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On the theory of line complexes of first and second order [1])

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One considers the *coordinates of the line in space* to be the relative values of the six two-rowed determinants that are constructed from the coordinates of two points or two planes. Between them, there exists identically a relation of second degree:

$$R = 0.$$

The fact that six arbitrarily chosen quantities that satisfy this equation can be regarded as coordinates of a line is to be expected from the way that the line coordinates came about as the coordinates of two points or planes, and the fact that the line coordinates can be considered to be autonomous homogeneous variables that have to satisfy an equation of second degree.

A further equation of second degree between them:

$$\Omega = 0$$

determines *line complex of second degree.*

This suggests the problem of converting the two equations R and Ω into two other equations that only include the squares of the variables by a linear substitution. Such a conversion is known to always be possible, and in just one way, if one assumes that the values of the roots that one obtains when the determinant of the form $\Omega + \lambda P$ is set equal to zero and solved for λ are all different from each other [2]). *The geometrical sense of this transformation shall be discussed in what follows.* Insofar as we exclude from consideration those complexes of second degree for which the conversion that we spoke of is not possible, we shall henceforth think of the two forms R and Ω as being given in the simplified form.

It is also emphasized that this form of equation is of great importance, not only for the complexes of second degree as such, but also for the *surfaces of fourth order and fourth class with 16 double planes and 16 double planes* that are closely related to these complexes.

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[1]) Excerpts previously published in the Göttinger Nachrichten, 1869, Session on 9 June, pp. 258.

[2]) In my Inaugural Dissertation: *Über die Transformation der allgemeinen Gleichung zweiten Grades zwischen Linienkoordinaten auf eine kanonische Form, Bonn,* 1868 (Abhandlung I of this collection), I have treated the algebraic nature of this transformation. There, I likewise brought under consideration the case that was excluded from the present paper of complexes of second degree, and presented the corresponding equation of the canonical form.

When the new variables that were introduced in place of the original line coordinates are set equal to zero they represent linear complexes that can be grouped with each other in a distinctive way. In regard to this, one arranges the lines in space into systems of 32, while the planes and points in it can be arranged into systems of 16 planes and 16 points, respectively. The relationship of the 16 planes and 16 points of such a system to each other is the same as that of the 16 double plane and 16 double points of any surface of fourth order and fourth class.

The fundamental meaning of these linear complexes for the complex of second degree is that the relationship with the complex of second degree is the same for all elements that are associated with each other by means of the linear complex. The same statement that is true for the complex of second degree is true for the surface of fourth order and fourth class that it determines. A series of theorems follows for any complex, as well as for these surfaces.

The algebraic representation of the picture that emerges from these geometric considerations is fashioned quite simply. In particular, the family of complexes of second degree that belong to the same surface of fourth order and fourth degree is represented by an arbitrary parameter in the same way as a system of confocal curves or surfaces of second degree.

Let it now be remarked that we shall mostly draw upon one of two mutually reciprocal theorems without expressly referring to the other one.

I.

Preliminary considerations.

1. Let the coordinates of two points of a line be denoted by:

$$x_1, x_2, x_3, x_4,$$
$$y_1, y_2, y_3, y_4,$$

and the coordinates of two planes through the same line by:

$$u_1, u_2, u_3, u_4,$$
$$v_1, v_2, v_3, v_4.$$

One then considers the coordinates of the line to be the determinants:

$$p_{ik} = x_i\, y_k - y_i\, x_k\,,$$

or the determinants:

$$q_{ik} = u_i\, v_k - v_i\, u_k\,.$$

One then has:

$$p_{ik} + p_{ki} = 0, \qquad q_{ik} + q_{ki} = 0.$$

Following Plücker, one calls the coordinates p_{ik} *ray coordinates* and the coordinates q_{ik} *axis coordinates.*

If we understand α, β, γ, δ to be the numbers 1, 2, 3, 4 in an arbitrary sequence then we have the following identities:

$$P \equiv p_{\alpha\beta}\, p_{\gamma\delta} + p_{\alpha\gamma}\, p_{\delta\beta} + p_{\alpha\delta}\, p_{\beta\gamma} = 0,$$
$$Q \equiv q_{\alpha\beta}\, q_{\gamma\delta} + q_{\alpha\gamma}\, q_{\delta\beta} + q_{\alpha\delta}\, q_{\beta\gamma} = 0,$$

which may collectively be denoted by the symbol:

$$R = 0.$$

In this notation, one has:

$$\rho\, p_{ik} = \frac{\partial Q}{\partial q_{ik}}, \qquad q_{ik} = \rho\, \frac{\partial P}{\partial p_{ik}},$$

where ρ means a proportionality factor.

A line whose coordinates are $p_{ik}^{(\alpha)}$, $q_{ik}^{(\alpha)}$ will be denoted by $(r^{(\alpha)})$ in what follows. We will write $r_{ik}^{(\alpha)}$ instead of $p_{ik}^{(\alpha)}$, $q_{ik}^{(\alpha)}$ in those cases where the difference between ray and axis coordinates is irrelevant.

2. Assuming these notations, one can write the condition for two lines (r), (r') to intersect in the following equivalent forms:

$$\sum p_{ik} \frac{\partial P'}{\partial p'_{ik}} = 0, \qquad \sum p'_{ik} \frac{\partial P}{\partial p_{ik}} = 0,$$
$$\sum q_{ik} \frac{\partial Q'}{\partial q'_{ik}} = 0, \qquad \sum q'_{ik} \frac{\partial Q}{\partial q_{ik}} = 0,$$
$$\sum p_{ik}\, q'_{ik} = 0, \qquad \sum p'_{ik}\, q_{ik} = 0.$$

Three lines (r), (r'), (r'') that intersect each other have either a point or a plane in common. Depending upon whether the one or the other situation exists, the second or the first factor of the four products:

$$\sum \pm p_{\alpha\beta}\, p'_{\alpha\gamma}\, p''_{\alpha\delta} \cdot \sum \pm p_{\gamma\delta}\, p'_{\delta\beta}\, p''_{\beta\gamma}$$

will vanish, respectively, products that can be represented in any of the following forms:

$$\sum \pm p_{\alpha\beta}\, p'_{\alpha\gamma}\, p''_{\alpha\delta} \cdot \sum \pm q_{\alpha\beta}\, q'_{\alpha\gamma}\, q''_{\alpha\delta},$$
$$\sum \pm q_{\gamma\delta}\, q'_{\delta\beta}\, q''_{\beta\gamma} \cdot \sum \pm q_{\alpha\beta}\, q'_{\alpha\gamma}\, q''_{\alpha\beta},$$
$$\sum \pm q_{\gamma\delta}\, q'_{\delta\beta}\, q''_{\beta\gamma} \cdot \sum \pm p_{\gamma\delta}\, p'_{\delta\beta}\, p''_{\beta\gamma}.$$

If (r), (r'), (r'') are lines that go through a point in the same plane then all of the coordinates of the same three-rowed determinants defined by it vanish, and one can set:

$$r_{ik} = \lambda r'_{ik} + \mu r''_{ik}.$$

Let (r), (r'), (r'') be lines that lie in a plane or go through a point. The coordinates of an arbitrary line (r) that lies in that plane and goes through the same point will be representable by:

$$r_{ik} = \lambda r'_{ik} + \mu r''_{ik} + \nu r'''_{ik}.$$

3. When the coordinates of a line in the expression R are replaced with the constants that enter into the equation of a complex of first degree, what results is an expression that does not generally vanish, which might be called the *invariant of the complex.* The vanishing of it expresses the idea that the complex subsumes the totality of all lines that cut a fixed line whose coordinates are the constants of the complex, so the complex is a so-called *special* complex.

