\mu\nu}$ seems to be a contragradient tensor; the complex of 16 quantities K_τ^σ is a mixed tensor.

We would further like to introduce some notations (when we think of the auxiliary vector p as always being transformed in the same way as the dw) ([7]).

When we now write A as:

([7]) [One will see the fact that ε^σ and η^σ are cogredient vectors most simply when one compares their expressions [formulas (10) and (11)] with formulas (8), (9), and (14) in *Hilbert's* first communication on the foundations of physics (*loc. cit.*). K.]

$$A = \iiiint \sum \left(\frac{\sqrt{g} \sum K_{\mu\nu} g_\tau^{\mu\nu} + 2 \sum \frac{\partial \sqrt{g}\, K_\tau^\sigma}{\partial v^2}}{\sqrt{g}} \cdot p^\tau \right) d\omega, \tag{23}$$

the system of quantities that is multiplied by p^τ will appear to be a contragedient vector. (In the sense of my previous Note, it is the "vectorial divergence" of the tensor $K_{\mu\nu}$).

We get a corresponding invariant from B:

$$\frac{1}{\sqrt{g}} \sum \frac{\partial \sqrt{g}\, \varepsilon^\sigma}{\partial w^\sigma}; \tag{24}$$

we will refer to it as the "scalar divergence" (again, in the sense of my previous Note) of the vector **ε** (that is constructed with the help of the vector p).

The two components of (24), viz.:

$$\frac{1}{\sqrt{g}} \sum \frac{\partial \sqrt{g}\, \eta^\sigma}{\partial w^\sigma} \qquad \text{and} \qquad \frac{1}{\sqrt{g}} \sum \frac{\partial \left(\sqrt{g}\, K_\tau^\sigma \varepsilon^\tau \right)}{\partial w^\sigma}, \tag{25}$$

are no less invariants in their own right.

However, how do things work for the U_τ^σ*, which were derived under the assumption of constant* p^τ *(=* $\underset{0}{p}^\tau$*)?*

Constant p^τ no longer remain constant under arbitrary transformations of the w, but only under the "affine" transformations:

$$\bar{w}^\rho = a_1^\rho \cdot w^I + \cdots + a_4^\rho \cdot w^{IV} + c^\rho .$$

One naturally thinks of the $g^{\mu\nu}$ are being transformed correspondingly (thus, linearly, with coefficient coefficients). It always happens that the individual $g^{\mu\nu}$ are functions of the w^ρ.

We can then say that:

U_τ^σ *is a mixed tensor of the affine group, thus-extended.*

This does not alter the fact that from our equations (14), (17), and the notation (23), the expression:

$$\frac{2}{\sqrt{g}} \sum \frac{\partial \left(\sqrt{g} (U_\tau^\sigma + K_\tau^\sigma) \right)}{\partial w^\sigma}, \tag{26}$$

which is independent of the p^τ, will be a contragredient vector of the *general* group.

This is a very remarkable state of affairs, and it will be fundamental to what we shall do later.

If one then takes any $\mathfrak{K}^*$, instead of $\mathfrak{K}$, from (21), and adds the assumption that the $\mathfrak{B}^I$, …, $\mathfrak{B}^{IV}$ that appear in (18) should be equal to the components of a vector W^I, …, W^{IV} (multiplied by $\sqrt{g}$) of the affine group:

$$\mathfrak{B}^\sigma = \sqrt{g}\,W^\sigma \tag{27}$$

then the same state of affairs as in (26) will be true for the general expressions:

$$\frac{2}{\sqrt{g}}\sum\frac{\partial\left(\sqrt{g}\,(U_\tau^{\bullet\sigma}+K_\tau^\sigma)\right)}{\partial w^\sigma}. \tag{28}$$

§ 5.

Identities that our K satisfies, as an invariant of the general group.

