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INTRODUCTION

This treatise is a contribution to the study of non-conservative dynamical systems for which the generalized force depends upon both the position and velocity parameters. Our goal is to generalize Hamilton's principle and to geometrically represent the trajectories as geodesics in appropriate spaces.

Continuing the work of E. Cartan, A. Lichnerowicz, and then F. Gallisot, one shows that one can base the dynamics of systems upon a 2-form that has the Lagrange equations of motion for its associated system. That 2-form is defined on the fiber bundle of tangent directions to the configuration space-time. We have been able to give it a form that is independent of the framing of space and time thanks to the introduction of an asymmetric tensor, namely, the *force tensor*, which we will substitute for the force vector. We have deduced that tensor from the force vector in order to remain in the case of classical mechanics. However, it is the opposite step that must be taken, since the force tensor permits one to characterize the dynamical state of a system of corpuscles more completely. The classical force vector is then defined to be the contracted product of the force tensor and the velocity vector.

In order to extend Hamilton's principle to non-conservative systems, one must generalize the classical variational calculus. In order to interpret the trajectories geometrically, one must generalize Finsler spaces. Those generalizations must involve an antisymmetric tensor of order two that analytical mechanics interprets as the force tensor of the dynamical system considered.

The first part of this work is devoted to differential geometry. Most of the results that will be pointed out can be interpreted in analytical mechanics immediately, and the second part will be devoted to that subject more especially.

In Chapter I, one studies the differential systems A (ω), E (ω), C (ω) that are called "associated," "extremal," and "characteristic," respectively, for a differential form ω of class C^∞ that is defined on a differentiable manifold V_n. One then gives a vector field X of class C^∞ on V_n. Since the trajectories of the field X are defined by the differential system S (X), one studies the fundamental forms that are attached to S (X), namely:

1) The invariant forms ω; i.e., the ones for which:

$$\theta(\mathrm{X})\ \omega = 0.$$

2) The forms ω that define an integral invariance relation:

$$i\ (\mathrm{X})\ \omega = 0.$$

3) The forms ω that define a relative integral invariant:

$$i\ (\mathrm{X})\ d\omega = 0.$$

4) The forms ω that define an absolute integral invariant:

$$i\,(\mathrm{X})\;\omega = 0, \qquad i\,(\mathrm{X})\;d\omega = 0.$$

In Chapter II, we will study the restricted forms that are defined on the space $\mathcal{V}$ of non-zero vectors y that are tangent to V_n or on the space W of tangent directions to V_n , which are forms whose coefficients are homogeneous with respect to the components of y ($\dot{h}$ forms, in what follows).

One defines the operator $\dot{d}$ on those forms such that:

$$\dot{d}\omega = dx^{\alpha} \wedge \partial_{\dot{\alpha}}\omega\,, \qquad \text{in which} \qquad \partial_{\dot{\alpha}}\omega = \frac{\partial\omega}{\partial y^{\alpha}}\,,$$

where the x^{α} are the coordinates of a point x of V_n , and the y^{α} are the components of a vector y in the tangent space T_x to V_n with respect to the natural frame.

In particular, one studies the differential algebra H of semi-basic forms, and one shows that a $\dot{d}$-closed form ω is the $\dot{d}$ differential of the form $\frac{1}{p+k}\,i\,(y)\;\omega$, where p is the degree of ω, and the coefficients of ω are $\dot{h}k$.

Chapter III is devoted to the classical variational calculus. One shows that a semi-basic form 1-form on W admits basic extremals if and only if it is $\dot{d}$–closed. One then studies the properties of Euler vectors and forms and establishes the Helmholtz conditions for a 2-form to be the Euler form of a function $L\,(x,\,y)$. The chapter concludes with some considerations on the geodesics of a Finsler space that are connected with the variational calculus.

In Chapter IV, one studies the generalizations of the classical variational calculus.

A first generalization is based upon considering the paths that are S-close to a basic path of W, which are paths that are defined when one is given an antisymmetric restricted tensor $S_{\alpha\beta}$ on W. In order to interpret the *S-extremals* of a function L geometrically, one introduces a general Finsler space or an *S-Finslerian* space. Such a space differs from a Finsler space by only the E convention of E. Cartan: The Riemannian torsion $S^{\alpha}_{\beta\gamma}$ is not zero, but is defined by the tensor $S_{\alpha\beta}$.

A second generalization is due to Lichnerowicz and is also based upon the consideration of special paths that are close to a given one. One defines non-holonomic functions and forms, as well as their exterior differentials.

Lichnerowicz's variational spaces (viz., $\mathcal{L}$ spaces) are the spaces that are defined by the same conventions as the Finsler spaces, but when one starts with a non-holonomic function L.

In Part Two, which is devoted more especially to the *analytical mechanics* of non-conservative dynamical systems, one considers time to be an $(n + 1)^{\text{th}}$ variable. Instead of the generalized force vector X, one considers the force tensor S that is defined by:

$$S_{\alpha\beta}\,dx^{\alpha} \wedge dx^{\beta} = \dot{d}\,(-\,X_{\alpha}\,dx^{\alpha})\;.$$

That tensor, along with the Lagrangian L, defines an S-Finslerian space, or an $\mathcal{L}$-space, whose geodesics are the trajectories of the dynamical system. One then shows that the system of equations of motion is the system that is associated with the 2-form:

$$\Omega = dl_\alpha \wedge dx^\alpha + \tfrac{1}{2} S_{\alpha\beta}\, dx^\alpha \wedge dx^\beta, \qquad \text{with} \qquad l_\alpha = \partial_{\dot{\alpha}} L .$$

That 2-form will play a fundamental role in what follows. The existence of Ω will imply a *theorem* that generalizes a theorem of E. Cartan: The difference of the circulations of a velocity vector $\boldsymbol{l}$ along the two 1-cycles C_0 and C_1 that surround the same tube of trajectories $\mathcal{T}$ is equal to the flux of the force tensor $S_{\alpha\beta}$ across the 2-chain of $\mathcal{T}$ whose boundary is $C_0 - C_1$.

One then studies the case in which the form Ω will be closed on W (i.e., the existence of global vector potential) or admits an integrating factor (viz., the simultaneous existence of a vector potential and a scalar potential).

The form Ω corresponds to the antisymmetric matrix $\begin{pmatrix} S & -I \\ I & 0 \end{pmatrix}$, where S is the matrix $(S_{\alpha\beta})$ and I is the identity matrix of order $n + 1$. One then has a matrix notation that is particularly convenient to the canonical equations.

The form Ω defines the structure of an almost-symplectic space, or "Lee space" structure, on $\mathcal{V}$. In particular, one can then deduce the condition that the force tensor must satisfy in order for Ω to admit an integrating factor.

Chapter VII is dedicated to non-holonomic dynamical systems. One introduces the concept of constraint tensor for them and studies, in particular, the perfect constraints, in the sense of Delassus, which are characterized by the condition that is called the "generalized principle of virtual work." The trajectories are interpreted as geodesics in S-Finslerian spaces or $\mathcal{L}$-spaces. A generalization of Meusnier's theorem shows that the trajectories of a dynamical system with perfect non-holonomic first-order constraints are characterized by the principle of least curvature. One shows its equivalence with the Gauss-Appell principle and then deduces the Appell equation in its homogeneous formulation.

The last chapter relates to some problems in regard to dynamical systems for which the notion of force tensor is imposed in particular. A change of frame introduces an antisymmetric tensor in an immediate way: viz., the centrifugal force tensor.

The dynamical systems with Appell constraints or gyroscopic constraints are characterized by a second-order antisymmetric tensor. The non-conservative dynamical systems admit a Painlevé integral H = const. that is independent of time such that the generalized force in the configuration space V_n will have components of the form:

$$Q_k = S_{km}\, x'^m, \qquad \text{with} \quad S_{km} = - S_{mk}$$

that satisfy a generalization of the Maupertuis principle that permits one to determine the trajectories independently of the timetable.

That chapter will conclude with some applications to general relativity.

One associates the force density vector K_α with a force tensor $s_{\alpha\beta}$ in a natural fashion such that the differential system of the streamlines is the associated system to the 2-form:

$$\Omega = dl_\alpha \wedge dx^\alpha + \tfrac{1}{2} s_{\alpha\beta}\, dx^\alpha \wedge dx^\beta .$$

The streamlines are the geodesics on an S-Riemannian space that is defined by the torsion forms:

$$\Sigma^\gamma = \left(\tfrac{1}{2} s_{\alpha\beta}\, dx^\alpha \wedge dx^\beta\right) l^\gamma .$$

The cases in which Ω is closed or admits an integrating factor relate to the classical models. The considerations of the preceding chapter will then, in turn, give (among other things) the now-classical results of general relativity that relate to the streamlines of charged perfect fluids.

That work was brought into play thanks to A. Lichnerowicz. I wish to acknowledge the profound admiration that I have for him here.

I would likewise like to express my respect and gratitude to:

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PART ONE

GENERALIZED VARIATIONAL GEOMETRY

CHAPTER I

REVIEW OF DIFFERENTIABLE MANIFOLDS

1. – Let V_n be an n-dimensional differentiable manifold of class C^{∞}, and let X be a differentiable vector field that is defined on V_n. That field generates a local group of local transformations of V_n by integrating the differential system:

$$\frac{dx(u)}{du} = \mathrm{X}_{x(u)} . \tag{1.1}$$

The solution to this system that issues from the point $x(0) = x$ will be denoted by ([1]):

$$x(u) = \exp(u\,\mathrm{X})\, x .$$

The differentiable map $\exp(u\,\mathrm{X})$ admits a tangent linear map, which is denoted by $\exp(u\,\mathrm{X})'$, of the vector space T_x that is tangent to V_n at x to the vector space $T_{x(u)}$ that is tangent to V_n at $x(u)$. One deduces a linear map from its reciprocal image that is denoted by $\exp(u\,\mathrm{X})^*$ and which takes the dual space $T^*_{x(u)}$ to $T_{x(u)}$ to the dual space T^*_x to T_x.

Let ω be a p-form of V_n. One calls the p-form $\theta(\mathrm{X})\,\omega$ that is defined by:

$$[\theta(\mathrm{X})\,\omega]_x = \lim_{u\to 0} \frac{\exp(u\,\mathrm{X})^*\omega_{x(u)} - \omega_x}{u},$$

the *infinitesimal transform* of ω by X or the *Lie derivative* of ω by X. Let $i(\mathrm{X})\,\omega$ be the interior product of ω by X.

We will then have the fundamental formula ([2]):

$$\theta(\mathrm{X})\,\omega = d\,i(\mathrm{X})\,\omega + i(\mathrm{X})\,d\omega . \tag{1.2}$$

Let Y be a second differentiable vector field that is defined on V_n. We set:

$$\theta(\mathrm{X})\,\mathrm{Y} = [\mathrm{X}, \mathrm{Y}] \tag{1.3}$$

and recall the formulas:

$$\theta([\mathrm{X}, \mathrm{Y}])\,\omega = \theta(\mathrm{X})\,\theta(\mathrm{Y})\,\omega - \theta(\mathrm{Y})\,\theta(\mathrm{X})\,\omega, \tag{1.4}$$

$$i\,[\mathrm{X}, \mathrm{Y}]\,\omega = \theta(\mathrm{X})\,i(\mathrm{Y})\,\omega - i(\mathrm{Y})\,\theta(\mathrm{X})\,\omega, \tag{1.5}$$

$$i(\mathrm{Y})\,i(\mathrm{X})\,d\omega = \theta(\mathrm{X})\,i(\mathrm{Y})\,\omega - i(\mathrm{Y})\,\theta(\mathrm{X})\,\omega - i\,[\mathrm{X}, \mathrm{Y}]\,\omega . \tag{1.6}$$

In what follows, we will be sometimes led to distinguish:

([1]) A. Lichnerowicz [**5**], pp. 15.

([2]) H. Cartan [**1**].

$$i\,(\mathrm{X})\ \omega = 0 \qquad \text{from} \qquad i\,(\mathrm{X})\ \omega \equiv 0.$$

The first relation is an exterior differential equation: A solution of that equation will be a $(p-1)$-vector $\mathrm{Y}_1 \wedge \ldots \wedge \mathrm{Y}_{p-1}$, where $\mathrm{Y}_1, \ldots, \mathrm{Y}_{p-1}$ are independent vectors in T_x such that one will have the numerical equality:

(1.7) $$i\,(\mathrm{X} \wedge \mathrm{Y}_1 \wedge \ldots \wedge \mathrm{Y}_{p-1})\ \omega = 0.$$

The second relation $i\,(\mathrm{X})\ \omega = 0$ expresses the idea that the equality (1.7) is verified for any vectors $\mathrm{Y}_1, \ldots, \mathrm{Y}_{p-1}$ in T_x ; i.e., that the point x considered is a zero of the form $i\,(\mathrm{X})\,\omega$.

A. – Some remarkable differential systems that are attached to an exterior form.

2. Associated system to an exterior form. – One calls a direction that is defined by a vector X such that:

$$i\,(\mathrm{X})\ \omega \equiv 0$$

an *associated direction* to ω at x. That identity indeed defines a direction because:

$$i\,(\lambda\,\mathrm{X}) = \lambda\, i\,(\mathrm{X}) \qquad \text{for any scalar } \lambda.$$

The corresponding linear differential system that is obtained by replacing X with dx in the equations that define the associated directions to ω at any point x of V_n is called the *associated system* $A\,(\omega)$ to ω. One obtains it by equating all of the derivatives of order $p-1$ in the form ω to zero.

Examples:

1) A 1-form:

$$\omega = a_i\,(x)\,dx^i,$$

with

$$i\,(\mathrm{X})\ \omega = a_i\,X^i = 0,$$

defines an $(n-1)$-dimensional planar manifold in the tangent vector space to V_n at any ordinary point of ω.

2) Let a 2-form be given by:

$$\omega = \tfrac{1}{2}\,a_{ij}\,dx^i \wedge dx^j,$$

with

$$i\,(\mathrm{X})\ \omega = \tfrac{1}{2}\,a_{ij}\,(X^i\,dx^j - X^j\,dx^i) \equiv 0,$$

which implies that:

$$a_{ij}\,X^i = 0\,.$$

If n is odd then that system will have rank at most $n - 1$, and one will get at least one associated direction at any point of V_n .

If n is even, and if the system has rank n then there will exist no associated direction at an ordinary point of V_n .

If the system has rank $2p < n$ then there will be an infinitude of associated directions that form an $(n - 2p)$-dimensional planar manifold in T_x .

A direction X that is associated with the 2-form ω is characterized by the property: The flux of the tensor a_{ij} across any 2-plane in T_x that contains X is zero.

3. Extremal system of an exterior differential form. – Let W be a local differential chain in V_n of dimension p, let X be a differentiable vector field, and let ω be a p-form in V_n . The point transformation exp (u X), where u is fixed, will transform the points of W into points of another chain that is denoted by:

$$W(u) = \exp(u\,\mathrm{X})\,W.$$

Consider the integral $I = \int_W \omega$.

The Lie derivative of I by X is, by definition, the scalar:

$$\theta(\mathrm{X})\int_W \omega = \lim_{u\to 0}\frac{1}{u}\left[\int_{W(u)} \omega_{x(u)} - \int_W \omega\right].$$

Make the change of variables in the right-hand side that transforms the coordinates of $x(u)$ into those of x.

We will then have:

$$\int_{W(u)} \omega_{x(u)} = \int_W \exp(u\,\mathrm{X})^*\,\omega_{x(u)}\,,$$

so

$$\theta(\mathrm{X})\int_W \omega = \int_W \lim_{u\to 0}\frac{1}{u}[\exp(u\,\mathrm{X})^*\,\omega_{x(u)} - \omega_x] = \int_W \theta(\mathrm{X})\,\omega\ .$$

We now apply formula (1.2) and get:

$$\theta(\mathrm{X})\int_W \omega = \int_W d\,i(\mathrm{X})\,\omega + \int_W i(\mathrm{X})\,d\omega\,,$$

and from Stokes's formula, when we let ∂W denote the boundary of W, we will get:

$$\theta(\mathrm{X})\int_W \omega = \int_{\partial W} i(\mathrm{X})\,\omega + \int_W i(\mathrm{X})\,d\omega, \tag{3.1}$$

We now suppose that the chain W is closed, and seek to determine the vector field X in such a fashion that for any W, we will have:

$$\theta(\mathrm{X})\int_W \omega = 0.$$

Let $Y_1, Y_2, \ldots, Y_p$ be p local vector fields that are tangent to V_n. In order to have:

$$\theta(X)\int_W \omega = \int_W i(X)\, d\omega = 0,$$

it is necessary and sufficient that one should have:

$$i\,(Y_1 \wedge Y_2 \wedge \ldots \wedge Y_p)\; i\,(X)\; d\omega = 0$$

for any $Y_1, Y_2, \ldots, Y_p$; i.e., that:

(3.2) $$i\,(X)\; d\omega \equiv 0.$$

By definition, a direction X that verifies the preceding identity is an *extremal direction* of ω; such a direction is nothing but an associated direction to $d\omega$.

The corresponding linear differential system is, by definition, the *extremal system* of ω, which is denoted by $E(\omega)$.

4. Characteristic system of a differential form. – Let X be a non-zero vector field on V_n. The field X defines a direction field that we further denote by X.

A direction field X is, by definition, a *characteristic field* for the p-form ω on an open subset U in V_n if one has:

(4.1) $$i\,(X)\;\omega \equiv 0$$

and

(4.2) $$i\,(X)\; d\omega \equiv \lambda\,\omega$$

at any point x of U, where λ is a numerical function of x.

The identity (4.2) can be replaced with:

(4.3) $$\theta(X)\; d\omega \equiv \lambda\,\omega.$$

That shows that when one is given an integral manifold W of the equation:

$$\omega = 0,$$

the manifold that is generated by the trajectories of a characteristic field that meets W will also be an integral manifold of that equation.

It is immediate that the definition of a characteristic field that was given above can be replaced with the following ones:

$$i\,(X)\;\omega \equiv 0$$

and

(4.4) $i\,(\mathrm{X})\; i\,(\mathrm{Y})\; d\omega \equiv 0$ for any Y such that $i\,(\mathrm{Y})\;\omega \equiv 0$.

From formula (1.6), the identity (4.4) is equivalent to:

$$i\,[\mathrm{X},\mathrm{Y}]\;\omega = 0 \qquad \text{for any Y such that} \qquad i\,(\mathrm{Y})\;\omega = 0\;.$$

The linear differential system that corresponds to defining X at any $x \in V_n$ is, by definition, the *characteristic system* $C\,(\omega)$ of the form ω.

If ω is a 1-form then the characteristic system will include the equation $\omega = 0$, and the differential system that is obtained by writing out the exterior equations:

$$i\,(\mathrm{Y})\; d\omega = 0 \quad \text{and} \quad i\,(\mathrm{Y})\;\omega = 0$$

will admit the same solutions as Y.

5. Integrability of the systems $A\,(\omega)$, $E\,(\omega)$, $C\,(\omega)$. – By definition, a linear differential system is called *completely integrable* on a domain D of V_n if the fact that X and Y are two integral direction fields on D implies that the bracket [X, Y] is also an integral direction field on D.

1. *Associated system* $A\,(\omega)$. – By hypothesis, $i\,(\mathrm{X})\;\omega \equiv 0$ and $i\,(\mathrm{Y})\;\omega \equiv 0$. It will result from formula (1.6) that:

$$i\,[\mathrm{X},\mathrm{Y}]\;\omega = i\,(\mathrm{X})\; i\,(\mathrm{Y})\; d\omega\,.$$

In general, the associated system will not be completely integrable then.

That will be true when $d\omega = 0$, and also when ω has a unique associated direction at any x of D .

2. *Extremal system* $E\,(\omega)$. – By hypothesis, $i\,(\mathrm{X})\; d\omega \equiv 0$ and $i\,(\mathrm{Y})\; d\omega \equiv 0$. Formula (1.6) will then imply that:

$$i\,[\mathrm{X},\mathrm{Y}] = i\,(\mathrm{X})\; i\,(\mathrm{Y})\; d\,(d\omega) \equiv 0\;.$$

The extremal system to ω will then be completely integrable.

3. *Characteriistic system* $C\,(\omega)$. – Its complete integrability follows directly from its definition.

B. – Some remarkable forms that are attached to a differential system $S\,(\mathrm{X})$.

6. – Let V_{n+1} be an $(n + 1)$-dimensional differentiable manifold of class C^∞. Let X be a non-zero vector field that is defined on a domain D_{n+1} of V_{n+1} , and let $T_x'^{*}$ be the vector subspace of T_x^{*} that is orthogonal to X.

Let $x^1, x^2, \ldots, x^{n+1}$ be a local coordinate system about a point x in an open subset of D_{n+1}, and let $X^1, X^2, \ldots, X^{n+1}$ be the components of the vector X in the associated natural frame. The differential system S (X) of the trajectories of the local group that is defined on U by X will then be:

$$\frac{dx^1}{X^1} = \frac{dx^2}{X^2} = \ldots = \frac{dx^{n+1}}{X^{n+1}}. \tag{6.1}$$

For a suitable D_{n+1} , the relation that expresses the idea that two points x and x' are on the same trajectory of S (X) is an equivalence relation R that is defined on that domain. D_{n+1} will then be fibered, and its base $I_n = D_{n+1} / R$ can be identified with the space of first integrals of S (X).

Let p denote the projection of D_{n+1} onto its base; any point x of D_{n+1} will then correspond to the point $y = p\, x$ in I_n .

The *first integrals* of S (X) are functions f (x) that are solutions to the first-order partial differential equation:

$$\theta(\mathrm{X})\, f = i\, (\mathrm{X})\, df = 0.$$

Locally, df is a closed form that is orthogonal to X at the point x . n independent first integrals f_1 , f_2 , …, f_n will represent a local coordinate system for the point $y = p\, x$. A local coordinate system for the point x that is adapted to the fiber structure of D_{n+1} will then be: f_1 , f_2 , …, f_n , and x^{n+1} when one supposes that $x^{n+1} = C$ is not a first integral of S(X).

Definitions:

1. A form ω is called *invariant* for S (X) if:

$$\theta(\mathrm{X})\ \omega \equiv 0 .$$

2. A form ω is called *semi-basic* for the space D_{n+1} that is fibered by X or defines an *integral invariance relation* for the differential system S(X) if:

$$i\, (\mathrm{X})\ \omega \equiv 0 .$$

3. A form ω defines a *relative integral invariant* for S (X) if:

$$i\, (\mathrm{X})\ d\omega \equiv 0 .$$

4. A form ω is called *basic* for the space D_{n+1} that is fibered by X or defines an *absolute integral invariant* for S (X) if:

$$i\, (\mathrm{X})\ \omega \equiv 0, \qquad \theta(\mathrm{X})\ \omega \equiv 0 .$$

7. Notion of an integral invariance relation (3). – Saying that the p-form ω defines an *integral invariance relation* for S (X) that is characterized by i (X) $\omega \equiv 0$ amounts to saying that X is an associated direction for ω at any point x in a domain D_{n+1} of V_{n+1} .

The identity i (X) $\omega \equiv 0$ expresses the idea that ω locally belongs to the vector space $\Lambda^p(T_x'^*)$, which is the p^{th} exterior power of $T_x'^*$, and a basis for the former space is $df_{i_1} \wedge df_{i_2} \wedge \ldots \wedge df_{i_p}$, where $f_1, f_2, \ldots, f_n$ are n independent first integrals of S (X) . We can then write ω in the following fashion:

$$\omega = \frac{1}{p!}\, a^{i_1 i_2 \cdots i_p}\, df_{i_1} \wedge df_{i_2} \wedge \ldots \wedge df_{i_p}, \tag{7.1}$$

in which the coefficients are functions of the variables:

$$x^\alpha, \qquad \text{with} \qquad \alpha = 1, \ldots, n + 1.$$

Let us now justify the expression "integral invariance relation." In order to do that, we consider a $(p - 1)$-dimensional chain W_0 that is or is not closed in a domain D of V_{n+1} . Let W_1 denote the chain $W_1 = \exp (u \text{ X})\, W_0$ that is the locus of points $x_1 = \exp (u \text{ X})\, x_0$, in which x_0 is an arbitrary point of W_0 , in which the parameter u has a suitable fixed value.

For a chain W_0 and a suitable parameter u, let $\mathcal{T}$ denote a "tube of trajectories," which is a p-dimensional chain in the domain D that is generated by the trajectories of the various points of W_0 and is bounded by W_0 and W_1 . The manifold that carries $\mathcal{T}$ admits a parametric representation of the form:

$$x^\alpha = f^{\,\alpha} (u, v^1, \ldots, v^{p-1}), \tag{7.2}$$

so

$$dx^\alpha = X^\alpha\, du + Y_1^\alpha dv^1 + \ldots + Y_{p-1}^\alpha\, dv^{p-1},$$

in which $Y_1, \ldots, Y_{p-1}$ are $p - 1$ are tangent vectors to $\mathcal{T}$.

Now suppose that ω is a p-form that is defined on the domain D of V_{n+1} to which $\mathcal{T}$ belongs.

The integral $I = \int_{\mathcal{T}} \omega$ then reduced to the multiple integral:

$$\begin{aligned} I &= \int_{\Delta_p} i(\mathrm{X} \wedge \mathrm{Y}_1 \wedge \cdots \wedge \mathrm{Y}_{p-1})\, \omega\, du\, dv^1 \cdots dv^{p-1} \\ &= \int_{\Delta_p} i(\mathrm{Y}_{p-1})\, i(\mathrm{Y}_{p-1}) \wedge \cdots \wedge i(\mathrm{X})\, \omega\, du\, dv^1 \cdots dv^{p-1}, \end{aligned}$$

(3) A. Lichnerowicz [**1**], pp. 1-8.

Δ_p is the domain on $\mathbb{R}^p$ whose image under the map f is $\mathcal{T}$.

In order to have $I = 0$ for any tube $\mathcal{T}$ – i.e., for any vectors $Y_1, \ldots, Y_{p-1}$ – it is necessary and sufficient that:

$$i\,(X)\,\omega \equiv 0,$$

hence, one has the:

Theorem:

In order for the integral $\int_{\mathcal{T}} \omega$ *to be zero for any tube* $\mathcal{T}$ *of trajectories for the differential system* S (X), *it is necessary and sufficient that:*

$$i\,(X)\,\omega \equiv 0\;;$$

i.e., that ω *should generate an integral invariance relation for* S (X) .

8. Notions of relative and absolute integral invariant. – Let ω be a p-form that is defined on V_{n+1}. Let W be an orientable p-dimensional chain in V_{n+1} that has ∂W for its boundary. We saw (**3.1**) that:

$$\theta(X)\int_W \omega = \int_{\partial W} i(X)\,\omega + \int_W i(X)\,d\omega.$$

1. Suppose that the chain W is closed (i.e., it is a cycle). The preceding formula reduces to:

$$\theta(X)\int_W \omega = \int_W i(X)\,d\omega. \tag{8.1}$$

In order for one to have $\theta(X)\int_W \omega = 0$ for any cycle W, it is necessary and sufficient (from § **7**) that one must have:

$$i\,(X)\,d\omega = 0\;; \tag{8.2}$$

i.e., that X must be an extremal direction to ω at any point x of V_{n+1}.

It then results that if W_0 and W_1 are two p-dimensional cycles in V_{n+1} such that $W_1 = \exp(u\,X)\,W_0$ then one will have:

$$\int_{W_1} \omega = \int_{W_0} \omega.$$

This equality, which one can deduce by applying Stokes's formula to the fact that $d\omega$ defines an integral invariance relation for S (X), justifies the term *relative integral invariant.*

2. Now suppose that the chain W has a boundary $\partial W \neq 0$. In order for one to have $\theta(\mathrm{X})\int_W \omega = 0$ for any W, it is necessary and sufficient that one should have both:

$$i\,(\mathrm{X})\,\omega \equiv 0 \qquad \text{and} \qquad i\,(\mathrm{X})\,d\omega \equiv 0\,. \tag{8.3}$$

Under those conditions:

$$\int_{W_1} \omega = \int_{W_0} \omega$$

for any p-dimensional chain W_0 in V_{n+1}, whether closed or not, where W_1 denotes the chain exp (u X) W_0, as always.

That equality, which justifies the expression *absolute integral invariant*, results from $\theta(\mathrm{X})\int_W \omega = 0$ and also the fact that the identities (8.3) imply that the form ω can be expressed solely in terms of n independent first integrals of S (X) and their differentials [which is a consequence of (7.1)].

Let us point out some theorems whose proofs are immediate:

Theorem:

1. *If a form ω defines a relative integral invariant for S* (X) *then the form $d\omega$ will define an absolute integral invariant for S* (X).

2. *If the form $d\omega$ generates an integral invariance relation for S* (X) *then the form ω will define an absolute integral invariant for S* (X), *and conversely.*

3. *If the forms ω and $d\omega$ generate integral invariance relations for S* (X) *then the form ω will define an absolute integral invariant for S* (X), *and conversely.*

4. *If the form ω generates an integral invariance relation for S* (X), *and if* Y *is an arbitrary vector field on V_{n+1} then the form i* (Y) *ω will also generate an integral invariance relation for S* (X).

Indeed: $i\,(\mathrm{X})\; i(\mathrm{Y})\;\omega = -\, i\,(\mathrm{Y})\; i(\mathrm{X})\;\omega = 0$.

9. One-parameter groups that leave the system S (X) **invariant.** – Let Y be a vector field that is tangent to V_{n+1} . That field will generate a local one-parameter group G_t of local transformations of V_{n+1} by integrating the differential system:

$$\frac{dx(t)}{dt} = \mathrm{Y}_{x(t)}$$

when one starts from an initial point $x\,(0) = x$.

The system S (X) is called *invariant under* G_t if the vector $X_{x(t)}$ is collinear with $(\exp t\,\mathrm{Y})'\mathrm{X}_x$ for any point x where G_t is defined and for t sufficiently small.

One can show ([4]) that this will be true if and only if:

$$\theta(\mathrm{Y})\,\mathrm{X} = [\mathrm{Y}, \mathrm{X}] = f\,\mathrm{X},$$

where f is a scalar function of x. In this case, one says that the system S (X) *admits an infinitesimal transformation* that is defined by Y. One will then have the following theorem:

Theorem 1:

If a form ω *generates an integral invariance relation for* S (X) *then so will the form* $\theta(\mathrm{X})\,\omega$.

Theorem 2:

If a form ω *defines a relative integral invariant for* S (X) *then so will the form* $\theta(\mathrm{X})\,\omega$.

Theorem 3:

If a form ω *defines an absolute integral invariant for* S (X) *then so will the forms* $\theta(\mathrm{X})\,\omega$ *and* i (Y) ω.

Proof:

If will suffice to establish the second part of Theorem **3**. One has:

$$i\,(\mathrm{X})\,di\,(\mathrm{Y})\,\omega = i\,(\mathrm{X})\,[\theta(\mathrm{Y})\,\omega - i\,(\mathrm{Y})\,d\omega],$$
$$i\,(\mathrm{X})\,di\,(\mathrm{Y})\,\omega = i\,(\mathrm{X})\,\theta(\mathrm{Y})\,\omega + i\,(\mathrm{Y})\,i\,(\mathrm{X})\,d\omega = 0,$$

which proves the property.

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([4]) H. Cartan [**2**], Chap. IV, pp. 3.

CHAPTER II

FIBER BUNDLES OF TANGENT VECTORS OR DIRECTIONS TO A DIFFERENTIABLE MANIFOLD

10. Definition of the fiber bundles $\mathcal{V}$ and *W*. – Let V_{n+1} be an $(n + 1)$-dimensional differentiable manifold of class C^{∞}. Let $\mathcal{V}$ be the fiber bundle of non-zero tangent vectors to V_{n+1}, whose structure group is $GL(n + 1, \mathbb{R})$ and whose fiber is isomorphic to $\mathbb{R}^{n+1}$ without its origin. Let Z be a point of $\mathcal{V}$, and let p be the canonical projection of Z onto its origin $x \in V_{n+1}$. Let x^{α} ($\alpha = 1, \ldots, n + 1$) be a local coordinate system of the point x of V_{n+1}, and let y^{α} be the components of a vector y of T_x in the associated natural frame R_x. The $2n + 2$ numbers x^{α}, y^{α} constitute a local coordinate system at a point Z of the fiber $\pi^{-1}x$. The change of coordinates on V_{n+1} that is defined by the functions $x^{\alpha'} = f^{\alpha'}(x^{\beta})$ implies the following change in the y:

$$y^{\alpha'} = \partial_{\beta} f^{\alpha'} y^{\beta} = A_{\beta}^{\alpha'} y^{\beta} . \tag{10.1}$$

Consider two points Z_1 and Z_2 in the fiber $\pi^{-1} x$ such that the corresponding vectors of T_x (namely, y_1 and y_2) are positively collinear ($y_2 = \lambda y_1$, $\lambda > 0$).

The relation thus-defined on $\mathcal{V}$ is an equivalence relation R. The quotient space $W = \mathcal{V} / R$ is, by definition, the space of oriented directions that are tangent to V_{n+1}. The space W can be endowed with the structure of a $(2n + 1)$-dimensional differentiable manifold. The projection p of each direction z onto its origin x endows W with the structure of a fiber bundle with base V_{n+1} whose fiber is homeomorphic to the sphere S_n and whose structure group of the group $GL(n + 1, \mathbb{R})$, or more precisely, the orthogonal group $O(n)$.

A local coordinate system at a point $z = p^{-1} x$ is once more the set of $2n + 2$ numbers x^{α}, y^{α}, where the $n + 1$ numbers y^{α} are defined only up to a positive proportionality factor.

11. Tensors and forms defined on $\mathcal{V}$ or *W*. – An affine tensor field relative to $\mathcal{V}$, in the usual sense, is a map t that makes any point Z in $\mathcal{V}$ correspond to an element of the affine tensor algebra that is constructed on T_Z. The tensors thus-defined relate to the linear group $GL(2n + 2, \mathbb{R})$.

However, $\mathcal{V}$ is a fiber bundle whose base is V_{n+1}, so the change of local chart on the base:

$$x^{\alpha} = f^{\alpha}(x^{\beta})$$

will induce the change of coframe in T_Z^* that is defined by:

$$\begin{cases} dx^\alpha = A^\alpha_{\beta'}\, dx^{\beta'}, \\ dy^\alpha = B^\alpha_{\beta'}\, dx^{\beta'} + A^\alpha_{\beta'}\, dy^{\beta'}, \end{cases}$$

with $A^\alpha_{\beta'} = \partial_{\beta'} f^{\,\alpha}$ and $B^\alpha_{\beta'} = y^{\gamma'}\partial_{\beta'} A^{\beta'}_{\gamma'}$.

