Gesammelte Abhandlungen, pp. 262-274.

L. On a geometric representation of the resolvents of algebraic equations.

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Translated by D. H. Delphenich

The general theory of the algebraic equations can be illustrated beautifully by a number of special geometric examples. Here, I recall only [1]) the problem of inflection points of curves of third order, the problem of the 28 double tangents to curves of fourth order, the problem of the 27 lines of a surface of third degree, etc., but then, especially, the subdivision of the circle, as well [2]).

The advanced uses of this example are based in the fact that, in and of itself, it exhibits a particularly abstract presentation of substitution theory in an intuitive way. It mostly relates to equations of a very special character, between whose roots, special groupings occur, and thus allow one to ignore how such special equations can appear. In the following, I would now like to exhibit a method, by means of which one obtains a geometric picture for *general* equations of an arbitrary degree – in particular, for those groupings of roots of an equation that one might use for the description of a resolvent. This method describes the n roots of an equation by n elements of a space of dimension $(n - 2)$ and replaces the permutation of roots with those linear transformations of the aforementioned space by which the n given elements will be taken to each other. By means of this representation, the theory of equations of n^{th} degree will be brought into a remarkable connection with the theory of covariants of n elements of a space of dimension $n - 2$, such that each of the two theories can be directly regarded as an image of the other one. – The essential aspect of this manner of presentation is that the permutation of the n roots amongst themselves will be replaced by a linear transformation of a continuous space in the geometrical picture. In a similar way, one can also make sensible equations of a particular type such that no longer all, but only the characteristic, permutations of their roots appear in the image as linear transformations of space. In the following, I will restrict myself to showing that just this character of the geometric picture is present for the inflection points of curves of third order and the equations of the subdivision of the circle. – Later on, I would then like to give a representation for the general equations of sixth degree that is based upon the same principles, which is drawn from line geometry, and by which, one can envision a closed system of 360 linear and 360 reciprocal transformations of three-dimensional space. Thus, in particular, one also

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[1]) Cf., Camille Jordan, Traité des Substitutions, Paris, 1870, pp. 301, *et seq.*

[2]) In contrast to the usual manner of expression, here the term "subdivision of the circle" refers to any "pure" equation $x^n - A$, where A is a parameter and $\varepsilon = e^{2\pi i/n}$ are adjoined as the roots of unity.

encounters the well-known resolvent of sixth degree of those equations that correspond to the special [3]) group of 120 substitutions that can be described by six elements and is not identical with the 120 substitutions of five elements.

The first inducement for me to pursue the matters that are suggested here was the geometric considerations that Clebsch had applied in Math. Annalen, Bd. 4 (1871), pp. 284, *et seq*., in regard to the discussion of equations of fifth degree, and which he was gracious enough to repeat in a personal communication to me. On the other hand, these things are closely connected with the consideration of linear transformations of geometric structures into themselves, as Lie and myself have set down in the article: "Über diejenigen ebenen Kurven, welche durch ein geschlossen System von einfach unendlich vielen vertauschbaren linearen Transformationen in sich übergehen," in the Math. Annalen, Bd. 4 (1871) [cf., Abh. XXVI in vol. 1 of this collection].

I.

Geometric representation of equations of n^{th} degree.

Let n elements (*or n plane manifolds of dimension* (*n* – *3*)) *be given in a space of dimension* (*n* – 2). *These elements go to each other by means of a closed system* [4]) *of n*! *linear transformations of the space in question.*

In general, one can, in fact, take *n* elements in such a space to *n* arbitrary elements by means of such a transformation; on the other hand, the transformation is determined completely when *n* mutually independent corresponding element pairs are given. In particular, one can now let *n* elements coincide with their *n* corresponding ones in an arbitrary sequence. There are thus just as many linear transformations of space, under which the arbitrarily chosen *n* elements go to each other, as there are permutations of *n* things, and therefore *n*! of them. These transformations define a closed system, since when arbitrarily many of them are combined with each other this again yields a linear transformation under which the totality of the *n* elements remains unchanged, and thus itself belongs to the given system.

Let us give an example: 3 points of a line go into each other by means of 6 linear transformations, 4 points of a plane go to each other by 24 linear transformations, and 5 points of space go to each other by 120 linear transformations of their respective carriers.

