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VIII. On line geometry and metric geometry.

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In a distinguished treatise [1]), Lie has, among other things, developed a fundamental analogy that exists between the geometry of linear complexes and ordinary metric geometry. This analogy comes back to the fact that one can map the linear complex in a single-valued way to the point-space [2]), where an isolated line enters into linear complexes as a fundamental structure, while in point-space, it is a conic section. Metric geometry is then nothing but the investigation of the projective properties of spatial structures on the basis of a conic section that is given once and for all, namely, the infinitely distant imaginary circle. The geometry of linear complexes is therefore linked with metric geometry by the map in question, in such a way that for a linear complex just one arbitrary line is distinguished. – The connection that is uncovered here between two realms of geometry that seem, on first glance, to be very heterogeneous must be tremendously fruitful for both of them. Here, it might suffice, in this regard, to refer to the rich contents of the aforementioned treatise, in particular, to the relation that is given in it between the problems of the principal tangent curves and curvature curves.

In connection with these investigations of Lie, to which I was directed by repeated, detailed communications from the same, I found that in precisely the same way that the geometry of linear complexes is connected with the metric geometry of ordinary spaces, *a connection exists between the entirety of line geometry and the metric geometry of spaces of dimension four* [3]). In this regard, I especially present a theorem of line

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[1]) [By this, we mean the great treatise of Lie that was recently published in Bd. 5 of Math. Ann.: "Über Komplexe, insbesondere Linien- and Kugelkomplexe, mit Anwendung auf die Theorie partieller Differentialgleichungen." It would certainly be desirable to clarify the close relationship that exists between Adhandlungen VIII and IX and this treatise of Lie, especially since the latter relates to numerous associated remarks of my own. However, it seems impossible to put this into the form of a brief, comprehensive commentary. Thus, one must occasionally turn to other references. We hope that the abundance of geometric ideas that are included in the treatise of Lie, which is presently difficult to fathom in its original form, a situation that was exacerbated by the inexcusable delay in the publication of Lie's work, which was prepared long ago, will become common knowledge to mathematicians! K.]

[2]) Nöther first commented upon this map: Zur Theorie algebraischer Funktionen. Gött. Nachrichten. 1869.

[3]) By the metric geometry of such a space, one must understand this to generally mean the investigation of the projective properties of that structure on the basis of a distinguished structure. If one determines, as one ordinarily does in metric investigations, the spatial element (the point) by rectangular coordinates, thus, by four coordinates x, y, z, t, then the structure in question consists of those infinitely distant elements for which $x^2 + y^2 + z^2 + t^2 = 0$.

geometry [4]) that built upon Dupin's theorem in ordinary metric geometry. I linked it with further considerations that are aimed at the objective of, on the one hand, transferring the entire content of metric geometry to line geometry, while, on the other hand, utilizing the algebraic methods that served successfully in line geometry for the treatment of metric problems. These considerations – which are, moreover, in close proximity to the ones that Lie set forth and grow out of them – will be represented in what follows, if only along general lines. Hopefully, the presentation given here will suffice to make it clear that in the manner described here, one obtains a deeper development of the two disciplines that are in question: *line geometry* and *metric geometry*.

In line geometry, as one knows, one prefers to define the line by six homogeneous coordinates p_{ik} that are linked by a condition equation of second degree:

$$P \equiv p_{12}\, p_{34} + p_{13}\, p_{42} + p_{14}\, p_{23} = 0.$$

If one regards, for the moment, the p_{ik} as independent variables then one constructs from them a manifold – or, as one often says, a space – of five dimensions. It shall be denoted by R_5 (more generally, a space of n dimensions will be denoted by R_n). The manifold of four dimensions of lines is singled out of this space by the aforementioned quadratic equation, in a manner that is similar to the way that the totality of points in ordinary space (R_3) is singled out by a quadratic equation that defines a surface of second degree. One is thus led to *treat line geometry analytically in a manner that is similar to the geometry of a surface of second degree.* The viewpoint thus suggested shall be given a basis and discussed more closely in § 1; furthermore, it is based upon all of my work on line geometry up to now.

We might likewise introduce a notation here that will be necessary in the sequel. We already denote the space of n dimensions by R_n . A structure that will now be singled out by μ equations, which will thus also define a manifold of $n - \mu$ dimensions, shall be denoted by $M_{n-\mu}$. Thus, upper right indices might give the degree of the μ equations through which the $M_{n-\mu}$ is determined. – The totality of straight lines defines an $M_4^{(2)}$ by means of this relationship, and it lies in the space R_5 . In a similar way, the lines of a linear complex define an $M_3^{(2)}$ in R_4 , the lines of a congruence, an $M_2^{(2)}$ in R_3 , and finally, the lines of a ruled surface define an $M_1^{(2)}$ in R_2 . (While this structure, which is regarded as being in R_5 , must preserve the relations $M_3^{(1,2)}$, $M_2^{(1,1,2)}$, $M_1^{(1,1,1,2)}$.) This notation is somewhat abstract, but it is not good to go around it in what follows.

The connection between line geometry and metric geometry for four variables now comes from a single-valued map of the $M_4^{(2)}$ in R_5 to R_4 . – It is well-known how one can map an $M_2^{(2)}$ in R_3 – say, for instance, a surface of second degree that lies in ordinary space – to R_2 – say, the plane – in a single-valued way. Geometrically speaking, this comes about by the process of stereographic projection. Thus, two fundamental points in the plane appear that are the images of the two generators that go through the point of projection. On the surface of second degree one finds a fundamental point, namely, the projection point. However, metric geometry uses a point-pair as the fundamental

[4]) Göttinger Nachrichten. 1871. No. 3. [Cf., Abh. VII of this collection.]

structure in the plane, namely, the two infinitely distant imaginary circle points. For that reason, one can say: The geometry on a surface of second degree and metric geometry in the plane correspond to each other as long as one distinguishes an (arbitrary) point on the surface. – By the map we mentioned of an $M_{n-1}^{(2)}$ in R_n onto R_{n-1}, we now find completely similar going on. Let an element of $M_{n-1}^{(2)}$ (which corresponds to the projection point) be distinguished. A fundamental structure then appears in R_{n-1}, since it also used the metric geometry of R_{n-1}, namely, an $M_{n-3}^{(1,2)}$. In general, one can then say:

The metric geometry of R_{n-1} can be regarded as the stereographic projection of an $M_{n-1}^{(2)}$ that lies in R_n [5]).