The corresponding matrix is $\begin{pmatrix} A & 0 \\ B & A \end{pmatrix}$, in which A and B are matrices of order $n + 1$ whose elements are $A^\alpha_{\beta'}$ and $B^\alpha_{\beta'}$, respectively; 0 is the zero matrix of order $n + 1$. The set of all those matrices is a subgroup of $GL\,(2n + 2, \mathbb{R})$ that we shall call the *prolongation of* $GL\,(n + 1, \mathbb{R})$ and denote by $\widetilde{GL}\,(n + 1, \mathbb{R})$. From now on, we shall call a tensor that relates to $\widetilde{GL}\,(n + 1, \mathbb{R})$ a tensor on $\mathcal{V}$ *in the large sense*.

One says that t is a tensor field of degree k on $\mathcal{V}$ *in the restricted sense* when one has:

$$t\,(Z') = \lambda^k\, t\,(Z)$$

for two points Z (x, y) and Z′ $(x, \lambda\, y)$ in the fiber $\pi^{-1} x$. A form ω on $\mathcal{V}$ in the large or restricted sense is an antisymmetric covariant tensor field on $\mathcal{V}$ in the large or restricted sense, resp.

Let ω be the 1-form that is represented by:

$$\omega = a_\alpha\,(x, y)\, dx^\alpha + b_\alpha\,(x, y)\, dy^\alpha$$

in a local coordinate domain.

At the point whose local coordinates are x^α, $\lambda\, y^\alpha$, where λ is an arbitrary positive function of the variables x^α, we will have:

$$\omega' = a_\alpha\,(x, \lambda\, y)\, dx^\alpha + b_\alpha\,(x, \lambda\, y)\,(\lambda\, dy^\alpha + y^\alpha\, d\lambda) .$$

In order to have $\omega' = \lambda^k\,\omega$ for any λ, it is necessary and sufficient that the a_α should be $\dot{h}\,k$ (i.e., homogeneous of degree k with respect to the y^α) , that the b_α should be $\dot{h}\,(k-1)$, and that:

$$b_\alpha\,(x, y)\, y^\alpha = 0.$$

One shows, more generally, that a p-form ω is restricted of degree k if the coefficients of the terms of degree $p - h$ with respect to the dx^α are $\dot{h}\,(k-h)$, and if:

$$y^{\alpha}\frac{\partial\omega}{\partial(dy^{\alpha})}\equiv 0.$$

By abuse of language, a restricted tensor (or a form) of degree 0 is said to be *defined on W*.

A *semi-basic* tensor field on $\mathcal{V}$ is a map t that makes an element of the affine tensor algebra that is constructed over $T_{\pi(\mathrm{Z})}$ correspond to any point Z of $\mathcal{V}$. In what follows, only the restricted semi-basic tensor fields of degree k will be used, and we shall refer to them as the $\dot{h}k$ tensors, since their components are homogeneous of degree k with respect to the variables y^{α}.

A semi-basic covariant antisymmetric tensor field of order p is, by definition, a semi-basic p-form. If the tensor is $\dot{h}k$ then the form will be called $\dot{h}k$ *semi-basic*.

The $\dot{h}k$ semi-basic p-forms on $\mathcal{V}$ define a module over the ring of functions on V_{n+1} with real values, which is a module that we shall denote by H_k^p.

The exterior algebra of restricted semi-basic forms that are defined on $\mathcal{V}$ is then a bi-graded algebra that we shall denote by $H(\mathcal{V})$.

12. Differential operators on $H(\mathcal{V})$. – Let t be a restricted tensor that is defined on an open subset U of $\mathcal{V}$. Let x^{α}, y^{α} be a local coordinate system about a point Z of U. If t is $\dot{h}k$ then the Euler identity:

$$k\,t = \partial_{\dot{\alpha}}t\,y^{\alpha}, \quad \text{in which} \qquad \partial_{\dot{\alpha}}t = \frac{\partial t}{\partial y^{\alpha}},$$

will show that the $\partial_{\dot{\alpha}}t$ define a restricted tensor of degree $k - 1$.

Now choose a form $\omega \in H_k^{(q)}(U)$. Its expression in local coordinates is:

$$\omega = \frac{1}{q!}a_{i_1\cdots i_q}dx^{i_1}\wedge\cdots\wedge dx^{i_q}.$$

The expression $dx^{\alpha}\wedge\partial_{\dot{\alpha}}\omega$ defines a semi-basic form of degree $q + 1$ and degree of homogeneity $k - 1$. From now on, we shall denote it by $\dot{d}\omega$. We then set:

(12.1) $$\dot{d}\omega = dx^{\alpha}\wedge\partial_{\dot{\alpha}}\omega,$$

by definition. The operator $\dot{d}$ is an endomorphism of $H(U)$ whose bi-degree is equal to $(1, -1)$; i.e., it is a map of the module H_k^q into the module H_{k-1}^{q+1}.

If we replace all of the dx with dy with the same index then the operator $\dot{d}$ will give an exterior differential in the fibers $\pi^{-1}x$; i.e., a differential with x fixed.

It then results that the operator $\dot{d}$ possess the following properties:

$$\dot{d}\,(\omega_1 + \omega_2) = \dot{d}\omega_1 + \dot{d}\omega_2\,,$$
$$\dot{d}\,(\omega_1 \wedge \omega_2) = \dot{d}\omega_1 \wedge \omega_2 + (-1)^{\deg\omega_1}\,\omega_1 \wedge \dot{d}\omega_2\,,$$
$$\dot{d}\,(\dot{d}\omega) = 0\,.$$

Let X be a restricted vector field that is defined on $\mathcal{V}$, and let ω be a restricted semi-basic q-form on $\mathcal{V}$, so the interior product of X by ω:

$$i\,(\mathrm{X})\,\omega = X^{\alpha}\frac{\partial\omega}{\partial(dx^{\alpha})}$$

will be a restricted semi-basic $(q - 1)$-form on $\mathcal{V}$.

Set:

$$\dot{\theta}(\mathrm{X})\,\omega = \dot{d}\,i\,(\mathrm{X})\,\omega + i\,(\mathrm{X})\,\dot{d}\omega\,.$$

The operator $\dot{\theta}(\mathrm{X})$ thus-defined is a derivation of degree 0; i.e.:

$$\dot{\theta}(\mathrm{X})\,(\omega_1 \wedge \omega_2) = \dot{\theta}(\mathrm{X})\,\omega_1 \wedge \omega_2 + \omega_1 \wedge \dot{\theta}(\mathrm{X})\,\omega_2\,.$$

One verifies that the operators $\dot{d}$ and $\dot{\theta}$ commute. If $f\,(x^{\alpha}, y^{\alpha})$ is an $\dot{h}k$ function that is defined on $\mathcal{V}$ then:

$$\dot{\theta}(\mathrm{X})\,f = X^{\alpha}\partial_{\dot{\alpha}}\,f = \langle\, \mathrm{X},\ \dot{d}f \,\rangle\,.$$

If $\omega = a_{\alpha}\,dx^{\alpha}$ then:

$$\dot{\theta}(\mathrm{X})\,\omega = (\partial_{\dot{\beta}}\,a_{\alpha}\,X^{\alpha} + a_{\beta}\,\partial_{\dot{\alpha}}\,X^{\beta})\,dx^{\alpha}\,.$$

13. $\dot{d}$-closed forms. – A form $\Omega \in H$ is *locally* $\dot{d}$*-closed* if $\dot{d}\,\Omega = 0$ on an open subset U of $\mathcal{V}$. From Poincaré's theorem, there will exist a form $\omega \in H$ on U such that:

$$\dot{d}\omega = \Omega\,.$$

We shall recover this result and specify the expression for ω by establishing a remarkable identity that is verified by any form in the algebra H.

Let ω be an $\dot{h}k$ semi-basic p-form that is defined on $\mathcal{V}$. Take the vector field to be the field y whose components are $X^\alpha = y^\alpha$ relative to the natural frame at the point Z whose local coordinates are x^α, y^α. Let us specify the operator $\dot{\theta}(y)\omega$. By definition:

$$\begin{aligned}
\dot{\theta}(y)\omega &= \dot{d}\,i(y)\omega + i(y)\dot{d}\omega \\
&= \dot{d}\left[y^\beta \frac{\partial\omega}{\partial(dx^\beta)}\right] + i(y)[dx^\alpha \wedge \partial_{\dot{\alpha}}\omega] \\
&= dx^\alpha \wedge \frac{\partial\omega}{\partial(dx^\alpha)} + dx^\alpha \wedge y^\beta \frac{\partial_{\dot{\alpha}}\omega}{\partial(dx^\beta)} + y^\alpha \partial_{\dot{\alpha}}\omega - dx^\alpha \wedge y^\beta \frac{\partial_{\dot{\alpha}}\omega}{\partial(dx^\beta)} \\
&= dx^\alpha \wedge \frac{\partial\omega}{\partial(dx^\alpha)} + y^\alpha \partial_{\dot{\alpha}}\omega\,.
\end{aligned}$$

The first expression on the right-hand side is equal to $p\ \omega$, because ω has degree p, and the second expression is equal to $k\ \omega$, because ω is $\dot{h}k$, so we will have the identity:

(13.1) $$\boxed{\dot{\theta}(y)\omega = (p+k)\omega}$$

or

(13.2) $$\dot{d}\,i(y)\omega + i(y)\dot{d}\omega = (p+k)\ \omega.$$

Consequence. – If an $\dot{h}k$ p-form ω is $\dot{d}$–closed then the identity (13.2) will reduce to:

(13.3) $$\dot{d}\,i(y)\omega = (p+k)\ \omega,$$

and $\omega = \dot{d}\dfrac{i(y)\omega}{p+k}$ if $p + k \neq 0$.

Theorem:

A $\dot{d}$-closed semi-basic p-form ω on $\mathcal{V}$ will be the $\dot{d}$ differential of the form $\dfrac{1}{p+k} i(y)\omega$ when $p + k \neq 0$.

Remark. – If ω is a p-form on $\mathbb{R}^{n+1}$ that takes the form:

$$\omega = \frac{1}{p!} a_{i_1\cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}$$

in canonical coordinates is closed, and if its coefficients are homogeneous functions on x^α of degree k then ω will be the exterior differential of the form:

$$\frac{1}{p+k} i(y)\,\omega \qquad \text{if} \qquad p+k \neq 0\,.$$

For $p + k > 0$, that result will be a consequence of the classical homotopy formula ([5]).

14. Special case of the algebra H (W). – The algebra H (W) is, by definition, the exterior algebra of restricted semi-basic forms on W; i.e., they are $\dot{h}0$.

If X is a restricted $\dot{h}0$ vector field (i.e., it is defined on W) then the algebra H (W) will be stable under the operators i (X) and θ (X), but not under the operator $\dot{d}$. That will permit one to deduce an element of H (W) from any $\dot{h}1$ semi-basic form.

The $\dot{h}1$ scalar function $\mathcal{L}$ (x, y) corresponds to the form on H (W):

$$\dot{d}\mathcal{L} = \partial_{\dot{\alpha}}\mathcal{L}\, dx^\alpha\,.$$

The $\dot{h}1$ 1-form $\omega = a_\alpha(x, y)\, dx^\alpha$ corresponds to the 2-form in H (W) :

$$\dot{d}\omega = \tfrac{1}{2}(\partial_{\dot{\alpha}}\, a_\beta - \partial_{\dot{\beta}}\, a_\alpha)\; dx^\alpha \wedge dx^\beta\,.$$

From the theorem in § **13**, any $\dot{d}$-closed semi-basic p-form ω on W is the $\dot{d}$-differential of the $(p - 1)$-form $\dfrac{1}{p}\, i\,(y)\;\omega$.

We verify that theorem by establishing, at the same time, some simpler necessary and sufficient conditions to have $\dot{d}\omega = 0$ for $p = 1$ and 2.

First of all, in order for an $\dot{h}0$ function $f(x, y)$ to be such that:

$$\dot{d}f = \partial_{\dot{\alpha}} f\, dx^\alpha = 0,$$

it is necessary and sufficient that f should be independent of the variables y^α.

Case of a 1-form. – Let an $\dot{h}0$ semi-basic 1-form be:

$$\omega = a_\alpha(x, y)\, dx^\alpha\,.$$

([5]) H. Cartan [**2**], Chap. III, pp. 18.

In order for ω to be $\dot{d}$–closed on an open subset U of W, it is necessary and sufficient that:

$$\partial_{\dot{\alpha}} a_\beta - \partial_{\dot{\beta}} a_\alpha = 0 \qquad \text{on } U . \tag{14.1}$$

Under those conditions:

$$\omega = \dot{d}\,(a_\alpha\, y^\alpha) .$$

Indeed:

$$\dot{d}\,(a_\alpha\, y^\alpha) = (a_\alpha + \partial_{\dot{\alpha}} a_\beta\, y^\beta)\, dx^\alpha .$$

Now, the relations (14.1) imply that:

$$\partial_{\dot{\alpha}} a_\beta\, y^\beta = 0 \qquad \text{on } U . \tag{14.2}$$

Conversely, when one differentiates the identities (14.2) with respect to y^β, that will imply:

$$\partial_{\dot{\alpha}} a_\beta + \partial_{\dot{\alpha}\dot{\beta}} a_\gamma\, y^\gamma = 0 .$$

Upon switching α and β and subtracting, one will get the identities (14.1), so one will have the:

Theorem:

In order for the $\dot{h}0$ form $\omega = a_\alpha(x, y)\, dx^\alpha$ to be $\dot{d}$–closed, it is necessary and sufficient that:

$$y^\beta \partial_{\dot{\alpha}} a_\beta = 0 .$$

Under those conditions, there will exist a unique $\dot{h}1$ function F such that:

$$\omega = \dot{d}F .$$

The function F is necessarily equal to $a_\alpha y^\alpha$.

Case of a 2-form. – Let:

$$\Omega = \tfrac{1}{2} a_{\alpha\beta}\, dx^\alpha \wedge dx^\beta$$

be an $\dot{h}0$ semi-basic 2-form. In order for Ω to be $\dot{d}$–closed on an open subset U of W, it is necessary and sufficient that one should have:

$$\partial_{\dot{\alpha}} a_{\beta\gamma} + \partial_{\dot{\beta}} a_{\gamma\alpha} + \partial_{\dot{\gamma}} a_{\alpha\beta} = 0 . \tag{14.3}$$

Under those conditions:

$$\Omega = \dot{d}\omega, \qquad \text{with} \quad \omega = \tfrac{1}{2} a_{\alpha\beta}\, y^\alpha\, dx^\beta.$$

Indeed:

$$\dot{d}\omega = \tfrac{1}{2} a_{\alpha\beta}\, dx^\alpha \wedge dx^\beta + \tfrac{1}{4} y^\gamma (\partial_{\dot{\alpha}}\, a_{\gamma\beta} - \partial_{\dot{\beta}}\, a_{\gamma\alpha})\; dx^\alpha \wedge dx^\beta.$$

However, the relations (14.3) imply the identities:

$$(14.4) \qquad y^\gamma (\partial_{\dot{\alpha}}\, a_{\gamma\beta} - \partial_{\dot{\beta}}\, a_{\gamma\alpha}) = 0,$$

and we indeed have:

$$\dot{d}\omega = \Omega\,.$$

The identities (14.4) are equivalent to (14.3), moreover. Indeed, upon differentiating (14.4) with respect to y^γ and then cyclically permuting α, β, γ and adding, we will get the identities (14.3), and therefore the theorem:

Theorem:

In order for the $\dot{h}0$ *semi-basic 2-form:*

$$\Omega = \tfrac{1}{2} a_{\alpha\beta}\, dx^\alpha \wedge dx^\beta$$

to be $\dot{d}$ *-closed, it is necessary and sufficient that one should have the identities:*

$$(\partial_{\dot{\alpha}}\, a_{\gamma\beta} - \partial_{\dot{\beta}}\, a_{\gamma\alpha})\, y^\gamma = 0.$$

Under those conditions, there will exist an $\dot{h}1$ *semi-basic 1-form* ω *such that:*

$$\Omega = \dot{d}\omega, \qquad \text{with} \quad \omega = \tfrac{1}{2} a_{\alpha\beta}\, y^\alpha dx^\beta + \dot{d}F$$

(*in which F is an arbitrary* $\dot{h}2$ *scalar function on* $\mathcal{V}$).

Remark. – On a well-defined neighborhood in a domain of local coordinates on $\mathcal{V}$ (x^α, y^α), it is sometimes convenient to set:

$$\dot{d}\omega = dx^\alpha \wedge \partial_{\dot{\alpha}}\omega$$

for an arbitrary $\dot{h}k$ p-form ω. The local operator thus-defined possesses the same properties as the operator $\dot{d}$ in the case of semi-basic forms. In addition, we have the formula:

$$d\dot{d}\omega = -\,\dot{d}d\omega\,.$$

A p-form ω is called $d\dot{d}$ *–closed* if:

$$d\dot{d}\omega = 0.$$

If ω is a 1-form that is defined on W and $d\dot{d}$ –closed then ω can be locally put into the form:

$$\omega = \dot{d}f + dg,$$

in which f (which is $\dot{h}1$) and g (which is $\dot{h}0$) are two scalar functions.

15. Prolonging a one-parameter group on V_{n+1} **to** $\mathcal{V}$. – Let C be a curve in V_{n+1} that has a parametric representation of the form:

$$x^\alpha = f^\alpha(u)$$

on an open subset U of V_{n+1} .

C corresponds to a curve $\pi^{-1}C$ in $\mathcal{V}$ that is defined in $\pi^{-1}U$ by:

$$x^\alpha = f^\alpha(u) \qquad \text{and} \qquad y^\alpha = \frac{df^\alpha}{du}.$$

If we change the parameter and set $u = \varphi(v)$ then the curve C will be represented by:

$$x^\alpha = f^\alpha[\varphi(v)] = F^\alpha(v),$$

and the curve $\pi^{-1}C$ will be represented by:

$$x^\alpha = F^\alpha(u), \quad y^\alpha = \frac{dF^\alpha}{dv} = \frac{df^\alpha}{du}\varphi'(v).$$

The coordinates y^α are all multiplied by $\varphi'(v)$. The curve $\pi^{-1}C$ will then depend upon the parameterization of the curve (C). By contrast, the curve $\pi^{-1}C$ in W is perfectly determined since the set x^α, λy^α defines a well-defined point of W for any λ . A curve in W that is deduced from a curve on V_{n+1} by means of p^{-1} will be referred to as a *basic curve* in W from now on.

We denote:

$$\bar{C} = p^{-1}C.$$

Let X be a vector field that is defined on U. It will generate a local one-parameter group of local transformations when one integrates the differential system S :

$$dx^\alpha = X^\alpha(x)\,du. \tag{15.1}$$

One and only one trajectory C of the group G passes through any point x_0 of U, which will be denoted by $x = \exp(u\,\mathrm{X})\,x_0$ and will be defined on local coordinates by:

$$x^\alpha = f^\alpha(x_0^\beta, u)\,. \tag{15.2}$$

The linear tangent map $(\exp u\,\mathrm{X})'$ makes the vector y_0 of T_{x_0} correspond to the vector y of T_x such that:

$$y^\alpha = \frac{\partial f^\alpha}{\partial x^\beta}(x_0, u)\,y_0^\beta\,. \tag{15.3}$$

If y denotes, in particular, the tangent vector at x to the curve C that is defined by $y^\alpha = \dot{x}^\alpha = \dfrac{dy^\alpha}{du} = X^\alpha(x)$ then we will have:

$$\frac{dy^\alpha}{du} = \partial_\beta X^\alpha y^\beta. \tag{15.4}$$

That system will admit not only the vector field that is tangent to C as a solution along C, but also any field y that is invariant under $\theta(\mathrm{X})$. Let us verify that.

The equalities:

$$[\theta(\mathrm{X})\,y]^\alpha = X^\beta\,\partial_\beta y^\alpha - y^\beta\,\partial_\beta X^\alpha = 0$$

are equivalent to:

$$\frac{dy^\alpha}{du} = \partial_\beta y^\alpha \frac{dy^\beta}{du} = \partial_\beta y^\alpha X^\beta = y^\beta\,\partial_\beta X^\alpha$$

along C.

We indeed recover equations (15.4).

Let $\bar{\mathrm{X}}$ denote the vector field on $\mathcal{V}$ that is the prolongation of the field X on V_{n+1} and is defined by the components:

$$X^\alpha \qquad \text{and} \qquad X^{\dot\alpha} = y^\beta\,\partial_\beta X^\alpha\,.$$

The field $\bar{\mathrm{X}}$ will generate a local one-parameter group $\bar{G}$ of local transformations of W when one integrates the system S that is defined by:

$$\frac{dx^\alpha}{du} = X^\alpha(u) \qquad \text{and} \qquad \frac{dy^\alpha}{du} = X^{\dot\alpha}\,.$$

That group $\bar{G}$, which is called the *prolongation* of G to $\mathcal{V}$, will admit trajectories that are curves in $\mathcal{V}$ that project onto V_{n+1} along the trajectories of G. The projections of those curves onto W are not basic, in general.

In order for the curve that passes through z_0 (x_0, y_0) to be basic, it is necessary and sufficient that $y_0 = \lambda\,\mathrm{X}(x_0)$.

If ω denotes a p-form that is defined on W (or more generally, a restricted p-form on $\mathcal{V}$) then its Lie derivative with respect to X will be, by definition:

$$\theta(\mathrm{X})\ \omega = \theta(\bar{\mathrm{X}})\,\omega,$$

and we will have the formula:

$$\theta(\mathrm{X})\ \omega = i(\bar{\mathrm{X}})\,d\omega + di(\bar{\mathrm{X}})\,\omega\,.$$

In particular, if $\mathcal{L}$ (x, y) is an $\dot{h}1$ scalar function then:

$$\theta(\mathrm{X})\ \mathcal{L} = X^{\alpha}\,\partial_{\alpha}\,\mathcal{L} + y^{\beta}\partial_{\beta}X^{\alpha}\partial_{\dot{\alpha}}\mathcal{L}\ .$$

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CHAPTER III

VARIATIONAL CALCULUS

16. Extremals of an integral. – Let $L\,(x, y)$ be an $\dot{h}1$ function of class C^2 on a domain U of $\mathcal{V}$. The 1-form $\omega = \dot{d}L$ will then be defined on W.

Let $f_0 = p^{-1}x_0$ and $f_1 = p^{-1}x_1$ be two fibers that belong to the domain $p^{-1}\pi U$ of W, and let x_0 and x_1 be two arbitrary points of πU.

Consider the integral $I\,(C) = \int_C \dot{d}L$, where C is an arbitrary differentiable path that joins a point of f_0 to a point of f_1. We call a curve C such that:

$$\theta(\mathrm{Z})\, I = 0$$

for any vector field Z that is tangent to W and verifies the relation $p\mathrm{Z} = 0$ at the points x_0 and x_1 an *extremal* of the integral $I\,(C)$.

From (1.2)

$$\theta(\mathrm{Z})\, I = \int_C \theta(\mathrm{Z})\, \dot{d}L = \int_C \theta(\mathrm{Z})\, \omega = \int_C i\,(\mathrm{Z})\, d\omega + \int_C di\,(\mathrm{Z})\, \omega .$$

The last integral is zero, because the form ω is semi-basic, and $p\mathrm{Z} = 0$ at x_0 and x_1.

If X^{α}, Y^{α} are the components of the vector Z in the natural frame at the point (x, y) of W then we will have:

$$i\,(\mathrm{Z})\; d\omega = X^{\alpha} \frac{\partial(d\omega)}{\partial(dx^{\alpha})} + Y^{\alpha} \frac{\partial(d\omega)}{\partial(dy^{\alpha})} .$$

In order for the integral $\int_C i\,(\mathrm{Z})\, d\omega$ to be zero for any field Z, it is necessary and sufficient that the path C should be such that one will have:

$$\frac{\partial(d\omega)}{\partial(dx^{\alpha})} = 0 \qquad \text{and} \qquad \frac{\partial(d\omega)}{\partial(dy^{\alpha})} = 0$$

along C. The preceding differential system is nothing more than the extremal system of the form ω, which is a completely integrable system.

Let us make that system more explicit. We have:

$$d\omega = \tfrac{1}{2}(\partial_{\alpha\dot{\beta}} L - \partial_{\beta\dot{\alpha}} L)\, dx^{\alpha} \wedge dx^{\beta} + \partial_{\dot{\alpha}\dot{\beta}} L\, dy^{\beta} \wedge dx^{\alpha} .$$

The extremal system is composed of the following $2\,(n + 1)$ equations:

(16.1) $$\frac{\partial(d\omega)}{\partial(dx^{\alpha})} = (\partial_{\alpha\dot{\beta}} L - \partial_{\beta\dot{\alpha}} L)\, dx^{\beta} - \partial_{\dot{\alpha}\dot{\beta}} L\, dy^{\beta} = 0,$$

(16.2) $$\frac{\partial(d\omega)}{\partial(dy^\alpha)} = \partial_{\dot\alpha\dot\beta} L\, dx^\beta = 0 \,.$$

Since L is $\dot{h}1$, we will have:

$$\partial_{\dot\alpha\dot\beta} L\, dy^\beta = 0$$

identically. The matrix $\left\| \partial_{\dot\alpha\dot\beta} L \right\|$ is singular then. By definition, the variational problem under study will be called *regular* if the matrix $\left\| \partial_{\dot\alpha\dot\beta} L \right\|$ has rank n. Under those conditions, the system (16.2) will show that the dx are proportional to the y with the same index. The extremals of the form $\dot{d}L$ will then be the *basic curves* of W.

Upon denoting an arbitrary parameter by u, we can set:

$$y^\alpha = \frac{dx^\alpha}{du} = \dot{x}^\alpha \,.$$

Equations (16.1) will then be written in the form:

(16.3) $$\partial_{\dot\alpha\dot\beta} L\, \ddot{x}^\beta - (\partial_{\alpha\dot\beta} L - \partial_{\beta\dot\alpha} L)\, \dot{x}^\beta = 0 \,,$$

and since:

$$\partial_{\alpha\dot\beta} L\, \dot{x}^\beta = \partial_\alpha L \,,$$

those equations, which define the projections of the extremals onto V_{n+1}, can be further written:

$$\frac{d}{du}\partial_{\dot\alpha} L - \partial_\alpha L = 0.$$

Those equations are the Euler equations that relate to the integral:

$$\int L\,(x, \dot{x})\, du \,.$$

We can then state the theorem:

Theorem:

The extremals of the integral $\int \dot{d}L$, *where* $L\,(x, y)$ *is an* $\dot{h}1$ *function on* $\mathcal{V}$, *are basic paths in* W *that project onto* V_{n+1} *along the extremals of the integral:*

$$\int L\,(x, \dot{x})\, du \,.$$

17. Extremal system of a 1-form ω defined on W. – Let ω be a 1-form that is defined on W. Its inverse image on $\mathcal{V}$, which we shall once more denote by ω, is written:

$$\omega = a_\alpha (x, y)\, dx^\alpha + b_\alpha (x, y)\, dy^\alpha$$

locally. Since that form is assumed to be defined on W, it will result that the a_α are $\dot{h}\,0$, while the b_α are $\dot{h}\,(-1)$, and that $b_\alpha\, y^\alpha = 0$.

Since the form ω is defined on W, the same thing will be true for its exterior differential:

$$d\omega = \tfrac{1}{2}\, a_{\alpha\beta}\, dx^\alpha \wedge dx^\beta + \tfrac{1}{2}\, b_{\alpha\beta}\, dy^\alpha \wedge dy^\beta + c_{\alpha\beta}\, dx^\alpha \wedge dy^\beta,$$

with

$$a_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\alpha a_\beta , \quad b_{\alpha\beta} = \partial_{\dot\alpha} b_\beta - \partial_{\dot\beta} b_\alpha , \quad c_{\alpha\beta} = \partial_\alpha b_\beta - \partial_{\dot\beta} a_\alpha .$$

It will then result that:

$$b_{\alpha\beta}\, y^\beta = 0 \qquad \text{and} \qquad c_{\alpha\beta}\, y^\beta = 0.$$

Now let us form the extremal system of ω. It is defined by:

$$\frac{\partial(d\omega)}{\partial(dx^\alpha)} = a_{\alpha\beta}\, dx^\beta + c_{\alpha\beta}\, dy^\beta = 0, \tag{17.1}$$

$$\frac{\partial(d\omega)}{\partial(dy^\alpha)} = -\, a_{\alpha\beta}\, dx^\beta + b_{\alpha\beta}\, dy^\beta = 0 . \tag{17.2}$$

This system, which is defined on W, is completely integrable. One and only one integral curve will pass through any point (x_0 , y_0) of W, which is defined on a neighborhood of (x_0 , y_0) by the equations:

$$x^\alpha = f^{\,\alpha} (x_0 , y_0 , u), \qquad y^\alpha = g^{\,\alpha} (x_0 , y_0 , u) .$$

These curves are not basic, in general. In order for that to be the case, it is necessary and sufficient that the following 2 $(n + 1)$ differential equations in $n + 1$ unknown functions (x^β):

$$a_{\alpha\beta}\, \dot{x}^\beta + c_{\alpha\beta}\, \ddot{x}^\beta = 0, \tag{17.3}$$

$$-\, c_{\alpha\beta}\, \dot{x}^\beta + b_{\alpha\beta}\, \ddot{x}^\beta = 0 \tag{17.4}$$

should be compatible.

That will be true if equations (17.4) are verified independently; i.e., if:

$$b_{\alpha\beta} = 0 \qquad \text{and if} \qquad c_{\beta\alpha}\, \dot{x}^\beta = 0.$$

In that case, there will locally exist an $\dot{h}0$ function F (x, y) such that $b_\alpha = \partial_{\dot{\alpha}} F$. The identities $c_{\beta\alpha}\,\dot{x}^\beta = 0$ will then be written in the form:

$$(\partial_{\dot{\beta}\alpha} F - \partial_{\dot{\beta}}\, a_\alpha)\,\dot{x}^\alpha = 0$$

or

$$\partial_{\dot{\beta}}\, A_\alpha\, \dot{x}^\alpha = 0 \quad \text{with} \quad A_\alpha = -\,\partial_\alpha F + a_\alpha\,.$$

The form $A_\alpha\, dx^\alpha$ is then $\dot{d}$ -closed. There will then exist an $\dot{h}1$ function L (x, y) such that $A_\alpha = \partial_{\dot{\alpha}} L$, or:

$$a_\alpha = \partial_{\dot{\alpha}} L + \partial_\alpha F\,.$$

The form ω is then written:

$$\omega = \partial_{\dot{\alpha}} L\, dx^\alpha + \partial_\alpha F\, dx^\alpha + \partial_{\dot{\alpha}} F\, dy^\alpha\,,$$

or

$$\omega = \dot{d}L + dF\,.$$

The preceding considerations are valid, in particular, for a semi-basic 1-form. Indeed, in that case, the $b_{\alpha\beta}$ will be identically zero.

Equations (17.4) can then be written:

$$\partial_{\dot{\alpha}}\, a_\beta\, \dot{x}^\beta = 0\,.$$

Since they must be verified identically, they constitute a necessary and sufficient condition (14.2) for the form:

$$\omega = a_\alpha\, dx^\alpha$$

to be $\dot{d}$ -closed. We can then state the theorem:

Theorem:

In order for a semi-basic form on W to admit basic extremals, it is necessary and sufficient that it should be $\dot{d}$ *-closed.*

It results from (I, § **8**) that the form $\omega = \dot{d}L$ will define a relative integral invariant and that its differential $d\omega$ will define an absolute integral invariant for the extremals of the form ω.

18. Euler vectors and forms. – Let C be a differentiable path in W that belongs to the same local coordinate domain U. Let:

$$x^\alpha = x^\alpha(u) \qquad \text{and} \qquad y^\alpha = y^\alpha(u)$$

be a parametric representation of *C*.

The $n + 1$ functions of u :

$$P_\alpha(L) = \partial_{\dot\alpha\dot\beta}L\frac{dy^\beta}{du} + (\partial_{\dot\alpha\beta}L - \partial_{\alpha\dot\beta}L)y^\beta \tag{18.1}$$

are the covariant components of an $\dot h 1$ restricted vector $P(L)$ that is defined at any point of *C*. Those various vectors are, by definition, the *Euler vectors* of the path *C* relative to *L*.

The extremals of $\int L(x, y)\,du$ are the paths in *W* along which the Euler vector $P(L)$ is zero.

A 2-form $\pi(L)$ is attached to the $\dot h 1$ function $L(x, y)$, which is defined on *W* by:

$$\pi(L) = d(\dot dL) = d(\partial_{\dot\alpha}L\,dx^\alpha) = \partial_{\dot\alpha\dot\beta}L\,dy^\alpha\wedge dx^\beta + \tfrac{1}{2}(\partial_{\dot\alpha\beta}L - \partial_{\alpha\dot\beta}L)\,dx^\alpha\wedge dx^\beta . \tag{18.2}$$

That 2-form $\pi(L)$ is, by definition, the *Euler form* that corresponds to the function L .

Let us point out some properties of the Euler forms and vectors that are attached to the same path of *W*.

1. The correspondence between L and $P(L)$ is linear: If L_1 and L_2 are two functions that are defined on $\mathcal{V}$ to be $\dot h 1$ and have class C_2 , and if k_1 and k_2 are two arbitrary constants then:

$$P(k_1 L_1 + k_2 L_2) = k_1 P(L_1) + k_2 P(L_2) .$$

2. We have:

$$P_\alpha(L)\,y^\alpha = 0$$

identically. Indeed, $\partial_{\dot\alpha\dot\beta}L\,y^\alpha = 0$, since $\partial_{\dot\beta}L$ is $\dot h 0$, and:

$$(\partial_{\dot\alpha\beta}L - \partial_{\alpha\dot\beta}L)y^\alpha y^\beta = 0,$$

from the antisymmetric of the expression in parentheses or by the use of the Euler identity.