Let the term *simultaneous invariant of two linear complexes* refer to the expression that arises when one introduces the constants of two linear complexes into the bilinear expression R.

The vanishing of the simultaneous invariant of two complexes expresses a relationship between them that might be referred to as *involution*.

If two linear complexes are special then the vanishing of the simultaneous invariant is the condition for the lines that represent them to intersect. If only one of the two complexes is special then the vanishing of the simultaneous invariant expresses that the lines that represents the one of them belongs to the other complex.

In the following, let it be assumed that none of the complexes under consideration is a special one.

All lines that belong to two linear complexes simultaneously intersect two fixed lines, namely, the *directrices of the congruence that is determined by the two complexes.* If the two complexes lie in involution then any two points that correspond to an arbitrary plane in them will be harmonic to the two points at which the plane cuts the two directrices. If one lets a plane rotate around a line that is common to both complexes then the point-pairs that correspond to the plane in its various positions are in involution on this line. Each of the two points that are determined by the two complexes in an arbitrary plane then corresponds to yet a second plane by them. This plane is the same for both points.

The planes and points of space can be arranged into groups of two planes and two points that lie on the intersection of them by means of two linear complexes that lie in involution.

By means of three linear complexes that are mutually in involution, the planes and points in space group together into tetrahedra that are conjugate to each other relative to the surface of second degree that is determined by the three line complexes. The three points that correspond to a face of such a tetrahedron in the three complexes are the three corner points of the tetrahedron that lie in them; conversely, the three planes that correspond to a corner point are the three faces that go through it.

4. The line coordinates r_{ik} represent the moments of the line that is to be determined with respect to the six edges of the coordinate tetrahedron when they are multiplied by certain (not completely arbitrary) constants. We ask what the meaning might be of a general linear transformation of the line coordinates.

The introduction of linear functions of the line coordinates in place of these coordinates comes down to considering the determining data of a line to be the moments of it relative to the six given linear complexes when they are multiplied by arbitrary constants. [3])

When one introduces the new variables that come about by a linear substitution into the identity that exists between the original line coordinates, one obtains a new expression of second degree in these variables that might be once more denoted by *R*, and whose vanishing is the necessary and sufficient condition that six, otherwise arbitrarily given, values of the variables can relate to a straight tline.

This expression *R* has entirely the same meaning as the one that is defined by the previous coordinates. Everything that was true for the previous ray and axis coordinates is now true for the new coordinates and partial differential quotients of *R* that one takes with respect to them.

The form of *R* immediately gives us information about the types and mutual positions of the complexes that are at the basis of the coordinate determination.

In particular, it is clear that when *R* includes only three terms, as would be the case for the original coordinates, the new variables would essentially be the moments of the line that they determine relative to the edges of a tetrahedron.

II.

The system of six fundamental complexes.

5. The detailed normal form for equation for the complexes of second degree that was mentioned above leads to the examination of those linear functions of the linear coordinates in which the equation of condition $R = 0$ can be written as the sum of squares that are multiplied by suitable constants. When set to zero, they represent six linear complexes that can be called *the six fundamental complexes.*

Let them be denoted by:

$$x_1 = 0, \qquad x_2 = 0, \qquad x_3 = 0, \qquad x_4 = 0, \qquad x_5 = 0, \qquad x_6 = 0$$

and the symbol *x* is thought of as being multiplied by constants such that the equation of condition can be written in the following form:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 = 0. \tag{1}$$

The system of variables depends upon 15 constants.

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[3]) [The formulation in this paper has been altered somewhat in hindsight, corresponding to the remarks that were added in the beginning of the next Abhandlung III, which we point out here. K]

The invariant of a linear complex:

$$a_1\, x_1 + a_2\, x_2 + \ldots + a_6\, x_6 = 0$$

is represented in it by:

$$a_1^2 + a_2^2 + \cdots + a_6^2\,,$$

and the simultaneous invariant of two linear complexes:

$$a_1\, x_1 + a_2\, x_2 + \ldots + a_6\, x_6 = 0,$$
$$b_1\, x_1 + b_2\, x_2 + \ldots + b_6\, x_6 = 0$$

is represented by:

$$a_1\, b_1 + a_2\, b_2 + \ldots + a_6\, b_6\,.$$

It next follows from this that the multipliers of the x can be chosen in such a way that the invariants of all fundamental complexes equal the positive unit. It further follows that the simultaneous invariant of two arbitrary fundamental complexes vanishes.

Any two of the six fundamental complexes lie in involution.

The equation of condition:

$$R = 0,$$

that one finds to exist between the original line coordinates subsumes the three products of any two of the six variables when they are grouped pair-wise. If they are transformed by means of a *real* linear substitution in such a way that it includes only the squares of the variables then one must find just as many positive and negative squares among them. Insofar as the sum of the squares of two conjugate imaginary expressions is equivalent to the sum of a positive and a negative real square, this yields the following [4])

There can be an arbitrary (even) number of the six fundamental complexes that are imaginary.

The symbols x that correspond to real fundamental complexes are chosen in such a way that half of them include real coefficients and half of them, pure imaginary ones.

From this, one gets:

The real fundamental complexes subdivide into two equally numerous groups. The complexes of the one group are right-handed and those of the other are left-handed [5])

The six fundamental complexes may be denoted simply by the numbers 1, 2, …, 6, and it remains undetermined whether one finds imaginaries among them or not.

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[4]) [In the presentation of this paper, it is not sufficiently clearly expressed that only such linear substitutions should be considered for which the complex conjugates enter into the newly introduced expressions at the same time.]

[5]) Plücker, *Neue Geometrie*, no. 47.

6. The directrices of the congruence of the two fundamental complexes (1, 2) obviously have the coordinates:

$$\rho x_1 = 1, \qquad \rho x_2 = \pm i, \qquad \rho x_3 = 0, \qquad \rho x_4 = 0, \qquad \rho x_5 = 0, \qquad \rho x_6 = 0.$$

The directrices of the congruence of two fundamental complexes belong to the remaining four fundamental complexes.

The totality of lines that cut an arbitrary one of the two directrices is represented by:

$$x_1^2 + x_2^2 = 0,$$

or, what amounts to the same thing, by:

$$x_3^2 + x_4^2 + x_5^2 + x_6^2 = 0.$$

The six fundamental complexes determine $6 \cdot 5/2 = 15$ linear congruences, whose 30 directrices are correspondingly grouped into a distinctive way. When the directrices of the congruence (1, 2) belong to the complexes 3, 4, 5, 6, they will be cut by the 12 directrices of the $4 \cdot 3/2 = 6$ congruences that are determined by them.