With this theorem, which will be established completely in § 2, the connection between line geometry and the metric geometry of R_4 is given completely. Naturally, one likewise has a connection between the geometry of linear complexes and the metric geometry of R_3. One can ultimately link the geometry of a linear congruence with the metric geometry of R_2, and the geometry of a ruled surface with the metric geometry of R_1 in the same sense. On the other hand, the theorem gives a basis for the treatment of the metric geometry of R_{n-1} that was previously suggested; this shall likewise be pursued a bit further in § 2. By it, one will be led, in the first place, to distinguish between those metric properties of R_{n-1} that, when carried over from the $M_{n-1}^{(2)}$ in R_n, imply a particular relationship to the projection point that is employed by the map, and the ones that do not. When $n = 5$, the latter yield general line-geometric theorems; the former yield ones in which an arbitrary line enters in fundamentally.

In order to show, at the least, an example of the fruitfulness of this approach, in § 3, I show the line-geometric analogue of the *orthogonal systems* of metric geometry. They are the systems of line complexes that I will refer to as *systems in involution*. A system in involution is a singly infinite system of complexes that depend upon a parameter of the fourth degree, such that four complexes of the system go through each line of space. These four complexes – and this actually constitutes the character of the systems we speak of – lie pair-wise in involution relative to that same line [6]). The involutory position of two complexes thus corresponds, on the side of line geometry, to the orthogonality of two surfaces in metric geometry. – For systems of complexes in involution one then has a theorem that is analogous to Dupin's theorem in ordinary metric geometry. As I will further discuss in § 4, to the extent that is fruitful, when one is given a system in involution, one may understand the principal tangent curves on a large number of

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[5]) Regarding the metric geometry of the plane as the stereographic projection of the geometry on a surface of second degree (in particular, a sphere) is a means that Chasles used in, in particular. The general notion suggested in the text was occasionally employed by Darboux in the theory of systems of orthogonal surfaces (Comptes rendus, t. 69, 1869, 2. Sur une nouvelle séries de systèmes othogonaux algébriques). As Darboux communicated to me personally, it is a general principle that he was led to in the presentation of his theorem on metric geometry.

[6]) I shall refer to the following relationship between two complexes relative to a common line as an *involutory position:* In each plane that goes through the line one finds, corresponding to each complex, a complex-curve that contacts the given line. The two contact points may be regarded as associated with each other. If one now rotates the plane then the two points describe collinear sequences of points. The complex is now said to be involutory when the relationship between the two sequences of points is involutory.

surfaces. In particular, we include the determination of the principal tangent curves of the Kummer surface of fourth order with sixteen nodes, as was derived from the investigations of Lie and myself [7]).

I will then expressly comment upon a distinction that exists between the things that are presented here and some chapters of the prior work of Lie, and thus likewise explain how, in connection with this distinction, Lie has developed a new transformation to be applied to metric geometry [8]). By means of the map we spoke of that takes linear complexes to ordinary point-space, Lie linked line geometry with the geometry whose element is the *sphere* of ordinary space. By contrast, here, line geometry will relate to the *point* geometry of the space *of four dimensions*. Whereas the latter relationship is one-to-one, the former is not, since each line generally corresponds to a sphere, but each sphere corresponds to a line-pair. Since both maps yield things that are of interest to line geometry, one can present the following method for the treatment of metric problems in which one ignores the considerations of line geometry as inessential: One relates a point of space of n dimensions to a sphere in the space of $n-1$ dimensions, in such a way that each point corresponds to a sphere, while each sphere, to a point-pair. This comes about simply when one lets the n coordinates of the points in R_n mean the $n-1$ coordinates of the center and the radius of a sphere in R_{n-1} . This is the method that Lie presented for linking the metric geometry of R_n and that of R_{n-1} . Not to be confused with this is a process that was presented by Darboux [9]) that likewise links the metric geometry of R_n with that of R_{n-1} . It essentially comes down to this: The metric geometry of R_n is carried over, by a spherical map, to a sphere in R_n and then, by stereographic projection onto R_{n-1}.

§ 1.

Line geometry is like the geometry of an $M_4^{(2)}$ in R_5 .

This statement finds its actual basis in the following behavior of the line coordinates p_{ik} . For the coordinates p_{ik} one has:

$$P \equiv p_{12}\, p_{34} + p_{13}\, p_{42} + p_{14}\, p_{23} = 0.$$

Now, in order for two lines p and p' to intersect, one must have:

$$\sum \frac{\partial P}{\partial p_{ik}} \cdot p'_{ik} = 0.$$

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[7]) Cf., a common communication in the Monatsberichten der Berliner Akademie. December 1870 [see Abh. VI of this collection, as well as what is included (explained, resp.) in the cited treatise [of S. Lie].

[8]) Göttinger Nachrichten. 1871. No. 7.

[9]) Cf., the note that was cited above: Sur une nouvelle séries de systèmes orthogonaux algébriques. Comptes rendus. 69. 1869.

As a result, one can present the following theorem, which I have already communicated on one occasion [10]):

If one replaces the line coordinates p_{ik} with arbitrary linear functions of them that shall satisfy only one condition in order to take the manifold $M_4^{(2)}$:

$$P = 0$$

into itself then one has a collinear or dualistic (reciprocal) conversion of the line space. On the other hand, one obtains all such collinear and reciprocal conversions in this way.