3. Suppose that L has the form:

$$L = A_\alpha(x)\,y^\alpha .$$

Under those conditions:

$$P_\alpha(L) = (\partial_\beta A_\alpha - \partial_\alpha A_\beta)\,y^\beta \tag{18.3}$$

and

$$\pi(L) = d(A_\alpha\,dx^\alpha) .$$

Suppose, more particularly, that the vector A whose covariant components are A_α is the gradient of a function $f(x)$; i.e.:

$$A_\alpha = \partial_\alpha f\,.$$

We then have:

$$P_\alpha(L) \equiv 0 \qquad \text{and} \qquad \pi(L) \equiv 0\,.$$

Conversely, if $\pi(L) \equiv 0$ then we will have:

$$\partial_{\dot\alpha\dot\beta} L = 0,$$

so

$$L = A_\alpha(x)\, y^\alpha \qquad\qquad \text{and} \qquad\qquad \partial_\alpha A_\beta - \partial_\beta A_\alpha = 0\,,$$

which implies that there locally exists a function $f(x)$ such that:

$$A_\alpha = \partial_{\dot\alpha} f\,.$$

We can then state the:

Theorem:

In order for two $\dot h 1$ *functions* $L(x, y)$ *and* $\bar L(x, y)$ *to admit identical Euler vectors, it is necessary and sufficient that one should have:*

$$\bar L - L = \partial_\alpha f\, y^\alpha$$

locally, where f *is an arbitrary function of the variables* x^α*.*

4. If $f(x)$ is an arbitrary differentiable function of the variables x^α and $L(x, y)$ is a twice-differentiable $\dot h 1$ function then we will have:

$$P_\alpha(f\,L) = f\,P_\alpha(L) + (\partial_\beta f\,\partial_{\dot\alpha} L - \partial_\alpha f\,\partial_{\dot\beta} L)\,y^\beta \tag{18.4}$$

and

$$\pi(f\,L) = f\,\pi(L) + df \wedge \dot d L\,. \tag{18.5}$$

In particular, if $L = \partial_\alpha g\, y^\alpha$, $g(x)$ is an arbitrary differentiable function of the variables x^α:

$$P_\alpha(f\,L) = (\partial_\beta f\,\partial_{\dot\alpha} L - \partial_\alpha f\,\partial_{\dot\beta} L)\,y^\beta$$

and

$$\pi(f\,L) = df \wedge dg\,.$$

19. Helmholtz conditions. – Let Ω be a 2-form that is defined on W that has the following expression in a domain U with local coordinates x^α, y^α:

$$\Omega = a_{\alpha\beta}\, dy^\alpha \wedge dx^\beta + \tfrac{1}{2} b_{\alpha\beta}\, dx^\alpha \wedge dx^\beta.$$

If Ω is the Euler form of a $\dot{h}1$ function L (x, y) then the $a_{\alpha\beta}$ will be symmetric, and $\Omega = 0$.

$$\omega = A_\alpha (x, y)\, dx^\alpha + B_\alpha (x, y)\, dy^\alpha,$$

such that:

$$\Omega = d\omega .$$

Since Ω does not contain any terms in $dy^\alpha \wedge dy^\beta$, the B_α will be independent of the variables y^α, and since ω is defined on W, the B_α will be $\dot{h}(-1)$. They will then be identically zero, and ω will be semi-basic. We will then have:

$$a_{\alpha\beta} = \partial_{\dot{\beta}} A_\alpha \qquad \text{and} \qquad b_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha .$$

Since the $a_{\alpha\beta}$ are symmetric, we deduce that:

$$\partial_{\dot{\beta}} A_\alpha - \partial_{\dot{\alpha}} A_\beta = 0,$$

which are relations that show that the form ω is $\dot{d}$-closed.

There will then locally exist a $\dot{h}1$ function L (x, y) such that:

$$A_\alpha = \partial_{\dot{\alpha}} L ,$$

and we will indeed have:

$$a_{\alpha\beta} = \partial_{\dot{\alpha}\dot{\beta}} L \qquad \text{and} \qquad b_{\alpha\beta} = \partial_{\alpha\dot{\beta}} L - \partial_{\beta\dot{\alpha}} L ,$$

and thus:

Theorem:

In order for the 2-form that is defined on W:

$$\Omega = a_{\alpha\beta}\, dy^\alpha \wedge dx^\beta + \tfrac{1}{2} b_{\alpha\beta}\, dx^\alpha \wedge dx^\beta$$

to be an Euler form, it is necessary and sufficient that Ω should be closed, since the coefficients $a_{\alpha\beta}$ are symmetric.

Upon making that condition more explicit, we will find the conditions that are called the *Helmholtz conditions*. Indeed, the preceding problem is equivalent to the one that was solved by Helmholtz and Mayer that relates to the existence of a function L (x^k, x'^k, t) such that system of n second-order differential equations:

$$G_i\, (x^k, x'^k, x''^k, t) = 0$$

can be put into the form:

$$\frac{d}{dt}\frac{\partial L}{\partial x'^i} - \frac{\partial L}{\partial x^i} = 0\ .$$

20. Extremals and geodesics. – Consider the Finsler space F that is defined on the manifold V_{n+1} by the $\dot{h}1$ function $L\,(x^\alpha, y^\alpha)$. With E. Cartan, set:

$$l^\alpha = \frac{1}{L}\,y^\alpha \qquad \text{and} \qquad l_\alpha = \partial_{\dot\alpha} L\ .$$

The geodesics of F are defined by the equations:

$$\frac{\nabla l^\alpha}{du} = \frac{dl^\alpha}{du} + \frac{1}{L}\Gamma^\alpha_{\beta\gamma}\, l^\beta\, l^\gamma = 0, \tag{20.1}$$

or by:

$$\frac{\nabla l_\alpha}{du} = \frac{dl_\alpha}{du} - \frac{1}{L}\Gamma^\beta_{\alpha\gamma}\, l_\beta\, l^\gamma = 0\ . \tag{20.2}$$

A classical calculation ([6]) shows that:

$$\frac{1}{L}\Gamma^\beta_{\alpha\gamma}\, l_\beta\, l^\gamma = \partial_\alpha L\ ,$$

and that:

$$\Gamma^\beta_{\alpha\gamma}\, y^\beta\, y^\gamma = 2\, G^\alpha,$$

with:

$$G^\alpha = g^{\alpha\beta}\, G_\beta \quad \text{and} \qquad 2\, G_\beta = \partial_{\dot\beta\lambda} F\, y^\lambda - \partial_\beta F \qquad \left(F = \tfrac{1}{2}L^2\right).$$

Equations (20.2) then show that the geodesics of F are identical to the extremals of the integral $\int L\, du$.

Those equations are equivalent to equations (20.1), which are written in the following form:

$$\frac{dl^\alpha}{du} + \frac{2}{L}G^\alpha = 0$$

or upon reverting to the variables y^α :

$$\frac{dy^\alpha}{du} + 2G^\alpha = y^\alpha\,\frac{dL}{L\,du} \qquad \text{with} \quad y^\alpha = \frac{dx^\alpha}{du}\ . \tag{20.3}$$

In what follows, we shall sometimes write that system in the form:

([6]) E. Cartan [**1**] ; J. Favard [**2**].

(20.4) $$\frac{\ddot{x}^1 + 2G^1}{\dot{x}^1} = \frac{\ddot{x}^2 + 2G^2}{\dot{x}^2} = \ldots = \frac{\ddot{x}^{n+1} + 2G^{n+1}}{\dot{x}^{n+1}}.$$

21. Geodesics map between two Finsler spaces. – Consider two Finsler spaces F and $\bar{F}$ that are defined on the same base manifold V_{n+1} when one is given the two $\dot{h}1$ functions $L(x, y)$ and $\bar{L}(x, y)$. The geodesics of F are defined by equations (20.4), while those of $\bar{F}$ are defined by equations that are obtained by starting with the preceding ones and replacing the G^α that relate to L with the $\bar{G}^\alpha$ that relate to $\bar{L}$. In order for those two system of equations to be equivalent, it is necessary and sufficient that there should exist an $\dot{h}1$ function $p(x, \dot{x})$ such that:

(21.1) $$\bar{G}^\alpha - G^\alpha \equiv p\,\dot{x}^\alpha .$$

The geodesics of $\bar{F}$ are defined by the Euler equations relative to the function $\bar{L}$:

$$\partial_{\dot{\alpha}\dot{\beta}}\bar{L}\,\ddot{x}^\beta + (\partial_{\dot{\alpha}\beta}L - \partial_{\alpha\dot{\beta}}\bar{L})\,\dot{x}^\beta = 0,$$

or, since $\partial_{\dot{\alpha}\dot{\beta}}\bar{L}\,\dot{x}^\beta \equiv 0$, by the equations:

(21.2) $$\partial_{\dot{\alpha}\dot{\beta}}\bar{L}\left(\ddot{x}^\beta - \frac{1}{L}\frac{dL}{du}\dot{x}^\beta\right) + (\partial_{\dot{\alpha}\beta}\bar{L} - \partial_{\alpha\dot{\beta}}\bar{L})\,\dot{x}^\beta = 0 .$$

In order for these geodesics to be the same as those of F, from (20.3), it is necessary and sufficient that the first expression in parentheses in (21.2) should be equal to G^β. One then has this result: The functions $\bar{L}$ that define the same geodesics as L are the $\dot{h}1$ functions that are solutions to the system of partial differential equations:

(21.3) $$\partial_{\dot{\alpha}\dot{\beta}}\bar{L}\,G^\beta + (\partial_{\dot{\alpha}\beta}\bar{L} - \partial_{\alpha\dot{\beta}}\bar{L})\,\dot{x}^\beta = 0.$$

As an application of the preceding considerations, we shall solve the following problem:

Problem. – If one is given an $\dot{h}1$ function $L(x, \dot{x})$ then does there exists a function $\bar{L}(x, y)$ of the form:

$$\bar{L}(x, y) = f(x)\;L(x, \dot{x})$$

such that L and $\bar{L}$ define the same geodesics?

We shall use an overbar to highlight everything that relates to the Finsler space $\bar{F}$ that is defined by $f(x)\,L$, where $f(x)$ is supposed to be known.

Set $\bar{F} = \frac{1}{2} f^2 L^2$; hence:

(21.4) $$2\bar{G}_\beta = \partial_{\dot{\beta}\dot{\alpha}}\bar{F}\,\dot{x}^\alpha - \partial_\beta \bar{F} = 2\,f^2\,G_\beta + 2f\,\partial_\alpha f\;L\,\partial_{\dot{\beta}}L\,\dot{x}^\alpha - f\,\partial_\beta f\,L^2.$$

In order for those relations to have the form (21.1) or the equivalent form:

(21.5) $$2\bar{G}_\beta = 2\,f^2\,G_\beta + 2\,p\,f^2\,L\,\partial_{\dot{\beta}}L\,,$$

it is necessary and sufficient that there should exist an $\dot{h}0$ function $g\,(x,\ \dot{x})$ such that:

(21.6) $$\partial_{\dot{\beta}}L = g\,(x,\ \dot{x})\,\partial_\beta f\,.$$

L will then have the form:

(21.7) $$L = g\,(x,\ \dot{x})\,\partial_\beta f\ \dot{x}^\beta\,.$$

However, in order for one to deduce (21.6) from that, it is necessary and sufficient that the function g should be independent of the $\dot{x}$. We can then state the:

Theorem:

In order for the functions L $(x,\ \dot{x})$ *and f* (x) *L* $(x,\ \dot{x})$ *to define the same geodesics, it is necessary and sufficient that the function L* $(x,\ \dot{x})$ *should have the form:*

$$L = g\,(x)\,\partial_\alpha f\,(x)\,\dot{x}^\alpha = g\,(x)\,\frac{df\,(x)}{du}\,.$$

This theorem is an immediate consequence of formula (18.4), moreover.

We remark that if L has the form (21.7) then not only $f\,L$, but any function of the form:

$$F\,(f)\,L \qquad \text{or} \qquad G\,(g)\,L\,,$$

will define the same extremals as L.

Furthermore, that will result directly from the definition of the extremal systems of the forms:

$$\omega = g\,\partial_\alpha f\,dx^\alpha = g\,df \quad \text{and} \quad \bar{\omega} = F\,(f)\,g\,df\,.$$

Indeed, $d\omega = dg \wedge df$ and $d\bar{\omega} = F\,(f)\,dg \wedge df$ admit the same associated system.

22. Extremals in Hamiltonian coordinates. – Consider the Finsler space that is defined on the manifold V_{n+1} by an $\dot{h}1$ function $L\,(x^\alpha, y^\alpha)$.

A vector y of the tangent space to V_{n+1} at x can be defined by either its contravariant components y^α with respect to the natural frame at the point x or by its covariant components $y_\alpha = g_{\alpha\beta}\, y^\beta$.

The point y whose origin is x corresponds to the point Z in the space $\mathcal{V}$ of vector tangent to V_{n+1} . We call the 2 (n + 1) numbers x^α and y_α the *Hamiltonian coordinates* of the point Z.

Since the Finslerian metric is assumed to be regular, the relations $y_\alpha = g_{\alpha\beta}\, y^\beta$ will permit one to calculate the y^β as functions of the x^α and the y_α , such that the expressions obtained will be homogeneous of first degree with respect to the y_α ($\underline{h}1$, in what follows) (†).

Upon replacing the y^α in L (x^α, y^α) with their expressions that are obtained in that way, L will become a function H of the x^α, y_α such that:

(22.1) $$H(x^\alpha, y_\alpha) = L(x^\alpha, g^{\alpha\beta}\, y_\beta) \qquad \text{and} \qquad H(x^\alpha, g_{\alpha\beta}\, y^\beta) = L(x^\alpha, y^\alpha)\,.$$

By definition, H (x^α, y_α) is the *Hamiltonian function* that corresponds to L . Since L is $\dot{h}1$, H will be $\underline{h}1$; i.e.:

(22.2) $$\partial^{\dot\alpha} H\, y_\alpha = H, \quad \text{with} \quad \partial^{\dot\alpha} H = \frac{\partial H}{\partial y_\alpha}\,.$$

The unit vector l has the same direction as y, so its contravariant components will be $l^\alpha = y^\alpha / L$, while its covariant components will be:

$$l_\alpha = \frac{y_\alpha}{H}\,.$$

The relation (22.1) will then show that:

$$l^\alpha = \partial^{\dot\alpha} H = \frac{y_\alpha}{H}\,.$$

We will then have:

$$y^\alpha = H\,\partial^{\dot\alpha} H = \partial^{\dot\alpha}\left(\tfrac{1}{2}H^2\right) = \partial^{\dot\alpha} K\,, \qquad \text{with} \qquad K = \tfrac{1}{2}H^2\,,$$

which are dual to:

$$y_\alpha = L\,\partial_{\dot\alpha} L = \partial_{\dot\alpha}\left(\tfrac{1}{2}L^2\right) = \partial_{\dot\alpha} F\,.$$

We shall now show that $\partial_\alpha H = -\,\partial_\alpha L$.

We differentiate the two sides of the identity:

(†) Translator: In the original, the notation was a dot beneath the h, but I do not have that option in my equation editor, so I substituted an underline.

$$H(x^\alpha, y_\alpha) = L(x^\alpha, g^{\alpha\beta} y_\beta) = L(x^\alpha, y^\alpha)$$

and get:

(22.3) $$dH = \partial_\alpha L \, dx^\alpha + \partial_{\dot\alpha} L \, dy^\alpha .$$

Now:

$$\partial_{\dot\alpha} L \, dy^\alpha = \frac{1}{H} y_\alpha \, dy^\alpha = 2\, dH - \frac{1}{H} y^\alpha \, dy_\alpha ,$$

from

$$y^\alpha y_\alpha = H^2 .$$

The expression (22.3) for dH will then become:

(22.4) $$dH = -\partial_\alpha L \, dx^\alpha + \frac{1}{H} y^\alpha \, dy_\alpha .$$

We then indeed deduce that $\partial_\alpha H = -\partial_\alpha L$, and we recover the fact that:

$$\partial^{\dot\alpha} H = \frac{1}{H} y^\alpha .$$

It will then be easy to write the fundamental formulas that relate to a Finsler space with the aid of the variables x^α, y_α, and the function H.

For example, we have the relations:

$$H^2 = L^2 = g_{\alpha\beta} y^\alpha y^\beta = g^{\alpha\beta} y_\alpha y_\beta .$$

The relations $y^\alpha \partial_{\dot\lambda} g_{\alpha\beta} = 0$ imply that $g_{\alpha\beta} = \partial_{\dot\alpha\dot\beta} F$.

One then shows that when one starts with $g^{\alpha\beta} g_{\alpha\beta} = \delta_\gamma^\beta$, one will get $y_\alpha \partial^\lambda g^{\alpha\beta} = 0$, which are relations that will then imply that:

$$g^{\alpha\beta} = \partial^{\dot\alpha\dot\beta} K .$$

Since the connection is Euclidian ($\nabla g^{\alpha\beta} = 0$) and special, in the Lichnerowicz sense, we will then deduce that the torsion tensor will have components:

$$T^{\alpha\beta\gamma} = -\tfrac{1}{2} \partial^{\dot\gamma} g^{\alpha\beta} = -\tfrac{1}{2} \partial^{\dot\alpha\dot\beta\dot\gamma} K .$$

However, we are more especially interested in the differential system for the geodesics of the Finsler space; they are defined by the Euler equations:

(22.5) $$\frac{d}{du} \partial_{\dot\alpha} L - \partial_\alpha L = 0, \qquad \text{with} \quad y^\alpha = \frac{dx^\alpha}{du} .$$

Now:

$$\partial_{\dot\alpha} L = \frac{1}{H} y_\alpha \quad \text{and} \quad \partial_\alpha L = - \partial_\alpha H .$$

Equations (22.5) can then be written in the form:

$$\frac{d}{du}\frac{y_\alpha}{H} + \partial_\alpha H = 0 \qquad \text{or} \qquad \frac{dy_\alpha}{du} - \frac{y_\alpha}{H}\frac{dH}{du} + H\, \partial_\alpha H = 0,$$

so the system of equations defines the geodesics of the Finsler space in Hamiltonian coordinates:

(22.6) $$\frac{dy_\alpha}{du} = - \partial_\alpha H + \lambda\, y_\alpha \quad \text{and} \quad \frac{dy_\alpha}{du} = \partial^{\dot\alpha} K ,$$

where λ is an $\underline{h}\,1$ function of the x^α, y_α.

In reality, the preceding equations define basic paths in *W* that project onto V_{n+1} along the geodesics in the Finsler space considered.

Instead of taking x^α, y_α to be the Hamiltonian coordinates on *W*, with the y_α being covariant components of an arbitrary vector in the tangent space T_x , take x^α and l_α, with $l_\alpha = y_\alpha / H$.

Those 2 $(n + 1)$ variables are no longer independent, because the l_α are the covariant components of a unit vector, so we will have:

$$H (x^\alpha, l_\alpha) = 1.$$

Take the parameter *u* to be the arc-length *s* of the geodesic, which is defined by:

$$ds = L (x^\alpha, dx^\alpha) = \partial_{\dot\alpha} L (x^\alpha, y^\alpha)\, dx^\alpha = l_\alpha\, dx^\alpha.$$

Under those conditions, equations (22.6) can be put into the form:

(22.7) $$\frac{dx^\alpha}{ds} = \partial^{\dot\alpha} H \quad \text{and} \quad \frac{dl_\alpha}{ds} = - \partial_\alpha H.$$

The preceding equations can be obtained directly. Indeed, they constitute the extremal system of the form:

$$\omega = \partial_{\dot\alpha} L\, dx^\alpha = l_\alpha\, dx^\alpha .$$

This extremal system is the associated system to the 2-form:

$$d\omega = dl_\alpha \wedge dx^\alpha.$$

Upon writing that:

$$i\,(Z)\, d\omega \equiv 0$$

for any vector Z that is tangent to W – i.e., such that:

$$i\,(\mathrm{Z})\,dH = 0,$$

we find that:

$$\frac{\partial\, d\omega}{\partial\, dx^\alpha} = \lambda \frac{\partial\, dH}{\partial\, dx^\alpha} \qquad \text{and} \qquad \frac{\partial\, d\omega}{\partial\, dl_\alpha} = \lambda \frac{\partial\, dH}{\partial\, dl_\alpha},$$

or

$$-\,dl_\alpha = \lambda\, \partial_\alpha H \qquad \text{and} \qquad dx^\alpha = \lambda\, \partial^{\dot\alpha} H\,.$$

Since $dx^\alpha / ds = l^\alpha = \partial^{\dot\alpha} H$, the proportionality factor is equal to ds, we will have equations (22.7)

23. Basic paths in W in Hamiltonian coordinates. – Consider a differentiable path in W. Such a path is defined parametrically by the equations:

$$x^\alpha = x^\alpha(u) \qquad \text{and} \qquad l_\alpha = l_\alpha(u)\,.$$

In order for that path to be basic, it is necessary and sufficient that there should exist a function $f(u)$ such that:

$$\frac{dx^\alpha}{ds} = f(u)\, l^\alpha, \qquad \text{where} \qquad l^\alpha = g^{\alpha\beta}\, l_\beta = \partial^{\dot\alpha} H\,,$$

upon supposing that the x^α and l_α that enter into $\partial^{\dot\alpha} H$ are expressed as functions of u.

A path is therefore basic if and only if one has:

$$(23.1) \qquad \frac{dx^1}{\partial^{\dot 1} H} = \frac{dx^2}{\partial^{\dot 2} H} = \ldots = \frac{dx^{n+1}}{\partial^{\dot n \dot + \dot 1} H}$$

along that path.

As an application of the preceding, consider the semi-basic form that is defined on W:

$$\omega = a_\alpha(x^\beta, l_\beta)\, dx^\alpha,$$

and look for the conditions under which the extremals of ω will be basic curves in W. We have:

$$d\omega = \tfrac{1}{2}(\partial_\alpha a_\beta - \partial_\beta a_\alpha)\, dx^\alpha \wedge dx^\beta + \partial^{\dot\beta} a_\alpha\, dl_\beta \wedge dx^\alpha\,.$$

Hence, one has the extremal system:

$$(23.2) \qquad (\partial_\alpha a_\beta - \partial_\beta a_\alpha)\frac{dx^\beta}{du} - \partial^{\dot\beta} a_\alpha \frac{dl_\beta}{du} = \lambda\, \partial_\alpha H\,,$$

(23.3) $$\partial^{\dot\alpha} a_\beta \frac{dx^\beta}{du} = \lambda\,\partial^{\dot\alpha} H\,.$$

Equations (23.3) must imply that:

$$\frac{dx^\beta}{du} = \mu\,\partial^{\dot\beta} H\,.$$

It will then be necessary that the coefficients $\partial^{\dot\alpha} a_\beta$ must have the form:

(23.4) $$\partial^{\dot\alpha} a_\beta = f(x, l)\ \delta^\alpha_\beta\,.$$

Upon supposing that $\alpha \neq \beta$, one will deduce that:

$$\frac{\partial^2 a_\beta}{\partial l_\alpha\, \partial l_\beta} = \frac{\partial f}{\partial l_\alpha} = 0.$$

The function f will then be independent of the l, and the a_α will necessarily have the form:

$$a_\alpha = f(x)\ l_\alpha + g_\alpha(x)\,.$$

The converse is immediate; hence:

Theorem:

In order for a semi-basic form that is defined on W:

$$\omega = a_\alpha (x^\beta, l_\beta)\ dx^\alpha$$

to admit basic extremals, it is necessary and sufficient that the a_α should have the form:

$$a_\alpha = f(x)\ l_\alpha + g_\alpha(x)\,,$$

in which $f(x)$ is a function of only the variables x^α, and the $g_\alpha(x)$ are the covariant components of a vector that is defined on V_{n+1}.

The result obtained indeed agrees with the one is § **17**. Indeed, when one passes to the variables x^α, y^α, ω will be put into the form:

$$\omega = [f(x)\,\partial_{\dot\alpha} L + g_\alpha(x)]\ dx^\alpha = \dot d\,(f\,L + g_\alpha\, y^\alpha)\,.$$

The preceding calculations also show that when one is given the 2-form Ω, which is defined on Ω by:

$$\Omega = \tfrac{1}{2} S_{\alpha\beta}\, dx^\alpha \wedge dx^\beta + a^\beta_\alpha\, dl_\beta \wedge dx^\alpha,$$

in order for the solutions to the associated system of Ω to be basic curves in W, it is necessary and sufficient that the coefficients a_α^β should have the form:

$$a_\alpha^\beta = f(x)\ \delta_\alpha^\beta\ ;$$

i.e., that one must have:

$$\Omega = \tfrac{1}{2} S_{\alpha\beta}\, dx^\alpha \wedge dx^\beta + f(x)\ dl_\beta \wedge dx^\alpha.$$

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CHAPTER IV

VARIATIONAL CALCULUS AND GENERALIZED FINSLER SPACE

24. ***S*-extremals of an integral.** – Recall the notations of § **16**. Let f_0 and f_1 be two fibers of W that belong to the same domain U of local coordinates on W. Let $x_0 = p\, f_0$ and $x_1 = p\, f_1$ be the corresponding points of V_{n+1} .

Let E be the set of differentiable paths in U that join a point of f_0 to a point of f_1 . Define one of those paths C by a representation of the form:

(24.1) $$x^\alpha = x^\alpha(u), \qquad y^\alpha = y^\alpha(u),$$

with

$$x_0 = x(u_0) \qquad \text{and} \qquad x_1 = x(u_1).$$

A path $\bar{C}$ in E that is close to C is defined by:

(24.2) $$x^\alpha = x^\alpha(u) + \delta x^\alpha, \qquad y^\alpha = y^\alpha(u) + \delta y^\alpha,$$

in which δx^α and δy^α are differentiable functions of u that have the form:

$$\delta x^\alpha = \varepsilon\, X^\alpha(u), \qquad \delta y^\alpha = \varepsilon\, Y^\alpha(u),$$

in which ε is an infinitely small number, and X^α, Y^α are the components of a tangent vector to W at the point $z(u)$ whose coordinates are $x^\alpha(u)$, $y^\alpha(u)$.

Suppose that T_α^β is a restricted $\dot{h}1$ tensor that is defined on $\mathcal{V}$. Paths $\bar{C}_T$ that are close to C and defined by arbitrary δx^α and:

(24.3) $$\delta y^\alpha = \frac{d}{du}\delta x^\alpha + T_\beta^\alpha \delta x^\beta$$

are said to be *T-close* to C .

Suppose that an $\dot{h}1$ function L of class C^2 is given on U. In local coordinates, it is expressed by $L(\delta x^\alpha, \delta y^\alpha)$. Set:

$$I(C) = \int_C L\, du .$$

Upon passing from C to $\bar{C}$, $I(C)$ will experience a variation ΔI whose principal part is:

(24.4) $$\delta I = \int_C (\partial_\alpha L\, \delta x^\alpha + \delta_{\dot\alpha} L\, \delta y^\alpha)\, du .$$

For a path $\bar{C}$ that is T-close to C, we will have:

$$\delta I = \int_C (\partial_\alpha L\,\delta x^\alpha + \partial_{\dot\alpha} L\, T^\alpha_\beta \,\delta x^\alpha)\,du + \int_C \partial_{\dot\alpha} L \frac{d\,\delta x^\alpha}{du}\,du\,.$$

We now integrate the last integral by parts. Since the δx^α are zero at the extremities of *C*, we will get:

$$\delta I = \int_C \left(\partial_\alpha L + \partial_{\dot\beta} L\, T^\beta_\alpha - \frac{d}{du}\partial_{\dot\alpha} L\right)\delta x^\alpha du\,.$$

From the fundamental lemma of the calculus of variations, in order to have $\delta I = 0$ for any δx^α, it is necessary and sufficient that one should have:

$$\partial_\alpha L + \partial_{\dot\beta} L\, T^\beta_\alpha - \frac{d}{du}\partial_{\dot\alpha} L = 0$$

or

(24.5) $$P_\alpha(L) \equiv \frac{d}{du}\partial_{\dot\alpha} L - \partial_\alpha L = \partial_{\dot\beta} L\, T^\beta_\alpha$$

along *C*.

We refer to the projection onto V_{n+1} of the basic paths that are solution to (24.5) as the *generalized extremals* of the integral $\int_{x_0}^{x_1} L(x, \dot x)\,du$; i.e., the solutions to the differential system:

(24.6) $$\begin{cases} \dfrac{d}{du}\partial_{\dot\alpha} L - \partial_\alpha L = \partial_{\dot\beta} L\, T^\beta_\alpha\,, \\ y^\alpha = \dfrac{dx^\alpha}{du}. \end{cases}$$

When the tensor T^β_α is zero at any point of $\mathcal{V}$, the *T*-extremals of *I* are the ordinary extremals of the integral:

$$\int_{x_0}^{x_1} L(x, \dot x)\,du\,.$$

Now, suppose that the manifold V_{n+1} is endowed with the Finslerian metric that is defined by $ds = L\,(x^\alpha, dx^\alpha)$.

Under those conditions, we can transform the right-hand side of (24.5).

Indeed:

$$\partial_{\dot\beta} L\, T^\beta_\alpha = g_{\beta\gamma}\, l^\gamma\, T^\beta_\alpha = T_{\alpha\gamma}\, l^\gamma \qquad (T_{\alpha\gamma} = g_{\beta\gamma}\, T^\beta_\alpha)\,.$$

Now, $l^\alpha P_\alpha(L) \equiv 0$, it will then result that:

$$T_{\alpha\gamma}\, l^\alpha\, l^\gamma = 0\,.$$

That condition is satisfied if the tensor $T_{\alpha\gamma}$ is antisymmetric, which we shall suppose in what follows.

In analytical mechanics, we are led to introduce the 2-form:

$$\omega = \tfrac{1}{2} S_{\alpha\beta}\, dx^{\alpha} \wedge dx^{\beta} = \tfrac{1}{2} \dot{d}\ \ (-X_{\alpha}\, dx^{\alpha}),$$

in which the X_α are the covariant components of the generalized force vector, and the $S_{\alpha\beta}$ are the $\dot{h}0$ components of a tensor that is called the *force tensor* and is defined on W. We will then be led to set:

$$T_{\alpha\beta} = L\ S_{\alpha\beta}\,.$$

We shall call the solutions to the differential system:

$$\begin{cases} \dfrac{d}{du}\partial_{\dot{\alpha}} L - \partial_{\alpha} L = S_{\alpha\beta}\, y^{\beta}, \\ y^{\alpha} = \dfrac{dx^{\alpha}}{du} \end{cases}$$

the *S-extremals* of the integral $\int_{x_0}^{x_1} L(x, \dot{x})\, du$.

We remark that this differential system is the associated system to the 2-form:

$$\begin{aligned} \Omega &= d(\partial_{\dot{\alpha}} L\, dx^{\alpha}) + \tfrac{1}{2} S_{\alpha\beta}\, dx^{\alpha} \wedge dx^{\beta} \\ &= d(\dot{d}L) + \dot{d}\left(-\tfrac{1}{2} X_{\alpha}\, dx^{\alpha}\right) \\ &= \dot{d}\left(-dL - \tfrac{1}{2} X_{\alpha}\, dx^{\alpha}\right). \end{aligned}$$

25. ***S*-Finslerian spaces**. – We propose to define a linear connection in the directions on V_{n+1} whose coefficients are determined when one is given the $\dot{h}1$ function L and the $\dot{h}0$ tensor $S_{\alpha\beta}$, and the geodesics on V_{n+1} relative to that connection are the S-extremals of the integral:

$$\int_{x_0}^{x_1} L(x, \dot{x})\, du$$

that was defined before. We shall adopt the style of presentation of A. Lichnerowicz ([7]).

Let E (V_{n+1}) be the principal fiber bundle of frames on V_{n+1}, and let p^{-1} E (V_{n+1}) be its inverse image over W. A linear connection on the direction on V_{n+1} is an infinitesimal connection on p^{-1} E (V_{n+1}).

Such a connection is defined when one is given a suitable 1-form ω of adjoint type with values in the Lie algebra of GL (n + 1, $\mathbb{R}$).

([7]) A. Lichnerowicz [**6**].

Let U be a local coordinate domain in V_{n+1}, while $p^{-1}U$ is the corresponding domain in $\mathcal{V}$. Let Z be a point of $\mathcal{V}$ such that $\pi\,\mathrm{Z} = x$. A local coordinate system for Z is the set of the coordinates x^α of x and the components y^α of a vector in T_x.

Now, take the coframe on T_x^* to be the 2 (n + 1) forms dx^α, dy^α. When referred to that coframe, the connection ω will be defined by its components ω^α_β, which have the form:

(25.1) $$\omega^\alpha_\beta = b^\alpha_{\beta\gamma}\, dx^\gamma + c^\alpha_{\beta\gamma}\, dy^\gamma .$$

Since the connection is defined on W, the $b^\alpha_{\beta\gamma}$ will be $\dot{h}0$, and the $c^\alpha_{\beta\gamma}$ will be $\dot{h}(-1)$, and we will have the identities:

$$c^\alpha_{\beta\gamma}\, y^\gamma = 0.$$

For an arbitrary restricted vector **X** that is defined on $\mathcal{V}$, we set:

$$\nabla X^\alpha = d\, X^\alpha + \omega^\alpha_\beta\, X^\beta .$$

In particular, consider the vector field that makes the point z of $\mathcal{V}$ correspond to the vector **z** in $T_{p\,z}$ whose components are y^α. Set:

(25.2) $$\theta^\alpha = \nabla y^\alpha = d\, y^\alpha + \omega^\alpha_\beta\, y^\beta .$$

Since the linear connection ω is assumed to be regular, the 2 (n + 1) forms dx^α and θ^α define a coframe on T_x^*. Relative to that coframe, set:

(25.3) $$\omega^\alpha_\beta = \Gamma^\alpha_{\beta\gamma}\, dx^\gamma + C^\alpha_{\beta\gamma}\, \theta^\gamma .$$

The forms ω^α_β are defined on W, so one will verify that the Γ are $\dot{h}0$, the C are $\dot{h}(-1)$, and that $C^\alpha_{\beta\gamma}\, y^\gamma = 0$.

The *Pfaffian derivatives* of a function f (x, y) relative to the coframe (dx^α, θ^α) are expressed in a simple fashion with the aid of the partial derivatives of f relative to the x and y. Indeed, upon denoting the Pfaffian derivatives by $\delta_\alpha f$ and $\delta_{\dot\alpha} f$, we will have:

$$df = \delta_\alpha f\, dx^\alpha + \delta_{\dot\alpha} f\, \theta^\alpha = \partial_\alpha f\, dx^\alpha + \partial_{\dot\alpha} f\, dy^\alpha ,$$

so, by identification:

(25.4) $$\delta_\alpha = \partial_\alpha - y^\gamma\, \Gamma^\beta_{\lambda\alpha}\, \partial_{\dot\beta} ,$$

(25.5) $$\delta_{\dot\alpha} = \partial_{\dot\alpha} - y^\gamma C^\beta_{\lambda\alpha}\partial_{\dot\beta}.$$

Now let us specify the torsion form of that connection. The *torsion 2-form* Σ is defined by:

$$\Sigma^\alpha = \omega^\alpha_\beta \wedge dx^\beta = \tfrac{1}{2} S^{\ \alpha}_{\beta\gamma}\, dx^\beta \wedge dx^\gamma - T^{\ \alpha}_{\beta\gamma}\, dx^\beta \wedge \theta^\gamma.$$

Upon replacing ω^α_β with its expression that we infer from (25.3), we will get:

(25.6) $$S^{\ \alpha}_{\beta\gamma} = -\ (\Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\beta})$$

and

(25.7) $$T^{\ \alpha}_{\beta\gamma} = C^{\ \alpha}_{\beta\gamma}.$$

Now, set $2F = L^2$, and:

$$g_{\alpha\beta} = \partial_{\dot\alpha\dot\beta} F.$$

Since the function F is $\dot h 2$, we will have:

$$2F = L^2 = g_{\alpha\beta}\, y^\alpha y^\beta,$$

which shows that the $g_{\alpha\beta}$ are the covariant components of an $\dot h 0$ symmetric tensor; i.e., they are defined on W.