I now think of the *n*! transformations as being applied to an arbitrary element of a space of dimension (*n* – 2), and which exchange the *n* given elements amongst each other. The elements then assume *n*! different position, in general. *The system of n*! *elements that are thus generated is the image of the Galois resolvent of the equation of* n^{th} *degree that is defined by the n given elements.*

[3]) Cf., Serret, Traité d'Algebre Supérieure, German edition, Leipzig, 1868, Bd. II, pp. 250.

[4]) Here, by the term "closed system of transformations," we shall understand this to mean, as already happens in the cited paper of Lie and myself, a system whose transformations, when combined with each other, always produce another transformation of the system [thus, in modern terminology, a *group*].

For special assumptions on the arbitrary elements the n! elements that emerge from them can coincide several times. The Galois resolvent will then be a power of an expression that will be referred to as a *special resolvent.*

The images of each special resolvent thus appear as those groups of elements that are included multiple times in the general group of n! *elements.*

These geometric definitions are capable of being outfitted in an analytical form that clearly explains the complete identity of them with the ordinary definition of the substitution theory. The n given elements may be described by their equations:

$$p = 0, \quad q = 0, \quad r = 0, \quad \ldots$$

A linear identity exists between the linear expressions $p, q, r, \ldots$ From now on, we would now like to think of the expressions $p, q, r, \ldots$ as being multiplied with such constants that the identity has the form:

$$0 = p + q + r + \ldots$$

By this assumption, the n! transformations of space are represented when one sets the new $p, q, r, \ldots$ equal to the previous ones in an arbitrary sequence. The linear transformations in question are then described in exactly the same way as the exchanges of n things $p, q, r, \ldots$

Furthermore, let an arbitrary element be given [where it must be regarded as inessential that we have restricted ourselves beforehand to elements that are represented by a linear equation in the coordinates; one already finds other Ansätze on pp. 269, et seq.]:

$$0 = ap + bq + cr + \ldots$$

The n! elements that emerge from these by the transformations in question are represented by all of those equations that can be derived from the foregoing ones by the exchanges of $p, q, r, \ldots$ or – what amounts to the same thing – the $a, b, c, \ldots$ The multiplication of all of these equations by each other yields the equation of the entire group of elements that is the image of the Galois resolvent. For special values of $a, b, c, \ldots$, this resolvent can then correspond to the powers of a lower-order expression.

We would like to illustrate this statement with the example of $n = 4$, and thus, the quadrangle – or, what might be more convenient – the tetrahedron in the plane [5]).

Let the four sides of it be described by:

$$p = 0, \quad q = 0, \quad r = 0, \quad s = 0,$$

for which, the identity exists:

$$p + q + r + s = 0.$$

[5]) [Cf., Clebsch, *loc. cit.*, § 4]

For each arbitrarily chosen line:

$$ap + bq + cr + ds = 0,$$

there exists a system 24 lines that are associated with it, in general. They can be easily constructed in the following way: The arbitrarily chosen line cuts the four faces of the tetrahedron in four points, which determine a certain double ratio with the three vertices of the tetrahedron that always lie on such a line. One now constructs those 24 points on each face (one of which is the current intersection point) that define one of the four double ratios, together with the three vertices that line on the face, which are chosen in an arbitrary sequence. These four times 24 points lie four times twenty-four times on a line; these 24 lines (one of which is the given one) are the desired ones.

If the arbitrarily chosen line goes through a vertex of the tetrahedron, in particular, then one obtains, as is easy to see, only twelve lines, which pair-wise go through the vertices of the tetrahedron. In fact, when the given line goes through a vertex, two of the coefficients a, b, c, d must be equal to each other. The Galois resolvent will correspondingly be the square of an equation of twelfth degree.

In particular, the given line goes through two opposing vertices of the given tetrahedron. One then obtains only systems of three lines, namely, the three diagonals of the tetrahedron. These are the image of a resolvent of third degree, and one will also be led to such things when one takes the a, b, c, d to be pair-wise equal, and thus, perhaps, as is asserted by the identity that exists between the p, q, r, s, chooses $a = b = 1$, $c = d = -1$.

II.

The covariants of *n* elements of a space of dimension $n - 2$.