Any two of the 30 directrices that are grouped together will be cut by 12 of the remaining ones.

The directrices of one of the three congruences (1, 2), (3, 4), (5, 6) will be cut by each of the directrices of the other two congruences.

The directrices of such three congruences that together depend upon all six fundamental complexes define the edges of a tetrahedron.

In harmony with this, the equation of condition:

$$R = 0$$

when it is written in the following variables:

$$\begin{aligned} y_1 &= x_1 + i\, x_2, & y_3 &= x_6 + i\, x_4, & y_5 &= x_5 + i\, x_6, \\ y_2 &= x_1 - i\, x_2, & y_4 &= x_3 - i\, x_4, & y_6 &= x_5 - i\, x_6, \end{aligned}$$

which, when set to zero, represent the directrices in question, takes the characteristic form for the edges of a tetrahedron:

$$y_1\, y_2 + y_3\, y_4 + y_5\, y_6 = 0.$$

The totality of lines that lie in a face of the tetrahedron or go through a corner point of it is represented by:

$$x_1^2 + x_2^2 = 0,$$
$$x_3^2 + x_4^2 = 0,$$
$$x_5^2 + x_6^2 = 0.$$

When one divides six elements in 15 different ways into three groups of two, the 30 directrices define the edges of 15 tetrahedra. The tetrahedra may be called the *fundamental tetrahedra*. The corner points and faces of these tetrahedra are all different.

Any two directrices of the same group belong to three of the fundamental tetrahedra as opposite edges. The twelve directices that cut the two in question are the remaining 3 · 4 edges of this tetrahedron. These three tetrahedra determine six pair-wise grouped points on each of the two directrices. When the fundamental complexes are mutually in involution, two arbitrarily-chosen pairs of the three are mutually harmonic. An analogous statement is true for the six faces of the tetrahedron that intersect an arbitrarily-chosen one of the two directrices [6]).

If one is given any of the 15 fundamental tetrahedra then the remaining fourteen of them divide into two groups of six and eight. The tetrahedra of the first group have two opposite edges in common, while those of the second group do not.

7. The six directrices (1, 2), (3, 4), (5, 6), which define a tetrahedron, have the following coordinates:

		x_1	x_2	x_3	x_4	x_5	x_6
(1, 2)	I	1	i	0	0	0	0
	II	1	$-i$	0	0	0	0
(3, 4)	III	0	0	1	i	0	0
	IV	0	0	1	$-i$	0	0
(5, 6)	V	0	0	0	0	1	i
	VI	0	0	0	0	1	$-i$

Which of these three mutually intersecting directrices have a point in common and which of them have a plane in common can only be decided when the explicit expression for the variables $x_1, \ldots, x_6$ is given in the original line coordinates. If, perhaps, I, III, V go through a corner point of the tetrahedron then II, IV, VI lie in the opposite face. Whether three mutually intersecting lines (x, x', x'') have a point or a plane in common is then

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[6]) It emerges from this that the three tetrahedra that have two opposite edges in common can never be real. Relative to the reality of the picture that enters in here, one must above all make the following remark: Either all of the six fundamental complexes are real or two, four, or all of them are imaginary. Corresponding to these assumptions:

18, 10, 6, 6

are the 30 directrices and:

6, 2, 1, 1

are the 15 real fundamental tetrahedra.

determined, as described the above, according to whether the first or the second of the following two expressions vanishes:

$$\begin{vmatrix} x_1 + ix_2 & x_3 + ix_4 & x_5 + ix_6 \\ x_1' + ix_2' & x_3' + ix_4' & x_5' + ix_6' \\ x_1'' + ix_2'' & x_3'' + ix_4'' & x_5'' + ix_6'' \end{vmatrix},$$

$$\begin{vmatrix} x_1 - ix_2 & x_3 - ix_4 & x_5 - ix_6 \\ x_1' - ix_2' & x_3' - ix_4' & x_5' - ix_6' \\ x_1'' - ix_2'' & x_3'' - ix_4'' & x_5'' - ix_6'' \end{vmatrix}.$$

One then has to change the sign of i in any two columns simultaneously.

One obtains similar criteria relative to each of the fourteen remaining fundamental tetrahedra.

8. Through each of the 60 corner points of the 15 fundamental tetrahedra go, in addition to the three associated faces, 12 more of the 60 faces, which are divided into three groups of four that intersect relative to one of the three directrices that goes through the corner point. Each of them intersects one of the three faces that are associated with the corner point in a new line. The point at which it encounters the directrix that lies on the third of these faces is one of the 59 other corner points.

One such line is the following:

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ \hline 0 & 0 & 1 & i & 1 & i \end{array}.$$

It is the connecting line between the two corner points:

$$\begin{array}{lll} (x_1 + ix_2, & x_3 + ix_4, & x_5 + ix_6), \\ (x_1 - ix_2, & x_3 + ix_4, & x_5 + ix_6), \end{array}$$

and the intersecting line of the two face planes:

$$\begin{array}{lll} (x_1 - ix_2, & x_3 + ix_4, & x_5 + ix_6), \\ (x_1 + ix_2, & x_3 + ix_4, & x_5 + ix_6). \end{array}$$

Such lines go through twelve of the assumed corner points. There are then:

$$\frac{12 \cdot 60}{2} = 360$$

of them, in all.

The 12 face planes, which go through one corner point in addition to the three associated ones, and which are divided into three bundles of four, intersect any three of the 16 remaining lines that include two corner points in addition to the assumed one. In fact, the line:

$$\begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ \hline 1 & i & 1 & i & 1 & i \end{array}$$

includes the three corner points:

$$\begin{array}{lll} (x_1 + ix_2, & x_3 + ix_4, & x_5 + ix_6), \\ (x_1 + ix_4, & x_3 + ix_6, & x_5 + ix_2), \\ (x_1 + ix_6, & x_3 + ix_2, & x_5 + ix_4), \end{array}$$

and lies in the face planes:

$$\begin{array}{lll} (x_1 + ix_2, & x_3 + ix_6, & x_5 + ix_4), \\ (x_1 + ix_4, & x_3 + ix_2, & x_5 + ix_6), \\ (x_1 + ix_6, & x_3 + ix_4, & x_5 + ix_2). \end{array}$$

There are $16 \cdot 60 / 3 = 320$ such lines.

In the foregoing considerations, the words "corner point" and "face plane" can be exchanged everywhere.

The 30 directrices of the 15 congruences that are determined by the 6 fundamental complexes are the edges of 15 (fundamental) tetrahedra.

15 face planes go through each of the 60 corner points of the fundamental tetrahedra; 15 corner points lie in each of the 60 face planes.

There are 360 lines that include two of the 60 corner points of the fundamental tetrahedra. These lines define the intersection of any two of the face planes.

There are 320 lines on which lie any three of the 60 corner points of the fundamental tetrahedra. Any three of the 60 face planes intersect along these lines.

The 30 directrices of the 15 congruences that are determined by the six fundamental complexes include any six of the 60 corner points and are the intersection of any six of the 60 face planes.

The six corner points, as well as the six face planes, are grouped with each other pair-wise. Any two pairs are mutually harmonic.