In order for the linear connection on the directions that is defined by ω to be naturally associated with a Euclidian connection on directions of the metric manifold that is defined on V_{n+1} by the tensor $g_{\alpha\beta}$, it is necessary and sufficient that one must have ([8]):

$$\nabla g_{\alpha\beta} = 0$$

for that connection, or more explicitly:

(25.8) $$dg_{\alpha\beta} - \omega^\lambda_\alpha\, g_{\lambda\beta} - \omega^\lambda_\beta\, g_{\lambda\alpha} = 0.$$

Set:

$$\omega_{\alpha\beta} = g_{\alpha\lambda}\,\omega^\lambda_\beta, \qquad \Gamma_{\alpha\beta\gamma} = g_{\alpha\lambda}\,\omega^\lambda_\beta, \qquad T_{\alpha\beta\gamma} = g_{\alpha\lambda}\, T^\lambda_{\beta\gamma}.$$

The relations (25.8) will then be equivalent to the following ones:

(25.9) $$\Gamma_{\alpha\beta\gamma} + \Gamma_{\beta\alpha\gamma} = \delta_\gamma\, g_{\alpha\beta},$$

(25.10) $$T_{\alpha\beta\gamma} + T_{\beta\alpha\gamma} = \delta_{\dot\gamma}\, g_{\alpha\beta}.$$

([8]) A. Lichnerowicz [**6**].

In order for the connection ω to be determined completely when one is given L and the tensor S, we shall make some supplementary hypotheses that relate to the torsion tensors and are analogous to the hypotheses that define the classes of *special connections*, in the sense of A. Lichnerowicz:

(25.11) 1. $T_{\alpha\beta\gamma} = T_{\beta\alpha\gamma}$,

(25.12) 2. $S^{\alpha}_{\beta\gamma} = -\, l^{\alpha} S_{\beta\gamma}$, with $l^{\alpha} = \dfrac{y^{\alpha}}{L}$.

If the tensor $S_{\alpha\beta}$ is zero over the entire manifold W then the hypotheses that were made will define one and only one connection: namely, the *Finslerian connection* on the manifold. That is the fundamental theorem of Finslerian geometry ([8]).

If the tensor $S_{\alpha\beta} \neq 0$ then we shall show that the preceding hypotheses further determine one and only one connection. The relations (25.10) and (25.11) show that one has:

$$T_{\alpha\beta\gamma} = \tfrac{1}{2}\delta_{\dot{\gamma}}\, g_{\alpha\beta}\,.$$

One infers from the expression for $g_{\alpha\beta}$ that:

$$y^{\beta}\partial_{\dot{\gamma}}\, g_{\alpha\beta} = 0.$$

Now, from (25.5):

$$\delta_{\dot{\gamma}}\, g_{\alpha\beta} = \partial_{\dot{\gamma}}\, g_{\alpha\beta} - y^{\lambda}T^{\rho}_{\lambda\gamma}\,\partial_{\dot{\rho}}\, g_{\alpha\beta}\,.$$

It will then result that:

$$y^{\beta}\delta_{\dot{\gamma}}\, g_{\alpha\beta} = 0.$$

That is:

$$y^{\beta}\, T_{\alpha\beta\gamma} = 0.$$

From (25.5), $\delta_{\dot{\gamma}} = \partial_{\dot{\gamma}}$, and therefore:

$$T_{\alpha\beta\gamma} = \tfrac{1}{2}\partial_{\dot{\gamma}}\, g_{\alpha\beta} = \tfrac{1}{2}\partial_{\dot{\alpha}\dot{\beta}\dot{\gamma}} F\,. \tag{25.13}$$

$T_{\alpha\beta\gamma}$ is then a tensor that is symmetric with respect to its three indices, and which satisfies:

$$T_{\alpha\beta\gamma}\, y^{\alpha} = T_{\alpha\beta\gamma}\, y^{\beta} = T_{\alpha\beta\gamma}\, y^{\gamma} = 0. \tag{25.14}$$

Calculating the $\Gamma_{\alpha\beta\gamma}$. – It remains for us to determine the coefficients $\Gamma_{\alpha\beta\gamma}$. From (25.9), (25.6), and (25.12), we have the relations:

$$\begin{cases} \Gamma_{\alpha\beta\gamma} + \Gamma_{\beta\alpha\gamma} = \delta_{\gamma}\, g_{\alpha\beta}\,, \\ \Gamma_{\alpha\beta\gamma} - \Gamma_{\beta\alpha\gamma} = l_{\alpha}\, S_{\beta\gamma}. \end{cases}$$

Write the four relations that are deduced from the preceding ones by cyclically permuting α, β, γ. With some obvious combinations, we will get:

(25.15) $$\Gamma_{\alpha\beta\gamma} = \tfrac{1}{2}(\delta_\gamma\, g_{\alpha\beta} + \delta_\beta\, g_{\alpha\gamma} - \delta_\alpha\, g_{\beta\gamma}) + \tfrac{1}{2}(l_\alpha\, S_{\beta\gamma} + l_\beta\, S_{\gamma\alpha} - l_\alpha\, S_{\alpha\beta}) \;.$$

Set:

$$\Gamma_{\alpha\beta\gamma} = \tfrac{1}{2}(l_\alpha\, S_{\beta\gamma} + l_\beta\, S_{\gamma\alpha} - l_\alpha\, S_{\alpha\beta}) \,,$$

to simplify, which is a tensor that is antisymmetric in the α and β.

Now, pass from the Pfaffian derivatives to the ordinary derivatives, so:

(25.16) $$\Gamma_{\alpha\beta\gamma} = [\beta\gamma, \alpha] - y^\lambda(\Gamma^\mu_{\lambda\gamma}\, T_{\mu\alpha\beta} + \Gamma^\mu_{\lambda\gamma}\, T_{\mu\gamma\alpha} - \Gamma^\mu_{\lambda\alpha}\, T_{\mu\beta\gamma}) + \Sigma_{\alpha\beta\gamma},$$

in which the $[\beta\gamma, \alpha]$ are the Christoffel symbols of the first kind.

Now, form $y^\beta\,\Gamma_{\alpha\beta\gamma}$ and $y^\beta\, y^\gamma\,\Gamma_{\alpha\beta\gamma}$, while taking (25.14) into account:

(25.17) $$y^\beta\,\Gamma_{\alpha\beta\gamma} = y^\beta\,[\beta\gamma, \alpha] - y^\beta\, y^\lambda\,\Gamma^\mu_{\lambda\beta}\, T_{\mu\gamma\alpha} + y^\beta\,\Sigma_{\alpha\beta\gamma}$$

and

(25.18) $$y^\beta\, y^\gamma\,\Gamma_{\alpha\beta\gamma} = y^\beta\, y^\gamma\,[\beta\gamma, \alpha] + y^\beta\, y^\gamma\,\Sigma_{\alpha\beta\gamma} \;.$$

With E. Cartan, we set:

$$y^\beta\, y^\gamma\,[\beta\gamma, \alpha] = 2\, G_\alpha = \partial_{\lambda\dot\alpha} F\, y^\lambda - \partial_\alpha F \;.$$

On the other hand:

$$y^\beta\, y^\gamma\,\Sigma_{\alpha\beta\gamma} = L\, S_{\alpha\beta}\, y^\gamma = -\, L\, X_\alpha \,,$$

upon setting:

$$X_\alpha = S_{\alpha\beta}\, y^\gamma \,.$$

The relations (25.18) then take the form:

$$y^\beta\, y^\gamma\,\Gamma_{\alpha\beta\gamma} = 2\, G_\alpha - L\, X_\alpha$$

or

$$y^\beta\, y^\gamma\,\Gamma^\alpha_{\beta\gamma} = 2\, G^\alpha - L\, X^\alpha \,.$$

Upon substituting this in (25.17), we will get:

$$y^\beta\,\Gamma_{\alpha\beta\gamma} = y^\beta\,[\beta\gamma, \alpha] - (2\, G^\beta - L\, X^\beta)\; T_{\alpha\beta\gamma} + y^\beta\,\Sigma_{\alpha\beta\gamma}$$

or

(25.19) $$y^\beta\,\Gamma_{\alpha\beta\gamma} = g_{\alpha\beta}\,\partial_{\dot\gamma} G^\beta + L\, X^\beta\; T_{\alpha\beta\gamma} + y^\beta\,\Sigma_{\alpha\beta\gamma},$$

upon remarking that:

$$\partial_{\dot{\gamma}} G_\alpha - 2\, G^\beta T_{\alpha\beta\gamma} = g_{\alpha\beta}\, \partial_{\dot{\gamma}} G^\beta .$$

We then transform (25.16) with the aid of (25.19) and get:

$$\text{(25.20)} \qquad \begin{aligned} \Gamma_{\alpha\beta\gamma} = [\beta\gamma,\, \alpha] & - (T_{\alpha\beta\lambda}\, \partial_{\dot{\gamma}} G^\lambda + T_{\gamma\alpha\lambda}\, \partial_{\dot{\beta}} G^\lambda - T_{\beta\gamma\lambda}\, \partial_{\dot{\alpha}} G^\lambda) \\ & - L\, X^\lambda\, (T_{\mu\lambda\gamma} T^{\mu}_{\alpha\beta} + T_{\mu\lambda\beta} T^{\mu}_{\gamma\alpha} - T_{\mu\lambda\alpha} T^{\mu}_{\beta\gamma}) \\ & - y^\lambda\, (\Sigma_{\mu\lambda\gamma} T^{\mu}_{\alpha\beta} + \Sigma_{\mu\lambda\beta} T^{\mu}_{\gamma\alpha} - \Sigma_{\mu\lambda\alpha} T^{\mu}_{\beta\gamma}) + \Sigma_{\alpha\beta\gamma}. \end{aligned}$$

The first row in the expression for $\Gamma_{\alpha\beta\gamma}$ represents the $\Gamma_{\alpha\beta\gamma}$ coefficients of the Finslerian connection, which are coefficients that we shall denote by $\dot{\Gamma}_{\alpha\beta\gamma}$. Upon specifying the various parentheses, we will find that:

$$\begin{aligned} \Gamma_{\alpha\beta\gamma} = \dot{\Gamma}_{\alpha\beta\gamma} & - L\, X^\lambda\, (T_{\mu\lambda\gamma} T^{\mu}_{\alpha\beta} + T_{\mu\lambda\beta} T^{\mu}_{\gamma\alpha} - T_{\mu\lambda\alpha} T^{\mu}_{\beta\gamma}) \\ & - \tfrac{1}{2} X_\lambda\, (l_\gamma T^{\lambda}_{\alpha\beta} + l_\beta T^{\lambda}_{\gamma\alpha} - l_\alpha T^{\lambda}_{\beta\gamma}) \\ & - \tfrac{1}{2} L\, (S_{\gamma\lambda} T^{\lambda}_{\alpha\beta} + S_{\beta\lambda} T^{\lambda}_{\gamma\alpha} - S_{\alpha\lambda} T^{\lambda}_{\beta\gamma}) \\ & + \tfrac{1}{2} (l_\alpha S_{\beta\gamma} + l_\beta S_{\gamma\alpha} - l_\gamma S_{\alpha\beta}) . \end{aligned}$$

We remark that the part of $\Gamma_{\alpha\beta\gamma}$ that is antisymmetric in β and γ is:

$$\Gamma_{\alpha[\beta\gamma]} = \tfrac{1}{2}\, l_\alpha\, S_{\beta\gamma}.$$

Calculating the $b_{\alpha\beta\gamma}$. – The coefficients $b_{\alpha\beta\gamma}$ are expressed simply as functions of the $\Gamma_{\alpha\beta\gamma}$.

Indeed, upon identifying the coefficients of the dx^γ in (25.1) and (25.3), we will get:

$$b_{\alpha\beta\gamma} = \Gamma_{\alpha\beta\gamma} + T_{\alpha\beta\mu}\, (\partial_{\dot{\gamma}} G^\mu + L\, X^\lambda T^{\mu}_{\lambda\gamma} + y^\lambda \Sigma^{\mu}_{\lambda\gamma}) .$$

Upon replacing $\Gamma_{\alpha\beta\gamma}$ with its expression that one infers from (25.20), we will find:

$$\begin{aligned} b_{\alpha\beta\gamma} = [\beta\gamma,\, \alpha] - (T_{\gamma\alpha\lambda}\, \partial_{\dot{\beta}} G^\lambda - T_{\beta\gamma\lambda}\, \partial_{\dot{\alpha}} G^\lambda) & - L\, X^\lambda\, (T_{\gamma\alpha\lambda} T^{\mu}_{\lambda\alpha} - T_{\mu\lambda\alpha} T^{\mu}_{\beta\gamma}) \\ & - y^\lambda\, (\Sigma_{\mu\lambda\beta} T^{\mu}_{\gamma\alpha} - \Sigma_{\mu\lambda\alpha} T^{\mu}_{\beta\gamma}) + \Sigma_{\alpha\beta\gamma} . \end{aligned}$$

The first two terms in the expression for $b_{\alpha\beta\gamma}$ represent the analogous coefficient for the Finslerian connection that we shall denote by $\dot{b}_{\alpha\beta\gamma}$. The coefficient of $L\, X^\lambda$ represents the curvature tensor $Q_{\lambda\gamma,\,\alpha\beta}$ ([9]), which is the same for the two connections, moreover.

We can then write:

([9]) Élie Cartan denoted this tensor by S .

$$b_{\alpha\beta\gamma} = \dot{b}_{\alpha\beta\gamma} - L\, X^{\lambda}\, Q_{\lambda\gamma,\,\alpha\beta} - y^{\lambda}(\Sigma_{\mu\lambda\beta}T^{\ \mu}_{\gamma\alpha} - \Sigma_{\mu\lambda\alpha}T^{\ \mu}_{\beta\gamma}) + \Sigma_{\alpha\beta\gamma}\,.$$

We can make this more explicit, moreover:

$$b_{\alpha\beta\gamma} = \dot{b}_{\alpha\beta\gamma} - L\, X_{\lambda}\, Q^{\ \lambda}_{\gamma,\alpha\beta} - \tfrac{1}{2}L\,(S_{\alpha\lambda}T^{\ \lambda}_{\beta\gamma} - S_{\beta\lambda}T^{\ \lambda}_{\gamma\alpha}) - \tfrac{1}{2}X_{\lambda}(l_{\alpha}T^{\ \lambda}_{\beta\gamma} - l_{\beta}T^{\ \lambda}_{\gamma\alpha}) + \Sigma_{\alpha\beta\gamma}\,.$$

Geodesics. – Let $l_{\alpha} = \partial_{\dot{\alpha}}L$. Let us calculate its absolute differential:

$$\begin{aligned}\nabla l_{\alpha} &= dl_{\alpha} - \omega_{\alpha}^{\beta}\, l_{\beta} \\ &= dl_{\alpha} - \Gamma^{\beta}_{\alpha\gamma}\, l_{\beta}\, dx^{\gamma} \\ &= dl_{\alpha} - \Gamma_{\beta\alpha\gamma}\, l^{\beta} dx^{\gamma}.\end{aligned}$$

Let us make the 2-form $\nabla l_{\alpha} \wedge dx^{\alpha}$ more explicit. We get:

$$\nabla l_{\alpha} \wedge dx^{\alpha} = dl_{\alpha} \wedge dx^{\alpha} + \tfrac{1}{2}l^{\beta}(\Gamma_{\beta\alpha\gamma} - \Gamma_{\beta\gamma\alpha})\, dx^{\alpha} \wedge dx^{\gamma}.$$

Now:

$$\Gamma_{\beta\alpha\gamma} - \Gamma_{\beta\gamma\alpha} = l_{\beta}\, S_{\alpha\gamma}\ .$$

Since $l_{\alpha} l^{\alpha} = 1$, we find:

$$\nabla l_{\alpha} \wedge dx^{\alpha} = dl_{\alpha} \wedge dx^{\alpha} + \tfrac{1}{2}S_{\alpha\beta}\, dx^{\alpha} \wedge dx^{\gamma}.$$

The 2-form thus-obtained is, as we will see, the fundamental 2-form of the dynamical system that is defined by L and $S_{\alpha\beta}$.

Along a geodesic, one will have $\nabla l_{\alpha} / du = 0$, where u is an arbitrary parameter. Upon setting $y^{\alpha} = d\,x^{\alpha} / du$, one will get the differential system:

$$\frac{\nabla l_{\alpha}}{du} = \frac{dl_{\alpha}}{du} - \Gamma_{\beta\alpha\gamma}\, l^{\beta} y^{\gamma} = 0.$$

Now:

$$\Gamma_{\beta\alpha\gamma}\, l^{\beta} y^{\gamma} = \dot{\Gamma}_{\beta\alpha\gamma}\, l^{\beta} y^{\gamma} + \Sigma_{\beta\alpha\gamma}\, l^{\beta} y^{\gamma}.$$

However:

$$\Sigma_{\beta\alpha\gamma}\, l^{\beta} y^{\gamma} = X_{\alpha}\,.$$

The differential system of the geodesics will then be the following one:

$$\frac{\nabla l_{\alpha}}{du} = \frac{\dot{\nabla} l_{\alpha}}{du} - X_{\alpha}$$

or

$$\frac{d}{du}\partial_{\dot{\alpha}}L - \partial_{\alpha}L = X_{\alpha}\,.$$

We indeed recover the *S*-extremals of the integral:

$$\int_{x_0}^{x_1} L(x, \dot{x})\, du .$$

The spaces that were just constructed differ from the Finsler spaces only by Élie Cartan's *E* convention: The $\Gamma_{\alpha\beta\gamma}$, which are denoted by in ([10]) and ([11]), are no longer symmetric in the β and γ, but are such that:

$$\Gamma_{\alpha\beta\gamma} - \Gamma_{\alpha\gamma\beta} = l_\alpha\, S_{\beta\gamma} .$$

It will then result that the map of an infinitesimal point-like cycle that is obtained by attaching a unit vector to each of its points by parallel displacing it from the origin of the cycle is no longer closed. The vector that joins the origin to the extremity will have the components:

$$\Sigma^\alpha = \left(\tfrac{1}{2} S_{\beta\gamma} dx^\beta \wedge dx^\gamma\right) l^\alpha .$$

We remark that we have been able to replace our hypothesis (25.12) with some others without modifying the geodesics. We point out the following two:

1. $$S_{\alpha\beta\gamma} = -\,(S_{\alpha\beta}\, l_\gamma - S_{\alpha\gamma}\, l_\beta) .$$

One then deduces that:

$$\Gamma_{\alpha\beta\gamma} = [\beta\gamma, \alpha]^\delta + S_{\alpha\beta}\, l_\gamma ,$$

where the index δ indicates that one is dealing with Pfaffian derivatives.

2. $$S_{\alpha\beta\gamma} = -\,(g_{\alpha\gamma}\, X_\beta - g_{\alpha\beta}\, X_\gamma) ,$$

as one has for Weyl spaces. One then deduces that:

$$\Gamma_{\alpha\beta\gamma} = [\beta\gamma, \alpha]^\delta + X_\alpha\, g_{\beta\gamma} - X_\beta\, g_{\alpha\gamma} .$$

However, in each of those two cases, we have found that:

$$\nabla l_\alpha \wedge dx^\alpha = d\, l_\alpha \wedge dx^\alpha + \tfrac{1}{2}(X_\alpha\, l_\beta - X_\beta\, l_\alpha)\, dx^\alpha \wedge dx^\beta .$$

That 2-form is not the fundamental 2-form for a dynamical system, in general.

Particular case: S-Riemannian spaces. – Suppose that L^2 is a quadratic form with respect to the variables *y*. Under those conditions, the $g_{\alpha\beta}$ will be independent of the *y*,

([10]) E. Cartan [**1**].
([11]) A. Lichnerowicz [**6**].

and the torsion tensor T will be zero. The coefficients of the connection will then reduce to:

$$\Gamma_{\alpha\beta\gamma} = [\beta\gamma, \alpha] + \Sigma_{\alpha\beta\gamma} .$$

Such a space will be called S-Riemannian. It can be used in general relativity, where $S_{\alpha\beta}$ are, for example, the components of the electromagnetic field tensor.

26. A. Lichnerowicz's generalized variational calculus ([12]). – As always, consider the differentiable manifold V_{n+1} , the manifold $\mathcal{V}$ of non-zero tangent vectors to V_{n+1} , and the manifold W of oriented directions tangent to V_{n+1} .

Let C be a differentiable path in V_{n+1} that is defined in a local coordinate domain U by the parametric representation:

$$x^\alpha = x^\alpha (v),$$

in which the functions $x^\alpha (v)$ have class C^2 on an interval (a, b) .

Let (u_0 , u_1) denote a sub-interval of (a, b) whose length is less than ε, where ε is a given, arbitrarily-small positive number, and let $y^\alpha (v, \varepsilon)$ be a set of $n + 1$ functions that are continuously-differentiable for any v that belongs to (u_0 , u_1) and zero for $v = u_0$ and $v = u_1$.

Set:

$$\delta x^\alpha = \varepsilon\, y^\alpha (v, \varepsilon) \qquad \text{and} \qquad \eta (\varepsilon) = \max |\, \delta x^\alpha (v) \,|, \tag{26.1}$$

for

$$v \in (u_0 , u_1) \qquad \text{and} \qquad \alpha = 1, 2, \ldots, n + 1.$$

Upon supposing that absolute values of the derivatives with respect to v of the functions $y^\alpha (v, \varepsilon)$ are bounded by a number K over (a, b), we will have:

$$\eta (\varepsilon) < K \varepsilon^2.$$

Sub-differential of a functional. – With Lichnerowicz, let $F_{u_0}^u [x^\alpha (v)]$ denote a functional that is attached to the arc (u_0 , u) of C and satisfies the following condition: For every u_0 in the interval (a, b), there exists an interval (u_0 , u) such that for any u in that interval, $F_{u_0}^u$ defines an integrable function of u.

One says the *sub-differential* δF of the functional F to mean a function of the $x^\alpha (v)$, their first derivatives $\dot{x}^\alpha (v)$, and the δx^α that is linear in the δx^α and such that:

$$\lim_{\varepsilon\to 0} \frac{\Delta F - \delta F}{\eta} = 0,$$

([12]) A. Lichnerowicz [**2**], pp. 343-350.

where ΔF denotes the increase in F that corresponds to the increases (26.1) in the x^α.

Example. – Let $f(x^\alpha, \dot{x}^\alpha)$ be an $\dot{h}1$ function of class C^2. Set:

$$F = \int_{u_0}^{u} f[x^\alpha(v), \dot{x}^\alpha(v)]\, dv\,. \tag{26.2}$$

The functional F admits the differential:

$$dF = \int_{u_0}^{u} \left[\partial_\alpha f - \frac{d}{dv}\partial_{\dot{\alpha}} f\right] dx^\alpha dv + \partial_{\dot{\alpha}} f[x^\alpha(u)]\, dx^\alpha(u)\,.$$

Lichnerowicz showed ([13]) that the sub-differential of F is:

$$\delta F = \partial_{\dot{\alpha}} f\, \delta x^\alpha\,, \tag{26.3}$$

where the δx^α are the increases (26.1) at the point u.

Sub-variation of an integral. – If $F_{u_0}^{u}$ denotes the integral (26.2) then we set:

$$J = \int_{u_0}^{u_1} H[F_{u_0}^{u}, x^\alpha(u), \dot{x}^\alpha(u)]\, du\,, \tag{26.4}$$

in which H is a continuous function of F, the x^α, and the $\dot{x}^\alpha$.

With Lichnerowicz, we shall say the *sub-variation* δJ of the integral J to mean an integral for the form:

$$\delta J = \int_{u_0}^{u_1} L[x^\alpha(u), \dot{x}^\alpha(u), \delta x^\alpha(u)]\, du\,,$$

in which L is a function that is linear in the δx^α, such that one will have:

$$\lim_{\varepsilon \to 0} \frac{\Delta J - \delta J}{\varepsilon \eta} = 0,$$

in which ΔJ denotes the increase in J that corresponds to the increases (26.1) in x^α.

The function H is supposed to have class C^3, so Lichnerowicz showed ([14]) that the sub-variation will have the following expression:

$$\delta J = \int_{u_0}^{u_1} \left[\partial_\alpha H(0) - \frac{d}{du}\partial_{\dot{\alpha}} H(0) + H'(0)\partial_{\dot{\alpha}} f - f\,\partial_{\dot{\alpha}} H'(0)\right] \delta x^\alpha\, du\,, \tag{26.5}$$

([13]) A. Lichnerowicz [**2**], pp. 346.
([14]) *Ibid.*, pp. 347-349.

in which H (0) is the function that is obtained by starting with H $(F, x^\alpha, \dot{x}^\alpha)$ and setting F = 0, and $H'(0)$ is the function of the x^α, $\dot{x}^\alpha$ that is obtained by annulling F in the partial derivative of H with respect to the argument F.

The generalized extremals of the integral J are, by definition, the curves (C) for which $\delta J = 0$ for any increases δx^α that are defined by (26.1). Those extremals are the solutions to the differential system:

$$\frac{d}{du}\partial_{\dot{\alpha}}H(0) - \partial_\alpha H(0) = H'(0)\partial_{\dot{\alpha}} f - f\partial_{\dot{\alpha}}H'(0) . \tag{26.2}$$

We remark that this differential system will remain invariant if one simultaneously changes H' into $-H'$ and f into $-f$, or H' into f and f into $-H'$. We will get the same differential system by replacing the function H $(F, x^\alpha, \dot{x}^\alpha)$ with the function:

$$H = H(0) + F\, H'(0)$$

that is obtained by replacing H (F) with its Taylor development to first order in a neighborhood of $F = 0$, while the variables x^α and $\dot{x}^\alpha$ are supposed to be fixed.

Generalization. – Suppose that k $\dot{h}1$ functions f^A (x, y) are given on $\mathcal{V}$ and set, as before:

$$F^A = \int_{u_0}^{u} f^A[x^\alpha(v), \dot{x}^\alpha(v)]\, dv , \quad \text{with} \quad A = 1, 2, \dots, k .$$

Now, let H be an $\dot{h}1$ function of the k functionals F^A, the x^α, and the $\dot{x}^\alpha$. Set:

$$J = \int_{u_0}^{u} H[F^A, x^\alpha(u), \dot{x}^\alpha(u)]\, du .$$

Some calculations that are analogous to the ones in Lichnerowicz (Lich. [**2**], pages 347 and 349) will show that the sub-variation of that integral has the following expression:

$$\delta J = \int_{u_0}^{u_1} \left[\partial_\alpha H(0) - \frac{d}{du}\partial_{\dot{\alpha}} H(0) + \partial_A H(0)\partial_{\dot{\alpha}} f^A - f^A \partial_{\dot{\alpha}A} H(0) \right] \delta x^\alpha \, du .$$

The generalized extremals of the integral J are the solutions to the differential system:

$$\frac{d}{du}\partial_{\dot{\alpha}}H(0) - \partial_\alpha H(0) = \partial_A H(0)\partial_{\dot{\alpha}} f^A - f^A\partial_{\dot{\alpha}A}H(0) . \tag{26.7}$$

27. Non-holonomic differential algebra and generalized extremals:

Non-holonomic functions. – Let U be a local coordinate domain on V_{n+1}, and let $p^{-1}U$ be the corresponding domain in W. In the neighborhood of any point x of U, consider a set of differentiable paths that connect x to any neighboring point x'.

Let $f(x, y)$ be an $\dot{h}1$ function that is defined on $\mathcal{V}$. Set:

$$F = \int_x^{x'} f(x,y)\,du\,, \qquad \text{with} \qquad y^\alpha = \frac{dx^\alpha}{du}\,,$$

where the integral is calculated along the path xx'.

The path xx' in V_{n+1} corresponds to a path zz' in W by way of p^{-1}, and one will have:

$$F = \int_z^{z'} \partial_{\dot\alpha} f(x,y)\,dx^\alpha\,. \tag{27.1}$$

Now let $\bar{L}$ be an $\dot{h}1$ function that is defined on $\mathcal{V}$ and has a value at the point z' whose coordinates are x'^α, y'^α that depends upon F, which we shall denote by $\bar{L}(F, x', y')$. Suppose that $\bar{L}$ is continuously differentiable with respect to all of its arguments, and let $L(x, y)$ denote the limit of $\bar{L}$ when z' tends to z along the arc zz'.

Set:

$$x'^\alpha = x^\alpha + \Delta x^\alpha, \qquad y'^\alpha = y^\alpha + \Delta y^\alpha,$$

and

$$\eta = \max\left(|\Delta x^\alpha|, |\Delta y^\alpha|\right), \qquad \text{for} \qquad \alpha = 1, 2, \ldots, n+1\,.$$

The difference $\Delta L = \bar{L}(F, x', y') - L(x, y)$ can be put into the form:

$$\Delta L = L'\,F + \partial_\alpha L + \partial_{\dot\alpha} L\,\Delta y^\alpha + \varepsilon\,\eta, \tag{27.2}$$

in which L' is the limit of the partial derivative of $\bar{L}$ with respect to F when z tends to z', and ε is a function of z' that tends to 0 when η tends to 0.

On the other hand, when z' is sufficiently close to z:

$$F = \partial_{\dot\alpha} L\,\Delta y^\alpha + \varepsilon'\,\eta\,, \qquad \varepsilon' \to 0 \text{ with } \eta\,.$$

Finally, we can put ΔL into the form:

$$\Delta L = (\partial_\alpha L + L'\partial_{\dot\alpha} f)\,\Delta x^\alpha + \partial_{\dot\alpha} L\,\Delta y^\alpha + \varepsilon''\,\eta\,, \tag{27.3}$$

with ε'' tending to 0 when η tends to 0.

We are led to associate the expression that we found for ΔL with the linear map on the vector space over $\mathbb{R}$ tangent to W at z that is defined by:

(27.4) $$(\partial_\alpha L + L'\partial_{\dot\alpha} f\,)\,dx^\alpha + \partial_{\dot\alpha} L\,dy^\alpha .$$

That linear form will be denoted by $d\bar{L}$ and will be, by definition, a *non-holonomic differential* of the function L (x, y). We also say that the set $L(x, y)$, $d\bar{L}$ defines a *non-holonomic differentiable function* $\bar{L}$ on $\mathcal{V}$.

In a more condensed form, we have:

(27.5) $$d\bar{L} = dL + L'\,\dot{d}f\ .$$

That expression for $d\bar{L}$ leads us to the following definitions for the partial derivatives of a non-holonomic function $\bar{L}$:

$$\partial_\alpha \bar{L} = \partial_\alpha L + L'\,\partial_{\dot\alpha} f\ ,$$

$$\partial_{\dot\alpha}\bar{L} = \partial_{\dot\alpha} L\ .$$

We remark that the exterior differential of $d\bar{L}$ is not zero. Indeed, one has:

$$d\,(d\bar{L}) = d^2\bar{L} = d\,(L'\,\dot{d}f\,),$$

but

$$d^2\bar{L} = d\,(d^2\bar{L}) = 0.$$

Non-holonomic forms. – Let ϖ be a differential form that is defined on $\mathcal{V}$ or W whose coefficients at z' close to z depend upon F. First, let a 1-form be defined by:

$$\varpi = \bar{a}_\alpha\,dx^\alpha + \bar{b}_\alpha\,dy^\alpha .$$

By definition, set:

$$d\varpi = d\bar{a}_\alpha \wedge dx^\alpha + d\bar{b}_\alpha \wedge dy^\alpha$$

at the point z . If we denote the limits of $\bar{a}_\alpha$, $\bar{b}_\alpha$, $\dfrac{\partial\bar{a}_\alpha}{\partial F}$, $\dfrac{\partial\bar{b}_\alpha}{\partial F}$, ϖ when z' tends to z by a_α , b_α , a'_α , b'_α , ω, respectively, then we will get:

$$d\varpi = da_\alpha \wedge dx^\alpha + db_\alpha \wedge dy^\alpha + \dot{d}f \wedge (a'_\alpha\,dx^\alpha + b'_\alpha\,dy^\alpha)$$

or

(27.6) $$d\varpi = d\omega + \dot{d}f \wedge \omega' .$$

By definition, we say that $d\varpi$ is the *exterior differential of the non-holonomic form* ϖ, which is equal to the form ω locally.

The same considerations will lead one to associate a non-holonomic p-form ϖ that is defined on $\mathcal{V}$ or W (i.e., a form whose coefficients at z' close to z depend upon F) with a (p+1)-form:

$$d\varpi = d\omega + \dot{d}f \wedge \omega'$$

that one calls the *exterior differential of the non-holonomic form* ϖ.

If ϖ_1 and ϖ_2 are two exterior forms of degrees p_1 and p_2, respectively, that are non-holonomic with respect to the same functional F then we will have:

$$d(\varpi_1 + \varpi_2) = d\varpi_1 + d\varpi_2 ,$$
$$d(\varpi_1 \wedge \varpi_2) = d\varpi_1 \wedge \varpi_2 + (-1)^{p_1} \varpi_1 \wedge d\varpi_2 .$$

Those formulas are immediate consequences of (27.6).

