"Über die Haupttangentenkurven der Kummerschen Fläche vierten Grades mit 16 Knotenpunkten," Sitz. d. Ber. Akad. (1870). Submitted by E. E. Kummer at the session on 15 December 1870. Reprinted in Math. Annalen **23** (1884); F. Klein, *Gesammelte mathematischen Werke*, art. VI.

On the principal tangent curves of the fourth-degree Kummer surface with 16 nodes.

By

Sophus Lie and Felix Klein

Translated by D. H. Delphenich

As is known ([1]), the fourth-degree **Kummer** surface with 16 nodes is the singularity surface for a simple infinitude of second-degree complexes; i.e., the surface that is the geometric locus of those points whose complex cone decomposes into two planes, or – what amounts to the same thing – that is enveloped by those planes whose complex curves resolve into two points. The consideration of these second-degree complexes leads almost immediately to the consideration of the principal tangent curves of the surface, as will be shown in what follows.

1. We choose a surface from the simple infinitude of them that belongs to a second-degree complex.

The complex curves that are inside of a tangential plane of the surface resolve into two points. Those two points are the ones at which the fourth-order curve that is contained in the tangential plane and is the intersection of the surface with a certain line that goes through the contact point (which is called its *associated singular line*) intersects it outside of that contact point.

One can now ask what those points of the surface are whose associated singular line is a principal tangent of the surface. The remaining tangents to the surface at such a point will obviously belong to the complex, as well. On the other hand, these latter complex lines are the only ones that contact the surface without at the same time being singular lines of the complex. We now consider the complex conic and the fourth-order curve of intersection with the surface in any plane. They contact at four points, and the tangents to those points will be the singular lines that lie in the plane ([2]). The two curves will have 16 common tangents, in addition to those doubly-counted tangents, since they have class

---

([1]) Cf., **Plücker**, *Neue Geometrie des Raumes, gegründet auf die Betrachtung der geraden Linie als Raumelement,* B. G. Teubner, 1868, 69, no. 310, *et seq.* Cf., also, here and in what follows **Klein:** "Zur Theorie der Komplexe des ersten and zweiten Grades," Math. Ann., Bd. [See art. II of this collection.]

([2]) **Plücker**, *Neue Geometrie,* no. 318.

two and twelve, resp. Their contact points with the fourth-order intersection curve will be points with the desired behavior.

The points of the Kummer surface whose associated singuilar lines are principal tangents to the surface then define a curve of order 16.

2. *Now, the curve thus-determined is a principal tangent curve to the surface.*

In order to prove that, we first remark that a projective correspondence exists between the planes that are laid through a complex line (which only needs to not be a singular line) and the contact points of the complex curves that are contained in them with that line. One excludes the case from this in which an infinitely small displacement of the point along the line would correspond to a rotation of the plane whose magnitude has the same order of infinite smallness.

Now, the connecting line of two consecutive points of the curve that was just determined is a complex line without, at the same time, being one of its singular lines. The two tangential planes at the two points contain the pencils of rays that belong to the complex whose vertices are those points. The two tangential planes are then two planes whose complex curves contact the chosen tangent at two consecutive points. From the foregoing remark, it will then follow from this that when one advances along the curve, the tangential plane to the surface will rotate around the tangents to the curve.

However, that is the characteristic property of the principal tangent curves of a surface; our assertion is then proved.

Since the concept of the principal tangent curve, as well of that of complex, is self-dual, it will then follow that the dualistically-complementary singularities of the curve are equal to each other. In particular, *their classes are equal to their orders, and thus* 16.

Since the curve is determined by the complex self-dually in a single way, like that complex, it will go to itself by a system of linear, as well as reciprocal, transformations ([3]). One excludes a series of properties from this that we cannot go into any further here.

3. In the manner that was just described, we will obtain a principal tangent curve that corresponds to each of the simple infinitude of second-degree complexes. However, in that way, one will have *all* principal tangent curves, provided that, say, enveloping curves of them do not exist, since one can give a line of the complex for every point of the surface that has one or the other of the two principal tangents at that point as its singular.

Among the second-degree complexes that belong to the surface, one will find six (doubly-counted) linear complexes. One regards the double tangents to the surface as their singular lines, in such a way that each of the six complexes actually belongs to one of the six systems that are defined by the double tangents. *There are six distinguished principal tangent curves that correspond to that complex.* They will only have order and class eight, as would be implied by the same considerations by which we determined the order and class of a general curve.