We shall now pursue the thought: *Since K is an invariant of our general group, it will follow that for arbitrary values of the p^τ*:

$$\delta I_1 = \delta\iiiint K\,d\omega = 0. \tag{29}$$

(Conversely, when the relation (29) is true for arbitrary p^τ, I_1, and therefore K, will be an invariant of the general group. All finite transformations of the w^τ will then be composed of infinitesimal $\delta w^\tau = p^\tau$.)

We then obtain a large number of differential relations that the invariant K (which is not at all specialized) must satisfy identically from the formulas of §§ 2, 3.

1. As in my previous Note, with no loss of generality, we take the p^τ to be such that the vector ε^σ, and therefore the associated boundary integral, drops out, as such. Obviously, this is associated with the fact that p^τ, $p^{\mu\nu}$, and $p_\rho^{\mu\nu}$ vanish along the boundary; i.e., the annulment of p^τ, $p^{\mu\nu}$, and $p_\rho^{\mu\nu}$. Thus, according to (13), one will then get $A = 0$; i.e., with for an arbitrary domain of integration and an arbitrary assumption on the p^τ in the interior of the domain. According to (23), we conclude that *the vectorial divergence of the tensor $K_{\mu\nu}$ must be zero identically.* In a formula:

$$\frac{\sqrt{g}\sum K_{\mu\nu}g_\tau^{\mu\nu}+2\sum\frac{\partial\sqrt{g}\,K_\tau^\sigma}{\partial v^2}}{\sqrt{g}}\cdot p^\tau \equiv 0 \qquad \text{(for } \tau = 1, 2, 3, 4). \tag{30}$$

These are the identities (12) of my previous Note that I shall now call *the A identities*. One must be clear: Since one is dealing with a vector, the left-hand side of (30), when exhibited for an arbitrary coordinate system, will be equal to well-known linear combinations of its original values. The vanishing of the transformed expressions will then say nothing else but the vanishing of the original expression.

2. As a result of the identities (30), the integral A will drop away for *arbitrary* p^τ. According to (13), (29), the integral B will also always vanish. We again argue that the domain of integration and the vector p^τ can be chosen entirely arbitrarily. It will then follow that the integrand of B – i.e., the scalar divergence of the vector $\boldsymbol{\varepsilon}$ – must be identically zero.

$$\frac{1}{\sqrt{g}}\sum\frac{\partial\sqrt{g}\,\varepsilon^n}{\partial w^n}\equiv 0, \tag{31}$$

or, what amounts to the same thing:

$$\frac{1}{\sqrt{g}}\sum\frac{\partial\sqrt{g}\left(\eta^\sigma+2\sum K_\tau^\sigma p^\tau\right)}{\partial w^\sigma}\equiv 0. \tag{31′}$$

Due to the arbitrariness in p^τ, very many individual equations are contained in this one formula (31) or (31′). One considers the terms that arise from $\sum K_\tau^\sigma p^\tau$ by differentiation, and argues that η^σ is constructed from terms that contain the $p^{\mu\nu}$ ($p_\rho^{\mu\nu}$, respectively) linearly, while the $p^{\mu\nu}$ itself are linear in the p and their differential quotients with respect to the w. However, the η^σ in (31) [(31′), resp.] will be differentiated with respect to the w^σ once more. We conclude that the left-hand side of (31) and (31′) are homogeneous, linear in the p^τ and their second and third differential quotients with respect to w. Since these can all be taken to be independent of each other, we will have:

$$4\left(1+4+\frac{4\cdot 5}{1\cdot 2}+\frac{4\cdot 5\cdot 6}{1\cdot 2\cdot 3}\right) = 140$$

equations, in all; I will call them the *B identities*.