Extremal system for a non-holonomic form. – By definition, the *extremal system for a non-holonomic form* ϖ is the associated system of its exterior differential $d\varpi$. For example, consider the semi-basic 1-form:

$$\varpi = \partial_{\dot{\alpha}}\bar{L}\ (F, x, y)\, dx^{\alpha},$$

in which L is an $\dot{h}1$ function that is defined on $\mathcal{V}$. We have:

$$d\varpi = d(\partial_{\dot{\alpha}} L\, dx^{\alpha}) + \dot{d}f \wedge \partial_{\dot{\alpha}} L'\, dx^{\alpha}$$

or

$$d\varpi = \partial_{\dot{\alpha}\dot{\beta}} L\, dy^{\beta} \wedge dx^{\alpha} + \tfrac{1}{2}(\partial_{\dot{\alpha}\beta} L - \partial_{\dot{\beta}\alpha} L)\, dx^{\alpha} \wedge dx_{\beta} + \tfrac{1}{2}(\partial_{\dot{\alpha}} f\, \partial_{\dot{\beta}} L' - \partial_{\dot{\beta}} f\, \partial_{\dot{\alpha}} L')\, dx^{\alpha} \wedge dx^{\beta} . \tag{27.7}$$

The associated system to $d\varpi$ is defined by the equations:

$$\partial_{\dot{\alpha}\dot{\beta}} L\, dy^{\beta} - (\partial_{\alpha\dot{\beta}} L - \partial_{\beta\dot{\alpha}} L)\, dx^{\beta} = (\partial_{\dot{\alpha}} f\, \partial_{\dot{\beta}} L - \partial_{\dot{\beta}} f\, \partial_{\dot{\alpha}} L)\, dx^{\beta} , \tag{27.8}$$

$$\partial_{\dot{\alpha}\dot{\beta}} L\, dx^{\alpha} = 0 . \tag{27.9}$$

Equations (27.9) show that the solutions to that system are the basic paths of W whose projections onto V_{n+1} of the solutions to the differential equations:

$$\partial_{\dot{\alpha}\dot{\beta}} L\, \ddot{x}^{\beta} - (\partial_{\alpha\dot{\beta}} L - \partial_{\beta\dot{\alpha}} L)\, \dot{x}^{\beta} = (\partial_{\dot{\alpha}} f\, \partial_{\dot{\beta}} L' - \partial_{\dot{\beta}} f\, \partial_{\dot{\alpha}} L')\, \dot{x}^{\beta}$$

or

$$\frac{d}{du}\partial_{\dot{\alpha}} L - \partial_{\alpha} L = L' \partial_{\dot{\alpha}} f - f \partial_{\dot{\alpha}} L' .$$

The curves thus-defined are the generalized extremals of the integral:

$$J = \int_{u_0}^{u_1} \bar{L}\,(F, x, \dot{x})\,du\,.$$

Hence, one has the:

Theorem:

The generalized extremals of the integral $J = \int_{u_0}^{u_1} \bar{L}\,(F, x, \dot{x})\,du$ *are the projections onto* V_{n+1} *of the solutions of the extremal system of the form:*

$$\varpi = \partial_{\dot{\alpha}} \bar{L}\;(F, x, y)\;dx^{\alpha}\,.$$

Generalization. – Now suppose that one is given $k\ \dot{h}1$ functions $f^A\,(x, y)$, with $A = 1, 2, \ldots, k$, that are defined on $\mathcal{V}$.

With the notations at the beginning of this paragraph, set:

$$F^A = \int_x^{x'} f^A(x, \dot{x})\,du \;=\; \int_x^{x'} \partial_{\dot{\alpha}} f^A(x, \dot{x})\,dx^{\alpha}\,.$$

Let $\bar{L}$ be a function of the k functionals F^A and the $2n + 2\ \dot{h}1$ variables x^{α}, y^{α}.

Let L denote the limit of $\bar{L}$ when z' tends to z.

By definition, we call the 1-form:

$$d\bar{L} = dL + \dot{d}f^A \partial_A L \tag{27.10}$$

the *differential of the non-holonomic function* $\bar{L}$, in which $\partial_A L$ is the limit of the partial derivative of $\bar{L}$ with respect to F^A when z' tends to z.

Now let ϖ be an arbitrary p-form that is defined on $\mathcal{V}$ or W and whose coefficients at z' are functions of the F^A. We let $d\varpi$ denote the $(p + 1)$-form that is defined at the point z by:

$$d\varpi = d\omega + \dot{d}f^A \wedge \partial_A \omega, \tag{27.11}$$

with

$$\omega = \lim_{z' \to z} \varpi \qquad \text{and} \qquad \partial_A\,\omega = \lim_{z' \to z} \frac{\partial \varpi}{\partial F^A}\,.$$

If ϖ_1 and ϖ_2 are two exterior forms defined on $\mathcal{V}$ or W whose degrees are p_1 and p_2, respectively, and they are non-holonomic with respect to the same functionals F^A then we will have some immediate consequences of the definition (27.11), namely:

$$d\,(\varpi_1 + \varpi_2) = d\varpi_1 + d\varpi_2\,,$$

$$d\,(\varpi_1 \wedge \varpi_2) = d\varpi_1 \wedge \varpi_2 + (-1)^{p_1}\,\varpi_1 \wedge d\varpi_2\,.$$

In particular, consider the form:

$$\varpi = \partial_{\dot\alpha} \bar{L}\,(F^A, x, y)\,dx^\alpha\,.$$

Its exterior differential is:

$$d\varpi = d\omega + \dot{d}f^A \wedge \dot{d}\,\partial_{\dot\alpha} L\,.$$

The *extremal system* of ϖ, which is, by definition, the associated system to $d\varpi$, is analogous to the system of equations (27.8) and (27.9). Its solutions are basic paths of W that project onto V_{n+1} along the solutions of the system:

$$\frac{d}{du}\partial_{\dot\alpha} L - \partial_\alpha L = \partial_A L\,\partial_{\dot\alpha} f^A - f^A\,\partial_{\dot\alpha A} L\,. \tag{27.12}$$

Those projections are the generalized extremals of the integral:

$$J = \int_{u_0}^{u_1} \bar{L}(F^A, x^\alpha, \dot{x}^\alpha)\,du\,.$$

28. Lichnerowicz spaces ([15]). – Recall the notations of § **25** and § **27**. Consider a non-holonomic function $\bar{L}\,(F, z)$ with $F = \int_z^{z'} \partial_{\dot\alpha} f(x, y,)\,dx^\alpha$ and such that $L = \bar{L}\,(0, z)$ is an $\dot{h}1$ function that is defined on $\mathcal{V}$.

We propose to define a linear connections on the directions on V_{n+1} that reduces to the Finslerian connection that is attached to L for $f = 0$, and is such that the geodesics of V_{n+1} relative to that connection are the generalized extremals of the integrals:

$$J = \int_{u_0}^{u_1} \bar{L}[(F, z(u)]\,du\,.$$

As for the S-Finslerian spaces, set:

$$\omega_\beta^\alpha = b_{\beta\gamma}^\alpha\,dx^\gamma + c_{\beta\gamma}^\alpha\,dy^\gamma$$

with respect to the coframe in T_z^* that is defined by the 2 $(n + 1)$-forms dx^α, dy^α, and:

$$\omega_\beta^\alpha = \Gamma_{\beta\gamma}^\alpha\,dx^\gamma + C_{\beta\gamma}^\alpha\,\theta^\gamma$$

([15]) A. Lichnerowicz [**2**], pp. 352-362.

with respect to the coframe in T_z^* that is defined by the 2 (n + 1) forms dx^α and $\theta^\alpha = \nabla y^\alpha$.

Since the connection form ω is supposed to be defined on W, the b and the Γ are $\dot{h}0$, while the c and the C are $\dot{h}(-1)$, and we will have:

$$c^\alpha_{\beta\gamma}\, y^\gamma = C^\alpha_{\beta\gamma}\, y^\gamma = 0$$

identically. The Pfaffian derivatives of a function G (x, y) relative to the coframe dx^α, θ^α are expressed as functions of the partial derivatives of G (x, y) by means of the formulas (25.4) and (25.5). If $\bar{G}$ is a non-holonomic function $\bar{G} = \bar{G}(F, x, y)$ then we will have:

(28.1)
$$\begin{cases} \delta_\alpha \bar{G} = \partial_\alpha \bar{G} - y^\lambda \Gamma^\beta_{\lambda\alpha} \partial_{\dot{\alpha}} \bar{G} = \partial_\alpha G + G' \partial_{\dot{\alpha}} f - y^\lambda \Gamma^\beta_{\lambda\alpha} \partial_{\dot{\alpha}} \bar{G}, \\ \delta_{\dot{\alpha}} \bar{G} = \partial_{\dot{\alpha}} G - y^\lambda T^\beta_{\lambda\alpha}\, \partial_{\dot{\beta}} G. \end{cases}$$

Take the metric tensor at z (x^α, y^α) to be the tensor whose non-holonomic components are:

$$\bar{g}_{\alpha\beta} = \partial_{\dot{\alpha}\dot{\beta}}\left(\tfrac{1}{2}\bar{L}^2\right) = \partial_{\dot{\alpha}\dot{\beta}}\left(\tfrac{1}{2}L^2\right) \equiv g_{\alpha\beta}\,,$$

such that

$$\partial_\gamma\, \bar{g}_{\alpha\beta} = \partial_\gamma g_{\alpha\beta} + g'_{\alpha\beta}\, \partial_{\dot{\gamma}} f\,,$$
$$\partial_{\dot{\gamma}}\, \bar{g}_{\alpha\beta} = \partial_\gamma\, g_{\alpha\beta}\,,$$

in which the notations are those of the preceding paragraph.

In order for the linear connection on directions that is defined by ω to be naturally associated with a Euclidian connection on directions that is defined on V_{n+1} by the tensor $\bar{g}_{\alpha\beta}$, it is necessary and sufficient that $\nabla\, \bar{g}_{\alpha\beta} = 0$; i.e., that:

(28.2)
$$d\,\bar{g}_{\alpha\beta} - \omega^\lambda_\alpha\, g_{\lambda\beta} - \omega^\lambda_\beta\, g_{\lambda\alpha} = 0.$$

Let us make this more explicit with respect to the coframe (dx^α, θ^α) ; we obtain:

(28.3)
$$\Gamma_{\alpha\beta\gamma} + \Gamma_{\beta\alpha\gamma} = \delta_\gamma\, \bar{g}_{\alpha\beta}\,,$$

(28.4)
$$T_{\alpha\beta\gamma} + T_{\beta\alpha\gamma} = \delta_{\dot{\gamma}}\, \bar{g}_{\alpha\beta}\,.$$

Now consider the torsion form:

$$\Sigma^\alpha = \omega^\alpha_\beta \wedge dx^\beta = \tfrac{1}{2} S^\alpha_{\beta\gamma}\, dx^\beta \wedge dx^\gamma - T^\alpha_{\beta\gamma}\, dx^\beta \wedge \theta^\gamma\,,$$

with

$$S^\alpha_{\beta\gamma} = -(\Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\beta})\,,$$

$$T^{\alpha}_{\beta\gamma} = C^{\alpha}_{\beta\gamma}.$$

Impose the condition on the connection ω that it must be special in the sense of A. Lichnerowicz. The torsion tensors must then verify the conditions:

(28.5) 1. $S_{\alpha\beta\gamma} = 0.$

(28.6) 2. $T_{\alpha\beta\gamma} = T_{\beta\alpha\gamma}.$

The conditions (28.5) imply that the coefficients $\Gamma_{\alpha\beta\gamma}$ are symmetric with respect to the last two indices.

The conditions (28.6) and (28.4) give us:

$$T_{\alpha\beta\gamma} = C_{\alpha\beta\gamma} = \tfrac{1}{2}\delta_{\dot{\gamma}}\, g_{\alpha\beta},$$

namely, with the same argument that was used to establish (25.13):

$$T_{\alpha\beta\gamma} = \tfrac{1}{2}\partial_{\dot{\gamma}}\, g_{\alpha\beta},$$

so the tensor $T_{\alpha\beta\gamma}$ is completely symmetric then.

Calculating the coefficients $\Gamma_{\alpha\beta\gamma}$. – The coefficients Γ are determined by means of the following system:

$$\begin{cases} \Gamma_{\alpha\beta\gamma} + \Gamma_{\beta\alpha\gamma} = \delta_{\gamma}\, \bar{g}_{\alpha\beta}, \\ \Gamma_{\alpha\beta\gamma} - \Gamma_{\beta\alpha\gamma} = 0. \end{cases}$$

We deduce from this that:

$$\Gamma_{\alpha\beta\gamma} = [\beta\,\gamma,\, \alpha]^{\delta},$$

in which the $[\beta\,\gamma,\, \alpha]^{\delta}$ are the Christoffel symbols, when they are expressed in terms of the Pfaffian derivatives.

Let us make things more explicit with the aid of formulas (28.1):

$$\delta_{\gamma}\, \bar{g}_{\alpha\beta} = \partial_{\gamma} g_{\alpha\beta} + g'_{\alpha\beta}\, \partial_{\dot{\beta}} f - 2y^{\lambda}\, \Gamma^{\mu}_{\lambda\gamma}\, T_{\mu\alpha\beta}\ . \tag{28.7}$$

Upon setting:

$$\Sigma_{\alpha\beta\gamma} = \tfrac{1}{2}(g'_{\alpha\gamma}\, \partial_{\dot{\beta}} f + g'_{\alpha\beta}\, \partial_{\dot{\gamma}} f - g'_{\beta\gamma}\, \partial_{\dot{\alpha}} f), \tag{28.8}$$

we will get:

$$\Gamma_{\alpha\beta\gamma} = [\beta\gamma,\, \alpha] - y^{\lambda}(\Gamma^{\mu}_{\lambda\gamma}\, T_{\mu\alpha\beta} + \Gamma^{\mu}_{\lambda\beta}\, T_{\mu\gamma\alpha} - \Gamma^{\mu}_{\lambda\alpha}\, T_{\mu\beta\gamma}) + \Sigma_{\alpha\beta\gamma}. \tag{28.9}$$

Set:

$$y^{\beta} y^{\gamma} [\beta\gamma, \alpha] = 2\, G_{\alpha} = \partial_{\lambda\dot{\alpha}} \left(\tfrac{1}{2} L^2\right) y^{\lambda} - \partial_{\alpha} \left(\tfrac{1}{2} L^2\right).$$

Let us calculate $y^{\beta} y^{\gamma} \Sigma_{\alpha\beta\gamma}$. Since the function f is supposed to be $\dot{h}1$, and $\frac{1}{2} g'_{\alpha\beta}\, y^{\alpha} y^{\beta}$ $= L\, L'$, we will get:

$$y^{\beta} y^{\gamma} \Sigma_{\alpha\beta\gamma} = f\, \partial_{\dot{\alpha}} (L\, L') - (L\, L') \partial_{\dot{\alpha}} f \;=\; f^2 \partial_{\dot{\alpha}} \varphi$$

upon setting:

$$\varphi = \frac{L\, L'}{f} \qquad (f \neq 0).$$

We then deduce that:

$$\begin{aligned} y^{\beta} y^{\gamma} \Gamma_{\alpha\beta\gamma} &= y^{\beta} y^{\gamma} [\beta\, \gamma, \alpha] + y^{\beta} y^{\gamma} \Sigma_{\alpha\beta\gamma} \\ &= 2\, G_{\alpha} + f^2 \partial_{\dot{\alpha}} \varphi\,. \end{aligned}$$

It is remarkable that the right-hand side is equal to $2\bar{G}_{\alpha}$, which is defined by:

$$2\bar{G}_{\alpha} = \partial_{\lambda\dot{\alpha}} \left(\tfrac{1}{2} \bar{L}^2\right) y^{\lambda} - \partial_{\alpha} \left(\tfrac{1}{2} \bar{L}^2\right).$$

Indeed:

$$\partial_{\lambda} \left(\tfrac{1}{2} \bar{L}^2\right) = \partial_{\lambda} \left(\tfrac{1}{2} L^2\right) + L\, L' \partial_{\lambda} f$$

and

$$\begin{aligned} 2\bar{G}_{\alpha} &= 2G_{\alpha} + \partial_{\dot{\alpha}} \left(L\, L' \partial_{\dot{\lambda}}\right) - L\, L' \partial_{\dot{\alpha}} f \\ &= 2G_{\alpha} + \partial_{\dot{\alpha}} (L\, L') f - L\, L' \partial_{\dot{\alpha}} f\,, \end{aligned}$$

because:

$$\partial_{\dot{\alpha}\dot{\beta}} f\, y^{\lambda} = 0\,.$$

We conclude with the calculations that we did in paragraph **25**. Furthermore, it will suffice to replace $L\, X_{\alpha}$ with $- f^2 \partial_{\dot{\alpha}} \varphi$ in the results obtained.

We then obtain:

$$\Gamma_{\alpha\beta\gamma} = [\beta\gamma, \alpha] - (T_{\alpha\beta\lambda}\, \partial_{\dot{\gamma}} \bar{G}^{\lambda} + T_{\alpha\gamma\lambda}\, \partial_{\dot{\beta}} \bar{G}^{\lambda} - T_{\beta\gamma\lambda}\, \partial_{\dot{\alpha}} \bar{G}^{\lambda}) + \Sigma_{\alpha\beta\gamma}\,,$$

or

$$\Gamma_{\alpha\beta\gamma} = \dot{\Gamma}_{\alpha\beta\gamma} + f^2\, \partial_{\dot{\lambda}}\, (T^{\lambda}_{\mu\gamma} T^{\mu}_{\alpha\beta} + T^{\lambda}_{\mu\beta} T^{\mu}_{\gamma\beta} - T^{\lambda}_{\mu\alpha} T^{\mu}_{\beta\gamma}) - y^{\lambda} (\Sigma_{\mu\lambda\gamma} T^{\mu}_{\alpha\beta} + \Sigma_{\mu\lambda\beta} T^{\mu}_{\gamma\alpha} - \Sigma_{\mu\lambda\alpha} T^{\mu}_{\beta\gamma}) + \Sigma_{\alpha\beta\gamma}\,.$$

In the latter expression, $\dot{\Gamma}_{\alpha\beta\gamma}$ represents the analogous coefficients for the Finslerian connection that is defined by L.

The Γ thus-defined are the coefficients $\bar{\Gamma}^*$ of A. Lichnerowicz's intermediate connection. One verifies that they are symmetric with respect to their last two indices.

Covariant derivation. – Let **X** be a restricted vector field that is defined on $\mathcal{V}$ or *W*. The components X^α are then functions of the x^α, y^α that are homogeneous with respect to the latter variables. Under those conditions, we will have:

$$\nabla X^\alpha = dX^\alpha + \omega_\beta^\alpha X^\beta .$$

Now suppose that **X** is a non-holonomic vector field; i.e., that is its components $\bar{X}^\alpha$ at a point z' that is close to z are functions of $\bar{L}$, and consequently of F. Its *absolute differential* will be then defined by:

$$\nabla \bar{X}^\alpha = d\bar{X}^\alpha + \omega_\beta^\alpha X^\beta ,$$

where $d\bar{X}^\alpha$ is the differential of the non-holonomic function $\bar{X}^\alpha$. The preceding considerations extend immediately to some arbitrary tensor fields, whether holonomic or not.

Geodesics. – Set $\bar{l}^\alpha = y^\alpha / \bar{L}$ and $\bar{l}_\alpha = \partial_{\dot\alpha}\bar{L}$. The geodesics of the space that was studied previously are defined by:

$$\frac{\nabla \bar{l}^\alpha}{du} = \frac{d\bar{l}^\alpha}{du} + \Gamma_{\beta\gamma}^\alpha l^\beta y^\gamma = 0$$

or by:

$$\frac{\nabla \bar{l}_\alpha}{du} = \frac{d\bar{l}_\alpha}{du} + \Gamma_{\beta\alpha\gamma} l^\beta y^\gamma = 0 .$$

Now:

$$\Gamma_{\beta\alpha\gamma} l^\beta y^\gamma = \dot{\Gamma}_{\beta\alpha\gamma} l^\beta y^\gamma + \Sigma_{\beta\alpha\gamma} l^\beta y^\gamma .$$

We next have:

$$\Sigma_{\beta\alpha\gamma} l^\beta y^\gamma = \tfrac{1}{2}(g'_{\beta\gamma}\partial_{\dot\alpha} f + g'_{\beta\alpha}\partial_{\dot\gamma} f - g'_{\alpha\gamma}\partial_{\dot\gamma} f) l^\beta y^\gamma = L'\partial_{\dot\alpha} f ,$$

$$\frac{d\bar{l}_\alpha}{du} = \frac{d\,l_\alpha}{du} + \partial_{\dot\alpha}(L'\partial_{\dot\beta} f) y^\beta = \frac{d\,l_\alpha}{du} + \partial_{\dot\alpha} L'\partial_{\dot\beta} f\, y^\beta = \frac{d\,l_\alpha}{du} + f\partial_{\dot\alpha} L' .$$

Finally, the geodesics of the space considered are defined by:

$$\frac{\nabla \bar{l}_\alpha}{du} = \frac{\nabla l_\alpha}{du} + f\,\partial_{\dot\alpha} L' - L'\partial_{\dot\alpha} f = 0$$

or by:

$$\frac{d}{du}\partial_{\dot\alpha} L - \partial_\alpha L = L'\partial_{\dot\alpha} f - f\,\partial_{\dot\alpha} L' .$$

The geodesics are indeed identical to the general extremals of the integral:

$$J = \int_{u_0}^{u_1} \bar{L}\,(F, z(u))\, du\,.$$

The space thus-constructed indeed satisfies the various conditions that have been imposed upon it.

Generalization. – Consider a non-holonomic function of several functionals $\bar{L}$ (F^A, z), with:

$$F^A = \int_z^{z'} \partial_{\dot{\alpha}}\, f^A(x, y)\, dx^\alpha = \int_z^{z'} \dot{d}f^A\,, \quad \text{where } A = 1, 2, \ldots, k\,,$$

such that $L = L\,(0, z)$ is an $\dot{h}1$ function that is defined on $\mathcal{V}$.

It is easy to extend the preceding considerations by defining a linear connection on the directions on V_{n+1} such that the geodesics of V_{n+1} relative to that connection are the generalized extremals of the integral:

$$J = \int_{u_0}^{u_1} \bar{L}[F^A, z(u)]\, du\,.$$

It suffices to replace $\Sigma_{\alpha\beta\gamma}$ with:

$$\Sigma_{\alpha\beta\gamma} = \tfrac{1}{2}(\partial_A g_{\alpha\gamma}\, \partial_{\dot{\beta}} f^A + \partial_A g_{\alpha\gamma}\, \partial_{\dot{\gamma}} f^A - \partial_A g_{\beta\gamma}\, \partial_{\dot{\alpha}} f^A)$$

and $2\bar{G}_\alpha$ with:

$$2\bar{G}_\alpha = 2\, G_\alpha + \partial_{\dot{\alpha}}(L\, \partial_A L)\, f^A - L\, \partial_A L\, \partial_{\dot{\alpha}} f^A$$

in the formulas that are obtained.

From now on, we shall refer to spaces of the preceding type as *Lichnerowicz spaces* or $\mathcal{L}$ *spaces*.

An $\mathcal{L}_1$ *space* corresponds to a function $\bar{L}$ that is non-holonomic with respect to only one functional F.

An $\mathcal{L}_k$ *space* corresponds to a function L that is non-holonomic with respect to k functionals F_A .

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PART TWO

MECHANICAL APPLICATIONS

CHAPTER V

DYNAMICAL SYSTEMS WITH HOLONOMIC CONSTRAINTS

29. Lagrange equations in the homogeneous formalism. – Let (*S*) be a non-conservative dynamical system with perfect, bilateral, holonomic constraints that admit n degrees of freedom.

Let V_{n+1} denote its configuration space-time.

Suppose that the configuration space of (*S*) is defined by the parameters x^k, where k = 1, 2, …, n. The parameters x^k and the time t define a local coordinate system for V_{n+1}. Set:

$$x'^k = \frac{dx^k}{dt}$$

and let $\mathcal{L}$ be the Lagrangian of the system (*S*) for the parameters x^k. The trajectories of (*S*) in V_{n+1} are defined by the n functions $x^k(t)$ that are solutions of the Lagrange equations:

$$\frac{d}{dt}\partial_{k'}\mathcal{L} - \partial_k\mathcal{L} = Q_k . \tag{29.1}$$

The Q_k are functions that are determined by the x^i, the x'^i, and time t.

Now set: $x^{n+1} = t$.

A local coordinate system at a point x of V_{n+1} is then x^α, where α = 1, 2, …, n, n + 1.

(In what follows, a Latin index can take the values 1, 2, …, n ; any Greek index will take the values 1, 2, …, n + 1.)

Let u be an arbitrary real parameter; set:

$$\dot{x}^\alpha = dx^\alpha / du,$$

so

$$x'^k = \dot{x}^k / \dot{x}^{n+1}.$$

The trajectories of the dynamical system (*S*) in V_{n+1} are then defined by functions $x^\alpha(u)$ that are solutions of a system of differential equations that is classical deduced from (29.1) ([16]).

Set $L(x^\alpha, \dot{x}^\alpha) = \mathcal{L}(x^\alpha, \dot{x}^k / \dot{x}^{n+1})\,\dot{x}^{n+1}$.

L, which is $\dot{h}(1)$, is the homogeneous Lagrangian of (*S*), by definition. We then deduce that:

$$\partial_{\dot{k}}L = \partial_{k'}L \qquad \text{and} \qquad \partial_{n+1}L = \mathcal{L} - x^{k'}\partial_{k'}\mathcal{L} = -\mathcal{H},$$

([16]) A. Lichnerowicz [**2**], pp. 375-365.

in which $\mathcal{H}$ denotes the Hamiltonian that corresponds to $\mathcal{L}$.

Equations (29.1) can then be put into the form:

$$\frac{d}{du}\partial_{\dot{k}} L - \partial_k L = Q_k \, \dot{x}^{n+1}, \tag{29.2}$$

or, with the notations of § **18** :

$$P_k(L) = X_k \qquad \text{upon setting} \quad X_k = Q_k \, \dot{x}^{n+1}.$$

One deduces from the identity:

$$P_\alpha(L)\,\dot{x}^\alpha \equiv 0$$

that:

$$P_{n+1}(L)\,\dot{x}^{n+1} = -P_k(L)\,\dot{x}^k = -X_k \dot{x}^k,$$

or

$$P_{n+1}(L) = X_{n+1} \qquad \text{upon setting} \quad X_{n+1} = -Q_k \dot{x}^k. \tag{29.3}$$

Finally, the functions $x^\alpha(u)$ are solutions to the system of Lagrange equations that relates to the homogeneous Lagrangian L :

$$P_\alpha(L) = \frac{d}{du}\partial_{\dot{\alpha}} L - \partial_\alpha L = X_\alpha. \tag{29.4}$$

The X_α, which are functions of the x^α and the $\dot{x}^\alpha$, are homogeneous of degree 1 with respect to the latter variables and are $(\dot{h}1)$ such that:

$$X_\alpha \dot{x}^\alpha \equiv 0.$$

They are the components of a vector that is called the *generalized force vector*. The $n + 1$ equations (29.4) are not independent, so one can give one of the functions $x^\alpha(u)$ arbitrarily, while the other n will be determined by equations (29.4), in general. Recall that the $(n + 1)^{\text{th}}$ equation, which is the equation:

$$P_{n+1}(L) = X^{n+1},$$

can be further written in the form:

$$-\frac{d\mathcal{H}}{du} - \frac{\partial L}{\partial t} = -Q_k \dot{x}^k,$$

or, upon setting $t = u$:

$$\frac{d\mathcal{H}}{dt} + \frac{\partial L}{\partial t} = Q_k x'^k.$$

That equation translates into the well-known Painlevé theorem. The Lagrange equations thus-defined have a form that is independent of any particular framing that was adopted for the configuration space-time.

30. Notion of generalized force tensor. – As in Part One, we let $\mathcal{V}$ denote the fiber bundle of non-zero vectors that are tangent to the differentiable manifold V_{n+1} and let W denote the fiber bundle of oriented directions that are tangent to V_{n+1}. The space W is referred to as the "state space" in mechanics or the "space-time of extension in phase." Consider the "elementary work" form:

$$\omega = X_\alpha \, dx^\alpha .$$

That form is a semi-basic $\dot{h}1$ form that is defined on $\mathcal{V}$. By way of the $\dot{d}$ operator, we make it correspond to an $\dot{h}0$ semi-basic 2-form that is defined on W:

$$\dot{d}\omega = \tfrac{1}{2}(\partial_{\dot{\alpha}} X_\beta - \partial_{\dot{\beta}} X_\alpha)\, dx^\alpha \wedge dx^\beta .$$

The components of that form are the components of a restricted, twice-covariant, antisymmetric $\dot{h}0$ tensor. In what follows, we shall refer to the tensor whose components are:

$$S_{\alpha\beta} = \tfrac{1}{2}(\partial_{\dot{\alpha}} X_\beta - \partial_{\dot{\beta}} X_\alpha)$$

as the *force tensor* that corresponds to the generalized force whose components are X_α.

The components $S_{\alpha\beta}$ are such that:

1. $$S_{\alpha\beta}\, \dot{x}^\beta = X_\alpha .$$

Indeed:

$$\partial_{\dot{\beta}} X_\alpha \, \dot{x}^\beta = X_\alpha ,$$

because the X_α are $\dot{h}1$, and on the other hand, the fact that:

$$X_\beta \, \dot{x}^\beta \equiv 0$$

will imply that:

$$\partial_{\dot{\alpha}} X_\beta \, \dot{x}^\beta = - X_\alpha$$

by partial differentiation. Since the form $\dot{d}\omega = - S_{\alpha\beta}\, dx^\alpha \wedge dx^\beta$ is $\dot{d}$–closed, we will have the identities:

$$\partial_{\dot{\alpha}} S_{\beta\gamma} + \partial_{\dot{\beta}} S_{\gamma\alpha} + \partial_{\dot{\gamma}} S_{\alpha\beta} = 0.$$

2. If the X^α are linear in the components of the velocity then the $S_{\alpha\beta}$ will be independent of the $\dot{x}$. In the space T_x that is tangent to the point x in V_{n+1}, the force vector X that corresponds to a given velocity vector $V(x)$ is deduced from the latter by the linear transformation that is defined by the matrix whose elements are the $S_{\alpha\beta}(x)$.

Geometric interpretation of the Lagrange equations.

31.-1. In a Finsler space. – Suppose that the differentiable manifold V_{n+1} is endowed with the Finslerian metric:

$$ds = L(x^\alpha, \dot{x}^\alpha)\, du .$$

Suppose, on the other hand, that the function L leads to a regular variational problem; i.e., that the matrix $\left\| \partial_{\dot{\alpha}\dot{\beta}} L \right\|$ has rank n on $\mathcal{V}$. The dynamical system (S) will then be called *regular*. Let (T) be an arbitrary trajectory of that system. A unit vector $\boldsymbol{l}$ that is tangent to (T) at an arbitrary point x of (T) will have the components:

$$l^\alpha = \frac{\dot{x}^\alpha}{L} \qquad \text{or} \qquad l_\alpha = \partial_{\dot{\alpha}} L .$$

The left-hand sides $P_\alpha(L)$ of the Lagrange equations are the components of the vector that is the covariant derivative of $\boldsymbol{l}$ with respect to u. Equations (29.4) then take the form:

$$\frac{\nabla l_\alpha}{du} = X_\alpha \tag{31.1}$$

or

$$\frac{\nabla \boldsymbol{l}}{du} = \mathbf{X} .$$

Take the parameter u to be the arc-length s of (T). The components of the generalized force vector are then X_α / L, and the left-hand sides of equations (31.1) are the components of the curvature vector (T) at x:

$$\frac{\nabla \boldsymbol{l}}{ds} = \frac{\mathbf{n}}{R} = \mathbf{C}, \qquad \frac{\mathbf{X}}{L} = \mathbf{F} .$$

Hence:

Theorem:

In a Finsler space that is defined on the configuration space-time of a dynamical system (S) *by:*

$$ds = L(x^\alpha, \dot{x}^\alpha)\, du ,$$

in which L is the homogeneous Lagrangian of (S)*, the trajectories will be the curves in that space such that the curvature vector is equal to the force vector at any point.*

Particular case. – If the force vector is zero at any point of V_{n+1} then the Lagrange equations can be written in the form:

$$\frac{\nabla l_\alpha}{du} = 0,$$

and they will express the idea that the trajectories are the geodesics of the Finsler space that is associated with the dynamical system. Those trajectories are the extremals of the form:

$$\omega = \partial_{\dot{\alpha}} L\, dx^\alpha$$

or the extremals of the integral:

$$I = \int_{u_0}^{u_1} L(x^\alpha, \dot{x}^\alpha)\, du\,.$$

(Hamilton's principle in its general form.)

That is equivalent to saying that those trajectories are characterized by the existence of one of E. Cartan's relative integral invariants:

$$\int \partial_{\dot{\alpha}} L\, dx^\alpha\,.$$

32.-2 In an *S*-Finslerian space. – Consider the *S*-Finslerian space (§ **25**) that is defined on V_{n+1} when one is given the $\dot{h}1$ scalar function L (x^α, $\dot{x}^\alpha$) and the restricted $\dot{h}\,0$ tensor $S_{\alpha\beta}$.

Recall that an *S*-Finslerian space differs from a Finslerian space by only the following convention:

$$\Gamma^\alpha_{\beta\gamma} - \Gamma^\alpha_{\gamma\beta} = S_{\beta\gamma}\, l^\alpha,$$

in which the $\Gamma^\alpha_{\beta\gamma}$ are defined by the connection forms:

$$\omega^\alpha_\beta = \Gamma^\alpha_{\beta\gamma}\, dx^\gamma + C^\alpha_{\beta\gamma}\, \nabla y^\gamma,$$

with

$$y^\gamma = \dot{x}^\gamma.$$

The differential system of the geodesics is:

$$\frac{d}{du}\partial_{\dot{\alpha}} L - \partial_\alpha L = S_{\alpha\beta}\, \dot{x}^\beta = X_\alpha\,.$$

It then results that the trajectories of the dynamical system S (L, $S_{\alpha\beta}$) are the geodesics of the *S*-Finslerian space that is defined by L and $S_{\alpha\beta}$.

Those trajectories are also the *S*-extremals (§ **24**) of the integral:

$$I = \int_{u_0}^{u_1} L(x, \dot{x})\, du\,.$$

Recall that an S-extremal of I is the projection onto V_{n+1} of a basic path on W that is defined by:

$$x^\alpha = x^\alpha(u), \qquad y^\alpha = y^\alpha(u) = \frac{dx^\alpha}{du}$$

for which I is an extremum, where the neighboring paths are defined by:

$$x^\alpha = x^\alpha(u) + \delta x^\alpha(u), \qquad\qquad y^\alpha = y^\alpha(u) + \delta y^\alpha(u),$$

with δx^α arbitrary, except at x_0 and x_1, where they are zero, and:

$$\delta y^\alpha = \frac{d}{du}\delta x^\alpha + L\, S^\alpha_\beta\, \delta x^\beta .$$

The theorem that is obtained in that way is a generalization of Hamilton's theorem that relates to conservative dynamical systems (case where $S_{\alpha\beta} = 0$).

We can then state:

Generalized Hamilton theorem:

The trajectories of a dynamical system S (L, $S_{\alpha\beta}$) *are the S-integrals of the integral* $\int_{u_0}^{u_1} L\, du$.

33.-3. In a Lichnerowicz space. – Consider the form:

$$\Omega = \tfrac{1}{2} S_{\alpha\beta}\, dx^\alpha \wedge dx^\beta .$$

Set $y^\alpha = \dot{x}^\alpha$ and associate Ω with the form:

$$\bar{\Omega} = \tfrac{1}{2} S_{\alpha\beta}\, dy^\alpha \wedge dy^\beta,$$

in which the variables x^α are supposed to be fixed. Since $\dot{d}\,\Omega = 0$, we then deduce that $d\,\bar{\Omega} = 0$.

Suppose the form $\bar{\Omega}$ has rank $2r$. Apply the theorem ([17]): Any closed exterior quadratic form of rank $2r$ can be put into the form:

$$dH_A \wedge dK^A \qquad (\text{with } A = 1, 2, \ldots, r),$$

in which the functions H_A and K^A constitute a system of independent first integrals of the characteristic system of that form.