The groups of $n!$ elements, which, from the foregoing, are associated with n given elements of a space of dimension $(n - 2)$, are obviously *covariants* of the systems of given elements, for which the absolute numerical values (double ratios) by which the element that enters in will be established relative to the n given ones, and which are unchanged by the linear transformations, serve as parameters.

The equations of these covariants have a remarkable property: *They are rationally composed from the symmetric functions of p, q, r,…* This emerges immediately from the formation of the these equations, which we obtain when we permute the p, q, r, … in an arbitrary linear equation in all possible ways and then multiply the resulting equations together.

From this itself, one understands that in the equations of the special element groups, which, corresponding to special resolvents, are included in general groups multiple times, the multiplicity must be expressed as a power in order for this representation to also find applications.

On the other hand, it is obvious that of the symmetric functions of the p, q, r, …, one of them – namely, their sum – corresponding to the identity:

$$0 = p + q + r + \ldots,$$

has been omitted.

Now, one may easily see *that the groups of* $n!$ *elements that were considered up to now are the only imaginable covariants of the given n elements that are constructed from separate individual elements, or, more generally, that each covariant of the given n elements:*

$$p = 0, \quad q = 0, \quad r = 0, \ldots$$

is rational, and composed entirely from the $(n - 1)$ *non-vanishing symmetric functions of the p, q, r, …*

In fact, each covariant must go to itself under these linear transformations, like the original structure. Its equation must then remain unchanged by the $n!$ permutations of the $p, q, r, \ldots$ in the present case, so, from known theorems, it must then be rationally expressible in terms of symmetric functions.

This reasoning admits an extension in the same sense as was necessary for the (multiply counted) covariants of less than $n!$ elements. In fact, the equation of the covariant does not need to remain completely unchanged under the permutations of the $p, q, r, \ldots$, since it can pick up a factor. However, this factor can only be a root of unity, since the repetition of a well-defined permutation finitely many times gives the identity, and thus, a well-defined power of the factor will be equal to unity. The corresponding power of the covariant equation then remains completely unchanged under the permutation of the $p, q, r, \ldots$; it is what we must call the actual covariant, and which can be rationally composed from the symmetric functions of the $p, q, r \ldots$

By the latter considerations, the theory of covariants of n elements in $(n - 2)$*-dimensional space is closely connected with the theory of equations of* n^{th} *degree.*

As an example of the applicability of such conclusions for the theory of covariants, what follows here is the treatment of the simplest case that arises from it, namely, $n = 3$, and thus, the treatment of *the binary cubic form.*

Let a cubic binary form f be given. Let it be described by three points on a line. One can then transform the line by six linear transformations (one of which is the identity) that will permute the three given points amongst themselves. By means of this, the points of the line define groups of six. *These groups of six points are covariants of the given cubic form; there are no other covariants,* in the sense that every covariant must be resolvable into a number of such groups.

It is now easy to give an account of the geometric character of these point groups, and thereby likewise settle the question of whether powers of lower groups are included in them. In fact, as would follow immediately from the way it was generated – and which, on the other hand, also suffices for its definition – a group of six points includes those six points that define the double ratio in question, along with the given three, when one permutes their sequence arbitrarily, which amounts to the same thing as saying that it includes such points that define six associated double ratios with the given three, the latter being thought of as having a fixed sequence. Therefore, each of those six double ratios that appear for altered sequences of four given points will be called *associated.*

It follows from this that among the single infinity of groups of six points, besides the ones that are described by $f = 0$ itself, when doubly counted, one finds two distinguished ones, corresponding to a harmonic and an equiharmonic ratio.

The group of points that lie harmonically consists of three doubly-counted points. They define a covariant of third degree that is ordinarily denoted by Q in the theory of binary cubic forms.

The group of points that lie equiharmomically encompasses only two triply-counted points. It constitutes the quadratic covariant Δ of the ordinary theory.