This suggests that these 140 equations can be classified, at least, schematically. I will not contradict the notation that was introduced above in (15) when I write:

$$\eta^\sigma = 2\left(\sum U_\tau^\sigma p^\tau+\sum U_\tau^{\sigma,\sigma'} p_{\sigma'}^\tau+\sum U_\tau^{\sigma,\sigma'\sigma''} p_{\sigma'\sigma''}^\tau\right) \tag{32}$$

(All indices that appear twice are summed over. Indices that are not separated by a comma can be permuted, but not the ones that are. Thus, there are 16 (1 + 4 + 10) = 240 U quantities.) When I drop the preceding factor of $2/\sqrt{g}$ and again write $\mathfrak{U}$ for $\sqrt{g}\,U$, equation (31′) will now be decomposed as follows:

1. 4 equations that correspond to terms in p^τ:

(33) $$\sum(\mathfrak{U}^{\sigma}_{\tau,\sigma} + \mathfrak{K}^{\sigma}_{\tau,\sigma}) \equiv 0.$$

2. 16 equations that corresponding to the terms in p^{τ}_{σ}:

(34) $$\mathfrak{U}^{\sigma}_{\tau} + \mathfrak{K}^{\sigma}_{\tau} + \sum_{\sigma'} \mathfrak{U}^{\sigma',\sigma}_{\tau,\sigma'} \equiv 0.$$

3. 40 equations that correspond to the terms in $p^{\tau}_{\sigma'\sigma''}$:

(35) $$\mathfrak{U}^{\sigma',\sigma''}_{\tau} + \mathfrak{U}^{\sigma''\sigma'}_{\tau} + \sum_{\sigma} \mathfrak{U}^{\sigma,\sigma',\sigma''}_{\tau,\sigma} \equiv 0.$$

4. 80 equations that belong to the terms in $p^{\tau}_{\sigma\sigma'\sigma''}$:

(36) $$\mathfrak{U}^{\sigma\sigma'\sigma''}_{\tau} + \mathfrak{U}^{\sigma',\sigma''\sigma}_{\tau} + \mathfrak{U}^{\sigma'',\sigma\sigma'}_{\tau} \equiv 0.$$

I have not examined the dependencies that might exist between these 140 B equations.

Moreover, we can now immediately reach the following conclusions:

a) The *A* identities [= (30)] and the *B* identities [= (33), (34), (35), (36)] collectively yield the *sufficient* conditions for a function *K* of $g^{\mu\nu}$, $g^{\mu\nu}_{\rho}$, $g^{\mu\nu}_{\rho\sigma}$ to be an invariant of our general group.

b) However, due to the main theorem of § 2, the left-hand sides of (33), when multiplied by $2/\sqrt{g}$, are directly identical to the left-hand side of (30).

c) Therefore, only the *B* are the sufficient conditions for the invariance of *K*.

d) However, that is not true for only the *A*. Equations *A* will also exist then when one replaces *K* with $K^{*} = K + \text{Div}$, and more generally, when *K* is a function such that it always gets increased by a divergence for arbitrary transformations of the *w*.

e) Therefore, the *B* identities are not generally derivable from the *A* identities.

However, with hindsight of the physical conclusions to be developed, the *A* belong in the first row for us. Therefore, from the foregoing, one might once more imagine that there are three forms that they can assume:

$$(37)\qquad \begin{cases} A\ identities: & \dfrac{2}{\sqrt{g}}\sum\dfrac{\partial\mathfrak{K}_\tau^\sigma}{\partial w^\sigma}+\sum K_{\mu\nu}g_\tau^{\mu\nu}\equiv 0, \\ A_\beta\ identities: & \dfrac{2}{\sqrt{g}}\sum\dfrac{\partial(\mathfrak{K}_\tau^\sigma+\mathfrak{U}_\tau^\sigma)}{\partial w^\sigma}\equiv 0, \qquad \text{for } \tau = 1, 2, 3, 4. \\ A_\gamma\ identities: & \dfrac{2}{\sqrt{g}}\sum\dfrac{\partial(\mathfrak{K}_\tau^\sigma+\mathfrak{U}_\tau^{*\sigma})}{\partial w^\sigma}\equiv 0, \end{cases}$$

In this, as would seem preferable, moreover, I have always emphasized only the terms that contain the $\mathfrak{K}_\tau^\sigma$, and have once more added numerical factors such that the same four vector components will always be the left-hand side.