([3]) Cf., the previously-cited treatise: "Zur Theorie, etc.," no. 13.

4. We now go on to the determination of the singularities of the principal tangent curves. We shall succeed in that when we borrow the following theorems from the general theory of such curves:

The principal tangent curves of an arbitrary surface have the same vertices at their nodes.

Above all, they have cusps at the points of the parabolic curve, assuming that it is not itself a principal tangent curve. In the latter case, it will be the enveloping curve for the remaining principal tangent curves. That is true especially when the parabolic curve consists of planar contact curves.

Furthermore, the principal tangent curves have stationary tangents at their points of intersection of the curve of four-point contact, provided that the curve of four-point contact is not at the same time a parabolic curve, which is a result that one would arrive at by a special consideration of the different cases that we do not regard as necessary here.

Finally, principal tangent curves can have no cusps and no stationary tangents except in the aforementioned cases.

The parabolic curve, which must be of order 32, consists of 16 contact conics in the 16 double tangential planes to the surface. It is then the enveloping curve of the principal tangent curves. The 16 planes are then stationary planes of these curves, since they are above all the planes of planar contact curves.

One now easily convinces oneself that the principal tangent curves have only a cusp at every node and have stationary contact with the double tangent planes only once. The curve can, in fact, have only 16 points in common with the double tangential planes. Four of them are due to the stationary contact, and twelve of them are due to the six cusps at the six nodes that lie in the plane.

The principal tangent curves then have 16 cusps (that fall upon the nodes of the surface) and 16 stationary planes (that are identical with its double tangential planes).

In our case, the curves of four-point contact consist of the 16 contact conics, which shall not come under any further consideration here, since they have been dealt with already. On the other hand, they consist of the six distinguished principal tangent curves of order eight that belong to the six linear complexes. It emerges from this that the singular lines of the complex will be double tangents to the surface, as we mentioned already. The curve of four-point contact encompasses no further curves, since the ones that were just enumerated collectively possess the correct order of 80.

We must now determine the number of points of intersection of a principal tangent curve with the six distinguished ones.

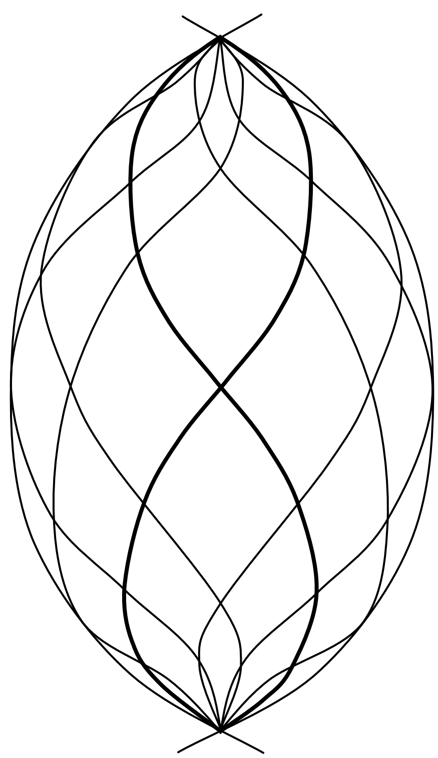
Those points of intersection are characterized by the fact that the four-point contacting principal tangent is a line of the second-degree complex that belongs to the given principal tangent curve. However, the principal tangents that have four-point contact at points of one of the six curves define a ruled surface of order eight, since its complete intersection with the **Kummer** surface consists of the chosen curve, which is counted four times. However, the second-degree complex will have 16 lines in common with such a surface. One will then obtain 16 points of intersection, corresponding to each of the six curves. We then have the theorem:

The principal tangent curves have $6 \cdot 16 = 96$ *stationary tangents.*

If we add that the principal tangent curves can possess no actual double points (and therefore no actual double osculating planes, either), since the two principal tangents of the complex cannot be associated as singular lines at any point of the **Kummer** surface that does not lie on the parabolic curve, then we can determine all of its singularities, which are dualistically equal and opposite, with no further discussion. In particular, we find: The rank = 48, the number of apparent double points = 72, the order of the double curves of the developables = 952, and the genus = 17.