It is quite intuitive that one finds the mutual relation of the forms f, Q, Δ when one interprets them, not as points of a line, but as rays of a pencil, and thus the two rays $\Delta = 0$ may go through the imaginary circle points in the infinitely distant line. $f = 0$ will then be defined by three rays that define equal angles $= 2/3\ R$ with each other. $Q = 0$ encompasses the bisectors of the angles defined by these rays. Finally, each six-element group consists of six rays that define angles $= \pm\ \varphi$ with the elements of $f = 0$, where φ denotes any inclination. The linear transformations by which $f = 0$ goes to itself – and therefore, also $Q = 0$ and $\Delta = 0$, as well as each six-element group – consist of, first, rotations of the pencil of rays in its plane, always around $2/3R$, and then a rotation of the pencil of rays around an element of $f = 0$ through $2R$, by which the plane of the pencil will be turned.

Upon establishing the factors of Δ as variables, one now easily recognizes that each six-element group is linearly and homogeneously composed of two of them. One thus has one homogeneous linear equation between any three six-element groups. In particular, such an equation will exist between f^2, Q^2, Δ^2, say:

$$\Delta^2 = \rho f^2 + \sigma Q^2.$$

It is known that the solution of cubic equations rests upon an identity of this form.

As was said before, the theory of the covariants of three points on a line was derived from the consideration of the permutation of three elements among them, so one can treat the covariants of the tetrahedron in the plane, the pentahedron in space, etc., in a completely similar way. On this subject, a remark must be made that is particular to the pentahedron, but which likewise finds application in the general case. The resolvents of the equation of fifth degree that can be represented by a pentahedron in space include not only a system of 120 associated planes or points, but even 120 associated (i.e., emerging from the application of the transformations) geometric structures, curves, surfaces, etc. Now, if perhaps a finite number of special curves lie on a covariant surface of the pentahedron then they will always group together into such resolvents. One can also express this as: *All equations that can give rise to covariant surfaces of the pentahedron may be decomposed into ones that are resolvents of one equation of fifth degree.*

An example of this is the following: Let five pentahedral planes be:

$$p = 0, \quad q = 0, \quad r = 0, \quad s = 0, \quad t = 0,$$

and, as always, let:

$$p + q + r + s + t = 0.$$

There is then a covariant surface of third degree [6]):

$$p^3 + q^3 + r^3 + s^3 + t^3 = 0.$$

As Clebsch (*loc. cit.*) has proved, the 27 lines of this surface decompose into two groups, with 15 in one of them (which are counted eight times as terms in a Galois resolvent) and 12 in the other (which are counted ten times).

III.

The equation for the inflection points of curves of third order.
The subdivision of the circle.

It was already suggested in the Introduction that the essential detail in the foregoing recitation of the equations of n^{th} degree is the fact that *the permutations of the roots amongst themselves can be replaced by linear transformations of space.* I would now like to show that the representation that certain equations of ninth degree find through the inflection points of the curves of third order possesses a similar character. The same is true for the equations of the subdivision of the circle. The situation is modified only in the two cases in which the geometrically represented equations no longer have the general degree, but groupings of their roots can come about. Correspondingly, in the geometric picture, not all of the permutations of the roots find their representation in terms of linear transformations, but only ones that are closely linked with the groupings of the roots.

As far as the equation of inflection points is concerned, it is easy to see that *a general plane curve of third order, and in particular, its inflection points, goes to itself under 18 linear transformations.* One produces them most simply when one starts with the canonical form for the equation of a curve that relates to an inflection point triangle. It is:

$$0 = a\left(x_1^3 + x_2^3 + x_3^3\right) + b\, x_1\, x_2\, x_3\,.$$

The linear transformations may be composed by permuting the x amongst themselves and then multiplying them with suitable cube roots of unity.

Under these transformations, not only the given curve f goes to itself, but also its Hessian determinant Δ *and any curve whatsoever of the pencil* $f + \lambda\,\Delta$.

The fact that these transformations simultaneously take a simple infinitude of curves of third order into themselves suggests the nonsense, which lies in the first counting, that a general curve of third order that depends upon nine constants should go to itself under a finite number of linear transformations, which indeed includes only eight parameters.

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[6]) My attention was first directed to the properties of these surfaces that go to themselves under linear transformation by Lie.