We then deduce that Ω can be put into the form:

([17]) E. Cartan [**2**], pp. 119-120.

$$\Omega = \dot{d}H_A \wedge \dot{d}K^A$$

in a neighborhood U of $\mathcal{V}$. The associated system to $\bar{\Omega}$ is homogeneous with respect to the y^α, so the first integrals H_A and K^A will be $\dot{h}$ functions. The sum of their degrees of homogeneity is two, so we can suppose that H_A and K^A are $\dot{h}1$. If that were not true then we could introduce a function that is $\dot{h}$ of a suitable degree and $\dot{d}$–closed, from the identity:

$$\dot{d}H_A \wedge \dot{d}K^A = \dot{d}(H\,f) \wedge \dot{d}\left(\frac{K}{f}\right).$$

We then obtain:

$$S_{\alpha\beta} = \partial_{\dot{\alpha}} H_A \, \partial_{\dot{\beta}} K^A - \partial_{\dot{\alpha}} K^A \, \partial_{\dot{\beta}} H_A$$

and

$$X_\alpha = S_{\alpha\beta} \, \dot{x}^\beta = K^A \, \partial_{\dot{\alpha}} H_A - H_A \, \partial_{\dot{\alpha}} K^A .$$

The Lagrange equations of the dynamical system S (L, $S_{\alpha\beta}$) show that the trajectories of S are the generalized extremals of the integral:

$$J = \int_{u_0}^{u_1} \left[L(x, \dot{x}) + K^A \int_{u_0}^{u} H_A dv \right] du .$$

Those trajectories are (§ **27**) the projections onto V_{n+1} of the extremals of the non-holonomic form:

$$\omega = \dot{d}L + \dot{d}K^A \int_z^{z'} \dot{d}H_A$$

or

$$\omega = \dot{d}L - \dot{d}H_A \int_z^{z'} \dot{d}K^A ,$$

in which z (x, y) and z'(x', y') are two neighboring points of W. Those trajectories are also geodesics of the space $\mathcal{L}_r$ that is defined by the non-holonomic function:

$$\bar{L} = L(x, \dot{x}) + K^A \int_z^{z'} \dot{d}H_A ;$$

i.e., by the $2r + 1$ functions: L, K^A, H_A .

Particular case of the space $\mathcal{L}_1$ ([18]).

In order to have $r = 1$, it is necessary and sufficient that the form:

([18]) J. Klein [**1**].

$$\Omega = \tfrac{1}{2} S_{\alpha\beta}\, dx^\alpha \wedge dx^\beta$$

should be decomposable or a monomial. In order for that to be true, it is necessary and sufficient ([19]) that the coefficients $S_{\alpha\beta}$ should verify the relations:

$$S_{\alpha\beta} S_{\gamma\delta} + S_{\alpha\gamma} S_{\delta\beta} + S_{\alpha\delta} S_{\beta\gamma} = 0.$$

Under those conditions, one can find two $\dot{h}\,1$ functions H and K such that one will locally have:

$$\Omega = \dot{d}H \wedge \dot{d}K .$$

We then have:

$$X_\alpha = K\,\partial_{\dot{\alpha}} H - H\,\partial_{\dot{\alpha}} K = K^2 \partial_{\dot{\alpha}} \frac{H}{K} \qquad (K \neq 0).$$

The trajectories of the corresponding dynamical system are then the geodesics of the space $\mathcal{L}_1$ that is defined by the three $\dot{h}\,1$ functions L, K, and H.

Examples:

1. Suppose that there exists a velocity potential – i.e., that the Q_k have the form:

$$Q_k = \partial_{k'} U(x^\alpha, x'^m) .$$

Replace x'^m with $\dot{x}^m / \dot{x}^{n+1}$ in U. The function U that is obtained in that way is $\dot{h}\,0$, so we will have the identity:

$$\partial_{\dot{\alpha}} U\, \dot{x}^\alpha = 0.$$

Now:

$$X_k = Q_k\, \dot{x}^{n+1} = (\dot{x}^{n+1})^2 \partial_{\dot{k}} U ,$$

so

$$X_{n+1} = -\,Q_k\, \dot{x}^k = (\dot{x}^{n+1})^2 \partial_{\dot{n+1}} U .$$

For any $\alpha = 1, \ldots, n + 1$, we will then have:

$$X_\alpha = (\dot{x}^{n+1})^2 \partial_{\dot{\alpha}} U .$$

The trajectories of the dynamical system considered are then the generalized extremals of the integral:

$$J = \int_{u_0}^{u_1} \left[L + \dot{x}^{n+1} \int_{u_0}^{u} U\, \dot{x}^{n+1} dv \right] du ,$$

([19]) E. Cartan [**3**], page 18.

or the geodesics of the space $\mathcal{L}_1$ that is defined by the Lagrangian L and the function $K = \dot{x}^{n+1}$, $H = U\,\dot{x}^{n+1}$.

This case includes the particular case in which the Q_i are independent of the velocity. It will then suffice to set $U = Q_k\,x'^k$; one will then deduce that $K = \dot{x}^{n+1}$, $H = Q_k\,\dot{x}^k$.

2. More generally, suppose that:

$$Q_k = f^2\,\partial_{k'}U\,,$$

in which f and U are two functions of x_k, t, and x'^k.

The preceding calculations will then show that:

$$X_\alpha = (f\,\dot{x}^{n+1})^2\,\partial_{\dot{\alpha}}U\,.$$

The trajectories of the corresponding dynamical system are then the geodesics of the space $\mathcal{L}_1$ that are defined by the Lagrangian L and the two functions:

$$K = f\,\dot{x}^{n+1} \qquad \text{and} \qquad H = U\,\dot{x}^{n+1}\,.$$

3. Suppose that we have:

$$Q_k = R_k\,(x^h, t) + S_{km}\,(x^h, t)\,x'^m,$$

with

$$S_{km} = -\,S_{km}\,.$$

Set:

$$R_k = S_{k,\,n+1} = -\,S_{n+1,\,k}\,.$$

Hence:

$$X_k = S_{k\alpha}\,\dot{x}^\alpha \qquad \text{and} \qquad X_{n+1} = -\,Q_k\,\dot{x}^k = -\,R_k\,\dot{x}^k = S_{n+1,\alpha}\,\dot{x}^\alpha\,.$$

Hence, for any $\alpha = 1, \ldots, n + 1$, we will have:

$$X_\alpha = S_{\alpha\beta}\,\dot{x}^\beta\,,$$

in which $S_{\alpha\beta}$ is a tensor on V_{n+1}.

In order for the corresponding Lichnerowicz space to have type $\mathcal{L}_1$, it is necessary and sufficient that one must have:

$$S_{\alpha\beta}\,S_{\gamma\delta} + S_{\alpha\gamma}\,S_{\delta\beta} + S_{\alpha\delta}\,S_{\beta\gamma} = 0\,;$$

i.e., that the tensor $S_{\alpha\beta}$ must be a bivector. There will then exist two vector fields whose covariant components $f_\alpha(x)$ and $g_\alpha(x)$ are such that one will have:

$$S_{\alpha\beta} = f_\alpha\,g_\beta - f_\beta\,g_\alpha$$

locally. The corresponding space $\mathcal{L}_1$ is defined by the Lagrangian L and the functions:

$$H = f_\alpha \dot{x}^\alpha \qquad \text{and} \qquad K = g_\alpha \dot{x}^\alpha .$$

34. The fundamental 2-form Ω. – When the force tensor of a dynamical system is zero, we have seen (§ **16**) that the system of Lagrange equations:

$$P_\alpha(L) = 0$$

is the extremal system of the form:

$$\omega = \partial_{\dot{\alpha}} L \, dx^\alpha$$

i.e., the associated system to the 2-form:

$$d\omega = d(\partial_{\dot{\alpha}} L) \wedge dx^\alpha .$$

Now, let S (L, $S_{\alpha\beta} \neq 0$) be a dynamical system, and consider the 2-form:

$$\Omega = d(\partial_{\dot{\alpha}} L) \wedge dx^\alpha + \tfrac{1}{2} S_{\alpha\beta}\, dx^\alpha \wedge dx^\beta . \tag{34.1}$$

The associated system to Ω is composed of $2n + 2$ Pfaff equations:

$$\begin{cases} -\pi_\alpha(L) + S_{\alpha\beta} dx^\beta = 0, \\ \partial_{\dot{\alpha}\dot{\beta}} L \, dx^\beta = 0, \end{cases} \tag{34.2}$$

with

$$\pi_\alpha(L) = \partial_{\dot{\alpha}\dot{\beta}} L \, d\dot{x}^\beta + (\partial_{\dot{\alpha}\beta} L - \partial_{\alpha\dot{\beta}} L)\, dx^\beta .$$

Since the dynamical system is supposed to be regular, the matrix $\left\| \partial_{\dot{\alpha}\dot{\beta}} L \right\|$ will have rank n. We then conclude, as in § **16**, that the system (34.2) defines some basic curves of W whose projections onto V_{n+1} are the solutions to the system:

$$P_\alpha(L) = \frac{\pi_\alpha(L)}{du} = S_{\alpha\beta}\, \dot{x}^\beta = X_\alpha ;$$

i.e., the trajectories of the dynamical system considered. One then has the theorem:

Theorem:

The trajectories of the dynamical system S (L, $S_{\alpha\beta}$) are the integral curves of the associated system to the 2-form:

$$\Omega = d(\partial_{\dot{\alpha}} L) \wedge dx^{\alpha} + \tfrac{1}{2} S_{\alpha\beta}\, dx^{\alpha} \wedge dx^{\beta},$$

in which we suppose that $\dot{x}^{\alpha} = dx^{\alpha} / du$.

Consequences. – The trajectories of the dynamical system S (L, $S_{\alpha\beta}$) are characterized by the property that they admit the integral invariance relation that is generated by the form Ω.

If $\mathcal{T}$ is a tube that is generated by a closed continuous series of trajectories of S that is bounded by two homotopic closed curves that surround that tube then we will have:

$$\int_{\mathcal{T}} \Omega = 0. \tag{34.3}$$

We shall deduce a fundamental relation from that property that directly generalizes Cartan's theorem that relates to the relative integral invariant $\int \partial_{\dot{\alpha}} L\, dx^{\alpha}$ and borrows from the notations of Lichnerowicz ([20]).

35. Lichnerowicz's theorem:

Let C_0 and C_1 be two closed homotopic paths that surround the same tube of trajectories in the configuration space-time V_{n+1} of a dynamical system S (L, $S_{\alpha\beta}$) . The difference between the circulations of the velocity vector $\partial_{\dot{\alpha}} L$ along the cycles C_0 and C_1 is equal to the flux across the portion of the tube whose boundary is $C_0 - C_1$ of the generalized force tensor $S_{\alpha\beta}$:

$$\int_{C_1} \partial_{\dot{\alpha}} L\, dx^{\alpha} - \int_{C_0} \partial_{\dot{\alpha}} L\, dx^{\alpha} = \iint_{\mathcal{T}_0^1} \tfrac{1}{2} S_{\alpha\beta}\, dx^{\alpha} \wedge dx^{\beta}.$$

The $\dot{x}^{\alpha}$ that appear in L and $S_{\alpha\beta}$ are the components at x of the velocity vector that is tangent to the trajectory that passes through x.

1st Proof:

Consider two homotopic closed paths C_0 and C_1 in V_{n+1} that surround the same tube of trajectories $\mathcal{T}$. Let $\mathcal{T}_0^1$ be the 2-chain whose support is $\mathcal{T}$ and whose boundary $C_0 - C_1$.

Set:

$$\omega = \partial_{\dot{\alpha}} L\, dx^{\alpha}.$$

Upon applying the Stokes's theorem, we will have:

$$\int_{C_0} \omega - \int_{C_1} \omega = \int_{\mathcal{T}_0^1} d\omega.$$

([20]) A. Lichnerowicz [**1**], pp. 8-10.

Now, from (34.3):

$$\int_{\mathcal{T}_0^1}\left(d\omega+\tfrac{1}{2}S_{\alpha\beta}\,dx^\alpha\wedge dx^\beta\right) = 0.$$

We will then deduce the formula that we proposed to prove:

$$\int_{C_1}\partial_{\dot\alpha}L\,dx^\alpha-\int_{C_0}\partial_{\dot\alpha}L\,dx^\alpha = \int_{\mathcal{T}_0^1}\tfrac{1}{2}S_{\alpha\beta}\,dx^\alpha\wedge dx^\beta \tag{35.1}$$

2nd proof ([21]):

Let $x_0\,x_1$ be an arc of the trajectory of (S) in V_{n+1}, where x_0 and x_1 correspond to the values u_0 and u_1, respectively, of the parameter u. Consider the action integral:

$$I = \int_{u_0}^{u_1}L\,du\,,$$

which is evaluated along the arc $x_0\,x_1$.

The variation δI of I that corresponds to some arbitrary δx^α at any point of $x_0\,x_1$, including the extremities, and has:

$$\delta\dot{x}^\alpha = \frac{d}{du}\delta x^\alpha$$

is given by the classical formula:

$$\delta I = [\partial_{\dot\alpha}L\,\delta x^\alpha]_{x_0}^{x_1}-\int_{u_0}^{u_1}P_\alpha(L)\,\delta x_x^\alpha\,du\,, \tag{35.2}$$

or, from the Lagrange equations:

$$\delta I = [\partial_{\dot\alpha}L\,\delta x^\alpha]_{x_0}^{x_1}-\int_{u_0}^{u_1}X_\alpha\,\delta x^\alpha\,du\,. \tag{35.3}$$

Since $X_\alpha\,du = S_{\alpha\beta}\,\dot{x}^\beta\,du = S_{\alpha\beta}\,dx^\beta$, we have:

$$\delta I = [\partial_{\dot\alpha}L\,\delta x^\alpha]_{x_0}^{x_1}-\int_{u_0}^{u_1}S_{\alpha\beta}\,\delta x^\alpha\,dx^\beta\,. \tag{35.4}$$

Integrate the sides of (35.4) over the closed, continuous sequence of trajectories that $\mathcal{T}_0^1$ defines; we will get:

$$\int_{C_1}\partial_{\dot\alpha}L\,dx^\alpha-\int_{C_0}\partial_{\dot\alpha}L\,dx^\alpha = \int_{\mathcal{T}_0^1}\tfrac{1}{2}S_{\alpha\beta}\,dx^\alpha\wedge dx^\beta\,.$$

As before, we deduce from that relation that:

([21]) A. Lichnerowicz [**1**], pp. 8-10.

$$\iint_{\mathcal{T}_0^1} \left(d\, \partial_{\dot\alpha} L \wedge dx^\alpha + \tfrac{1}{2} S_{\alpha\beta}\, dx^\alpha \wedge dx^\beta \right) = 0$$

by an application of Stokes's formula. We have then proved directly that the form:

$$\Omega = d(\partial_{\dot\alpha} L) \wedge dx^\alpha + \tfrac{1}{2} S_{\alpha\beta}\, dx^\alpha \wedge dx^\beta$$

defines an integral invariance relation for the trajectories of (S).

36. Case in which the form Ω is closed. – Let $\Omega = d(\partial_{\dot\alpha} L) \wedge dx^\alpha + \tfrac{1}{2} S_{\alpha\beta}\, dx^\alpha \wedge dx^\beta$.

We then deduce that:

(36.1) $$d\,\Omega = \frac{1}{3!} K_{\alpha\beta\gamma}\, dx^\alpha \wedge dx^\beta \wedge dx^\gamma + \frac{1}{2} \partial_{\dot\gamma} S_{\alpha\beta}\, dx^\alpha \wedge dx^\beta \wedge d\dot{x}^\gamma$$

with

$$K_{\alpha\beta\gamma} = \partial_\alpha S_{\beta\gamma} + \partial_\beta S_{\gamma\alpha} + \partial_\gamma S_{\alpha\beta}\,.$$

In order to have $d\,\Omega = 0$, it is necessary and sufficient that one should have:

1. $\partial_{\dot\gamma} S_{\alpha\beta} = 0$ for any α, β, γ; i.e., that the tensor $S_{\alpha\beta}$ must be independent of the $\dot{x}$.

2. $K_{\alpha\beta\gamma} = 0$; i.e., that $S_{\alpha\beta}$ should locally be a rotational tensor.

There will then exist a local vector field **A** whose covariant components are A_α such that:

(36.2) $$S_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha.$$

We let **A** denote the vector-potential of the dynamical system S (L, $S_{\alpha\beta}$) . Upon remarking that under these conditions:

(36.3) $$\Omega = d(\partial_{\dot\alpha} L\, dx^\alpha + A_\alpha dx^\alpha)\,,$$

we can state the theorem:

Theorem:

In order for the fundamental 2-form Ω of a dynamical system S (L, $S_{\alpha\beta}$) *to be closed, it is necessary and sufficient that the tensor $S_{\alpha\beta}$ should be derived from a vector-potential* **A** (A_α) *whose components are independent of the velocity; i.e., that:*

$$S_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha.$$

The trajectories of the dynamical system are then characterized by the existence of the relative integral invariant that is defined by:

$$\omega = (\partial_{\dot{\alpha}} L + A_a)\, dx^{\alpha} ;$$

i.e., they are the extremals of the integral:

$$I = \int_{u_0}^{u_1} (L + A_\alpha \dot{x}^\alpha)\, du .$$

Those trajectories are also the geodesics of the Finsler space that is defined on the configuration space-time V_{n+1} by the function:

$$L + A_\alpha \dot{x}^\alpha .$$

37. Example: "centrifugal force" tensor. – Consider a system of N material points M_k whose coordinates are X_k , Y_k , Z_k with respect to an orthonormal frame (R) in Euclidian space E_3 . Suppose that the frame (R) is in motion with respect to an orthonormal frame (R_0) . Let a', b', c' be the components with respect to (R) of the velocity vector of the origin of that frame, and let p, q, r be the components with respect to (R) of the instantaneous rotation of (R) with respect to to R_0 .

The absolute *vis viva* of the system of points M_k is then:

$$2T_a = \sum_{k=1}^{N} m_k \left[(X'_k + a' + q\, Z_k - r\, Y_k)^2 + (Y'_k + b' + r\, X_k - p\, Z_k)^2 + (Z'_k + c' + p\, Y_k - q\, X_2)^2 \right],$$

while the relative *vis viva* will reduce to:

$$2T_r = \sum_{k=1}^{N} m_k (X_k'^2 + Y_k'^2 + Z_k'^2) .$$

Suppose that the system considered admits n degrees of freedom x^i.

The relative *vis viva* and the absolute *vis viva* have expressions of the form:

$$2T_r = a_{ij}\, x'^i\, x'^j,$$

$$2T_a = a_{ij}\, x'^i\, x'^j + 2b_i\, x'^i + c,$$

resp., in which the a_{ij} , b_i , and c are functions of x^k and time.

Suppose that the frame R coincides with the fixed frame R_0 at the instant t.

The Lagrange equations that relate to R_0 will then be:

(37.1) $$\frac{d}{dt}(a_{ij}\,x'^j) - \partial_i\,a_{jk}x'^j\,x'^k = Q_i\,.$$

Q_i denotes the i^{th} component of the generalized force vector.

The Lagrange equations that relate to R are:

(37.2) $$\frac{d}{dt}(a_{ij}\,x'^j) - \partial_i\,a_{jk}x'^j\,x'^k = Q_i - \frac{d}{dt}b_i + \partial_i(b_j\,x'^j + c)\,.$$

The supplementary terms that belong to the right-hand side, namely:

$$(\partial_i\,b_j - \partial_j\,b_i)\,x'^j + \partial_i c + \frac{\partial b_i}{\partial t},$$

represent the set of inertial, or "centrifugal," forces.

Pass to the homogeneous formalism by setting $x^{n+1} = t$, $\mathcal{T} = T_r\,\dot{x}^{n+1}$. Equations (36.5) will then become (§ **29**):

$$\frac{d}{du}\partial_{\dot{\alpha}}\mathcal{T} - \partial_\alpha\mathcal{T} = X_\alpha + (\partial_\alpha\,\varphi_\beta - \partial_\beta\,\varphi_\alpha)\,\dot{x}^\beta$$

upon setting $\varphi_i = b_i$ and $\varphi_{n+1} = c$.

The tensor $\partial_\alpha\,\varphi_\beta - \partial_\beta\,\varphi_\alpha$ that we call the *centrifugal force tensor* is derived from the potential vector φ_α. It is indeed of a tensor of the preceding type (37.2).

Upon setting $L = \mathcal{T} + \varphi_\alpha\,\dot{x}^\alpha$, the equations of motion will become:

$$\frac{d}{du}\partial_{\dot{\alpha}}L - \partial_\alpha L = X_\alpha\,.$$

Conversely, suppose that one is given a dynamical system $(L, S_{\alpha\beta})$. The equations of motion will be:

$$\frac{d}{du}\partial_{\dot{\alpha}}L - \partial_\alpha L = S_{\alpha\beta}\,\dot{x}^\beta\,.$$

We say that the tensor $S_{\alpha\beta}$ has *centrifugal force type* if there exists a global vector potential φ_α that depends upon only x^β, and not on the $\dot{x}^\beta$, such that:

$$S_{\alpha\beta} = \partial_\alpha\,\varphi_\beta - \partial_\beta\,\varphi_\alpha\,.$$

In order for that to be true, it is necessary and sufficient that the form:

$$\Omega = d(\dot{d}L) + \tfrac{1}{2}S_{\alpha\beta}\,dx^\alpha \wedge dx^\beta$$

should be closed; i.e., that $\partial_{\dot{\alpha}}\,S_{\beta\gamma} = 0$, and:

$$\partial_\alpha S_{\beta\gamma} + \partial_\beta S_{\gamma\alpha} + \partial_\gamma S_{\alpha\beta} = 0\ .$$

The same thing will be true for the electromagnetic field tensor in general relativity (§ **50**).

38. Case in which Ω admits an integrating factor. – By definition, Ω admits an integrating factor if there exists a differentiable function $f\,(x,\ \dot{x}) \neq 0$ such that the form $f\Omega$ is closed. The integrating factor f is such that:

$$d\,(f\,\Omega) = df \wedge \Omega + f\,d\,\Omega = 0$$

or

$$d\,\Omega = -\,\frac{df}{f} \wedge \Omega$$

or

$$d\,\Omega = d\varphi \wedge \Omega\ , \tag{38.1}$$

when one sets $f = e^{-\varphi}$.

Assume the existence of φ and make the identity (38.1) more explicit:

$$\begin{aligned}&\tfrac{1}{6}K_{\alpha\beta\gamma}\,dx^\alpha \wedge dx^\beta \wedge dx^\gamma + \tfrac{1}{2}\partial_{\dot\gamma}\,S_{\alpha\beta}\,dx^\alpha \wedge dx^\beta \wedge d\dot{x}^\gamma \\ &= (\partial_\gamma\varphi\,dx^\gamma + \partial_{\dot\gamma}\varphi\,d\dot{x}^\gamma) \wedge \left(\partial_{\dot\alpha\dot\beta}\,L\,d\dot{x}^\beta \wedge dx^\alpha + \tfrac{1}{2}R_{\alpha\beta}\,dx^\alpha \wedge dx^\beta\right),\end{aligned} \tag{38.2}$$

with:

$$K_{\alpha\beta\gamma} = \partial_\alpha S_{\beta\gamma} + \partial_\beta S_{\gamma\alpha} + \partial_\gamma S_{\alpha\beta} \qquad \text{and} \qquad R_{\alpha\beta} = \partial_{\alpha\dot\beta}\,L - \partial_{\dot\beta\alpha}\,L\ + S_{\alpha\beta}\,.$$

Upon identifying the various coefficients of the two sides of (38.2), we will get three systems of relations:

$$\partial_{\dot\gamma}\varphi\,\partial_{\dot\alpha\dot\beta}L - \partial_{\dot\beta}\varphi\,\partial_{\dot\alpha\dot\gamma}L\ = 0, \tag{38.3}$$

$$\partial_\beta\varphi\,\partial_{\dot\alpha\dot\gamma}L - \partial_\alpha\varphi\,\partial_{\dot\beta\dot\gamma}L + \partial_{\dot\gamma}\varphi\,R_{\alpha\beta} = \partial_{\dot\gamma}\,S_{\alpha\beta}\,, \tag{38.4}$$

$$R_{\alpha\beta}\,\partial_\gamma\,\varphi + R_{\beta\gamma}\partial_\alpha\,\varphi + R_{\alpha\gamma}\partial_\beta\,\varphi\ = K_{\alpha\beta\gamma}\,. \tag{38.5}$$

The relations (38.3) imply that the function φ is independent of the x because if that were not true then $\partial_{\dot\alpha\dot\beta}L$ would have the form:

$$\lambda\,\partial_{\dot\alpha}\varphi\,\partial_{\dot\beta}\varphi$$

and the matrix $\|\partial_{\dot\alpha\dot\beta}L\|$ would have rank 1, which is absurd, since the dynamical system is supposed to be regular.

When one integrates the relations (38.4) over $\dot x^\gamma$, where γ is arbitrary, that will then imply that:

$$S_{\alpha\beta} = \partial_\beta\varphi\,\partial_{\dot\alpha}L - \partial_\alpha\varphi\,\partial_{\dot\beta}L + T_{\alpha\beta}(x),$$

in which the $T_{\alpha\beta}$ depend upon only the variables x^γ.

Now make (38.5) more specific; after reductions, one will get:

$$\partial_\alpha T_{\beta\gamma} + \partial_\beta T_{\gamma\alpha} + \partial_\gamma T_{\alpha\beta} = \partial_\alpha\varphi\,T_{\beta\gamma} + \partial_\beta\varphi\,T_{\gamma\alpha} + \partial_\gamma\varphi\,T_{\alpha\beta}.$$

Those relations express the idea that:

$$d\left(\tfrac{1}{2}T_{\alpha\beta}\,dx^\alpha\wedge dx^\beta\right) = d\varphi\wedge\left(\tfrac{1}{2}T_{\alpha\beta}\,dx^\alpha\wedge dx^\beta\right);$$

i.e., that the form:

$$e^{-\varphi}\,T_{\alpha\beta}\,dx^\alpha\wedge dx^\beta$$

is closed.

There will then locally exist a vector field **A** (x) whose covariant components are A_α such that:

$$e^{-\varphi}\,T_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha.$$

Therefore, if the form Ω admits an integrating factor then the tensor $S_{\alpha\beta}$ will necessarily have components of the form:

$$S_{\alpha\beta} = \partial_{\dot\alpha}L\,\partial_\beta\varphi - \partial_{\dot\beta}L\,\partial_\alpha\varphi + e^{\varphi}(\partial_\alpha A_\beta - \partial_\beta A_\alpha) \tag{38.6}$$

then functions φ and A_α will depend upon only the variables x^α.

Conversely, if the tensor $S_{\alpha\beta}$ has the preceding form then we will have:

$$e^{-\varphi}\,\Omega = e^{-\varphi}d(\dot dL) + e^{-\varphi}\,\dot dL\wedge d\varphi + d(A_\alpha\,dx^\alpha) = d(e^{-\varphi}\dot dL + A_\alpha\,dx^\alpha),$$

in which $f = e^{-\varphi}$ is indeed an integrating factor for Ω.

Upon remarking that the forms Ω and $f\,\Omega$ admit the same associated system, we can then state the theorem:

Theorem:

In order for the fundamental 2-form Ω of a dynamical system S (L, $S_{\alpha\beta}$) to admit an integrating factor, it is necessary and sufficient that the force tensor should have components that can be put into the form:

$$S_{\alpha\beta} = \partial_{\dot\alpha}L\,\partial_\beta\varphi - \partial_{\dot\beta}L\,\partial_\alpha\varphi + e^{\varphi}(\partial_\alpha A_\beta - \partial_\beta A_\alpha).$$

$\varphi(x)$ and $A_\alpha(x)\,dx^\alpha$ are a scalar function and a 1-form, respectively, that are defined on the configuration space-time V_{n+1} .

Under those conditions, the trajectories are characterized by the existence of the relative integral invariant that is defined by:

$$\omega = (e^{-\varphi}\partial_{\dot\alpha}L + A_\alpha)\,dx^\alpha ;$$

i.e., they are the extremals of the integral:

$$I = \int_{u_0}^{u_1} (e^{-\varphi}L + A_\alpha\,\dot x^\alpha)\,du .$$

Those trajectories are also the geodesics of the Finsler space that is defined on V_{n+1} by the function:

$$e^{-\varphi}L + A_\alpha\,\dot x^\alpha .$$

Particular cases:

1. $\varphi = 0$. In this case, $S_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. The form Ω will then be closed and equal to:

$$d\,(\partial_{\dot\alpha}L\,dx^\alpha + A_\alpha\,dx^\alpha) ,$$

as we found before.

2. $\mathbf{A} = 0$. In this case:

$$S_{\alpha\beta} = \partial_{\dot\alpha}L\,\partial_\beta\varphi - \partial_{\dot\beta}L\,\partial_\alpha\varphi .$$

The trajectories will then be the extremals of the integral:

$$I = \int_{u_0}^{u_1} e^{-\varphi}L\,du .$$

In that case, as in the general case, there exists a Finsler space that admits the same geodesics as the S-Finslerian space that is defined by L and $S_{\alpha\beta}$.

39. Canonical equations. – Let $S\,(L, S_{\alpha\beta})$ be a dynamical system, and let $\mathcal{F}$ be the associated Finsler space; i.e., the Finsler space that is defined on the configuration space-time by:

$$ds = L\,(x^\alpha,\ \dot x^\alpha)\,du .$$

As in § **22**, set:

$$y_\alpha = g_{\alpha\beta}\,y^\beta \qquad \text{with} \qquad y^\beta = dx^\beta / du .$$

The Lagrangian $L\,(x^\alpha, y^\alpha)$ will then correspond to the Hamiltonian $H\,(x^\alpha, y_\alpha)$ such that:

$$H(x^\alpha, y_\alpha) = H(x^\alpha, g_{\alpha\beta}\, y^\beta) = L(x^\alpha, y^\alpha)\,.$$

Recall that the function H is $\dot{h}1$ and that:

(39.1) $$\partial_\alpha L = -\,\partial_\alpha H, \qquad\qquad y^\alpha = H\,\partial^\alpha H\,.$$

If we take the parameter u to be the arc-length s of the trajectory then y^α and y_α will be the contravariant and covariant components, respectively, of a unit vector that is tangent to $\mathcal{F}$.

Upon setting $l^\alpha = dx^\alpha / ds$, $l_\alpha = g_{\alpha\beta}\, l^\beta = \partial_{\dot{\alpha}} L$, we will have:

(39.2) $$L(x^\alpha, l^\alpha) = H(x^\alpha, l_\alpha) = 1.$$

The $2n + 2$ numbers x^α and l_α are supposed to be independent, so they can be considered to be a local coordinate system at a point z in the fiber bundle $\mathcal{V}$ of tangent vectors to V_{n+1}.

Since the numbers x^α and l_α are coupled by the relation:

$$H(x^\alpha, l_\alpha) = 1,$$

they define a point in the state space W.

The trajectories in W of the dynamical system (S) are then defined by the formulas:

$$x^\alpha = x^\alpha(s), \qquad l_\alpha = l_\alpha(s)\,,$$

such that:

$$\frac{dx^\alpha}{ds} = \partial^\alpha H \quad \text{and} \quad \frac{dl_\alpha}{ds} + \partial_\alpha H = X_\alpha\,.$$

The latter equations are the Lagrangian equations of the dynamical system, when written in terms of the variables x and l.

Upon setting $X_\alpha = S_{\alpha\beta}\,\partial^\beta H$, where the $S_{\alpha\beta}$ are the components of the force tensor, we will get the system of canonical equations in the form:

(39.3) $$\begin{cases} \dfrac{dx^\alpha}{ds} = \partial^\alpha H, \\[2mm] \dfrac{dl_\alpha}{ds} = -\partial_\alpha H + S_{\alpha\beta}\,\partial^\beta H, \end{cases}$$

or even in the form:

(39.4)
$$\begin{cases} \dfrac{dx^\alpha}{ds} = \partial^\alpha H, \\ \dfrac{dl_\alpha}{ds} - S_{\alpha\beta}\dfrac{dx^\beta}{ds} = -\partial_\alpha H. \end{cases}$$

Since $X_\alpha \dfrac{dx^\alpha}{ds} = S_{\alpha\beta}\dfrac{dx^\alpha}{ds}\dfrac{dx^\beta}{ds} = 0$, we will see that the system of canonical equations admits the first integral:

$$H(x^\alpha, l_\alpha) = \text{const.}$$

We can then state:

Theorem:

The trajectories of the dynamical system S with the Hamiltonian $H(x^\alpha, l_\alpha)$ *and force tensor* $S_{\alpha\beta}(x, l)$ *are the integral curves of the system* (39.3) *that verify the initial condition:*

$$H[(x^\alpha)_0, (l_\alpha)_0] = 1.$$

We remark that the system of canonical equations is once more the associated system of the 2-form Ω that is written here:

(39.5)
$$\Omega = dl_\alpha \wedge dx^\alpha + \tfrac{1}{2} S_{\alpha\beta}\, dx^\alpha \wedge dx^\beta .$$

Indeed, the associated system to Ω is obtained by writing out that the relation:

$$i(Z)\,\Omega = 0$$

is verified for any vector Z that is tangent to W; i.e., such that:

$$i(Z)\, dH = 0 .$$

Upon writing that:

$$\frac{\partial\Omega}{\partial(dx^\alpha)} = \lambda\frac{\partial(dH)}{\partial(dx^\alpha)},$$

$$\frac{\partial\Omega}{\partial(dl_\alpha)} = \lambda\frac{\partial(dH)}{\partial(dl_\alpha)},$$

and upon remarking that $\lambda = ds$, we will get the canonical relations in the form (39.4).

The first $n + 1$ equations show that the integral curves of the associated system to Ω are basic curves on W. From the results of **23**, that should be obvious, moreover, from the expression (39.5) for Ω.

We have that H = const. is a first integral of the canonical system. Any first integral F = const. is a solution to the partial differential equation:

$$\theta(\mathrm{Z})\, F = 0,$$

in which Z is the vector whose components are the right-hand sides of equations (39.1), so:

$$\partial^\alpha H\, \partial_\alpha F + (-\,\partial_\alpha H + X_\alpha)\, \partial^\alpha F = 0,$$

or rather:

$$(F, H) + X_\alpha\, \partial^\alpha F = 0,$$

in which (F, H) denotes the Poisson bracket of the functions F and H relative to the variables x^α and l_α.