§ 6.

Transition to the conservation laws.

What I had to say just now in pursuit of the systematic train of thought regarding the special type of construction of the invariant *K*, as it is based in the modern theory of gravitation, is so closely linked with *Einstein’s* own relevant investigations that I prefer to move it to the following paragraphs, and what follows here will be the principal transition to the differential laws of conservation of impulse and energy and an oversight into the different forms in which these laws appear in the literature. The ten gravitational equations for the material field that we will deal with in each case will be especially simple in our notation, as I already pointed out in the introduction under (3). Here, I will likewise replace the Latin symbols with German ones and then have:

$$(38)\qquad \mathfrak{K}_{\mu\nu} - \kappa\mathfrak{T}_{\mu\nu} = 0.$$

Naturally, I can also write the 16 equations:

$$(39)\qquad \mathfrak{K}_\tau^\sigma - \kappa\mathfrak{T}_\tau^\sigma = 0$$

in place of these. *All that we have to do now is to replace the values of* $\mathfrak{K}_{\mu\nu}$ ($\mathfrak{K}_\tau^\sigma$, resp.) *that follow from this in the identities that are posed for the invariant K.* Things are so simple that I can summarize and explain the results in a table.

I begin with the identities A_α to A_γ (37):

1. It follows from the A_α, after dividing by $2\kappa/\sqrt{g}$, that:

$$(40)\qquad \sum\frac{\partial\mathfrak{T}_\tau^\sigma}{\partial w^\sigma}+\frac{1}{2}\sum\mathfrak{T}_{\mu\nu}g_\tau^{\mu\nu} = 0.$$

These are the conservation laws for the energy components of the material field in the form that is found everywhere in the literature.

2. Naturally, I can also write (and this is true, to a certain extent):

$$\sum \frac{\partial \mathfrak{T}_\tau^\sigma}{\partial w^\sigma} + \frac{1}{2\kappa} \sum \mathfrak{K}_{\mu\nu} g_\tau^{\mu\nu} = 0. \tag{41}$$

3. If I refer to the A_β then this will be completely equivalent to:

$$\sum \frac{\partial \left(\mathfrak{T}_\tau^\sigma + \frac{1}{\kappa} \mathfrak{U}_\tau^\sigma \right)}{\partial w^\sigma} = 0. \tag{42}$$

This is the essence of the conservation laws that *Lorentz* presented in Part III of his series of articles that was cited in the introduction; cf., *ibid.*, pp. 482, formula (79). [The direct identification is somewhat long-winded, insofar as *Lorentz* did not initially associate the δI_1 with the $\delta g^{\mu\nu}$, but with the $\delta g_{\mu\nu}$. However, one cannot doubt the agreement, since he started with the same infinitesimal transformation $\delta w^\tau = p^\tau$ (with constant p^τ) in order to achieve his goal that led us to the A_β identities ([8])].

4. Finally, according to the A_γ, the same relations can also be written:

$$\sum \frac{\partial \left(\mathfrak{T}_\tau^\sigma + \frac{1}{\kappa} \mathfrak{U}_\tau^{*\sigma} \right)}{\partial w^\sigma} = 0. \tag{43}$$

One has to specialize the K^* (formula (21)), and in connection with that, the $\mathfrak{U}_\tau^{*\sigma}$, only as would be expedient in order to obtain the well-known *Einstein* formulas:

$$\sum \frac{\partial \left(\mathfrak{T}_\tau^\sigma + \mathfrak{t}_\tau^\sigma \right)}{\partial w^\sigma} = 0. \tag{44}$$

The details will be revealed in the following paragraphs. In any event, one already understands here that the left-hand sides of the *Einstein* relations, when multiplied by $\sqrt{g}$, likewise represent vector components, like the left-hand sided of (41), (42), which agree with them precisely. I bring this up only because this state of affairs does not seem to be clearly understood, in all respects.