As a geometric picture of the equation of ninth degree, we now consider, not a curve of third order that possesses inflection points, but *the inflection points themselves and the cycle of transformations under which they are permuted amongst themselves.*

Any equation that can give rise to a curve of third order, although a foreign element would be employed in it, must be decomposable into resolvents of the inflection point equation. One seeks, e.g., such triangles whose sides always contact the C_3 in a vertex. The representation of the C_3 by elliptical functions shows immediately that there are 24 such triangles. The discovery of them, in fact, came out of the subdivision of the elliptic functions into nine; of the 81 values that are produced in this way, nine of them relate to the inflection point itself, and the remaining 72 of them yield three triangles that are always the same. However, one now writes the C_3, relative to one such triangle that is considered to be the fundamental triangle, in the following way:

$$0 = a\left(x_2 x_3^2 + x_3 x_1^2 + x_1 x_2^2\right) + b\, x_1\, x_2\, x_3\,.$$

When one cyclically permutes x_1, x_2, x_3, this equation and the triangle remain unchanged. These permutations correspond to linear transformations that are included in the aforementioned 18; only these then possess the property of taking C_3 into itself. The triangle thus remains unchanged under three of the 18 transformations under which C_3 goes to itself. Thus, six triangles always define an unchanging group – a resolvent of the inflection point equation. The solution of the 24th degree equation that determines the triangles next requires the solution of an equation of fourth degree for the determination of the groups of six associated triangles, and then only the solution of the inflection point equation. – One naturally comes to the same result when one treats the subdivision of elliptic functions into nine.

Here, one occasionally remarks that 18 reciprocal transformations are closely linked with the 18 linear transformations that were considered here. One obtains them from the 18 linear ones when one switches the x_i and the u_i, where one understands x_i, u_i to mean the point and line coordinates that relate to an inflection point triangle. In place of the inflection points then, there harmonic polars enter in, and in place of the pencil $f + \lambda\,\Delta$, the pencil of polar contacting curves of third class, etc. From this standpoint, the examination of the 18 linear and 18 reciprocal transformations seems to be the main problem; at the same time, one thus disposes of the theory of curves of third order or third class and their root-related associations.

As far as the *equations of the circle subdivision* or the projective generalizations of them, the *equations of the cyclic projectivity* [7]), are concerned, one immediately recognizes in what way the characteristic permutations of their roots can be replaced with linear transformations (rotations of the plane around the midpoint of the circle).

[7]) Clebsch, in Crelle's Journal, Bd. 63 (1863/64), pp. 120.

IV.

Geometric representation of the general equation of sixth degree.

I now turn to the discussion of the particular geometric representation that one can present for the equations of sixth degree. *One represents the roots of the equation by six linear complexes that lie pair-wise in involution; the permutations of them amongst themselves correspond to linear transformations of point space.*

The geometric concepts that come under consideration are, for the most part, the same as the ones that I set down in the article "Zur Theorie der Linienkomplexe des erstens und zweitens Grades" in the Math. Annalen, Bd. 2 (1870) [cf., Abh. II in Band 1 of this collection]. According to the discussion therein, between six linear complexes:

$$x_1 = 0,\ x_2 = 0,\ \ldots,\ x_6 = 0$$

that lie pair-wise in involution with each other (cf., the cited article), there exists an identity of the form:

$$0 = x_1^2 + x_2^2 + \cdots + x_6^2 .$$

I now further employ a theorem of line geometry that I spoke of, in a somewhat less general form, in my inaugural dissertation [8]). It reads as follows:

The coordinate determination of lines can be based upon six arbitrary linear complexes; they will satisfy an identity of second degree:

$$R = 0.$$

A collinear or reciprocal transformation of space corresponds to a linear transformation of line coordinates, under which R goes to a multiple of itself. Conversely, if one sets, in place of the line coordinates, linear expressions such that R goes to a multiple of itself then they correspond to a collinear or reciprocal transformation of space.

In the case at hand, the identity that exists between the x will not change its form under a permutation of them. Any permutation of the x then corresponds to a collinear or reciprocal transformation of space, and indeed is collinear or reciprocal, depending upon whether the permutation of the x is composed of an even or odd number of transpositions.

The 720 permutations of the six complexes x amongst themselves or, what is equivalent, the 360 collinear and 360 reciprocal transformations of space group together into, on the one hand, 720 lines, and, on the other, 360 points and 360 planes; any such group is the image of the Galois resolvent of the equation of sixth degree that is described by the six complexes.