As far as the first part of this theorem is concerned, all lines will obviously go to lines under the transformations in question, and intersecting lines will go to intersecting lines. The totality of the two-fold infinitude of lines that go through a point (and thus intersect it) generally correspond to a two-fold infinitude of lines that intersect it. Thus, there remains the double possibility that they either again go through a point or that they represent the totality of lines that lie in a plane [11]). In the former case, one has a spatial transformation that takes each line into a line and each point into a point, and it is obviously a collinear conversion. In the latter case, by comparison, one has a spatial transformation that takes each line into a line and each point into a plane. It is therefore a dualistic conversion.

However, one will conversely have that each collinear and each dualistic conversion will be represented in line coordinates in the aforementioned way. By such a conversion, the point coordinates will then be replaced with linear functions of the point or plane coordinates. As a result, in place of the previous line coordinates p_{ik} , which can be represented equivalently as two-rowed determinants of the point coordinates or as plane coordinates, linear functions of them enter in. These linear functions also have the property of taking the $M_4^{(2)}$:

$$P = 0$$

into itself, since indeed straight lines remain straight lines under it, and thus the line manifold that is represented by the aforementioned equation does not change.

With this, the aforementioned theorem is proved completely. This theorem now gives rise to the following treatment of line-geometric problems: The newer geometry examines all spatial structures – in particular, the line structure – only insofar as they remain unchanged under collinear or dualistic transformations, or, if one prefers, they lead all other properties back to properties of this sort. We bring precisely the same class of transformations under consideration when we regard the line space as an $M_4^{(2)}$ in R_5 and examine the projective properties of R_5 that relate to the $M_4^{(2)}$. All of line geometry will then come down to the following problem:

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[10]) Math. Ann., Bd. 2 (1870) (Geometrisches über Resolventen…) [See Bd. 2 of this collection. This theorem was already proved in my Dissertation (Abh. I of this collection) by means of computation. (No. 6-8) K.]

[11]) In a similar way, one divides the linear transformations that take an $M_{n-1}^{(2)}$ in R_n into itself into two families in the event that n is an odd number. Cf., the article: "Über die sogennante Nicht-Euklidische Geometrie," § 16. [Math. Annalen, Bd. 4 (1872).] [See Abh. XVI of this collection.]

Investigate the projective content of the $M_4^{(2)}$ *that lie in* R_5. *Then convert the results into the language of line geometry.*

I have explained how one goes about doing this more thoroughly, in all due brevity, in the essay: "Die allgemeine lineare Transformation der Linienkoordinaten." (Math. Annalen, Bd. 2 [see Abh. III of this collection]) A linear equation (or, we can say: a plane in R_5) represents a linear complex that will be a special case when the plane contacts the $M_4^{(2)}$. If two planes are conjugate relative to the $M_4^{(2)}$ then one says that the components are involution. If the intersection of the planes contacts the $M_4^{(2)}$ then it contacts the two complexes. (The common congruence to them then has two coinciding directrices.)

We shall not go further into these matters here [12]); the following remarks might still find a place here, though. Line geometry is ultimately nothing but projective space geometry. The aforementioned theorem thus forms the basis for a peculiar treatment of the geometry of R_3 in which the linear and dualistic transformations of R_3 will be replaced by the linear transformations of a higher space such that a structure that lies in this space remains unchanged. One can pose the question of whether an analogous treatment of spaces other than R_3 is possible. This is generally the case, but only in special spaces. Thus, one can treat R_1 as a conic section in R_2 or as a space curve of third order in R_3, etc. Then the straight line R_1 may be related to a conic section (a space curve of third order, resp.) in such a way that its three-fold infinitude of linear transformations correspond to the equally numerous linear transformations take a conic section in the plane (a space curve of third order in space, resp.) to itself. On this, rests the conversion principle published by Hesse (Borchards Journal, Bd. 66, 1866). In particular, Hesse expressed the relationship between the straight lines and the conic sections in the plane and showed how the projective geometry of the plane yielded a geometry of point-pairs on the line by this conversion [13]).

§ 2.

Connection between the metric geometric of $(n-1)$ variables and the geometry of an $M_{n-1}^{(2)}$ in R_n.

Let an $M_{n-1}^{(2)}$ in R_n be given. By a suitable choice of homogeneous variables $x_1, \ldots, x_{n+1}$ one can generally bring its equation into the form:

$$0 = x_1^2 + x_2^2 + \cdots + x_{n-1}^2 + x_n^2 + x_{n+1}^2 .$$

We now set:

[12]) [This method of examination was later systematically developed by C. Segre. (Mem. della R. Acc. di Torino, Ser. II, t. 36 (1885).]

[13]) One can again connect with this, when one, like Clebsch and Gordan, for the purpose of carrying out the typical representation of even binary forms, determines the points of the line by means of three homogeneous coordinates, between which one condition equation of second order exists; cf., Clebsch: Theorie der binären Formen (Leipzig 1871). Ninth section.

$$p = x_n + ix_{n+1} ,$$
$$q = x_n - ix_{n+1} ,$$

then we get:

$$0 = x_1^2 + x_2^2 + \cdots + x_{n-1}^2 + pq .$$

Starting from this form of equation, one can map the $M_{n-1}^{(2)}$ to R_{n-1} with no further assumptions. For that purpose, one needs only to set:

$$\begin{aligned}
\rho\, x_1 &= y_1 y_n , \\
\rho\, x_2 &= y_2 y_n , \\
&\cdots \\
\rho\, x_{n-1} &= y_{n-1} y_n , \\
\rho\, p &= y_n y_n , \\
\rho\, q &= -(y_1^2 + y_2^2 + \cdots + y_{n-1}^2) .
\end{aligned}$$

For $n = 3$, these are the well-known formula that express the stereographic projection of a surface of second degree onto the plane.