We now return to the original summary (31), (31′) of the *B* identities:

([8]) [In fact, *Vermeil* later verified the identity of both kinds of results by direct calculation. K.]

$$\left[\frac{1}{\sqrt{g}}\sum\frac{\partial\sqrt{g}\,\varepsilon^\sigma}{\partial w^\sigma}=\frac{1}{\sqrt{g}}\sum\frac{\partial\sqrt{g}\left(\eta^\sigma+2\sum K_\tau^\sigma p^\tau\right)}{\partial w^\sigma}\right]\equiv 0.$$

If we replace the K_τ^σ with the values κT_τ^σ that follow from the gravitational equations of the field then a new vector will enter in place of the vector ε^σ, which might be called e^σ:

(45) $$e^\sigma = \eta^\sigma + 2\kappa\sum T_\tau^\sigma p^\tau .$$

I maintain that *this new vector is precisely the one that Hilbert referred to in his Note – while restricting to the electromagnetic case – as the energy vector* (such that conservation laws for *Hilbert* are summarized in the one equation:

(46) $$\frac{1}{\sqrt{g}}\sum\frac{\partial\sqrt{g}\,\varepsilon^\sigma}{\partial w^\sigma}=0.)$$

In order to prove this, I remark that:

a) As far as the part in (45) that originates from the "matter," and thus, the term $2\kappa\sum T_\tau^\sigma p^\tau$, is concerned, it will agree with the given one with no further assumptions when I, leaning upon *Hilbert*, set κ equal to 1, after an associated conversion of the notation that *Hilbert* made in formula (19) of his Note, and in the theorems that were connected with it.

b) As far as the "gravitational part" of the η^σ is concerned, some time ago, *Freedericks* already calculated the next non-obvious term for me, which *Hilbert* (*loc. cit.*) gave in formulas (8), (9), and (14), and thus came to precisely the expression that I introduced in (11) as η^σ ([9]).

Now, formula (46) initially appears to be quite different from formulas (42), (43), even when I remove the factor $2\kappa/\sqrt{g}$. However, the proper relationship will become entirely clear when I separate (46) into 4 + 16 + 40 + 80 equations according the schema (33) to (36).

The first four equations will read:

(47) $$\sum(\mathfrak{U}_{\tau,\sigma}^\sigma+\kappa\mathfrak{T}_{\tau,\sigma}^\sigma)=0,$$

and thus agree with equations (42).

The following 16 equations will read:

(48) $$\mathfrak{U}_\tau^\sigma+\kappa\mathfrak{T}_\tau^\sigma+\sum_{\sigma'}\mathfrak{U}_{\tau,\sigma'}^{\sigma',\sigma}=0.$$

([9]) [Obviously, *Hilbert* has chosen his initially very complicated-appearing representation of ε^σ in order to allow the vector character of that quantity to emerge from the outset. K.]

This is only a special notation for the field equations (39), so, from (34), $\mathfrak{U}_{\tau}^{\sigma} + \sum_{\sigma'} \mathfrak{U}_{\tau,\sigma'}^{\sigma',\sigma}$ is identical to $-\mathfrak{K}_{\tau}^{\sigma}$.

The still-remaining 40 + 80 equations, however, agree with the identities (35), (36) with no further analysis; they have nothing at all to do with the material field that we are considering here.

Basically, *Hilbert's* claim (46) then reduces to the conservation laws (42); all that come about are known equations, anyway. The statement has the advantage that it not only asserts something that is simple in an invariant theoretic way in itself, but also that the quantities e^{σ} that appear in it can be briefly characterized by saying: *It is a cogredient vector that contains the auxiliary vector p^{τ}, but depends upon the $T_{\mu\nu}$, the $K_{\mu\nu}$, and their differential quotients, moreover.*

As one sees, what was said explicitly in the foregoing paragraph on the different forms of the conservations laws extends what was expressed in numbers **6** to **8** of my previous Note in only an indeterminate way.