9. Any three of the fundamental complexes – for example, 1, 2, 3 – determine a surface of second degree by means of the lines of their one generator. The directrices of the congruences (2, 3), (3, 1), (1, 2) are lines of the second generator. Since these directrices belong to the complexes 4, 5, 6, it is clear that the complexes 4, 5, 6 determine the same surface of second order by means of the lines of its other generator.

The six complexes may be divided into two groups of three in $\frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3} \cdot \frac{1}{2} = 10$ ways. *Any two associated groups determine the same surface of second degree by means of their different generators.*

The ten surfaces thus defined may be called the ten *fundamental surfaces.*

Any two associated directrices of the 30 total belong to four of the fundamental surfaces as generators. The pair (1, 2) of directrices then lies on the surfaces (1, 2, 3), (1, 2, 4), (1, 2, 5), (1, 2, 6). As for the remaining six fundamental surfaces, the directrices (1, 2) are mutually conjugate polars.

The fundamental surfaces divide into two groups relative to one of the fundamental tetrahedra. The six surfaces of the one group include any four of the six tetrahedral edges, so the tetrahedron is conjugate to itself relative to the surfaces of the other group.

In order to represent the fundamental surfaces – say, (1, 2, 3) ≡ (4, 5, 6) – one can use the condition that says that a line must contact the surface; in other words, the complex equation of the surface [7]).

This will be:

$$x_1^2 + x_2^2 + x_3^2 = 0,$$

or, what is obviously the same thing:

$$x_4^2 + x_5^2 + x_6^2 = 0.$$

10. *The lines of space and the planes and points in it may be grouped into closed systems* relative to the six fundamental complexes, in a way that is similar to the way that was the case for the planes and points relative to two or three complexes that lie in involution.

Now, let a line be given whose coordinates are:

$$a_1, a_2, \ldots, a_6 .$$

Therefore:

$$a_1^2 + a_2^2 + \cdots + a_6^2 = 0.$$

Since this relation remains fulfilled for any arbitrary choice of signs of the coordinates, one finds that each of the $2^5 = 32$ sign combinations:

$$\pm a_1, \pm a_2, \ldots, \pm a_6$$

corresponds to a line. The relationship between the 32 lines and each other is obviously mutual.

The lines of space group together into 32 groups relative to the six fundamental complexes.

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[7]) If f_1, f_2, f_3 are three linear complexes, A_{11}, A_{22}, A_{33}, their invariants, and A_{21}, etc., their simultaneous invariants then the complex equation that they determine is the hyperboloid:

$$0 = \begin{vmatrix} 0 & f_1 & f_2 & f_3 \\ f_1 & A_{11} & A_{12} & A_{13} \\ f_2 & A_{21} & A_{22} & A_{23} \\ f_3 & A_{31} & A_{32} & A_{33} \end{vmatrix}.$$

When one forms the two-rowed determinants according to the schema:

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \\ \pm a_1 & \pm a_2 & \pm a_3 & \pm a_4 & \pm a_5 & \pm a_6 \end{vmatrix}$$

and then sets them equal to zero, one foresees immediately that of the 32 lines:

$2 \cdot 15$ (?) 16 belong to a complex,
$4 \cdot 20$ (?) 8 belong to a congruence,
$8 \cdot 15$ (?) 4 belong to a surface of second degree,

from which, each of the 32 lines lie on 15 of the complexes, 20 of the congruences, and 15 of the surfaces of second degree.

The 32 lines divide into two groups of 16, according to whether their coordinates possess an even or odd number of equal signs, respectively. When a plane curve is generated by a line of one of the two groups, an analogous situation prevails for the remaining 15 lines; the 16 lines of the other group generate cones.

By means of an arbitrary line of the one group, the lines in the other group divide into ones that are its conjugate polars relative to the six fundamental complexes and the ones that are its conjugate polars relative to the ten fundamental surfaces. The coordinates of the former six differ from the coordinates of the chosen line by a change of sign, and those of the latter ten, by three.

When the six fundamental complexes are found from an equation of sixth degree, the equation of degree 32 by which one determines a system of lines, such as we have considered here, requires only the solution of equations of second degree.

The system of 32 associated lines simplifies as long as one or more of the coordinates a are equal to zero. In particular, the lines that cut an associated pair of the 30 directrices correspond to 8, those that are generators of one of the ten fundamental surfaces, to 4, and finally the 30 directrices themselves, to 2.

11. Let the equation of the projection of a point into an arbitrarily chosen plane of the complex:

$$x_k = 0$$

that corresponds to one in the coordinate planes be:

$$a_k\, u + b_k\, v + c_k\, w = 0.$$

Then, as a result of the equation of condition:

$$x_1^2 + x_2^2 + \cdots + x_6^2 = 0$$

the expression:

$$\sum_{1\cdots 6} (a_k u + b_k v + c_k w)^2$$

vanishes identically. Therefore, the following determinant:

$$\begin{vmatrix} a_1^2 & b_1^2 & c_1^2 & b_1c_1 & c_1a_1 & a_1b_1 \\ a_2^2 & b_2^2 & c_2^2 & b_2c_2 & c_2a_2 & a_2b_2 \\ \bullet & \bullet & \bullet & \bullet\bullet & \bullet\bullet & \bullet\bullet \\ \bullet & \bullet & \bullet & \bullet\bullet & \bullet\bullet & \bullet\bullet \\ \bullet & \bullet & \bullet & \bullet\bullet & \bullet\bullet & \bullet\bullet \\ a_6^2 & b_6^2 & c_6^2 & b_6c_6 & c_6a_6 & a_6b_6 \end{vmatrix},$$

also vanishes, which says that the six points 1, 2, …, 6 lie on a conic section:

The six points that correspond to an arbitrary plane in the six fundamental complexes lie on a curve of second order.

The six planes that correspond to an arbitrary point in the six fundamental complexes envelop a cone of second class.

When the arbitrarily chosen plane goes through one of the 60 corner points of the 15 fundamental tetrahedra, the hexagon that is defined by six of the points that correspond to the fundamental complexes will become a Brianchon hexagon. A tetrahedral point becomes a Brianchon point. The hexagon includes two (three, resp.) Brianchon points that lie in a line when the arbitrarily chosen plane contains 360 (320, resp.) of the lines that belong to the system of fundamental tetrahedra. The number of Brianchon points becomes four when the plane is laid through one of the 360 lines and one of the 320 lines that intersect it.

When the arbitrarily chosen plane contacts one of the ten fundamental surfaces then the six points that correspond to the fundamental complexes lie in such a way that three of them lie on each of two lines – namely, the two generators of the fundamental surface that includes the plane. If the plane goes through one of the 30 directrices then four of six points move onto the directrices, while the other two come together at the intersection point with the associated directrix. Finally, if the plane overlaps one of the faces of the fundamental tetrahedron then the six points move pair-wise into three associated tetrahedral points.