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[8]) "Über die Transformation der allgemeinen Gleichung des zweitens Grades zwischen Linienkoordinaten auf eine kanonische Form," Bonn 1868. C. Georgi [Cf., Abh. I in Bd. 1 of this collection].

It is not my intent here to examine these groups more closely, which is, moreover, rather simple in connection with the line coordinate determination; I would not like to go into the subject here of how the system of lines that are common to 2, 3, 4 of the complexes, respectively [cf., the cited Abh. II], define examples of particular resolvents.

Any two of the six given complexes have a congruence in common, and it possesses two directrices. There are $6 \cdot 5 / 2 = 15$ such directrix pairs. These directrix pairs are likewise the line-pairs that are always common to the remaining complexes.

The 15 directrix pairs are the image of a resolvent of fifteenth degree.

The 15 directrix pairs now define the edges of 15 tetrahedra (corresponding to the fact that one can divide them into three groups of two in 15 ways).

These 15 tetrahedra represent a second resolvent of fifteenth degree.

From the 15 tetrahedra, one can now look for five of them that collectively have all 30 directrices for edges in six ways.

These groups of five tetrahedra represent a resolvent of sixth degree.

This is the resolvent of sixth degree that differs from the given equation that was already mentioned in the introduction.

Any three of the given six complexes have the lines of one generator of a hyperboloid in common, while the lines of the other generator of that hyperboloid belong to the remaining three complexes. There are ten such hyperboloids, corresponding to the ten possible ways of dividing six things into two groups of three.

The hyperboloids define a resolvent of tenth degree.

I would thus like to expressly emphasize that the equation of sixteenth degree depends upon the determination of the singularities of the Kummer surface of fourth degree with 16 nodes [9]), and which, as I showed in *loc. cit.*, has an immediate relationship to a system of six linear complexes of the type that were considered here, *is not* a resolvent of the equation of sixth degree that is represented by the complex. Moreover, its relationship to the equation of sixth degree is such that one can represent its 16 roots by the symbol:

$$(a_1 x_1 \pm a_2 x_2 \pm \ldots \pm a_6 x_6)^2,$$

where the sign of a should be chosen in such a way that the number of equal signs is always even [10]).

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[9]) The fact that the solution of this equation requires only the solution of a general equation of sixth degree and some quadratic ones was first established by Camille Jordan in the treatise "Sur une équation du 16ème degré" (Crelle's Journal, Bd. 70 (1869).

[10]) This equation of sixteenth degree is closely related to a second one of the same degree, viz., the one that is determined by 16 lines of an f_3 with a double conic section (or also the 16 lines of an f_3 that meet a fixed conic section). The latter equation demands only one equation of fifth degree and some quadratic

In conclusion, I would still like to demonstrate how the four geometric representations of the equations of sixth degree that were examined here can illuminate the algebraic character of some of the problems that are included in the general problem: *Find those rational transformations, under which a general equation of sixth degree goes to another one and which possess some well-defined invariant property.* Admittedly, one method for treating this problem has already been suggested by Clebsch in Math. Annalen, Bd. 4, pp. 289 to 290, not only for the equations of sixth degree, but for those of arbitrary degree; however, it is perhaps always interesting to see how these things behave for the geometric representations that were applied here.

In an arbitrary plane of space, the six complexes x correspond to six points that lie on a conic section [cf., the cited Abh. II]. These six points should describe the given equation of sixth degree. If one now gives that plane any other position then the given equation of sixth degree goes to another one by a rational substitution. In particular, one can give the plane such positions that the equation takes on distinguished invariant properties.

For example, if one lays the plane through one of the four vertices of the aforementioned 15 tetrahedra then the invariant R vanishes for the resulting equation; the six corresponding points define an involution.

If the plane falls on one of the 60 faces of the 15 tetrahedra then the six points in it reduce to three doubly counted points.

Finally, if the plane contacts one of the ten aforementioned hyperboloids then the conic section that includes the six points decomposes into two lines, upon which, each of three points lie. The equation of sixth degree is then solvable through one quadratic equation and two cubic ones.

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ones for its solution. This is to be regarded as one of the equations of sixteenth degree that were considered in the text for which one must adjoin root of the equation of sixth degree that is to be solved.