§ 7.

Details of Einstein's formulation of the energy theorems.

I now have to add how one must specialize the quantity that I denoted by K^* in order to come to *Einstein's* final formula:

(44) $$\sum \frac{\partial\left(\mathfrak{T}_{\tau}^{\sigma} + \mathfrak{t}_{\tau}^{\sigma}\right)}{\partial w^{\sigma}} = 0,$$

and also say a few things about the simplification that is thus achieved.

I would thus do best to refer to *Einstein's* aforementioned presentation in the Sitzungsberichten der Berliner Akademie on October 1916. In it, *Einstein* started from the fact that the invariant K (which he called G) contains the second differential quotient of the $g^{\mu\nu}$ only linearly, and multiplied by functions of the $g^{\mu\nu}$ themselves. One can then eliminate the stated differential quotients from the integral $I_1 = \iiiint K\, d\omega$ by partial integration, so:

(49) $$K = G^* + \text{Div},$$

in which G^* is a function of only the first differential quotients. In particular, *Einstein* gave G^* the value ([10]):

(50) $$G^* = \sum_{\mu\nu\rho\sigma} g^{\mu\nu}\left\{\Gamma_{\mu\rho}^{\sigma}\Gamma_{\nu\sigma}^{\rho} - \Gamma_{\mu\nu}^{\rho}\Gamma_{\rho\sigma}^{\sigma}\right\},$$

([10]) [Here and in what follows, the signs of G^* and $\mathfrak{G}^*$ were altered during reprinting, corresponding to the situation that was considered sufficiently in the first printing that, in agreement with *Einstein*, one takes the sign of ds^2 in the manner that was described in footnote ([4]) on page 569 of that treatise; *Einstein's G* will then be identical with *Hilbert's K*. K.]

in which, we understand $-\Gamma_{\mu\nu}^{\rho}$ to mean the so-called symbol of the second kind ([11]):

(51) $$-\Gamma_{\mu\nu}^{\rho} = \sum_{\tau} \frac{g^{\rho\tau}}{2}\left(\frac{\partial g_{\mu\tau}}{\partial w^{\nu}} + \frac{\partial g_{\nu\tau}}{\partial w^{\mu}} - \frac{\partial g_{\mu\nu}}{\partial w^{\tau}}\right).$$

This G^* is obviously invariant under affine transformations of the w.

Einstein's remaining final formulas will now follow with no further analysis from our earlier Ansätzen when we set:

(52) $$K^* = G^*, \qquad \text{so} \qquad \mathfrak{K}^* = \mathfrak{G}^*.$$

In order to understand things completely, we must only set:

$$\kappa = 1$$

afterwards.

One must actually deal with only two points:

a) From (21), we have:

(53) $$\mathfrak{K}_{\mu\nu} \equiv \mathfrak{G}_{\mu\nu}^{*}.$$

However, since $\mathfrak{G}^*$ contains only first-order differential quotients of the $g^{\mu\nu}$, $\mathfrak{G}_{\mu\nu}^{*}$ can be formally represented more simply than $\mathfrak{K}_{\mu\nu}$:

(54) $$\mathfrak{G}_{\mu\nu}^{*} = \frac{\partial \mathfrak{G}^*}{\partial g^{\mu\nu}} - \sum \frac{\partial\left(\dfrac{\partial \mathfrak{G}^*}{\partial g_{\rho}^{\mu\nu}}\right)}{\partial w^{\rho}}.$$

As such, for *Einstein*, the $\mathfrak{G}_{\mu\nu}^{*}$, in fact, appear in place of the $\mathfrak{K}_{\mu\nu}$ in the field equations (formula (17) of his article). One must say that a special property of the $\mathfrak{K}_{\mu\nu}$ will be made visibly prominent upon introducing the $\mathfrak{G}_{\mu\nu}^{*}$, namely, that they include no differential quotients of the $g^{\mu\nu}$ of order higher than two,.