12. Let the six points that correspond to a given plane in the six fundamental complexes be denoted by:

$$1, 2, 3, 4, 5, 6.$$

Each of these points corresponds to five planes, in addition to the given one. As long as the plane that corresponds to 1 in x_2 agrees with the plane that belongs to 2 in x_1, there are 15 new planes in all that cut the given one along the 15 connecting lines of the six points with each other. The three planes (2, 3), (3, 1), (1, 2) intersect (cf., no. 3) at the pole of the given plane relative to the fundamental surface (1, 2, 3). Since this surface is identical to the surface (4, 5, 6), the planes (5, 6), (6, 4), (4, 5) intersect at the same point. The six planes that correspond to this point in the six fundamental complexes:

$$x_1, x_2, x_3, x_4, x_5, x_6$$

coincide with the planes:

$$(2, 3), (3, 1), (1, 2), (5, 6), (6, 4), (4, 5),$$

which in fact envelop a cone of the second class, as would emerge from the consideration of the hexagon 123456.

Relative to the fundamental complexes, the planes and points of space group into closed systems of 16 planes and 16 points. Six of the 16 points lie in each of the 16 planes, and six of the 16 planes go through each of the 16 points. The six points in a plane lie on a curve of second order, while the six planes through a point envelop a cone of second class.

When one of the 16 planes is given, one finds the 16 points when one constructs the points that correspond to it in the six fundamental complexes and the poles that are conjugate to it relative to the ten fundamental surfaces.

The system of 16 double planes and 16 double points of the surfaces of fourth order and fourth class that Kummer investigated is of the type that is considered here [8]).

If one of the 32 lines that are associated relative to the fundamental complex lie in one of the 16 planes of such a system then the 15 lines of that same group distribute onto the 15 other planes and the 16 lines of the other group onto the 16 points. In particular, if the chosen line contacts the conic section that lies in the plane in question then one finds it among the 15 lines of the same group, and the 16 lines of the other group are faces of the cone that go through the 16 points.

The 32 associated lines are distinguished by the signs of their coordinates. This remark immediately gives the notation for them in terms of five indices that are chosen from two different values. The 16 planes and 16 points of the system considered here can be denoted in a similar way. The 16 planes correspond to the lines of the one group, while the 16 points correspond to the lines of the other. It emerges from this that the equation of degree 16 that determines the 16 planes is distinguished from the equation of degree 32 that we just considered only by the fact that a square root in it is assumed to be known.

If one of the 16 points of the system considered here lies in one of the 16 planes of an arbitrary system, then an analogous statement is true for the remaining 15 points, and each of the 16 planes includes one of the 16 points of the second system.

The 16 planes of a system intersect in $16 \cdot 15 / 2 = 120$ lines, which are likewise the connecting lines for the 16 points. They divide into 15 groups of eight each. The lines of one group belong to the same two fundamental complexes and thus both of the corresponding directrices have common transversals. This gives one the means to construct the 30 directrices and the 15 fundamental tetrahedra from the system of 16 planes and 16 points.

The 16 planes of the system intersect in three of the 240 points, which lie on six of the 120 lines of intersection, in addition to the 16 points of the system. Likewise, there

[8]) Monatsberichte der Berliner Akademie, 1864.

are 240 planes that include three of the 16 points of the system. They intersect in six of the 120 lines of it.

If one of the 16 planes of the systems has a distinguished position with respect to the six fundamental complexes then an analogous statement is true for the remaining 15 planes and 16 points. We especially emphasize the system that arises when one of the planes includes one of the 60 corner points of the fundamental tetrahedra. Then, four of the 16 planes go through the corner points of the tetrahedra in question and four of the 16 points lie in the faces of them. The system becomes the system of singularities of a tetrahedroid [9]). The equation of degree 16 that determines the planes of the system is algebraically soluble here, since all that is required here is the solution of a biquadratic equation and several quadratic ones.

III.

The Kummer surface and its connection with complexes of second degree.

12. Let the following equation be given as the equation of the complex of second degree to be examined:

(2) $$k_1x_1^2 + k_2x_2^2 + \cdots + k_6x_6^2 = 0.$$

Therefore, one has:

(1) $$x_1^2 + x_2^2 + \cdots + x_6^2 = 0,$$

such that the complex remains unchanged when one generally writes $k_\alpha + \lambda$ instead of k_α. The four constants that are thus included in equation (2), together with the 15 constants of the fundamental complex, give the 19 constants of the complex of second degree.

The for of equation (2) says that *the given complex and all of the geometric structures that immediately depend upon it correspond to each other relative to the system of six fundamental complexes* [10]).

Therefore, the lines of the complex then group into systems of 32. 16 complex curves and 16 complex cones belong together, etc.

From this last theorem, one derives the theorem that will discussed in the sequel on the basis of the properties of the complexes of second degree that were developed by Plücker [11]).

14. Those points whose complex cone decomposes into a plane pair – the so-called *singular points* – define a surface of fourth order and fourth class with 16 double points

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[9]) Cayley, in Liouville's Journal, 11 (1846). (Coll. Papers, v. I, 302-306.)

[10]) This mutual correspondence can also be regarded being mediated by the ten fundamental surfaces, instead of the six fundamental complexes.

[11]) Julius Plücker, *Neue Geometrie des Raumes, gegründet auf die Betrachtung der geraden Lineie als Raumelement.* B. G. Teubner, Leipzig, 1868, 1869.

and 16 double planes. This surface will be enveloped by *singular planes*, which are planes whose complex curve has resolved to two points in the system [12]).

Such a surface shall be called a *Kummer surface* in what follows. It is called the *singularity surface* in relation to the complex.

The reasoning that follows immediately examines the Kummer surface as such, regardless of its relation to the given complex.

A Kummer surface corresponds to itself relative to the system of six fundamental complexes.

Let the relevant fundamental complexes [13]) be denoted by $x_1, x_2, \ldots, x_6$, as before.

An equation of fourth degree serves to determine the tangential planes that go through a line:

$$a_1, a_2, \ldots, a_6$$

on a given Kummer surface. That equation can include only the squares of the coordinates a [14]). Therefore, the four tangential planes that go through any of the 32 lines:

$$\pm a_1, \pm a_2, \ldots, \pm a_6,$$

are all determined by the same biquadratic equation.

The four tangential planes, which can go through the given lines that lie on the surface, are reciprocally associated by the surface with the four intersection points of any of the 16 lines of the other group. It follows from this and the foregoing that the same equation determines the tangential planes that go through any line and the intersection points that lie on it.

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[12]) Plücker, *Neue Geometrie*, no. 311, 320.