40. Canonical equations in matrix form ([22]). – Let $\left(\dfrac{dz}{ds}\right)$ be the column matrix that is composed of the derivatives of the x^α and l_α with respect to s, let (grad$_z$ H) be the column matrix of the partial derivatives of H with respect to the x^α and l_α, let E_S be the antisymmetric matrix $\begin{pmatrix} 0 & I \\ -I & S \end{pmatrix}$, and let J_S be the antisymmetric matrix $\begin{pmatrix} S & -I \\ I & 0 \end{pmatrix}$, where S is the matrix whose elements are $S_{\alpha\beta}$ and I and 0 are the identity matrix and zero matrix, respectively, of order $n + 1$.

The system of canonical equations (39.3) can be put into the form:

$$\left(\frac{dz}{ds}\right) = E_s\, (\mathrm{grad}_z\, H)\,. \tag{40.1}$$

Since the matrix J_S is the inverse of the matrix E_S, the canonical system can be put into the equivalent form:

$$J_S \left(\frac{dz}{ds}\right) = (\mathrm{grad}_z\, H)\,, \tag{40.2}$$

which corresponds to equations (39.4).

We remark that J_S is the matrix associated with the form Ω; i.e., the matrix of the coefficients of:

$$\frac{\partial\Omega}{\partial(dx^\alpha)} \qquad \text{and} \qquad \frac{\partial\Omega}{\partial(dl_\alpha)}\,.$$

We can also say that J_S is the matrix of the coefficients of the alternating bilinear form $f(\Omega)$ that is associated with Ω; indeed, one has:

([22]) Y. Thiry [**2**], pp. 206-212.

$$f(\Omega) = {}^t(dz)\, J_S\, (\delta z),$$

in which ${}^t(dz)$ is the row matrix that has the elements dx^α, dl_α. (δz) is the column matrix whose elements are δx^α, δl_α, while (dz) and (δz) correspond to two arbitrary vectors $d\,\mathbf{z}$ and $\delta\mathbf{z}$, resp., in the space T_z that is tangent to the point $z\,(x^\alpha, l_\alpha)$ of $\mathcal{V}$.

In what follows, it will sometimes be convenient to set: $l_\alpha = x^{\alpha *}$, with $\alpha^* = \alpha + n + 1$, and to denote an index that takes the values 1, 2, …, $2n + 2$ by an uppercase Latin letter. When one lets a_{AB} denote the element of the A^th row and B^th column of J_S, the 2-form Ω can then be written:

$$\Omega = \tfrac{1}{2}\, a_{\mathrm{AB}}\, dx^{\mathrm{A}} \wedge dx^{\mathrm{B}}, \tag{40.3}$$

and the system of canonical equations (40.2) will become:

$$a_{\mathrm{AB}} \frac{dx^{\mathrm{B}}}{ds} = \partial_{\mathrm{A}} H\,. \tag{40.4}$$

41. Change of variables. – Consider the change of variables on $\mathcal{V}$ that is defined by:

$$x^{\mathrm{A}} = \mathrm{X}^{\mathrm{A}}\,(x^{\mathrm{B}'}), \tag{41.1}$$

where the $2n + 2$ functions X^{A} are supposed to be differentiable with respect to the new variables $x^{\mathrm{A}'}$.

We deduce the following formulas by differentiating (41.1):

$$dx^{\mathrm{A}} = \partial_{\mathrm{B}'}\, \mathrm{X}^{\mathrm{A}}\, dx^{\mathrm{B}'}\,. \tag{41.2}$$

Let M denote the Jacobian matrix whose elements are $\mathrm{X}^{\mathrm{A}}_{\mathrm{B}'} = \partial_{\mathrm{B}'}\, \mathrm{X}^{\mathrm{A}}$, where A is the row index and B′ is the column index. We can then write the relations (41.2) in the matrix form:

$$(dz) = M\,(dz')\,.$$

The bilinear form $f(\Omega) = {}^t(dz)\, J_S\, (\delta z)$ transforms into:

$$f(\Omega') = {}^t(dz')\, {}^tM\, J_S\, M\, (\delta z)\,,$$

in which ${}^tM$ is the matrix transpose of M.

The bilinear form $f(\Omega')$ is once more alternating because the matrix:

$$K_S = {}^tM\, J_S\, M$$

is antisymmetric.

The associated system to the corresponding form Ω' is:

(41.3) $$K_S\left(\frac{dz'}{ds}\right) = (\text{grad}_{z'}\,H')\,,$$

in which $(\text{grad}_{z'}\,H')$ is the column matrix whose partial derivatives with respect to $x^{A'}$ of the Hamiltonian H, which is assumed to be expressed in terms of the new variables.

The canonical system is then written in the following form in terms of the new variables:

(41.4) $$a_{A'B'}\frac{dx^{B'}}{ds} = \partial_{A'}\,H',$$

in which the matrix whose elements are $a_{A'B'}$ is the matrix ${}^tM\,J_S\,M$, J_S is the matrix in the old canonical system, M is the Jacobian matrix of the change of variables, and:

$$H'(x^{A'}) = H\,[\mathrm{X}^A\,(x^{B'})] = H\,(x^A).$$

42. Canonical transformations. – We say that the transformation that is defined by the matrix M is canonical if the matrix:

$$K_S = {}^tM\,J_S\,M$$

has the form:

$$K_S = \begin{pmatrix} S' & -I \\ I & 0 \end{pmatrix} = J_{S'}$$

in which S' is an antisymmetric matrix.

We say that the transformation that is defined by the matrix M is *pseudo-canonical* if:

$$K_S = f\,J_{S'}$$

for any antisymmetric matrix S, where f is a scalar function of the variables $x^{A'}$.

One easily shows that the set of canonical or pseudo-canonical transformations has a multiplicative structure group that locally admits the symplectic group $Sp\,(n + 1, \mathbb{R})$ as a subgroup.

Let us try to characterize the matrices M that define the canonical transformations. In order to do that, we return to the notations x^α, l_α, and denote the new variables by $x^{\alpha'}$, $l_{\alpha'}$.

Set:

$$x^\alpha = \mathrm{X}^\alpha\,(x^{\beta'}, l_{\beta'}), \qquad l_\alpha = \mathrm{L}_\alpha\,(x^{\beta'}, l_{\beta'})\,,$$

so

$$dx^\alpha = \mathrm{X}^\alpha_{\beta'}\,dx^{\beta'} + \mathrm{X}^{\alpha\beta'}dl_{\beta'}\,,$$
$$dl_\alpha = \mathrm{L}_{\alpha\beta'}\,dx^{\beta'} + \mathrm{L}^{\beta'}_\alpha\,dl_{\beta'}\,,$$

with

$$\mathrm{X}^\alpha_{\beta'} = \partial_{\beta'}\mathrm{X}^\alpha, \qquad \mathrm{X}^{\alpha\beta'} = \partial^{\beta'}\mathrm{X}^\alpha,$$
$$\mathrm{L}_{\alpha\beta'} = \partial_{\beta'}\mathrm{L}_\alpha, \qquad \mathrm{L}^{\beta'}_\alpha = \partial^{\beta'}\mathrm{L}_\alpha.$$

The matrix M then has the form:

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

in which A, B, C, D are the submatrices of M of order $n + 1$ whose elements are $\mathrm{X}^{\alpha}_{\beta'}$, $\mathrm{X}^{\alpha\beta'}$, $\mathrm{L}_{\alpha\beta'}$, $\mathrm{L}^{\beta'}_{\alpha}$, respectively, and the first index is the column index.

More explicitly:

$$K_S = \begin{pmatrix} {}^tA & {}^tC \\ {}^tB & {}^tD \end{pmatrix}\begin{pmatrix} S & -I \\ I & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} {}^tASA + {}^tCA - {}^tAC & {}^tASB + {}^tCB - {}^tAD \\ {}^tBSA + {}^tDA - {}^tBC & {}^tBSB + {}^tDB - {}^tBD \end{pmatrix}.$$

In order for K_S to have the form J_S for any S, it is necessary that:

$${}^tDA - {}^tBC = 1 \quad \text{and} \quad {}^tDB - {}^tBD = 0 .$$

Suppose that the matrix A is regular. The conditions:

$${}^tBSA = {}^tBSB = 0$$

will then be equivalent to the condition:

$${}^tBS = 0,$$

which will be verified for any S only when:

$$B = 0.$$

We deduce the following result from this: In order for the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$, in which A is supposed to be regular, to define a canonical transformation, it is necessary and sufficient that:

1. $B = 0$ and 2. ${}^tDA = I,$

or rather that the variables x^{α} must be independent of the new variables $l_{\alpha'}$ and that:

$$\partial^{\gamma'} \mathrm{L}_{\alpha}\, \partial_{\gamma'} \mathrm{X}^{\beta} = \delta^{\beta}_{\alpha} .$$

Under those conditions, the form:

$$\Omega = dl_{\alpha} \wedge dx^{\alpha} + \tfrac{1}{2} S_{\alpha\beta}\, dx^{\alpha} \wedge dx^{\beta}$$

will have the form:

$$\Omega' = dl_{\alpha'} \wedge dx^{\alpha'} + \tfrac{1}{2} S'_{\alpha'\beta'}\, dx^{\alpha'} \wedge dx^{\beta'}$$

for its transform, in which the matrix whose elements are the $S'_{\alpha'\beta'}$ is the matrix:

$$S' = {}^tASA + {}^tCA - {}^tAC .$$

The canonical equations of the dynamical system considered relative to the new variables will then be:

$$\begin{cases} \dfrac{dx^{\alpha'}}{ds} = \partial^{\alpha'} H', \\ \dfrac{dl_{\alpha'}}{ds} = -\partial_{\alpha'} H' + S'_{\alpha'\beta'}\, \partial^{\beta'} H', \end{cases}$$

with $H'(x^{\alpha'}, l_{\alpha'}) = H(\mathrm{X}^{\alpha}, \mathrm{L}_{\alpha})$.

For a fixed tensor $S_{\alpha\beta}$, one can find some other transformations that respect the form of the canonical system. The corresponding matrices A, B, C, D verify the relations:

$$\begin{gathered} {}^tBSB + {}^tDB - {}^tBD = 0, \\ {}^tBSA + {}^tDA - {}^tBC = f\, I, \end{gathered}$$

in which f is an arbitrary scalar function of the new variables.

Particular cases:

1. *Change of variables that leaves the form* Ω *invariant.*

We say that the form Ω is invariant under the change of variables that is defined by M if the transformed form is:

$$\Omega' = dl_{\alpha'} \wedge dx^{\alpha'} + \tfrac{1}{2} S_{\alpha'\beta'}\; dx^{\alpha'} \wedge dx^{\beta'},$$

in which $S_{\alpha'\beta'}$ is obtained by replacing the x^{α} in $S_{\alpha\beta}$ with $\mathrm{X}^{\alpha}(x', l')$ and the l_{α} with $\mathrm{L}_{\alpha}(x', l')$.

In order for that to be true for any S, it is necessary and sufficient that when one supposes that A is regular, one should have:

1. $B = 0$,

2. ${}^tDA = I$,

3. ${}^tASA + {}^tCA - {}^tAC = S$ for any S ; i.e., that $A = I$ and the ${}^tC = C$.

The matrices thus-obtained have the form: $M_C = \begin{pmatrix} I & 0 \\ C & I \end{pmatrix}$, where C is a symmetric matrix.

Those matrices form a subgroup of the multiplicative group of canonical matrices that are isomorphic to the additive group of symmetric matrices of order $n + 1$.

The matrix C is symmetric, so we will have the identity:

$$\partial_{\alpha'} \mathrm{L}_{\beta} - \partial_{\beta'} \mathrm{L}_{\alpha} = 0 .$$

There will then be a function $F(x')$ such that:

$$\mathrm{L}_\alpha = l_{\alpha'} + \partial_{\alpha'} F(x^{\beta'}) .$$

The change of variables considered is then defined by:

$$x^\alpha = x^{\alpha'} + a^\alpha ,$$

in which the a^α are constants, and:

$$l_\alpha = l_{\alpha'} + \partial_{\alpha'} F .$$

It will then result that:

$$dl_\alpha \wedge dx^\alpha = dl_{\alpha'} \wedge dx^{\alpha'}$$

and

$$dx^\alpha \wedge dx^\beta = dx^{\alpha'} \wedge dx^{\beta'}.$$

2. *Change of variables on V_{n+1}, prolonged to $\mathcal{V}$.* – Define a change of variables on V_{n+1} by:

$$x^\alpha = \mathrm{X}^\alpha (x^{\beta'}) .$$

From the tensorial nature of l_α and $S_{\alpha\beta}$, it is obvious that this change of variables is canonical. Let us verify that. By differentiation, we will get:

$$dx^\alpha = \partial_{\beta'} \mathrm{X}^\alpha \, dx^{\beta'} \qquad \text{or} \qquad l^\alpha = \partial_{\beta'} \mathrm{X}^\alpha \, l^{\beta'}.$$

The matrix A has elements $\mathrm{X}^\alpha_{\beta'} = \partial_{\beta'} \mathrm{X}^\alpha$, where the α is the row index and β' is the column index. The matrix B is zero.

The covariant components l_α transforms according to the law:

$$l_{\beta'} = \mathrm{X}^\alpha_{\beta'} \, l_\alpha \qquad \text{or} \qquad l_\alpha = \mathrm{X}^{\beta'}_\alpha \, l_{\beta'} .$$

By differentiation, we get:

$$dl_\alpha = \partial_{\beta'} \mathrm{X}^\gamma_{\beta'} \, l_{\gamma'} \, dx^{\beta'} + \mathrm{X}^{\beta'}_\alpha \, dl_{\beta'} .$$

The matrix D whose elements are $\mathrm{X}^{\beta'}_\alpha$, where the index α represents the rows, is the inverse of the matrix whose elements are $\mathrm{X}^\alpha_{\beta'}$, where the index α represents the columns; i.e., ${}^tA$. We then have:

$$D = {}^tA^{-1}.$$

The transformation $(x^\alpha, l_\alpha) \rightarrow (x^{\alpha'}, l_{\alpha'})$ thus-defined is indeed a canonical transformation.

The matrix $K_S = {}^tM \, J_S \, M$ then has the form: $\begin{pmatrix} {}^tA S A & -I \\ I & 0 \end{pmatrix}$, because one can show directly from the expression for C that:

$${}^tCA - {}^tAC = 0,$$

but that results from the fact that the change of variables considered is such that:

$$l_\alpha \, dx^\alpha = l_{\alpha'} \, dx^{\alpha'} \qquad \text{or} \qquad dl_\alpha \wedge dx^\alpha = dl_{\alpha'} \wedge dx^{\alpha'}.$$

Remark. – It is easy to recover the results of § **38** by the matrix method in the case where Ω is closed or admits an integrating factor.

43. Lee space defined by the 2-form Ω. – Consider the manifold $\mathcal{V}$ whose point z admits the $2n + 2$ numbers x^α and l_α, which are assumed to be independent, as a local coordinate system.

When one is given the 2-form on $\mathcal{V}$:

$$\Omega = dl_\alpha \wedge dx^\alpha + \tfrac{1}{2} S_{\alpha\beta} \, dx^\alpha \wedge dx^\beta, \tag{43.1}$$

that will define an *almost-symplectic structure* on $\mathcal{V}$, or rather, a *Lee space structure* ([23]).

Indeed, since the variables x^α and l_α are assumed to be independent, the form Ω will have maximum rank $2n + 2$; the associated matrix to Ω is $J_S = \begin{pmatrix} S & -I \\ I & 0 \end{pmatrix}$, which is a regular matrix, so its inverse will be the matrix $E_S = \begin{pmatrix} 0 & I \\ -I & S \end{pmatrix}$.

As in § **40**, set $l_\alpha = x^{\overset{*}{\alpha}}$, with $\overset{*}{\alpha} = \alpha + n + 1$.

The 2-form Ω is then written:

$$\Omega = \tfrac{1}{2} a_{\mathrm{AB}} \, dx^{\mathrm{A}} \wedge dx^{\mathrm{B}},$$

where a_{AB} is the element of J_S that is in row A and column B.

Recall that any uppercase Latin index can take the values $1, 2, \ldots, 2n + 2$.

Let a^{AB} denote the elements of the matrix E_S that is inverse to J_S . With Lee, introduce the following four tensors:

1. The *curvature tensor*, whose components:

$$K_{\mathrm{ABC}} = \partial_{\mathrm{A}} \, a_{\mathrm{BC}} + \partial_{\mathrm{B}} \, a_{\mathrm{CB}} + \partial_{\mathrm{C}} \, a_{\mathrm{AB}} \tag{43.2}$$

are such that:

$$d\,\Omega = \tfrac{1}{6} \, K_{\mathrm{ABC}} \, dx^{\mathrm{A}} \wedge dx^{\mathrm{B}} \wedge dx^{\mathrm{C}} .$$

2. The *covariant curvature vector*, whose components are:

([23]) H. C. Lee, "A kind of even-dimensional geometry and its application to exterior calculus," Amer. J. Math. **55** (1943), 433-438.

(43.3) $$K_{\mathrm{A}} = K_{\mathrm{ABC}}\, a^{\mathrm{BC}},$$

in which the form $K_{\mathrm{A}}\, dx^{\mathrm{A}}$ is the codifferential $(\delta\Omega)$ of Ω with respect to itself.

3. The *first conformal curvature tensor*, whose components are:

(43.4) $$b_{\mathrm{AB}} = \partial_{\mathrm{A}} K_{\mathrm{B}} - \partial_{\mathrm{B}} K_{\mathrm{A}}\ .$$

One has:

$$d\,(\delta\Omega) = \tfrac{1}{2}\, b_{\mathrm{AB}}\, dx^{\mathrm{A}} \wedge dx^{\mathrm{B}}.$$

4. The *second conformal curvature tensor*, whose components are:

(43.5) $$C_{\mathrm{ABC}} = K_{\mathrm{ABC}} + \frac{1}{2n}(K_{\mathrm{A}}\, a_{\mathrm{BC}} + K_{\mathrm{B}}\, a_{\mathrm{CA}} + K_{\mathrm{C}}\, a_{\mathrm{AB}})\ .$$

That tensor is such that:

$$\tfrac{1}{6}\, C_{\mathrm{ABC}}\, dx^{\mathrm{A}} \wedge dx^{\mathrm{B}} \wedge dx^{\mathrm{C}} = d\,\Omega - \frac{1}{n}\,\delta\Omega \wedge \Omega\ .$$

Let us now specify the components of the various tensors.

We find that for the curvature tensor, we have:

(43.6) $$K_{\alpha\beta\gamma} = \partial_\alpha S_{\beta\gamma} + \partial_\beta S_{\gamma\alpha} + \partial_\gamma S_{\alpha\beta},$$

(43.7) $$K_{\alpha^*\beta\gamma} = \partial_{\alpha^*} S_{\beta\gamma} = \partial^\alpha S_{\beta\gamma}, \qquad K_{\alpha\beta^*\gamma} = \partial_{\beta^*} S_{\gamma\alpha}, \qquad K_{\alpha\beta\gamma^*} = \partial_{\gamma^*} S_{\alpha\beta}.$$

The components that admit more than one starred index are zero.

We find that for the curvature vector, we have:

$$K_\alpha = K_{\alpha\beta\gamma}\, a^{\beta\gamma} + K_{\alpha\beta^*\gamma}\, a^{\beta^*\gamma} + K_{\alpha\beta\gamma^*}\, a^{\beta\gamma^*} = \partial_{\gamma^*} S_{\alpha\beta}\, \delta^{\beta\gamma} - \partial_{\beta^*} S_{\gamma\alpha}\, \delta^{\beta\gamma},$$

so

(43.8) $$K_\alpha = 2\sum_\beta \partial_{\beta^*} S_{\alpha\beta} = 2\,\partial^\beta S_{\alpha\beta},$$

with summation over β.

$$K_{\alpha^*} = K_{\alpha^*\beta\gamma}\, a^{\beta\gamma} = 0.$$

We will then have:

$$\delta\Omega = 2\,\partial^\beta S_{\alpha\beta}\, dx^\alpha.$$

For the second curvature tensor, we will have:

$$C_{\alpha\beta\gamma} = K_{\alpha\beta\gamma} + \frac{1}{2n}(K_\alpha S_{\beta\gamma} + K_\beta S_{\alpha\gamma} + K_\gamma S_{\alpha\beta}),$$

or

(43.9) $$C_{\alpha\beta\gamma} = \mathrm{S}\left(\partial_\alpha S_{\beta\lambda} + \frac{1}{n}\partial^\lambda S_{\alpha\lambda}\, S_{\beta\gamma}\right),$$

in which S indicates that one must combine the terms in the parentheses with the ones that one deduces from them by cyclically permuting α, β, γ:

$$C_{\alpha^*\beta\gamma} = K_{\alpha^*\beta\gamma} + \frac{1}{2n}(K_{\alpha^*}\, S_{\beta\gamma} - K_\beta\, S_{\alpha\gamma} + K_\gamma S_{\alpha\beta}) .$$

If all three of α, β, γ are different then we will get:

(43.10) $$C_{\alpha^*\beta\gamma} = \partial_{\alpha^*}\, S_{\beta\gamma} = \partial^\alpha\, S_{\beta\gamma^*} .$$

If $\beta = \gamma$ then the corresponding coefficients C will be zero. If $\alpha = \beta \neq \gamma$ then we will get:

(43.11) $$C_{\alpha^*\alpha\gamma} = \partial_{\alpha^*}\, S_{\alpha\gamma} + \frac{1}{n}\sum_\lambda \partial_{\lambda^*}\, S_{\gamma\lambda} .$$

All of the coefficients C that have more than one starred index are zero.

44. Necessary and sufficient conditions for Ω to admit an integrating factor. – The remarkable Lee spaces are the ones that Lee called "flat," which are the ones for which Ω is closed, and the ones that are "conformally flat," for which the form Ω admits an integrating factor. We are then reduced to the fundamental case that was studied on § **38**.

In order for the form Ω to be closed, it is necessary and sufficient that the curvature tensor should be zero. As we have seen, that condition is equivalent to the existence of a vector potential **A** (x) such that locally:

$$S_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha .$$

In order for the form Ω to admit an integrating factor, it is necessary and sufficient that the second conformal curvature tensor should be zero. That theorem is due to Lee, and was rediscovered and completed by C. Ehresmann and P. Libermann ([24]), who supposed that $n > 1$; that is obviously the case for the 2-form Ω of a dynamical system.

Now write down that the tensor $C_{\mathrm{ABC}} = 0$. From (43.10), $S_{\beta\gamma}$ depends upon only the variables x, l_β, and l_γ. From (43.11), $\partial_{\alpha^*}\, S_{\alpha\gamma}$ has a value that is independent of α. Then set:

$$\partial_{\alpha^*}\, S_{\alpha\gamma} = \varphi_\gamma ,$$

and the relations $C_{\alpha^*\alpha\gamma} = 0$ will then be verified. Since $\partial_{\alpha^*\gamma^*}\, S_{\alpha\gamma} = 0$, the functions φ_γ depend upon only the variables x.

([24]) C. Ehresmann and P. Libermann, C. R. Acad. Sci. **227** (1948), 420-421; *ibid.*, **229** (1949), 697-699.

$S_{\alpha\beta}$ then has the form:

(44.1) $$S_{\alpha\beta} = l_\alpha\, \varphi_\beta\,(x) - l_\beta\, \varphi_\alpha\,(x) + T_{\alpha\beta}\,(x)\ .$$

Let us now specify the conditions:

$$C_{\alpha\beta\gamma} = 0\ .$$

Upon remarking that:

$$\frac{1}{n}\partial^\lambda S_{\alpha\lambda} = -\,\varphi_\alpha\,(x)\ ,$$

we will get:

(44.2) $$\mathrm{S}\,(l_\alpha\,\partial_\gamma\,\varphi_\beta - l_\alpha\,\partial_\gamma\,\varphi_\beta + \partial_\gamma\,T_{\alpha\beta} - \varphi_\gamma\,T_{\alpha\beta}) = 0\ ,$$

after reductions. Those conditions are equivalent to:

(44.3) $$\partial_\gamma\,\varphi_\beta - \partial_\beta\,\varphi_\gamma = 0$$

and

(44.4) $$\mathrm{S}\,(\partial_\gamma\,T_{\alpha\beta} - \varphi_\gamma\,T_{\alpha\beta}) = 0.$$

The conditions (44.3) express the idea that the form $\varphi_\alpha\,dx^\alpha$ is closed on V_{n+1} ; there will then exist a function $\varphi\,(x)$ such that locally:

$$\varphi_\alpha = \partial_\alpha\,\varphi\ .$$

The conditions (43.4) express the idea that the 2-form:

$$\tfrac{1}{2}\,T_{\alpha\beta}\,dx^\alpha \wedge dx^\beta$$

admits $e^{-\varphi}$ as an integrating factor.

There will then exist a vector potential $A_\alpha\,(x)$ such that locally one has:

$$T_{\alpha\beta} = e^\varphi\,(\partial_\alpha\,A_\beta - \partial_\beta\,A_\alpha)\ .$$

As in § **37**, we find that $S_{\alpha\beta}$ has the form:

(44.5) $$S_{\alpha\beta} = l_\alpha\,\partial_\beta\,\varphi - l_\beta\,\partial_\alpha\,\varphi + e^\varphi\,(\partial_\alpha\,A_\beta - \partial_\beta\,A_\alpha)\ .$$

We can then state the theorem:

Theorem:

In order for the fundamental 2-form of a dynamical system:

$$\Omega = dl_\alpha \wedge dx^\alpha + S_{\alpha\beta}\,dx^\alpha \wedge dx^\beta$$

to admit an integrating factor, it is necessary and sufficient that the force tensor $S_{\alpha\beta}$ *should verify the following conditions for any* α, β, γ:

1. $$\frac{\partial S_{\beta\gamma}}{\partial l_\alpha} = 0 \qquad \text{for} \qquad \alpha \neq \beta \quad \text{and} \quad \alpha \neq \gamma.$$

2. $$\frac{\partial S_{\beta\gamma}}{\partial l_\beta} = \frac{\partial S_{\alpha\gamma}}{\partial l_\alpha}.$$

3. $$\mathrm{S}\left(\frac{\partial S_{\beta\gamma}}{\partial x^\alpha} + \frac{1}{n}\sum_\lambda \frac{\partial S_{\alpha\lambda}}{\partial l_\lambda} S_{\beta\gamma}\right) = 0.$$

Under those conditions, there will exist a scalar function $\varphi(x)$ and a covariant field $A_\alpha(x)$ that is defined on V_{n+1} such that locally one has:

$$S_{\alpha\beta} = l_\alpha \partial_\beta \varphi - l_\beta \partial_\alpha \varphi + e^\varphi (\partial_\alpha A_\beta - \partial_\beta A_\alpha) .$$

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CHAPTER VI

DYNAMICAL SYSTEMS WITH NON-HOLONOMIC CONSTRAINTS

45. – First-order constraints in the homogeneous formalism. – Let (S_0) be a dynamical system with perfect holonomic constraints and n degrees of freedom x^k. Recall the notations of § **29**; in the inhomogeneous formalism, the equations of motion are:

$$P_k(\mathcal{L}) = Q_k . \tag{45.1}$$

Let $a(x^k, t, x'^k)$ be a function of $2n + 1$ variables x^k, t, and x'^k that is not the total derivative with respect to time of a function $A(x^k, t)$, and is such that a = const. is not a first integral of the equations of motion of (S_0).

Imposing a first-order non-holonomic constraint on the dynamical system S_0 :

$$a(x^k, t, x'^k) = 0$$

amounts to adding a function R_k of x^i, t, and x'^i to each right-hand side of equations (45.1) in such a fashion that the motion of the new dynamical system (S) will be defined in configuration space-time V_{n+1} by:

$$P_k(\mathcal{L}) = Q_k + R_k ; \tag{45.2}$$

these equations admit $a = 0$ as a first integral ([25]).

We now pass to the homogeneous formalism, as in § **29**.

Set:

$$Y_k = R_k\,\dot{x}^{n+1}, \qquad\qquad Y_{n+1} = -R_k\,\dot{x}^k .$$

The Y_α are the covariant components of a vector that is called the *constraint force*.

The constraint relation will then be written:

$$a(x^\alpha, \dot{x}^\alpha) = 0.$$

The function a is $\dot{h}$. Since $\lambda\, a = 0$ defines the same constraint on $\mathcal{V}$ when $\lambda \neq 0$, we can fix the degree of homogeneity of a arbitrarily.

In general, suppose that a is $\dot{h}$ 0.

The homogeneous Lagrangian L defines a Finsler space structure on V_{n+1} . We suppose that the space is regular and that the metric tensor $g_{\alpha\beta}$ is such that:

([25]) F. Gallisot [**1**], pp. 45.

$$ds^2 = g_{\alpha\beta}\, dx^\alpha\, dx^\beta$$

is a positive-definite quadratic form at any point of $\mathcal{V}$.

The trajectories of (S) in configuration space-time are then defined by the n + 1 equations:

(45.3) $$P_\alpha (L) = X_\alpha + Y_\alpha .$$

The canonical equations of the trajectories in phase space W are:

(45.4) $$\begin{cases} \dfrac{dx^\alpha}{ds} = \partial^\alpha H, \\ \dfrac{dl_\alpha}{ds} = -\partial^\alpha H + X_\alpha + Y_\alpha , \end{cases}$$

in which the functions H, X_α , Y_α are supposed to be expressed with the aid of the variables x^α and l_α .

The function $a\,(x^\alpha,\ \dot{x}^\alpha)$ corresponds to the function $\bar{a}\,(x^\alpha, l_\alpha)$ such that:

$$\bar{a}\,(x^\alpha,\ \partial_{\dot{\alpha}} L\,) = a\,(x^\alpha,\ \dot{x}^\alpha)\ .$$

Let us now translate the fact that $\bar{a}\,(x^\alpha, l_\alpha) = 0$ is a first integral of equations (45.4).

Upon writing that:

$$i\,(\mathrm{Z})\ d\,\bar{a}\ = 0,$$

in which Z is the tangent vector to W whose components are the right-hand sides of equations (45.4), we will get:

(45.5) $$(\bar{a}, H) + \partial^\alpha \bar{a}\,(X_\alpha + Y_\alpha) = 0.$$

($\bar{a}$, H) denotes the Poisson bracket of the functions $\bar{a}$ and H.

We associate the constraint force Y, whose components Y_α are $\dot{h}1$ and such that:

$$Y_\alpha\, \dot{x}^\alpha = 0,$$

with the tensor $T_{\alpha\beta}$, which we call the *constraint tensor*, and which is defined by:

(45.6) $$T_{\alpha\beta} = \tfrac{1}{2}(\partial_{\dot{\beta}}\, Y_\alpha - \partial_{\dot{\alpha}}\, Y_\beta)\ .$$

The components $T_{\alpha\beta}$ are $\dot{h}0$ and such that:

$$T_{\alpha\beta}\, \dot{x}^\beta = Y_\alpha\, .$$

The considerations of the preceding chapter will then show that the system (45.4) is the associated system of the 2-form:

$$\Omega = dl_\alpha \wedge dx^\alpha + \tfrac{1}{2}(S_{\alpha\beta} + T_{\alpha\beta})\, dx^\alpha \wedge dx^\beta , \tag{45.7}$$

in which the variables l_α and x^α are coupled by the relation:

$$H(x^\alpha, l_\alpha) = 1.$$

The trajectories of the dynamical system (S) can be considered to be the geodesics of the S-Finslerian space that is defined by the Lagrangian L and tensor $S_{\alpha\beta} + T_{\alpha\beta}$.

Being given the constraint relation $a = 0$ will not determine the constraint force; it will depend upon the manner by which the constraint is realized, in addition. In what follows, we shall study the constraints that are realized perfectly, in the Delassus sense, or "perfect constraints."

46. – Perfect constraints. – The constraint relation:

$$a(x^\alpha, y^\alpha) = 0 , \qquad \text{with} \quad y^\alpha = \frac{dx^\alpha}{du} , \tag{46.1}$$

defines a cone of directions C_x at each point x of V_{n+1} that is situated in the space T_x that is tangent to V_{n+1} at x.

That cone will reduce to a plane if the constraint is linearly non-holonomic; i.e., if the relation (46.1) can be put into the form:

$$a_\alpha(x)\, y^\alpha = 0 .$$

In this case, the constraint is called *perfect* if the constraint force Y_α is perpendicular to that plane in the sense of the Finslerian metric.

In the general case, associate each generator G of the cone C_x with the tangent plane along that generator.

If we let y_0^α denote the components of a vector that is carried by G then the equation of that tangent plane will be:

$$\partial_{\dot{\alpha}} a(x^\alpha, y_0^\alpha)\, y^a = 0. \tag{46.2}$$

The relation (46.2) defines what we call the *linear constraint that is tangent to the given constraint* at the point x^α, y_0^α of W.

The constraint is called *perfect* if the constraint force at the point x^α, y_0^α is perpendicular to the tangent plane to the cone C_x along the generator y_0^α. The constraint forces then verify the condition that is called *the generalized virtual work condition:* The virtual work $Y_\alpha\, \delta x^\alpha$ done by the constraint force that corresponds to the line element (x_0^α, y_0^α) is zero is zero for any virtual displacement δx^α that is permitted by the linear constraint that is tangent to the given constraint at x_0^α, y_0^α.

In the case of a perfect constraint, the components of the constraint force have the form:

$$Y_\alpha = \lambda\, \partial_{\dot{\alpha}}\, a\,, \tag{46.3}$$

in which λ is a function of the x^α, y^α that is $\dot{h}1$ if a is $\dot{h}1$ and $\dot{h}2$ if a is $\dot{h}0$. Conversely, if the components of **Y** have the form (46.3) then the constraint considered will be perfect.

The statement of the generalized virtual work condition that was given above is equivalent to the following statement: At a given instant, the virtual work done by the constraint forces $Y_k\, \delta x^k$ is zero for any virtual displacement δx^k that is compatible with the constraint that is independent of time that coincides with the given constraint at the instant considered.

Indeed, that condition will imply that:

$$Y_k = \lambda\, \partial_{\dot{k}}\, a\,.$$

Since $Y_\alpha\, \dot{x}^\alpha = 0$ and $\partial_{\dot{\alpha}}\, a\, \dot{x}^\alpha = 0$, where a is supposed to $\dot{h}0$, we deduce from this that:

$$Y_{n+1} = \lambda\, \partial_{n+1}\, a\,.$$

The perfect first-order constraints include, in particular, the linearly non-holonomic constraints, and even the holonomic constraints, with the condition that one must replace the constraint relation:

$$A\,(x^\alpha) = 0$$

with the first-order relation:

$$a = \partial_{\dot{\alpha}} A\, \dot{x}^\alpha = 0.$$

The constraint force in the case of a perfect constraint is determined when one is given the constraint relation $a = 0$.

The coefficient λ is determined when one expresses the idea that $a = 0$ is a first integral of the equations of motion.

In order to explain the calculations, suppose that the function a is $\dot{h}0$ or that it can be expressed as a function of the variables x^α and l_α.