As a fundamental structure, what appears under this map is:

1. In R_{n-1} , the $M_{n-3}^{(1,2)}$ that will be expressed by the pair of equations:

$$0 = y_n ,$$
$$0 = y_1^2 + y_2^2 + \cdots + y_{n-1}^2 .$$

Each element of it corresponds, not to one element of the given $M_{n-1}^{(2)}$, but to a one-fold infinitude of them.

2. On the $M_{n-1}^{(2)}$, a single element (the projection point):

$$x_1 = 0,\ x_2 = 0,\ \ldots,\ x_{n-1} = 0, \quad p = 0.$$

It corresponds to the linear manifold of $(n - 1)$ dimensions:

$$y_n = 0.$$

Now, it was already remarked that the metric geometry of R_{n-1} employs precisely a $M_{n-3}^{(1,2)}$ as a fundamental structure. By our map, as we asserted, the metric geometry of R_{n-1} will then be related with the geometry of $M_{n-1}^{(2)}$ in R_n by the establishment of a distinguished element.

The type of this relationship will be represented by the following theorem, which singles out the relationship as an essential element:

The linear transformations of R_{n-1} , which leave the fundamental $M_{n-3}^{(1,2)}$ unchanged, correspond to those linear transformations of the R_n that do not change the given M_{n-1} and the (arbitrarily chosen) projection point that one finds on it.

In fact, if we set, instead of $y_1, \ldots, y_n$, linear functions of them that do not change the fundamental $M_{n-3}^{(1,2)}$:

$$0 = y_n ,$$
$$0 = y_1^2 + y_2^2 + \cdots + y_{n-1}^2 ,$$

then the formulas yield the validity of the theorem with no further assumptions.

The former transformations are, however, the ones that one considers in the metric geometry of R_{n-1}; i.e., they are the conversion that do not change the metric properties of R_{n-1}. As an example, if $n = 4$ then R_{n-1} is the ordinary point space. The fundamental $M_{n-3}^{(1,2)}$ is the infinitely distant imaginary circle. The linear transformations of point space that do not change the latter are the ones that one refers to as motions, similarity transformations, and reflections. Under these transformations, however, all metric relationships between spatial figures remain unchanged. – On the other hand, one would have to bring under consideration the corresponding cycle of linear transformations of x, when one asks about those properties of the structures in R_n that relate to the given $M_{n-1}^{(2)}$ and the projection point one finds on it.

One can now pose the question: Which transformations of R_{n-1} correspond to those linear transformations of R_n that leave only the given $M_{n-1}^{(2)}$ unchanged, but not the projection point itself, as well? Before we answer this question, we would like to alter the mapping formulas thus employed in such a way that a formula appears for the connection with the ordinary representation of the metric geometry of R_n (where rectangular coordinates will be used). To that end, it suffices to set $y_n = 1$ and to regard the $y_1, \ldots, y_n$ that are thus absolutely determined as rectangular coordinates. $y_n = 0$ is then the location of the infinitely distant element of R_{n-1} (the infinitely distant plane). In $y_n = 0$, one finds the fundamental $M_{n-3}^{(1,2)}$ that is singled out from it by the equation:

$$0 = y_1^2 + y_2^2 + \cdots + y_{n-1}^2 .$$

It will now be confirmed that the $M_{n-2}^{(2)}$, which is represented by the following equation:

$$(y_1 - \alpha_1)^2 + (y_2 - \alpha_2)^2 + \ldots + (y_{n-1} - \alpha_{n-1})^2 = r^2,$$

distinguishes a *sphere* in R_{n-1}, by analogy with ordinary space geometry. $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$ are the coordinates of its center and r is its radius. Such a sphere is the image of a plane intersection that is given in R_n and projects onto $M_{n-1}^{(2)}$ in R_{n-1}. The equation of the sphere is then the general linear relation between the given mapping functions that represent $M_{n-1}^{(2)}$. Among the spheres, one finds, in particular, ones with infinitely large radius – i.e., planes; they are the images of those plane intersections in the given $M_{n-1}^{(2)}$ that go through the projection point [14]).

[14]) One makes sense of this by the ordinary stereographic projection of a F_2. Any plane intersection maps to a circle; in particular, when it includes the projection point it maps to the projection point.

We now consider how one changes the map of the given $M_{n-1}^{(2)}$ when one takes $M_{n-1}^{(2)}$ into itself by linear transformations of R_n . We have already examined those transformations among them that do not change the projection point. They correspond to the motions and the similarity transformations of R_{n-1} . All other transformations, however, are obviously compositions of transformations of this sort and transformations that correspond to a relocation of the projection point to the given $M_{n-1}^{(2)}$. However, if we exchange the projection point we have used up to now:

$$x_1 = 0,\ x_2 = 0,\ \ldots,\ x_{n-1} = 0, \quad p = 0$$

with another one which we, regardless of generality, would like to make:

$$x_1 = 0,\ x_2 = 0,\ \ldots,\ x_{n-1} = 0, \quad q = 0,$$

then what comes out of this is that the quantities in R_{n-1} :

$$y_1 , y_2 , \ldots, y_{n-1} ,$$

must be replaced with the following ones:

$$\frac{y_1}{\rho^2}, \frac{y_2}{\rho^2}, \quad \ldots, \frac{y_{n-1}}{\rho^2},$$

where ρ^2 refers to the expression:

$$\rho^2 = y_1^2 + y_2^2 + \cdots + y_{n-1}^2 .$$

Such a transformation shall be called, by analogy with the corresponding transformation of two and three variables, a transformation *through reciprocal radius vectors*. We can now state the theorem:

The totality of the linear transformations of R_n that take the given $M_{n-1}^{(2)}$ into itself correspond to a cycle of transformations in R_{n-1} that can be composed of motions, similarity transformations, and transformations through reciprocal radii.