b) Moreover, from (16), (22), we have only the following simple formulas for the $\mathfrak{U}_{\tau}^{*\sigma}$:

(55) $$\mathfrak{U}_{\tau}^{*\sigma} = \frac{1}{2}\left(\mathfrak{G}^{*}\delta_{\tau}^{\sigma} - \sum \frac{\partial \mathfrak{G}^*}{\partial g_{\sigma}^{\mu\nu}}\, g_{\tau}^{\mu\nu}\right).$$

These $\mathfrak{U}_{\tau}^{*\sigma}$ are actually shorter than then the general $\mathfrak{U}_{\tau}^{\sigma}$, but the result of the divergence map:

([11]) Cf., the details of the intermediate calculations on pp. 110, 191 of *Weyl's* book.

$$\sum \frac{\partial \mathfrak{U}_\tau^{*\sigma}}{\partial w^\sigma}$$

is once more the same. Thus, the reduction of the formulas also only makes the simplification more clearly intuitive, which the final result that comes under consideration for us will possess anyway as a result of the way that K was constructed.

The $\mathfrak{U}_\tau^{*\sigma}$ *that are defined by* (55), *when divided by* κ, *are now, with no further analysis, Einstein's* $\mathfrak{t}_\tau^\sigma$:

(56) $$\frac{1}{\kappa}\mathfrak{U}_\tau^{*\sigma} = \mathfrak{t}_\tau^\sigma .$$

In fact, *Einstein* gave precisely the same value for his $\mathfrak{t}_\tau^\sigma$ that is found in the right-hand side of (55) in formula (20) of his treatise – on the basis of an entirely different set of calculations – when he took $\kappa = 1$.

If we correspondingly substitute the $\mathfrak{t}_\tau^\sigma$ for the $\frac{1}{\kappa}\mathfrak{U}_\tau^{*\sigma}$ in (43) then we will obtain equations (44). Q. E. D.

I wish to add a small supplement to these developments. It is well-known that *Einstein*, in his "Kosmologische Betrachtungen zur allgemeinen Relativitätstheorie" ([12]), made a suggestion that one should modify the fundamental field equations of gravitation in such a way that – in our notation – one would write:

(57) $$K_{\mu\nu} - \lambda\, g_{\mu\nu} - \kappa T_{\mu\nu} = 0,$$

instead of (3), in which we understand λ to mean a constant. Since:

$$\frac{\partial\sqrt{g}}{\partial g^{\mu\nu}} : \sqrt{g} = -\tfrac{1}{2}\, g_{\mu\nu},$$

we can also write (57) as:

(58) $$\bar{K}_{\mu\nu} - \kappa T_{\mu\nu} = 0, \qquad \text{or also} \qquad \bar{K}_\tau^\sigma - \kappa T_\tau^\sigma = 0,$$

where:

(59) $$\bar{K} = K + 2\lambda.$$

All of the assumptions that we used in paragraphs **2** *to* **5** *as the basis for the construction of the identities for K are now true for this* $\bar{K}$. For example, we can then

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([12]) Sitzungsberichte der Berliner Akademie of 8 February 1917.

write down the identities (37) for $\bar{K}$, in which we only have to replace the U_τ^σ with the $\bar{U}_\tau^\sigma$, which, according to (16), one will have:

$$\bar{U}_\tau^\sigma = U_\tau^\sigma + \lambda\delta_\tau^\sigma . \tag{60}$$

As before, we thus also come to the same conservation laws, perhaps corresponding to formula (42):

$$\sum \frac{\partial\left(\mathfrak{T}_\tau^\sigma + \frac{1}{\kappa}\bar{\mathfrak{U}}_\tau^\sigma\right)}{\partial w^\sigma} = 0, \tag{61}$$