Those variables are coupled by the relation:

$$H\,(x^\alpha, l_\alpha) = 1,$$

in which H is the Hamiltonian that corresponds to the Lagrangian $L\,(x^\alpha, y^\alpha)$.

Let $\mathcal{F}$ be the Finsler space that is defined on V_{n+1} by:

$$ds = L(x^\alpha, dx^\alpha) .$$

Replace l_α with $\partial_{\dot\alpha} L$ in $\bar{a}(x^\alpha, l_\alpha)$. Upon partially differentiating this, we will get:

$$\begin{aligned}\partial_{\dot\alpha} a &= \partial^\beta \bar{a}\, \partial_{\dot\alpha\dot\beta} L \\ &= \partial^\beta \bar{a}\,(g_{\alpha\beta} - l_\alpha l_\beta) \\ &= \partial^\beta \bar{a}\, g_{\alpha\beta} ,\end{aligned}$$

because $\partial_{\dot\alpha} \bar{a}\, l^\alpha = \partial^\beta \bar{a}\, l_\beta = 0$, since a is $\dot{h}0$.

We then deduce that the components of the constraint force are:

$$Y_\alpha = \lambda \partial_{\dot\alpha} \bar{a} \qquad \text{and} \qquad Y^\alpha = \lambda \partial^\alpha \bar{a} .$$

The relation (45.5), which expresses the idea that $\bar{a} = 0$ is a first integral of the equations of motion, will then become:

$$(\bar{a}, H) + \partial^\alpha \bar{a}\, X_\alpha + \lambda\, g_{\alpha\beta}\, \partial^\alpha \bar{a}\, \partial^\beta \bar{a} = 0 . \tag{46.4}$$

The coefficient of λ, which represents the square of the norm of the vector $\partial^\alpha \bar{a}$, is positive; that relation will then determine λ .

Example of a perfect non-holonomic constraint. – One launches a projectile with an initial velocity $\mathbf{V}_0$. One makes a force **F** act on that projectile in such a fashion that the motion is uniform and planar.

With respect to an orthonormal frame *Ox*, *Oy* that is situated in the trajectory plane, the constraint relation is:

$$a = x'^2 + y'^2 - v^2 = 0$$

or

$$a = \frac{\dot{x}^2 + \dot{y}^2}{\dot{t}^2} - v^2 = 0 .$$

Making the hypothesis that the constraint is perfect amounts to supposing that the constraint force **F** is collinear with the velocity vector.

Geometric interpretation of the trajectories. – Consider a dynamical system (*S*) with a Lagrangian *L* that is subject to a perfect constraint that is defined by:

$$a(x^\alpha, \dot{x}^\alpha) = 0,$$

with the function *a* being $\dot{h}0$.

On the other hand, suppose that the tensor $S_{\alpha\beta} = 0$, in such a way that the equations of the trajectories in the configuration space-time V_{n+1} must be defined by the system:

$$P_\alpha(L) = \lambda \partial_{\dot{\alpha}} a, \qquad \text{in which } \lambda \text{ is } \dot{h}2.$$

The corresponding constraint tensor is:

$$T_{\alpha\beta} = \tfrac{1}{2}(\partial_{\dot{\alpha}} a \, \partial_{\dot{\beta}} \lambda - \partial_{\dot{\beta}} \lambda \, \partial_{\dot{\alpha}} a).$$

The trajectories admit the integral invariance relation that is defined by:

$$\Omega = d(\partial_{\dot{\alpha}} L) \wedge dx^\alpha + \tfrac{1}{2} \dot{d}a \wedge \dot{d}\lambda.$$

Set $\lambda = K^2$ and $a = H / K$, which is always possible by changing the signs of a and λ, if necessary. We will then have:

$$\lambda \partial_{\dot{\alpha}} a = K \, \partial_{\dot{\alpha}} H - H \, \partial_{\dot{\alpha}} K.$$

Since the functions H and K are $\dot{h}1$, the trajectories of (S) will be the generalized extremals of the integral:

$$I = \int_{u_0}^{u_1} \left(L + K \int_{u_0}^{u_1} H \, dv \right) du.$$

Those trajectories are also the geodesics of the space $\mathcal{L}_1$ that is defined by the functions L, K, and H, so one will have the theorem:

Theorem:

The trajectories of a dynamical system whose given forces are derived from a force function and which are subject to a perfect constraint a $(x^\alpha, \dot{x}^\alpha) = 0$ *can be considered to be the geodesics of a space* $\mathcal{L}_1$.

Case of several perfect constraints. – The preceding can be generalized immediately to the case of several perfect constraints that are defined by $k < n$ compatible constraints:

$$a_A(x^\alpha, \dot{x}^\alpha) = 0 \qquad \text{with} \quad A = 1, 2, \ldots, k.$$

The equations of motion are:

$$P_\alpha(L) = X_\alpha + \lambda^A \partial_{\dot{\alpha}} a_A.$$

If $X_\alpha = 0$ then the trajectories can be considered to be the geodesics of a space $\mathcal{L}_k$.

47. Principle of least curvature. – Consider the phase space W – i.e., the fiber bundle of oriented directions that are tangent to the configuration space-time V_{n+1} of a dynamical system S (L, X_α) .

A point z of W can be defined by the $2n + 2$ numbers x^α, l^α, which are coupled by:

$$L\,(x^\alpha, l^\alpha) = 1\ .$$

The locus of points z of W whose coordinates verify the relation:

$$a\,(x^\alpha, l^\alpha) = 0,$$

in which a is $\dot{h}\,0$, defines a submanifold U of W.

Consider the basic curves Γ of U that all pass through the same point z . Their projections γ onto V_{n+1} will all have the same tangent at the point x .

Compare their curvature vectors at x :

$$\mathbf{C} = \frac{\nabla \boldsymbol{l}}{ds}\,.$$

A basic curve Γ of U is defined by:

$$x^\alpha = x^\alpha(s), \qquad l^\alpha = l^\alpha(s) \qquad \text{with} \qquad l^\alpha = \frac{dx^\alpha}{ds}\,.$$

Since the functions x^α and l^α of s verify:

$$a\,(x^\alpha, l^\alpha) = 1$$

identically, upon differentiating this with respect to the arc-length s of γ, we will get:

(47.1) $$\partial_{\dot\alpha} a\, l^\alpha + \partial_{\dot\alpha} a \frac{dl^\alpha}{ds} = 0.$$

Now:

$$\frac{dl^\alpha}{ds} = \frac{\nabla l^\alpha}{ds} - 2G^\alpha(x, l) = C^\alpha - 2G^\alpha\,.$$

The relation (47.1) will then become:

(47.2) $$\partial_{\dot\alpha} a\, l^\alpha + \partial_{\dot\alpha} a\,(C^\alpha - 2G^\alpha) = 0.$$

Let Γ' be another basic curve in U that passes through z and whose projection γ' onto V_{n+1} admits a parametric representation as a function of the arc-length s'. We will then obtain the following relation for Γ':

(47.3) $$\partial_{\dot{\alpha}} a\, l^{\alpha} + \partial_{\dot{\alpha}} a\, (C'^{\alpha} - 2G^{\alpha}) = 0.$$

Upon comparing the relations (47.2) and (47.3), we will get:

(47.4) $$\partial_{\dot{\alpha}} a\, (C'^{\alpha} - C^{\alpha}) = 0$$

at the point z. That relation translates into a generalization of Meusnier's theorem that we can state in the following way:

Theorem:

All of the curves in an $(n + 1)$*-dimensional Finsler space whose line elements* (x, l) *all verify the same relation:*

$$a\,(x, l) = 0$$

and are tangent to the same point x admit curvature vectors at that point whose extremities are in an $(n - 1)$*-dimensional planar manifold in* T_x *(viz., the tangent to* V_{n+1} *at x) that is the intersection of the normal plane to* $\mathbf{l}$ *at x and the plane perpendicular to the vector whose components are* $\partial_{\dot{\alpha}} a$ *at x.*

That theorem generalizes immediately to the case in which U is a submanifold of W that is defined $k < n$ relations:

$$a_A\,(x^{\alpha}, \dot{x}^{\alpha}) = 0 \qquad \text{with} \quad A = 1, 2, \ldots, k\,.$$

The relation (47.4) is replaced with the k relations:

(47.5) $$\partial_{\dot{\alpha}} a_A\, (C^{\alpha} - C'^{\alpha}) = 0\,.$$

Proof of the least-curvature theorem. – Extend Synge's theorem ([26]) that relates to dynamical systems with linearly-non-holonomic constraints that are independent of time to an arbitrary Lagrangian $L\,(x^{\alpha}, l^{\alpha})$ that is subject to k perfect constraints:

$$a_A\,(x^{\alpha}, l^{\alpha}) = 0\,.$$

Let S be a given dynamical system. Its trajectories in phase space belong to a submanifold U of W that is defined by the k relations:

$$a_A\,(x, l) = 0\,.$$

Let (S_0) be the free dynamical system that is associated with (S) ; i.e., the one that is deduced from (S) by suppressing the k constraints.

([26]) J. L. Synge [**1**].

Then consider the three basic paths Γ, Γ', Γ'' that pass through the same point z (x, l) of W:

1. The path Γ is the trajectory of (S) that passes through z .

2. The path Γ' is an arbitrary basic curve in U that passes through z .

3. The path Γ'' is the trajectory of (S_0) that passes through z .

Let γ, γ', γ'' be the projections of Γ, Γ', Γ'', resp. onto V_{n+1} , and let **C**, **C**$'$, **C**$''$, resp., be the curvature vectors of those curves at the point x . Let the curvatures of γ and γ' relative to γ'' be c and c', which are the norms of the vectors **C** – **C**$''$ and **C**$'$ – **C**$''$, resp.; we will then have:

$$c^2 = (C_\alpha - C''_\alpha)\,(C^\alpha - C''^\alpha),$$
$$c'^2 = (C'^\alpha - C''_\alpha)\,(C'^\alpha - C''^\alpha)\,.$$

From a classical identity, we can write:

$$c'^2 - c^2 = (C_\alpha - C'_\alpha)\,(C^\alpha - C'^\alpha) - 2\,(C^\alpha - C'^\alpha)(C_\alpha - C'_\alpha)\,. \tag{47.6}$$

Now:

$$C'_\alpha = X_\alpha \qquad \text{and} \qquad C_\alpha = X_\alpha + \lambda^A \partial_{\dot\alpha} a_A\,,$$

since the constraints are assumed to be perfect.

We will then have:

$$(C^\alpha - C'^\alpha)(C_\alpha - C'_\alpha) = \lambda^A \partial_{\dot\alpha} a_A\,(C^\alpha - C'^\alpha). \tag{47.7}$$

However, from Meusnier's theorem, that expression is zero. The relation (47.6) will then reduce to:

$$c'^2 - c^2 = (C_\alpha - C'_\alpha)\,(C^\alpha - C'^\alpha)\,. \tag{47.8}$$

Since the space $\mathcal{F}$ is properly Finslerian, the right-hand side of this, which is the square of the normal of the vector **C** – **C**$'$, will be positive. We then deduce the following inequality from this:

(47.9): $$c'^2 \geq c^2,$$

which will then give the theorem:

Theorem of least curvature:

Let a dynamical system S (L, X_α) *be subject to perfect, first-order, non-holonomic constraints. Let S_0* (L, X_α) *be the associated free system, and let $\mathcal{F}$ be the Finsler space that is defined on the configuration space-time by the Lagrangian L.*

Among all curves in $\mathcal{F}$ that are tangent to the same point x and satisfy the constraint relations, the trajectory of (S) *is the one that has the least curvature at x with respect to the associated free trajectory.*

48. Consequences of the principle of least curvature. –

1. *Gauss-Appell principle.* – Consider the function:

$$2R = \left(\frac{\nabla l_\alpha}{ds} - X_\alpha\right)\left(\frac{\nabla l^\alpha}{ds} - X^\alpha\right). \tag{48.1}$$

That function of the x^α, $\dot{x}^\alpha = dx^\alpha / ds = l^\alpha$, $\ddot{x}^\alpha = dl^\alpha / ds$ is equal to the square of the relative curvature at a point x of an arbitrary curve γ in configuration space-time V_{n+1} with respect to the free trajectory γ'' that is tangent to it at x.

In the phase space W, the trajectory of the dynamical system (S) that passes through $z(x^\alpha, l^\alpha)$ is the curve in the submanifold U of W (viz., the constraint space) for which the function R is a minimum at z .

The principle of least curvature then translates analytically in the following way, where u is a new arbitrary parameter:

The trajectories of the system (S) are defined by the functions $x^\alpha = x^\alpha(u)$ for which:

$$\partial_{\ddot{\alpha}} R = 0,$$

when one takes the constraint relations into account, and the function R is defined by:

$$R = \frac{1}{2L^2}\left(\frac{\nabla l_\alpha}{du} - X_\alpha\right)\left(\frac{\nabla l^\alpha}{du} - X^\alpha\right).$$

In the preceding form, the principle of least curvature seems to be a generalization of least constraint. In fact, the preceding function R is the one that Appell ([27]) introduced, when it is extended to configuration space-time.

2. *Converse.* – Suppose that a dynamical system S (L, X_α) is subject to k first-order non-holonomic constraints whose constraint relations are:

([27]) P. Appell [**1**], t. II, pp. 392.

$$a_A(x^\alpha,\ \dot{x}^\alpha) = 0, \qquad A = 1, 2, \ldots, k\,.$$

Suppose that the functions a are homogeneous of order 0 with respect to:

$$\dot{x}^\alpha = dx^\alpha / dx\,.$$

Now show that the constraints are perfect constraints if the trajectories of (S) in the configuration space verify the principle of least curvature.

Partially differentiate R with respect to $\ddot{x}^\alpha$, while taking into account the fact that:

$$\frac{\nabla l^\alpha}{ds} = \ddot{x}^\alpha + 2G^\alpha(x, l), \qquad \text{with} \qquad \ddot{x}^\alpha = dl^\alpha / ds\,.$$

We then get:

(48.2) $$\partial_{\ddot{\alpha}} R = \frac{\nabla l_\alpha}{ds} - X_\alpha\,.$$

Let Γ and Γ' be two curves of W that pass through the same point z (x, l) . Upon passing from Γ to Γ', the function R will experience a variation at z that takes the form:

(48.3) $$\delta R = \left(\frac{\nabla l_\alpha}{ds} - X_\alpha\right)\delta\ddot{x}^\alpha\,.$$

If Γ is the trajectory of (S) that passes through z then be must have $\delta R = 0$ for all of the $\delta\ddot{x}^\alpha$ that are permitted by the constraints; i.e., the ones for which:

(48.4) $$\partial_{\dot{\alpha}} a\, \delta\ddot{x}^\alpha = 0,$$

and that verify the relation:

(48.5) $$\partial_{\dot{\alpha}} L\, \delta\ddot{x}^\alpha = 0,$$

in addition, which is a consequence of L (x^α, l^α) = 1.

We will then have:

(48.6) $$\frac{\nabla l_\alpha}{ds} - X_\alpha = \lambda^A \partial_{\dot{\alpha}} a_A + \mu\, \partial_{\dot{\alpha}} L$$

along Γ. The contracted product with l^α will then give $\mu = 0$.

Equations (48.6) then show that the constraints considered are perfect constraints. We can then state:

Theorem:

In order for a dynamical system that is subject to first-order constraints to verify the principle of least curvature, it is necessary and sufficient that those constraints should be perfect.

3. *Appell equations.* – For the free system S_0 (L, X_α), the equations of the trajectories in configuration space-time can be put into the form that was indicated by Appell:

(48.7) $$\partial_{\ddot{\alpha}} R = 0.$$

Those equations are a direct consequence of the relations (48.2)

If we let A denote the energy of acceleration in V_{n+1} – i.e., if we set:

$$A = \frac{1}{2}\frac{\nabla l_\alpha}{ds}\frac{\nabla l^\alpha}{ds},$$

then we can put the Appell equations (48.7) into the form:

(48.8) $$\partial_{\ddot{\alpha}} A = X_\alpha .$$

For the bound system (S), the Appell equations of the trajectories in V_{n+1} are deduced from (48.6). They are written in the form:

(48.9) $$\partial_{\ddot{\alpha}} R = \lambda^A \partial_{\ddot{\alpha}} a_A , \qquad A = 1, \dots, k,$$

when the k perfect constraints are defined by the relations:

$$a_A (x^\alpha, \dot{x}^\alpha) = 0.$$

We deduce from those relations by derivation that:

(48.10) $$\partial_{\alpha} a_A \dot{x}^\alpha + \partial_{\dot{\alpha}} a_A \ddot{x}^\alpha = 0.$$

Suppose that these relations permit us to calculate k of the $\ddot{x}^\alpha$ (for example, the first k) as functions of the other ones. R will then become a function of the x^α, $\dot{x}^\alpha$, and the $n + 1 - k$ second derivatives $\ddot{x}^{(\alpha)}$, where $(a) = k + 1, \dots, n + 1$.

Since we must have:

$$\delta R = 0 \qquad \text{for any } \ddot{x}^{(\alpha)},$$

we will get the $n + 1 - k$ Appell equations:

$$\partial_{(\ddot{\alpha})} R = 0,$$

which will define the trajectories of (S) in V_{n+1} when they are combined with the k constraint relations:

$$a_A\,(x^\alpha,\ \dot{x}^\alpha) = 0.$$

Like the Lagrange equations or the canonical equations, the Appell equations have a form that is independent of any way that one frames the configuration space-time.

CHAPTER VII

APPLICATIONS

49. Dynamical systems that admit a Painlevé first integral. – Let (S) be a dynamical system with perfect bilateral constraints and n degrees of freedom that are characterized by the parameters x^k.

Suppose that there exists a one-parameter group on the configuration space-time of (S) that takes t to $t + h$ and leaves (S) invariant.

Under some general hypotheses, one can pass to the quotient and define a configuration space V_n that corresponds to V_{n+1}. The Lagrangian $\mathcal{L} = T + U$, as well as the functions Q_k will then be independent of time.

The Lagrangian $\mathcal{L}$ corresponds to the homogeneous Lagrangian L (§ **29**) that is independent of $x^{n+1} = t$. The last Lagrange equation will then reduce to:

$$\frac{d}{du}\frac{\partial L}{\partial \dot{x}^{n+1}} = -\,Q_k\,\dot{x}^k\,.$$

Suppose that the generalized force Q_k has zero power; i.e., that $Q_k\,\dot{x}^k = 0$ over the entire trajectory.

Under those conditions, the system of Lagrange equations of (S) will admit the first integral:

$$\frac{\partial L}{\partial \dot{x}^{n+1}} = -\,\mathcal{H} = h, \qquad \text{where } h \text{ is a constant.}$$

Consider the set of trajectories (T) of (S) that correspond to a well-defined value of h. The fundamental 2-form Ω can be written as follows for those trajectories:

$$\Omega = d\partial_{\dot{k}} L \wedge dx^k + \tfrac{1}{2} S_{\alpha\beta}\, dx^\alpha \wedge dx^\beta.$$

From the well-known descent process ([28]), one can replace the Lagrangian L with a Lagrangian L_1 that is independent of time $\dot{x}^{n+1}$. That Lagrangian is defined by:

$$L_1 = L\,[x^k,\ \dot{x}^k,\ \varphi(x^k,\ \dot{x}^k, h)] - h\,\varphi(x^k,\ \dot{x}^k, h),$$

in which $\varphi = \dot{x}^{n+1}$ is obtained by solving $\dfrac{\partial L}{\partial \dot{x}^{n+1}} = h$ for $\dot{x}^{n+1}$. One indeed verifies that for the trajectories (T), one has:

$$\partial_{\dot{k}} L_1 = \partial_{\dot{k}} L\,.$$

Now, set:

([28]) Y. Thiry [**1**], Chap. I.

$$X_k = Q_k\, \dot{x}^k$$

and suppose that the X_k are independent of $\dot{x}^{n+1}$; since:

$$\frac{\partial X_k}{\partial \dot{x}^{n+1}} = Q_k - \partial_{m'} Q_k\, x'^m,$$

that will amount to supposing that the components of the generalized force Q_k are homogeneous and of first degree with respect to the components of the velocity x'^k (viz., $h'1$) .

Under those conditions, the 2-form Ω will become:

$$\Omega_1 = d\partial_{\dot{k}} L_1 \wedge dx^k + \tfrac{1}{2} S_{jk}\, dx^j \wedge dx^k.$$

The trajectories of the dynamical system (S) that correspond to a well-defined value of h can then be obtained in the configuration space V_n independently of the time parameterization as solutions to the associated system of Ω_1 .

We can then state the following theorem, which takes the form of a generalization of Maupertuis's principle:

Theorem:

Let S be a dynamical system with n degrees of freedom x^k that admits the Painlevé first integral:

$$\mathcal{H} = -h = \text{const.}$$

and is such that the generalized force has components of the form:

$$Q_k = S_{km}\, x'^m, \qquad \text{with} \qquad S_{km} = -S_{mk}\,,$$

in which the S_{km} are $h'0$ functions of x^k and x'^k .

The trajectories of S have a well-defined total energy E and are defined in configuration space V_n independently of the law of traversal as the S-extremals of the integral:

$$I = \int_{x_0}^{x_1} L_1(x^h, \dot{x}^k, h)\, du\,,$$

in which u is an arbitrary parameter, $\dot{x}^k = dx^k / du$, and $h = -E$. The Lagrangian L_1 is defined by starting from the homogeneous Lagrangian L with:

$$L_1 = L\,(x^k,\ \dot{x}^k,\ \varphi) - h\,\varphi\,,$$

with $\dot{x}^{n+1} = \varphi\,(x^k,\ \dot{x}^k,\ h)$, which is a relation that is equivalent to $\mathcal{H} = -h$.

Particular cases:

1. Ω_1 *is closed.* – Under those conditions, there exists a vector potential whose components $A_l(x^k)$ are such that:

$$S_{kl} = \partial_k A_l - \partial_l A_k .$$

The trajectories of S have constant energy $E = - h$, and are then the extremals of the integrals:

$$I = \int_{x_0}^{x_0} L_2 \, du ,$$

with $L_2 = L_1 + A_k \dot{x}^k$.

Those trajectories are also geodesics in the Finsler space that is defined on V_n by $ds = L_2\, du$.

If we take:

$$L = a_{ij} \frac{\dot{x}^i \dot{x}^j}{2\dot{x}^{n+1}} + b\,\dot{x}^i + (T_0 + u)\,\dot{x}^{n+1}$$

then we will get the following expression for L_2 :

$$L_2 = \sqrt{2(T_0 + U - h)\, a_{ij} \dot{x}^i \dot{x}^j} + (b_i + A_i)\,\dot{x}^i .$$

Under those conditions, we know that the trajectories T in V_n can be considered to be the projections onto V_n of geodesics of a Riemannian space in $n + 1$ variables, where the $(n + 1)^{\text{th}}$ variable x_0 is no longer time.

2. Ω_1 *admits an integrating factor.* – That will be true if the components of the force tensor have the form:

$$S_{lk} = \partial_i L_1 \, \partial_k \varphi - \partial_k L_1 \, \partial_l \varphi + e^{\varphi} (\partial_l A_k - \partial_k A_l) ,$$

in which φ, A_k are $n + 1$ arbitrary functions of the variables x^k. The trajectories T are then the extremals of the integrals:

$$I = \int_{x_0}^{x_1} (e^{-\varphi} L_1 + A_k \, \dot{x}^k)\, du .$$

Examples:

1. *Appell constraints* ([29]). – Suppose that the dynamical system S is subject to the Appell constraint that is defined by:

$$a(x^k, x'^k) = 0, \qquad \text{in which} \qquad a \text{ is } h'1,$$

and the constraint force has components of the form:

([29]) P. Appell, C. R. Acad. Sci. **152** (1911), pp. 1197.

$$Q_k = \lambda\, \partial_{k'} a\,, \qquad\qquad \lambda \text{ is } h'1.$$

We are indeed under the conditions for applying the generalized Maupertius's principle, because $Q_k\, x'^k = \lambda\, a = 0$ along any trajectory.

2. *Gyroscopic coupling.* – Let S be a dynamical system that is composed of n linear oscillators. Since the position of an oscillator over its support D_k, which is supposed to be fixed, is defined by its abscissa x_k, the Lagrange system of equations will have the form:

$$x_k'' + \lambda_k^2\, x_k = 0\ .$$

A gyroscopic coupling between those n oscillators translates into the presence of a generalized force vector in the right-hand side of this that has components of the form ([30]):

$$Q_k = \sum_{m=1}^{n} S_{km}\, x_m'\ .$$

The tensor S_{km}, which is supposed to be antisymmetric, depends upon only the x_k; by definition, it is the "gyroscopic coupling" tensor.

50. Applications to general relativity. – Let V_4 be the space-time of general relativity that is endowed with the Riemannian metric that is defined by:

$$ds^2 = g_{\alpha\beta}(x)\, dx^\alpha\, dx^\beta\ .$$

If u is an arbitrary parameter then set:

$$L^2 = g_{\alpha\beta}\, \dot{x}^\alpha \dot{x}^\beta\,, \qquad\qquad \dot{x}^\alpha = dx^\alpha / du\,,$$

and

$$l_\alpha = \partial_{\dot{\alpha}} L = g_{\alpha\beta}\frac{\dot{x}^\beta}{L} = g_{\alpha\beta}\, l^\beta.$$

Suppose that an energetic distribution is defined on a domain D in V_4 that corresponds to the energy-impulse tensor $T_{\alpha\beta}$. Set ([31]):

$$T_{\alpha\beta} = r\, l_\alpha\, l_\beta - \theta_{\alpha\beta} \qquad \text{and} \qquad \nabla_\alpha \theta^\alpha_\beta = r\, K_\beta\,,$$

in which the $\theta_{\alpha\beta}$ depend upon both the x and the l.

([30]) Y. Rocard, "Dynamique générale des vibrations," Masson, 1943, pp. 114-124.
([31]) A. Lichnerowicz [**3**], Chap. II, **17**.

The K_α are $\dot{h}0$ and define the force density vector that is associated with the energy tensor $\theta_{\alpha\beta}$ and the scalar r (which is a pseudo-density) .

The conservation equations:

$$\nabla_\alpha T^\alpha_\beta = 0$$

give, on the one hand, the equation of continuity, and on the other, the differential system of the streamlines, which are tangent to the unit velocity vector l^α everywhere.

The differential system can be put into the form:

$$\text{(50.1)} \qquad \frac{\nabla l_\alpha}{ds} = (K_\alpha\, l_\beta - K_\beta\, l_\alpha)\, l^\beta \qquad \text{or} \qquad \frac{\nabla l_\alpha}{du} = (K_\alpha\, l_\beta - K_\beta\, l_\alpha)\, \dot{x}^\beta .$$

Set:

$$X_\alpha = (K_\alpha\, l_\beta - K_\beta\, l_\alpha)\ \dot{x}^\beta = L\, K_\alpha - K\, L_\alpha \qquad \text{with} \quad K = K_\alpha\, \dot{x}^\alpha .$$

The force vector X_α is associated with the antisymmetric tensor:

$$\begin{aligned} s_{\alpha\beta} &= \tfrac{1}{2}(\partial_{\dot\beta} X_\alpha - \partial_{\dot\alpha} X_\beta) \\ &= K_\alpha\, l_\beta - K_\beta\, l_\alpha + \tfrac{1}{2} L\,(\partial_{\dot\beta} K_\alpha - \partial_{\dot\alpha} K_\beta) + \tfrac{1}{2}\dot{x}^\lambda (l_\beta\, \partial_{\dot\alpha} K_\lambda - l_\alpha \partial_{\dot\beta} K_\lambda)\ . \end{aligned}$$

When the force density vector K_α is independent of the velocity, the tensor $s_{\alpha\beta}$ will reduce to $K_\alpha\, l_\beta - K_\beta\, l_\alpha$.

In the general case, we will have the identities:

$$s_{\alpha\beta}\, \dot{x}^\beta = X_\alpha \quad \text{and} \quad \partial_{\dot\alpha} s_{\beta\gamma} + \partial_{\dot\beta} s_{\gamma\alpha} + \partial_{\dot\gamma} s_{\alpha\beta} = 0.$$

The symmetric tensor $\theta_{\alpha\beta}$, which is a pseudo-tensor, then corresponds to an antisymmetric tensor $s_{\alpha\beta}$ that is called the *force tensor*, such that the differential equations of the streamlines will be:

$$\text{(50.2)} \qquad l^\beta\, \nabla_\beta\, l_\alpha = s_{\alpha\beta}\, l^\beta \qquad \text{or} \qquad l^\beta\, (\nabla_\beta\, l_\alpha - \nabla_\alpha\, l_\beta) = s_{\alpha\beta}\, l^\beta$$

The general results that were previously obtained permit one to state the following equivalent theorems:

Theorem:

1. *The differential system of the streamlines of an arbitrary energy tensor schema is the associated system to the 2-form:*

$$\text{(50.3)} \qquad \Omega = dl_\alpha \wedge dx^\alpha + \tfrac{1}{2} s_{\alpha\beta}\, dx^\alpha \wedge dx^\beta .$$

The l_α are the covariant components of the unit velocity vector: $l_\alpha = \partial_{\dot{\alpha}} L$, with $L^2 = g_{\alpha\beta}\,\dot{x}^\alpha \dot{x}^\beta$, where the $s_{\alpha\beta}$ are the components of the force tensor that is associated with the energy tensor for the schema in question.

2. *The form Ω defines an integral invariance relation for the streamlines.*

3. *Let C_0 and C_1 be two homotopic cycles in dimension one that surround the same tube of streamlines. The difference between the circulations of the unit velocity vector around C_0 and C_1 is equal to the flux of the force tensor across the portion of the tube that is bounded by C_0 and C_1.*

4. *The streamlines are the s-extremals of the integral:*

$$I = \int_{u_0}^{u_1} L\,du\,.$$

5. *The streamlines are the geodesics of the s-Riemannian space that is defined by L and the force tensor $s_{\alpha\beta}$ (§ **25**).*

Recall the definition of such a space:

One considers a Riemannian metric on V_4 that is defined by:

$$ds^2 = g_{\alpha\beta}(x)\,dx^\alpha\,dx^\beta\,,$$

and the Euclidian connection on the directions will correspond to the torsion forms:

$$\Sigma^\alpha = \left(\tfrac{1}{2} s_{\beta\gamma}\,dx^\beta \wedge dx^\gamma\right) l^\alpha\,, \tag{50.4}$$

in which the expression in the parentheses represents the 2-form that is associated with the force tensor.

Under these conditions, the connection will be defined by:

$$\Gamma_{\alpha\beta\gamma} = [\beta\gamma, \alpha] + (l_\alpha\,s_{\beta\gamma} + l_\beta\,s_{\gamma\alpha} - l_\gamma\,s_{\alpha\beta})\,, \tag{50.5}$$

which will reduce to:

$$\Gamma_{\alpha\beta\gamma} = [\beta\gamma, \alpha] - s_{\alpha\beta}\,l_\gamma$$

when the K_α are independent of the $\dot{x}^\gamma$.

Example:

Charged perfect fluid schema ([32]). – The most interesting results that are obtained in general relativity are the ones that correspond to a fundamental 2-form Ω that is either

([32]) A. Lichnerowicz [**3**], Chap. IV, §§ 34-37.

closed or admits an integrating factor. Let us study the schema that encompasses all of the other ones, namely, the homogeneously-charged perfect fluid.

The energy-impulse tensor of such a schema is defined in a domain D of V_4 by:

$$T_{\alpha\beta} = (\rho + p)\, l_\alpha\, l_\beta - p\, g_{\alpha\beta} + \tau_{\alpha\beta}\,.$$

ρ is the proper density, p is the proper pressure, and $\tau_{\alpha\beta}$ is the impulse-energy tensor of the electromagnetic field that is defined by the tensor $F_{\alpha\beta}$.

The differential system of the streamlines is:

$$\frac{\nabla l_\alpha}{ds} = (g_\alpha^\beta - l^\beta l_\alpha)\left(\frac{\partial_\beta p}{\rho + p} + \frac{\mu}{\rho + p} F_{\beta\lambda} l^\lambda\right),$$

in which μ is the proper charge density of the fluid.

We can further write those equations in the form:

(50.6) $$\frac{\nabla l_\alpha}{ds} = \frac{l^\beta}{\rho + p}(\partial_\alpha p\; l_\beta - \partial_\beta p\; l_\alpha + \mu\, F_{\alpha\beta})\,.$$

Suppose that there exists a state equation $\rho = f(p)$, so the index F of the fluid can be defined by:

$$F = e^\varphi \qquad \text{with} \qquad \varphi = \int_{p_0}^{p} \frac{dp}{\rho + p}\,.$$

We then deduce that:

$$\frac{\partial_\alpha p}{\rho + p} = \partial_\alpha\, \varphi\,.$$

On the other hand, suppose that the fluid is charged homogeneously.

Under those conditions, $k = \dfrac{\mu F}{\rho + p}$ will be constant over the entire domain D of V_4 .

Let us make a final hypothesis: There exists a global vector potential **A** in D such that:

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha\,.$$

Under those conditions, the force tensor will have components of the form:

(50.9) $$s_{\alpha\beta} = l_\beta\, \partial_\alpha\, \varphi - l_\alpha\, \partial_\beta\, \varphi + k\, e^{-\varphi}\, (\partial_\alpha A_\beta - \partial_\beta A_\alpha)\,.$$

Those components characterize a 2-form Ω that admits an integrating factor [relations (38.6) of § **38**] .

We have thus obtained following classical results directly:

Theorem:

For any motion of a homogeneously-charged perfect fluid, the streamlines are the extremals of the form:

$$\omega = (F\, l_\alpha + k\, A_\alpha)\, dx^\alpha$$

or the integral:

$$I = \int_{u_0}^{u_1} (F\, L + k\, A_\alpha \dot{x}^\alpha)\, du\,.$$

Those streamlines are characterized by the existence of a relative integral invariant that is defined by:

$$\omega = (F\, l_\alpha + k\, A_\alpha)\, dx^\alpha.$$

Those streamlines are also geodesics in the Finsler space that is defined on V_4 by:

$$ds = (F\, L + k\, A_\alpha\, \dot{x}^\alpha)\, du\,.$$

Particular cases:

1. Pure matter schema: $s_{\alpha\beta} = 0$, $\Omega = dl_\alpha \wedge dx^\alpha$.

2. Matter-electromagnetic field schema (homogeneous case):

$$s_{\alpha\beta} = k\, F_{\alpha\beta} \qquad \text{and} \qquad \Omega = d\,(l_\alpha + k\, A_\alpha) \wedge dx^\alpha.$$

3. Holonomic medium schema:

$$s_{\alpha\beta} = l_\beta\, \partial_\alpha\, \varphi - l_\alpha\, \partial_\beta\, \varphi\,,$$

$$\Omega = l_\alpha \wedge dx^\alpha + d\varphi \wedge dL\,,$$

which admits e^φ as an integrating factor.

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