Here, one now links this treatment of metric geometry in R_{n-1} with the one that we spoke of in the introduction. Next, one will divide the entirety of metric geometry into two parts. One will distinguish those relationships that carry over to $M_{n-1}^{(2)}$, in which the chosen projection point is implicit, and the ones for which this is not the case. The latter are, as one now sees, all of the ones that remain unchanged under the inversion through reciprocal radii. For their treatment, it must be preferable to also treat the R_{n-1} algebraically as an $M_{n-1}^{(2)}$ in R_n . This means: One will, by their treatment, determine the element of R_{n-1} , not by $n - 1$ absolute coordinates, but by $n + 1$ homogeneous ones, between which there exists a condition equation of second degree. (Since the latter, when set to zero, describes a plane section of $M_{n-1}^{(2)}$, it represents a sphere in R_{n-1} .)

One thus determines – for example – the points of ordinary space, not by three absolute coordinates, but by five homogeneous ones:

$$s_1 , s_2 , s_3 , s_4 , s_5 ,$$

which, when set to zero, describe a circle. Geometrically, this comes down to establishing the points through the relative values of the powers of these coordinates, when multiplied by certain constants, relative to five given spheres. Between the five s there exists a condition equation of second degree:

$$\Omega = 0.$$

The entire part of metric geometry that remains unchanged under reciprocal radii is present in the discussion of this equation, and in the same sense, as one links all of line geometry to the discussion of the corresponding equation $P = 0$. That this treatment of metric problems can be a great advantage might be mentioned here only by an example. Lie was led to this example in his study of the structure of linear complexes when he gave line-geometric reasons that I had given in a previous treatise [15]), and which he had carried over to corresponding metric notions. On the other hand, this example was, for me, the impetus to present the more general ideas set down here. Namely, one determines a point of space by five coordinates s_1 , …, s_5 , which, when set to zero, describe spheres that intersect orthogonally. Ω then has the form:

$$s_1^2 + s_2^2 + s_3^2 + s_4^2 + s_5^2 = 0.$$

If one now writes the equation:

$$\frac{s_1^2}{k_1+\lambda} + \frac{s_2^2}{k_2+\lambda} + \cdots + \frac{s_5^2}{k_5+\lambda} = 0,$$

where λ is a parameter, then one has, with no further assumptions, the system of orthogonal surfaces before one that Darboux and Moutard had found, and which is constructed from surfaces of fourth order that include the imaginary circle doubly [16]). – This form corresponds, up to the number of variables, precisely to the form that I have given in *loc. cit.* for the equation of the complex of second degree with the same singularity surface; one therefore also finds the same sort of discussion applied to it that Lie carried out in the aforementioned treatise [Math. Ann., Bd. 5].

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[15]) Math. Annalen, Bd. 2 (1870). Zur Theorie der Komplexe ersten und zweiten Grades. [See Abh. II of this collection.]

[16]) Cf., Lie. Göttinger Nachrichten. 1871. No. 7, or the aforementioned treatise of S. Lie (Math. Ann., Bd. 5). – Darboux was already led to the same form of equation previously. He had developed it in a treatise that he submitted to the Paris Academy, but which has, however, not been published yet. Cf., a recent Note in the Compte rendus, Sept. 1871, where Darboux cited some results obtained by his general treatment. [The treatise of Darboux that is mentioned here has since then appeared in extended form as a book: Sur une classe remarquable des courbes et des surfaces. Paris, 1873.]

However, for the geometry of the $M_{n-1}^{(2)}$ in R_n, the way that it is linked with the metric geometry here is also not without importance. Among many similar considerations, I will bring up only one of them that will be employed in line geometry in the sequel. The ∞^{n-2} different directions of advance that lead an element of the metric space R_{n-1} to a neighboring element together define an *angle*; e.g., two such directions of advance can be perpendicular to each other. As is well-known, this angle remains unchanged under inversions through reciprocal radii. When an $M_{n-1}^{(2)}$ in R_n is given, one may thus also speak of angles that are defined by the directions of advance of an element of $M_{n-1}^{(2)}$ to a neighboring element of $M_{n-1}^{(2)}$. For the determination of this angle, only the projective properties of $M_{n-1}^{(2)}$ itself come under consideration, but not, perhaps, fundamental structures that lie outside of R_n. One sees this clearly when one takes $n = 3$, so the $M_{n-1}^{(2)}$ can mean a surface of second degree. Two generators go through each point of the surface; the fundamental line-pair [17]) that relates to them is the angle that exists between the directions of advance to neighboring points. In particular, one can take two directions of advance to be perpendicular to each other when they lie harmonically to the two generators. The analytic expression for this is obviously this one: Let $\Omega = 0$ be the surface of second degree. We then define the expression:

$$2\,\Omega_{xy} = \frac{\partial\Omega}{\partial x_1}\cdot y_1 + \frac{\partial\Omega}{\partial x_2}\cdot y_2 + \frac{\partial\Omega}{\partial x_3}\cdot y_3 + \frac{\partial\Omega}{\partial x_4}\cdot y_4 .$$

If we substitute dx_1, dx_2, dx_3, dx_4 in it for x_1, x_2, x_3, x_4, resp. and $d'x_1$, $d'x_2$, $d'x_3$, $d'x_4$ in it for y_1, y_2, y_3, y_4, resp. then the vanishing of the expression means that the two directions of advance are perpendicular to each other in the stated sense [18]). The corresponding formula will also remain valid in n dimension, and shall be employed for line geometry in the following paragraphs.

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[17]) Cf., "Über die sogenannte Nicht-Euklidische Geometrie." Math. Annalen, Bd. 4 (1871) [see Abh. XVI of this collection] § 2, 3.