in which we might now alter the $\bar{\mathfrak{U}}_\tau^\sigma$ by setting:

$$\bar{K}^* = \bar{K} + \mathfrak{Div}, \tag{62}$$

in place of $\bar{K}$. In particular, we would like to take $\bar{K}^*$ to be:

$$\bar{G}^* = G^* + 2\lambda, \qquad \text{i.e.,} \qquad \bar{\mathfrak{G}}^* = \mathfrak{G}^* + 2\lambda\sqrt{g}\,, \tag{63}$$

according to our most recent developments. If we then write:

$$\bar{\mathfrak{t}}_\tau^\sigma = \frac{1}{2\kappa}\left(\bar{\mathfrak{G}}^*\delta_\tau^\sigma - \sum\frac{\partial\bar{\mathfrak{G}}^*}{\partial g_\sigma^{\mu\nu}}\,g_\tau^{\mu\nu}\right), \tag{64}$$

while we include (55), (56), then we will now have:

$$\sum \frac{\partial\left(\mathfrak{T}_\tau^\sigma + \bar{\mathfrak{t}}_\tau^\sigma\right)}{\partial w^\sigma} = 0. \tag{65}$$

This corresponds to the statement that *Einstein* made in his most recent publication ([13]).

([13]) Sitzungsberichte der Berliner Akademie on May 1918, pp. 456.

§ 8.

Concluding remarks.

The relationships that have been given so far in the developments in the papers that I cited of *Einstein*, *Hilbert*, *Lorentz*, and *Weyl* are individually much narrower in scope than what would emerge from a mere comparison of the final results. Many formulas that occurred to me in my subsequent research are also found there, but not in the unified train of thought that I adhered to. It is very interesting to pursue that in detail. My developments are, in fact, closest to those of *Lorentz*, who nonetheless restricted himself to infinitesimal transformations $\delta w^\tau = p^\tau$ for which p^τ was independent of *w*. *Einstein* considered those p^τ that corresponded to affine transformation of the *w*, and *Weyl* (as I myself did in my previous Note) considered p^τ that were arbitrary, moreover, but vanished on the boundary of the domain of integration in a suitable way ([14]).

I must also not fail to thank *Nöther* once more for her helpful participation in my recent papers, as she worked out the mathematical ideas that I employed while adapting to the physical way of posing the integral I_1 in general, and those ideas will, in turn, be presented in a Note that will be published in the next of these Nachrichten ([15]).

([14]) As I published in an essay "Zur Gravitationstheorie" (Bd. 54 of the Annalen der Physik) that was inserted before my Note, but was first published by him.

([15]) [I presented *Nöther's* main theorems to the Gesellschaft der Wissenschaften on 26 July. The Note itself appeared later on in the Göttinger Nachrichten 1918, pp. 235-257, under the title "Invariante Variationsprobleme." –

The foregoing "main theorem" that was presented in § 2 is a special case of the following, far-reaching theorem of *Nöther*, which was proved in the cited place:

"If an integral *I* is invariant under a G_ρ (i.e., a continuous group with ρ essential parameters) then there will be ρ linearly-independent combinations of the *Lagrangian* expressions that become divergences."

However, as far as the assertion of *Hilbert* that was contained in XXXI (see pps. 561 and 565 of the previous article) is concerned, in particular, according to *Nöther*, its exact formulation will give the following:

"If an integral *I* admits the displacement group then the energy relations will become improper if and only if *I* is invariant under an infinite group that contains the displacement group as a subgroup."

Moreover, the theorem of *Hilbert* (XXXI, resp.) that four relations exist between the field equations of the theory of relativity also gets generalized by *Nöther*. His theorem then reads:

"If the integral *I* is invariant under a group with ρ arbitrary functions in which these functions appear up to their derivatives of order σ then there will exist ρ identity relations between the *Lagrangian* expressions and their derivatives up to order σ."

K.]