[13]) For the Fresnel wave surface, which is derived from the ellipsoid:

$$a^2 x^2 + b^2 y^2 + c^2 z^2 = 1,$$

the fundamental complexes are the following ones:

$$\begin{aligned}
&(yz' - y'z) + a\sqrt{-1}\,(x - x') = 0, && (yz' - y'z) - a\sqrt{-1}\,(x - x') = 0,\\
&(zx' - z'x) + b\sqrt{-1}\,(y - y') = 0, && (zx' - z'x) - b\sqrt{-1}\,(y - y') = 0,\\
&(xy' - x'y) + c\sqrt{-1}\,(z - z') = 0, && (xy' - x'y) - c\sqrt{-1}\,(z - z') = 0.
\end{aligned}$$

[14]) This assertion is true in its own right, although it was not established in the present article. The basis for it must be derived tediously form the algebraic developments in section IV, so it is omitted here. – I originally proved the theorem of the equality of anharmonic ratios, to which the argument is directed, by the same method that was later developed by A. Voss in the treatise: "Über Komplexe und Kongruenzen," Math. Annalen, Bd. 9 (1876), and then went on to show that, from (8) to (10), pp. 79, an arbitrary line belongs to one (and, in turn, four) complexes of second degree that have the same singularity surface. In my note: "Über Plückersche Komplexfläche" (see Abhandlung XI of this collection), I gave another proof that connects up with the properties of the general complex surfaces that Plücker himself found in a more elementary way. – As for the discovery of the theorem, moreover, this brings up the fact that v. Staudt has presented a similar theorem for the tetrahedron, and one can regard the tetrahedron (as the totality of four planes and four corners) as the most degenerate case of a Kummer surface. K]

The anharmonic ratio of the four tangential planes that can go through a line on a Kummer surface is equal to the anharmonic ratio of the four intersection points of that line with the surface.

15. Let a point of a Kummer surface be given. One derives a system of 16 points and 16 planes from it by means of the six corresponding fundamental complexes. The points are points of the surface and the planes are planes of it. Likewise, the tangential planes to the given points correspond to a system of 16 planes and 16 points of the surface. The two systems have the reciprocal relationship that a point of one of them lies in each plane of the other that is the associated contact point. It follows from this that the six lines along which a plane of one of the systems will intersect the six associated contact points in the planes of the other system, not only in all of the common points but will also contact one of those six points that corresponds to the chosen plane in the fundamental complexes. These lines are also double tangents to the surface. This yields the following theorem:

Once the fundamental complex that is associated with a Kummer surface is determined by an equation of the sixth degree, one can rationally derive the coordinates of 32 points, 32 planes, and 96 double tangents to the surface from the coordinates of a point (or plane) of the surface.

The six tangents that can be drawn from the contact points of a plane in the Kummer surface to the intersection curve that lies in it contact it at the six points that lie on a conic section that corresponds to the chosen plane in the six fundamental complexes.

The 28 double tangents to any plane intersection curve of a Kummer surface divide into two groups of 16 and 12. The double tangents of the first group are the intersections of the plane of the curve with the 16 double planes of the surface. The 12 double tangents of the second group separate into groups of two. The six points at which the various pairs intersect relative to the lines are the six points that lie on a conic section that corresponds to the curve in the six fundamental complexes.

The double tangents of a Kummer surface define six different congruences of the second order and second class, each of which belongs to the six fundamental complexes [15]).

16. Distinguished among the systems of 16 points and 16 planes of the Kummer surface that are associated with the six fundamental complexes is the system of 16 double points and 16 double planes of the surface. In this case, the system of 16 contact curves and 16 contact cones enters in place of the associated second system. The 96 double tangents will be replaced with the 96 bundles of lines that go through one of the double points in one of the double planes.

The determination of the singularities of a Kummer surface depends upon the solution of an equation of sixth degree and several quadratic equations [16]).

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[15]) Cf., Kummer, Abhandl. der Ber. Akad., 1866.
[16]) C. Jordan in Crelle's Journal, Bd. 70 (1869).

In order to find the fundamental tetrahedra from the singularity system of a Kummer surface, one must construct the 30 lines that cut eight of the 120 intersection lines of the 16 double planes.

The surface may be constructed when one of the double planes of the Kummer surface is known in addition to the six fundamental complexes [17]). Then, when all of the 16 double planes of the fundamental complex and the contact curves in them are given, one knows, by the construction of any plane intersection curve of the surface, 16 double tangents and the contact points on it.

If the given double plane contains one of the 60 corner points of the fundamental tetrahedra then the associated Kummer surface becomes a tetrahedroid.

A tetrahedroid is characterized by the fact that the six double points that lie in a double plane define a Brianchon hexagon.

The singularities of a tetrahedroid are algebraically determinate.

17. We now return to the consideration of the complexes of second degree.

Those lines that are the intersection lines of two planes into which a complex cone has resolved, or – what amounts to the same thing – those lines that are the connecting lines of two points into which a complex curve decomposes, are the *singular lines* of the complex. They define a congruence of fourth order and fourth class. The singular lines contact the singularity surface of the complex. The contact point is called the *associated singular point* and the contact plane is the *associated singular plane.* The complex cone whose center is the associated singular point resolves to the two tangential planes of the singularity surface that go through the singular line, in addition to the doubly counted associated singular planes. Correspondingly, the complex curve decomposes in the associated singular plane of the system into two points that are common to the singular line and the singularity surface, along with the doubly counted associated singular points [18]).

The complex curve that lies in an arbitrary plane contacts the intersection curve of fourth order of the plane with the singularity surface in four points. Common tangents to both curves at these points are the four singular lines that lie in the plane [19]).

Which of the tangents to the singularity surface at a given point of it belongs to the given complex as a singular line is not determined by the surface itself. It can be chosen arbitrarily from the simple infinitude of tangents as a singular line; this would then leave an associated complex uniquely determined. From the associated singular planes, one derives a system of 16 singular planes by means of the six fundamental complexes. As long as the two points into which the complex curve decomposes for a singular plane are determined by solving a quadratic equation, the corresponding points in the remaining planes are known. Six of the complex lines that go through one of the intersection points of three of the 16 planes are then given, and for that reason the complex cone for these points is linearly constructible. When these intersection points lie on six of the 120

[17]) If one lays the double plane through one of the 320 lines that include three of the 60 corner points of the 15 fundamental tetrahedra then one obtains a surface that corresponds to the model in the Kummer citation (Monatsberichte der Berl. Akad., 1864).

[18]) Plücker, *Neue Geometrie*, no. 317.

[19]) Plücker, *Neue Geometrie*, no. 318.

intersection lines to two of the 16 planes, one knows the complex surfaces that are associated with these lines. By the construction of the complex curve that lies in an arbitrary plane, one can then have 240 tangents at one's disposal.

If a Kummer surface and a line that lies on it are given then one can construct a unique complex of second order that has the surface for its singularity surface and the line for its singular line.

A Kummer surface is the singularity surface for a simply infinite family of complexes of second order.

A Kummer surface depends upon 18 constants [20]).

If the given line is a double tangent to the surface then the associated complex degenerates into the doubly counted fundamental linear complex that belongs to the double tangents.

The doubly counted six fundamental linear complexes also belong to the family of complexes of second degree that have a given Kummer surface for the singularity surface. One can regard the double tangents of the surface that associated with such a complex as its singular lines.

18. Among the singular lines of the given complex, the ones that osculate the singularity surface are distinguished. The tangents to the contact points all belong to the given complex.

If a Kummer surface and a line that contacts it is given then there are, in addition to the complex that was just constructed, two more complexes that have the surface for singularity surface and include the line (but not as a singular line). Singular lines at the contact point with the given surface are the two principal tangents at this point for this complex.

From the construction of such a complex, one can next determine the two principal tangents at the contact points by a quadratic equation. The two points into which the complex curve resolves in the associated singular plane are then given linearly.