[18]) In general, the angle in question will be given by the following expression: Let Ω_{xy} be the expression described in the text; Ω_{xx} and Ω_{yy} mean the equation Ω describes with the x or the y, resp. The desired angle is then equal to:

$$\arccos \frac{\Omega_{xy}}{\sqrt{\Omega_{xx}}\sqrt{\Omega_{yy}}},$$

in which dx and $d'x$ are to be inserted in place of x and y, resp.

Here, it is obvious how this determination of the angle is connected with the more general projective metric that Cayley described, which employed an F_2 as a fundamental structure. (Phil. Trans., t. 149. A sixth Memoir upon Quantics [(1859) Coll. Papers, Bd. II]. Cf., also the author: "Über die sogenannte Nicht-Euklidische Geometrie," loc. cit.) In it, the connection that is also true for arbitrarily many dimensions is not pursued further. [F. Lindemann went deeper into the notion of the angle between two linear complexes in his dissertation: Über unendliche kleine Bewegungen und über Kraftsysteme bei allgemeiner projektivischer Massbestimmung. Erlangen, 1873 (Math. Ann., Bd. 7).]

§ 3.

Transferring the study of curvature curves and orthogonal systems to line geometry.

The study of curvature curves and orthogonal systems constitute one part of metric geometry that remains unchanged under reciprocal radii. We shall now seek the corresponding notion in line geometry as an example of such a transfer. To this end, the ordinary theory might first be formulated in such a manner that its independence of the transformation through reciprocal radii becomes evident.

Let a surface be given in ordinary space R_3 . At each point, it will contact a one-fold infinitude of spheres. Among them, there are now always two of them that are distinguished and which also contact at a neighboring point – the so-called *principal spheres.* Their existence is intimately linked with the definition of the curvature curves. One obtains a curvature curve when one proceeds from a chosen point to a neighboring point to which one of the principal spheres is in contact. Curvature curves are then the curves that lie in a surface in whose consecutive points the surface will contact the same sphere [19]). Two curvature curves go through each point; they are perpendicular to each other.

We now go to metric spaces of four, or even $(n - 1)$, dimensions. A surface in one of them – i.e., a manifold in it that is singled out by one equation – will again contact a one-fold infinitude of spheres at each point (by a sphere, we mean, as before, a particular $M_{n-2}^{(2)}$). Among them, one finds $(n - 2)$ of them in stationary contact – i.e., ones that also contact at a neighboring point. One can now proceed from the arbitrarily chosen point to a neighboring point, and so forth. One then obtains a corresponding curvature curve in R_3, a one-fold infinitude in the surface that lies in the manifold that has the property that *the given surface will contact the stated sphere at two consecutive points.* Such manifolds [20]) go through each point of the surface $n - 2$; their directions of advance are perpendicular to each other.

We now go over to line geometry. We then must set n equal to 5. In place of the surface in R_4, the line complex enters in here. Instead of points of the surface, we speak of lines in the complex; in place of spheres in R_4 , linear complexes (that indeed intersect the $M_4^{(2)}$ in R_5 that describes the line space). We thus obtain the following:

Let a line complex be given. It will contact a one-fold infinitude of linear complexes at an (arbitrarily chosen) line; these are the so-called linear tangential complexes of Plücker. Among them, three are distinguished that further contact at a neighboring line. If one proceeds from the chosen lines to one of these neighboring lines and further from there in the same sense then one describes a line surface that is associated with the

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[19]) The ordinary definition of curvature curves – that the surface normals intersect at consecutive points of a curvature curve – is a consequence of the one presented here. By the application of reciprocal radii it goes to a more general one in which the (randomly chosen) inversion center enters in; for that reason, we prefer the definition that is given in the text.

[20]) One can again define them by saying that the surface normals that are directed from each of their consecutive points intersect. – The curvature theory of a space of arbitrarily many dimensions has been the object of repeated representations in recent times. One then mostly starts with the latter definition.

complex – in the following, it shall be called a *principal surface of the complex* – which has the property that the complex will be contacted at two consecutive generators of the stated linear complex. Three principal surfaces go through each line of the complex; its directions of advance are perpendicular to each other in a sequential sense.

We now have to discuss what sort of metric sense this perpendicularity possesses. Since, on the basis of ordinary line coordinates, the condition equation for it has the form:

$$P \equiv p_{12}\, p_{34} + p_{13}\, p_{43} + p_{14}\, p_{23} = 0,$$

from § 2, two directions of advance dp and $d'p$ can be said to be perpendicular to each other when:

$$0 = dp_{12}\, d'p_{34} + dp_{13}\, d'p_{43} + dp_{14}\, d'p_{23} + d'p_{12}\, dp_{34} + d'p_{13}\, dp_{43} + d'p_{14}\, dp_{23}\,.$$

However, there then exists a relationship between the given line p and its two neighbors $p + dp$ and $p + d'p$ that I shall refer to as *two neighboring lines in an involutory position* [21]) and has the following geometric content:

A neighboring line $p + dp$ always associates the planes that go through p projectively with the points found on p. Each plane that goes through p intersects $p + dp$ at a point that goes back to p itself when one takes the limit. One sees this clearly when one considers p and $p + dp$ as consecutive generators of a line surface. Any plane that goes through p then corresponds to a point that lies on p and is determined by $p + dp$: the contact point with the surface.

If two neighboring lines $p + dp$ and $p + d'p$ are now given then one always regards two points to be corresponding when a plane through p is associated with respect to the two neighboring lines. One then obtains two mutually collinear sequences of points on p.

The two neighboring lines are now said to lie in involution when the two point sequences define an involution [22]). [Cf., the completely analogous definition of the involutory position of complexes in the introduction, pp. 3]

We now once again go back to the metric space R_3 and consider an orthogonal system in it. This is a one-fold infinitude of surfaces, three of which go through each point of space. They intersect each other perpendicularly. For such systems of orthogonal surfaces, one has Dupin's theorem: *Any two surfaces of the system intersect each other along a common curvature curve.*

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[21]) One can denote the *angle* between two neighboring points by:

$$\arccos \frac{dp_{12} d'p_{34} + d'p_{12} dp_{34} + \cdots}{2\sqrt{dp_{12} dp_{34} + \cdots}\sqrt{d'p_{12} d'p_{34} + \cdots}}\,.$$

[22]) One sees, with no further assumptions, the validity of the following theorem: The lines that belong to a line complex in the vicinity of one of its lines p lie in involution to a line that neighbors a certain p, which generally does not itself belong to the complex. Conversely, all such neighboring lines belong to the complex. – Two complexes are now said to be in involution with respect to a common line p when the neighboring lines $p + dp$ and $p + d'p$ lie in involution, which are associated with the line p in the sense that was explained here, resp.

One can consider similar systems of surfaces in R_{n-1} ; there is a theorem that is true for them that corresponds to Dupin's. These systems of surfaces are again one-fold infinite and $n - 1$ surfaces go through every point of R_{n-1} . They intersect each other in a mutually perpendicular way. For this system, one then has the theorem: *Any $n - 2$ surfaces intersect along a common curvature curve*, and by "curvature curve," we mean the one-fold infinitude of manifold we just considered.

In line space, we will have to define the concept of *system of complexes in involution*. It is a one-fold infinitude of system of complexes. Each line of space belongs to four of the complexes and indeed the four complexes that belong to a line always lie in involution pair-wise with respect to this line.

For this system of complexes in involution, one then again has a theorem that corresponds to Dupin's: *Any three complexes intersect on a common principal surface.*

It is well-known how yet another general theorem can be presented for irreducible orthogonal systems. Kummer then showed that the curves of an irreducible system of orthogonal curves are necessarily confocal; Darboux [23]) extended this theorem to systems of orthogonal surfaces on R_3 and added further properties that first appeared in R_3 . It is obvious that analogous properties exist for irreducible orthogonal systems in ordinary space to the ones that exist in line space for irreducible systems in involution. I shall not go into this here, but I will only remark that the theorem of the confocality of orthogonal surfaces corresponds to line-geometric theorem: *Complexes in an irreducible system in involution have a common singularity surface* [24]).

The simplest example of an irreducible system in involution then also gives the one-fold infinitude of complexes of second degree with a common singularity surface. One can apply the following algebraic representation for them, as I have shown in the previously-cited work: "Zur Theorie, etc." (Math. Annalen, Bd. 2 (1870) [see Abh. II of this collection]: Let $x_1 , \ldots, x_5$ be homogeneous functions of p_{ik} for which:

$$x_1^2 + x_2^2 + \cdots + x_6^2 = 0.$$

The complexes are then represented by:

$$\frac{x_1^2}{k_1 - \lambda} + \frac{x_2^2}{k_2 - \lambda} + \cdots + \frac{x_6^2}{k_6 - \lambda} = 0,$$

where λ refers to a parameter. In fact, the parameter λ appears in the fourth degree by means of the relation $\sum x^2 = 0$. Any line of the space thus belongs to four complexes of the system. However, any two complexes $\lambda = \lambda_1$ and $\lambda = \lambda_2$ also lie in involution with respect to the same line. The condition for two complexes:

$$\varphi = 0, \quad \psi = 0$$

[23]) Annales Scientifiques de l'École Normale Supérieure. t. 2. 1865.

[24]) For Plücker, the singularity surface is defined only for complexes of degree two. This lacuna was filled in by Pasch in his Habilitationsschrift: "Zur Theorie der Komplexe und Kongruenzen von Geraden," Giessen 1870.

to be in involution with respect to a common line is, in the chosen coordinate representation:

$$\sum \frac{\partial\varphi}{\partial x_a} \cdot \frac{\partial\psi}{\partial x_a} = 0 \qquad \{\text{by means of } \varphi = 0,\ \psi = 0\}.$$

However, this expression always vanishes when φ, ψ are two complexes of the system in question. It will then equal:

$$4\left[\frac{x_1^2}{(k_1-\lambda_1)(k_1-\lambda_2)} + \frac{x_2^2}{(k_2-\lambda_1)(k_2-\lambda_2)} + \cdots + \frac{x_6^2}{(k_6-\lambda_1)(k_6-\lambda_2)}\right],$$

which is, by a partial fraction expansion, equal to:

$$\frac{4}{\lambda_1-\lambda_2}(\varphi-\psi),$$

so it vanishes with φ and ψ.

Any three complexes of the system then have line surfaces of the sixth degree in common as a complex of second degree; *the principal surfaces of the complex of the second degree are then of sixth degree*. I shall not further into a discussion of these surfaces, which encompass a great number of special surfaces when one specializes the complex.

§ 4.

Further considerations on the principal surfaces of the complex.

In these last paragraphs we might develop even more properties of the principal surfaces of complexes, systems in involution, etc., and indeed, purely line-geometric considerations. The latter again naturally carry over to the metric geometry of R_4 , which will not, however, be pursued further.

Lie has found the remarkable theorem (which came about in a much different context for him) that *on any line surface that belongs to a linear complex one knows a principal tangent curve*. Namely, there are two points on any generator of the line surface whose tangential plane is likewise the plane that corresponds with in the linear complex. The totality of these points defines the principal tangent curve in question.

One can prove this quite simply. All of the tangents to the surface at such a point belong to the complex. The points thus define a curve whose tangents belong to the complex, namely, a complex curve. However, a complex curve has the property that each of its points possesses the corresponding plane in the complex for its osculating plane. On the other hand, this plane is, in any case, the tangential plane to the surface, by assumption. The curve is thus a principal tangent curve.

When a line surface is given as belonging to any complex, one can determine a curve on it in a similar manner; however, in general it is not a principal tangent curve. Namely, one seeks those two points on any generators at which the surface will contact the complex cone. The sequence of these point-pairs constitutes the curve in question.