In addition to the four doubly-counted singular lines, the complex curve in an arbitrary plane has $2 \cdot 4 \cdot 3 - 2 \cdot 4 = 16$ tangents in common with the intersection curve of fourth order of the singularity surface that lies in the same plane. The contact points of it with the intersection curve of the singularity surface are those points at which the chosen planes of the curve will be cut by those points of the singularity surface at which the associated singular line coincides with a principal tangent.

The curve of the singular point whose associated singular lines osculate the singularity surface is of order 16.

19. Let a Kummer surface and an arbitrary line be given. Four planes of the surface go through the line and four points of the planes lie on the line. The biquadratic equation

[20]) This agrees with the enumeration that was given by Kummer.

that determines the four planes is the same as the one that determines the four points. Correspondingly, one can associate the four planes with the four individual points, and indeed, in four different ways. One deduces which type of association one should choose in one of the planes by the contact point with the Kummer surface and the associated point of a line. The complexes that have the given Kummer surface for the singularity surface and the line that was constructed for their singularity line obviously include that given line.

One may construct four complexes that have a given Kummer surface for their singularity surface and which contain a given line, moreover.

The two points that lie on the singular line thus constructed can be linearly determined when one of them is known to be an intersection point of the given line with the surface.

If the given line contacts the singularity surface then two of the four previously constructed complexes merge together into ones that have the given line for singular line.

The tangents of the singularity surface are the lines for which the biquadratic equation that determines the four complexes that are associated with four given lines has a double root.

The complex equation of the singularity surface has the form of a discriminant.

20. The simply-infinite family of the complexes of second order that are associated with a given Kummer surface determines a system of conic sections in each plane of space that contact the intersection curve of fourth order of the Kummer surface with the plane four times. The system is of fourth class. Since the degenerate conic section can be regarded as the six corresponding points of the fundamental complex with their pairs of double tangents, the system is of order $2 \cdot 4 - 6 = 2$.

A system of conic sections of fourth class and second order is determined by means of a Kummer surface in any plane of space.

21. The lines of the complexes that run inside of a *double plane* of the singularity surface intersect at a point of the contact curve. They are all singular lines. This point can be chosen arbitrarily on the contact curve; an associated complex is then linearly determined. If one lets the point go to one of the six double points that lie along the contact curve then the complex degenerates into those fundamental complexes that belong to the bundle of lines that go through the double point in the chosen plane. In a similar way, each *double point* of the surface corresponds to a plane that contacts the tangential cone at the double point [21]).

The 16 points and 16 planes correspond to each other relative to the six fundamental complexes.

In general, the lines of the given complexes are not double tangents of the singularity surface. This is the case only for the 96 bundles of singular lines that go through a double point inside of a double plane of the surface.

[21]) Plücker, *Neue Geometrie*, no. 321.

Those 16 singular lines that lie in a double plane of the singularity surface and contact the contact curve of the double plane are the only complex lines that contact the singularity surface at four points. The 16 corresponding singular lines that go through the double points of the singularity surface are the only complex lines that possess the dualistically reciprocal property.

22. One of the lines corresponds to a second line as its *polar* relative to a complex of second order. It has a double relationship with the latter, namely, first, it is the geometric locus of the poles of the first line with respect to all curves that are enveloped by the lines of the complex in the planes that go through them, and second, they are enveloped by the polar planes of the first line relative to all cones that are composed of lines of the complex at the points of it. This relationship between the two lines is therefore not invertible. Except for the lines of the complex that are polar conjugates to themselves, there is only a finite number of lines that are again the polars of their polars [22]).

The polars of the diagonals of the quadrilateral that is defined by the four singular lines that lie in an arbitrary plane intersect each other at a point. This point will be called the *pole* of the plane with respect to the complex. It coincides with the pole of the plane relative to the singularity surface. – In a similar way, each point corresponds to a *polar plane* relative to the complex.

The pole of a singular plane is its contact point with the singularity surface, so the polar plane of this point is generally the given singular plane.

However, in general, the relationship between planes and poles, points and polar planes is not an invertible one. Except for the singular planes and points, it exists for only a finite number of planes and points [23]).

23. A given complex of the second degree can be referred to any of the 15 fundamental tetrahedra. Its equation then assumes the following form [24]):

$$\sum a_{ik} r_{ik}^2 + 2A\, r_{\alpha\beta} r_{\gamma\delta} + 2B\, r_{\alpha\gamma} r_{\delta\beta} + 2C\, r_{\alpha\delta} r_{\beta\gamma} = 0,$$

where r_{ik} refer to either ray or axis coordinates.

If the equation of the given complex of second degree were written as the coordinate tetrahedron relative to one of the 15 fundamental tetrahedra then, in addition to the squares of the variables, only the products of those variables that relate to the opposite edges of the tetrahedon would appear.

It is easy to see that this form of an equation for the fundamental tetrahedron is characteristic. One further deduces:

If the given complex relates to an arbitrary coordinate tetrahedron and in the corresponding equation two variables that refer to the opposite edges of the coordinate

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[22]) Plücker, *Neue Geometrie*¸ no. 299.
[23]) Plücker, *Neue Geometrie*¸ no. 3238, 330, 337.
[24]) Cf., no. 6.

tetrahedron appear only in the opposite connection except as squares, then the edges in question are two associated ones from the system of fundamental tetrahedra.

The foregoing form of the equation shows that the opposite edges of the basic tetrahedron are mutually corresponding polars relative to the given complex.

Of the 30 edges of the fundamental tetrahedra, the ones that are associated relative to the complex correspond reciprocally as polars.

The 30 edges of the fundamental tetrahedra are, except for lines of the complexes, the only lines that possess these properties.

From this, one infers:

Of the 60 corner points and 60 faces of the fundamental tetrahedra, the associated ones correspond reciprocally as poles and polar planes relative to the complex.

Except for the singular points and planes, there are no other points and planes that are reciprocally associated relative to the complex.

Insofar as the behavior of planes, poles, points, and polar planes can be regarded as mediated by the singularity surface, the foregoing two theorems may also be regarded as expressing properties of the Kummer surface.

24. Investigations that are similar to the foregoing are already contained in the treatise on complexes of second degree by Battaglini [25]). However, the assumptions that he based it on are generally not sufficient. If we understand r_{ik} to mean either ray or axis coordinates then he gave the complex of second degree by the following equation:

$$\sum a_{ik} r_{ik}^2 = 0,$$

which possesses three terms less than the previous one, in which the complex relates to one of the fundamental tetrahedra. In fact, it contains only 17 constants, while the complex depends upon 19 constants. Correspondingly, two of the simultaneous invariants vanish that can be derived from them and the equation of condition that exists between the line coordinates.

The complex that Battaglini examined is specified by the fact that for it the quantities $k_1, k_2, \ldots, k_6$, when augmented by a suitable constant, are equal and opposite to each other. As a result, one of the fundamental tetrahedra is the one on which the complex takes on its simplest form, so it is distinguished from the other ones, and the singularity surface becomes a tetrahedroid that belongs to this tetrahedron (no. 16).

[25]) Atti della Reale Accademia di Napoli, 3 (1866), as well as Giornale di Matematiche, Napoli, v. 6 (1868).

IV.

Algebraic representation.