However, if the line surface is, in particular, a principal surface of the given complex then *the curve thus constructed is a principal tangent curve of it* [25]).

The proof follows immediately from the definition of a principal surface by means of Lie's theorem. The given complex will be contacted at any two consecutive generators of the stated linear complex. The linear complex in question thus includes three consecutive generators of the line surface and determines the stated two point-pairs by the first two of them, like the given complex itself.

Now, one can also formulate Lie's theorem as follows: If a linear complex includes three consecutive generators of a line surface then it is determined by the first two point-pairs, which belong to the principal tangent curve itself. With this, the proof of our theorem is given.

We now consider a system in involution. A single line *p* belongs to four of the complexes, which may be denoted by *a*, *b*, *c*, *d*. They lie pair-wise in involution with each other. As a consequence, one next has the following theorem, for whose proof, I refer one to the paper: "Zur Theorie, etc." (Math. Annalen, Bd. 2 (1870) [see Abh. II of this collection] [26]).

1. The line surface that three of the complexes – say, *a*, *b*, *c* – belong to will be contacted at all points of *p* of the complex cone of *d* in question.

2. The line *p* contacts the caustic of the congruence of two complexes *a*, *b* in the stated two points, at which it contacts the caustic of the congruence of the other two complexes *c*, *d*.

3. The three point-pairs (*ab*, *cd*), (*ac*, *bd*), (*ad*, *bc*) that arise in this way on the lines are harmonic to each other [27]).

We now consider a principal surface that is common to three of the complexes – say, *a*, *b*, *c*. On it, corresponding to each of the three complexes, one knows a principal tangent curve that intersects each generator twice. Now, however, the theorem of Paul Serret is true for principal tangent curves of the line complex: The generators of the line surface will be projectively divided among the principal tangent curves. From this, one will once conclude that the six points at which each generator of the three principal tangent curves *a*, *b*, *c* will intersect, are six fixed elements projectively. This is, as mentioned before (Theorem 3), in fact the case. On the other hand, one will infer that *on the principal surface in question one can determine all principal tangent curves by a purely algebraic process*. One then obtains all principal tangent curves when one moves a point on the surface in such a way that, in any case, it defines a fixed double ratio with three of the six points (and thus, with all of them) on all generators, and only algebraic operations are necessary for this.

The two points at which the principal curve *a* involves a generator – say, *p* – are, I now assert, the contact point of *p* with the caustic of the congruence common to *b* and *c*. The complex cone of *a* then contacts surface at these points, and therefore also, from Theorem 1, the complex cone of *d*. For this reason, these points are the contact points of

[25]) The converse of this theorem is also valid.

[26]) There, it is only proved for linear complexes. It is therefore also valid for the linear tangential complexes of complexes given here, and thus, for the latter itself.

[27]) This is linked with the further theorem: If one lays a plane through *p* then, corresponding to *a*, *b*, *c*, *d*, it includes each complex curve. The four contact points of the four curves, along with *p*, define a four-parameter point-group whose covariant of the sixth degree will be represented by the three point-pairs of the text.

p with the caustic *ad* and thus also, from Theorem 2, with the caustic *bc*, which was to be proved. One then has the theorem:

The principal surface abc contacts the caustic bc along the principal tangent curve a, the caustic ca, along the curve b, and the caustic ab, along the curve c.

From this, one may further infer:

One also knows the principal tangent curves of the congruence of any two complexes on the caustic.

They are the curves that are principal tangent curves of the principal surface. In fact, from the totality of contact curves of the caustic with the principal surfaces, one also obtains the totality of the principal tangent curves of the caustic. Then the line that contacts the caustic *ab* at a point, and which contains *a* and *b*, simultaneously belongs to two more complexes *c* and *d*. One thus obtains, in the contact curves of the caustic with the principal surfaces *abc*, *abd*, *both* of the principal tangent curves that go through the chosen point [28]).

For the system in involution of the complex of second degree, one finds, in particular: The principal tangent curves to the principal surface, along which the caustic contacts, are of order and class 32. The caustics themselves are likewise the principal surfaces of sixteenth order and class.

By a suitable particularization, one obtains from the consideration of these caustics and principal surfaces the determination of the principal tangent curves of a great number of special surfaces. Let only one such particularization be chosen here. The two complexes of the system, which together determine the congruence, and, by it, the caustic, might coincide. Then, the congruence will be the congruence of singular lines of the complex in question. Its caustic decomposes into the singularity surface that is common to all complexes, which is of fourth order and class, and a further surface of twelfth order and class. On both of them, one obtains the principal tangent curves, which will now be of sixteenth order. Now, for the general complex of second degree the singularity surface is a Kummer surface of fourth degree with sixteen nodes. One thus obtains a determination of the principal tangent curves of this surface that goes thus: They are the contact curves of the surface with those line surfaces, which belong to an arbitrary (but chosen once and for all) associated complex as singular lines and, moreover, yet a second (variable) one that belongs to the associated complex. Thus, one finds this to be in agreement with a determination of the principal tangent curves of the Kummer surface that was given by Lie and myself in a common paper in the Monatsberichten der Berliner Akademie, December 1870. [See Abh. VI of this collection.] The content of this Note can be regarded as closer analysis of some of the concepts that were published there.

Göttingen, in October 1871.

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[28]) I have summarized the various theorems presented here, along with theorem 3: "that any three complexes of a system in involution intersect along a common principal surface," in the previously-cited Note in the Gött. Nachrichten, No. 3 (1871) [see Abh. VII of this collection.] as the theorem: *The line surface that is common to three complexes of a system in involution contacts the caustic of any two of them along a principal tangent curve*, and proved this analytically in that version. The heterogeneous component that this theorem includes seems to be separated in the text. I am obliged to Lie for the fact that he drew my attention to the possibility of this separation.