25. Let the given complex be determined, as before, by the equation:

(2) $$k_1 x_1^2 + k_2 x_2^2 + \cdots + k_6 x_6^2 = 0,$$

where:

(1) $$x_1^2 + x_2^2 + \cdots + x_6^2 = 0.$$

By means of equation (1), it is permissible for the quantities k to increase by an arbitrary constant with changing the complex.

The *singular lines* of the complexes are then represented [26]) by (1), (2), and the following equation:

(3) $$k_1^2 x_1^2 + k_2^2 x_2^2 + \cdots + k_6^2 x_6^2 = 0.$$

Let (x) mean an arbitrary singular line, so the coordinates (y) of any line that goes through the associated singular point in the singular plane that is associated with (x) will have the form:

(4) $$\rho\, y_\alpha = (k_\alpha + \sigma)\, x_\alpha,$$

where ρ is a proportionality factor and σ means a constant [27]). The lines that are represented by (4) may be called the *associated lines* to the singular line (x). The totality of associated lines to all singular lines coincides with the totality of all tangents to the singularity surface.

If the singular line (x) is determined in such a way that the associated lines belong to the given complex (2) then it osculates the singularity surface. One then finds the representation of the *osculating singular lines* in not only (1), (2), (3), but also the equation:

(5) $$k_1^3 x_1^2 + k_2^3 x_2^2 + \cdots + k_6^3 x_6^2 = 0.$$

The line surface that is defined by the osculating singular lines is of order 16 and class 16.

The 32 *distinguished singular lines* that lie in one of the double planes of the singularity surface and contact it along the contact curve that is contained in it, or go through a double point of the singularity surface and are generators of the tangent cone at it, are determined by the condition that their associated singular lines are themselves again singular lines. Their coordinates thus satisfy, in addition to equations (1), (2), (3), (5), also the following equation:

(6) $$k_1^4 x_1^2 + k_2^4 x_2^2 + \cdots + k_6^4 x_6^2 = 0.$$

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[26]) Plücker, *Neue Geometrie*, no. 300.

[27]) Plücker, *Neue Geometrie*, *ibid.*

Upon solving this, one finds that:

(7) $$\rho x_1^2 = \frac{1}{(k_2 - k_1)(k_3 - k_1)\cdots(k_6 - k_1)}, \text{ etc.}$$

26. If $x_1, x_2, \ldots, x_6$, and σ refer to arbitrary parameters that are subject to equations (1), (2), (3) then an arbitrary *tangent to the singularity surface* is given by equation (4). By elimination of $x_1, x_2, \ldots, x_6$, this yields:

(8) $$y_1^2 + y_2^2 + \ldots + y_6^2 = 0,$$

(9) $$\frac{y_1^2}{k_1+\sigma} + \frac{y_2^2}{k_2+\sigma} + \ldots + \frac{y_6^2}{k_6+\sigma} = 0,$$

(10) $$\frac{y_1^2}{(k_1+\sigma)^2} + \frac{y_2^2}{(k_2+\sigma)^2} + \cdots + \frac{y_6^2}{(k_6+\sigma)^2} = 0.$$

Equation (9) is an equation of fourth degree for the determination of σ. Equation (10) says that the differential quotient of equation (9) with respect to σ vanishes.

The complex equation of the singularity surface is the discriminant of equation (9) *with respect to* σ.

As it must be, this complex equation is of degree 12.

If we understand σ to mean an arbitrary quantity then equation (9) represents a complex of second degree. It has the singularity surface in common with the given equation (2). The system of equations (8), (9), (10) then remains unchanged when k_α is generally replaced by $\dfrac{1}{k_\alpha + \sigma}$.

Equation (9) *represents the system of complexes of second degree that is associated with the singularity surface of the given complex.*

Equation (9) is completely analogous to the equations that appear in the determination of confocal curves or surfaces of second order.

If (y) means a given line then equation (9) determines the four associated values of σ. Of them, two will be equal to each other when (y) is a tangent of the singularity surface.

The *principal tangents* to the singularity surface are characterized by the fact that three roots of equation (9) are equal; they are then given by (8), (9), (10), and the following equation:

(11) $$\frac{y_1^2}{(k_1+\sigma)^3} + \frac{y_2^2}{(k_2+\sigma)^3} + \cdots + \frac{y_6^2}{(k_6+\sigma)^3} = 0.$$

Finally, the lines that envelop the contact curves in the double planes of the singularity surface, relative to the contact cone at the double points that generate it, are determined by (8), (9), (10), (11), and the following equation:

$$\frac{y_1^2}{(k_1+\sigma)^4}+\frac{y_2^2}{(k_2+\sigma)^4}+\cdots+\frac{y_6^2}{(k_6+\sigma)^4} = 0. \tag{12}$$

Let any value of σ that corresponds to the system of equations (8), (9), (10), (11), (12) be obtained. The lines in the corresponding complex (9) that are associated with the lines thus determined are themselves again singular lines. Their associated lines define *the totality of the lines that lie in the double planes of the singularity surfaces and go through the double points of them.* The coordinates of such a line may then be written in the form:

$$\rho\, x_\alpha = \left(\lambda+\frac{\mu}{k_\alpha+\sigma}+\frac{\nu}{(k_\alpha+\sigma)^2}\right) y_\alpha, \tag{13}$$

where y_α is a solution of the equations (8), (9), (10), (11), (12). This form remains unchanged for any value of σ that one might ascribe to it.

It remains for us to determine the *double tangents* of the singularity surface. For the six congruences defined by it, one obtains from (9) and (10), when one sets σ equal to $-k_1$, $-k_2$, etc., in turn:

$$\begin{cases} y_1 = 0, \quad \dfrac{y_2^2}{k_2-k_1}+\cdots+\dfrac{y_6^2}{k_6-k_1}=0, \\ \quad\cdots \qquad\qquad\qquad \cdots \\ y_6 = 0, \quad \dfrac{y_1^2}{k_1-k_6}+\cdots+\dfrac{y_5^2}{k_5-k_1}=0. \end{cases} \tag{14}$$

In the same equations, one arrives at (4) when one sets σ equal to $-k_1$, $-k_2$, etc., and eliminates x_α from (1), (2), (3).

The conversion of the complex equation that was assumed in the foregoing considerations is, in many cases, not possible. Among the complexes that were excluded here are the ones that Th. Reye considered, whose lines are the intersection of corresponding planes of two collinear spatial systems [28]). Such complexes are distinguished by the appearance of double lines, excluded points, and excluded planes [29]). I think that I will give an overview of the cases that were distinguished by him at the soonest opportunity.

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[28]) Reye, *Die Geometrie der Lage*, C. Rümpler, Hannover, 1868.
[29]) Plücker, *Neue Geometrie*, no. 313.

[Perhaps it would be useful to abstract the following rule from the formulas of section IV:

While the individual complexes $\sum k_i x_i^2$ remain unchanged when one sets the k_i equal to any $k_i' = a\, k_i + b$, the entire family of complexes, and therefore, the Kummer surface that is associated with it, remains unchanged when one likewise replaces the k_i' with any sort of linear functions of the k_i: $k_i' = \frac{\alpha k_i + \beta}{\gamma k_i + \delta}$. K]

Göttingen, 14 June 1869.