5. The number of cusps and stationary osculating planes will be zero for the six distinguished principal tangent curves. Such a curve would, in fact, go through each of the double points simply and would have one of the six double tangent planes at that point that contain the curve as an osculating plane. One must exhibit the continual transition between the general curves and those curves especially such that the latter are counted twice, and that the union of any two branches that come together at a cusp will imply the remaining ones. The order and class will then drop by one-half. From this, the rank must also be half as large as the other one, and thus, equal to 24. However, one also finds that when one calculates the number of stationary tangents. Namely, one now comes to 40 of them, since the curve intersects each of the other ones, no longer 16 times, but only 8 times, since it is counted twice, and that will happen only five times, not six.

We further find: The number of apparent double points equals 16, the order of the double curve of the developable equals 200, and the genus is equal to 5.



6. The way that one must imagine the sequence of principal tangent curve is represented schematically in the illustration above for the case in which the six associated linear complexes are real.

Namely, in that case, the parts of the surface for which the principal tangents are real will have the form of a segment that is limited by two conic sections that go from one node to the other. The two limiting curve segments belong to the contact conic at the two double tangents of the surface which contain both nodes at the same time.

Now, two of the six distinguished principal tangent curves will run inside of such a segment. They belong to those two of the six linear complexes that correspond to the two double tangential planes at the two nodes between which the segment extend. The curves in question are drawn darker in the figure; they have the form of an "S." They go from the one node to the other one, at which they contact one of the two limiting curves. Except for the two nodes, they intersect at an inflection point that is common to both and defines the midpoint of the illustration. Moreover, these curves can be continued beyond the two nodes to further segments of the surface that have a similar form.

Of the remaining principal tangent curves (there being three of them), one knows that they have a cusp at the nodes, that they contact each of the two limiting conic sections once, and that possess an inflection point wherever they meet the two distinguished curves outside of the two nodes. With that, it will be easy to follow their course in the figure.

---

7. The determination of the principal tangent curves of the Kummer surface that was given in the foregoing, which we have linked with the consideration of the associated second-degree complex, can be regarded from a different viewpoint when one starts from one of the six linear complexes that are found among them. Namely, the surface will be a focal surface of ray system that belongs to that complex: viz., the ray system of its double tangents. We would now like to show that *the problem of determining the principal tangent curves of the focal surface of a ray system that belongs to a linear complex is identical to the problem of finding the curvature curves of a certain surface.* In our case, that surface will be the fourth-order surface that contains the imaginary circle at infinity twice, and since one knows its curvature curves, one will obtain a determination of the principal tangent curves of the **Kummer** surface that naturally coincides with the one that was given above.

Namely, one refers the lines of the given linear complex uniquely to the points of space when one regards two of the six line coordinates, which refer to two of the intersecting edges of the tetrahedron, as functions of the four remaining ones by means of the given linear equation and the identity that exists between the line coordinates and interprets the latter coordinates as point coordinates ([4]).

One then finds that all lines of the complex that go through a point will correspond to the points of a straight line, and that this straight line will cut a fixed conic that is fundamental for the map. The ray system that the linear complex has in common with a complex of degree n maps to a surface of degree $2n$ that contains the conic n times. In

([4]) This mapping process was already given occasionally by **Nöther**: "Zur Theorie der algebraischen Funktionen mehrerer komplexer Veränderlichen," Gött. Nachrichten, 1869.

particular, the image of a straight line – i.e., the complex line that intersects it – is a second-degree surface that goes through the conic.

From now on, we would like to choose the imaginary circle at infinity for our fundamental conic, such that image of a straight line would be a circle.

Now, let an arbitrary surface be given and a curvature curve on it. The surface is the image of a ray system that belongs to the linear complex, and the curve is the image of a rectilinear surface that is contained in it. We assert that *this rectilinear surface contacts the focal surface of the ray system along a principal tangent curve.*

In order to prove this, we next remark that, inversely, the image of this focal surface will be the ray system whose lines simultaneously contact the given surface and cut the imaginary circle at infinity. Any straight line that contacts the focal surface will then correspond to a sphere that contacts the given surface. In particular, a principal tangent will correspond to a sphere with stationary contact.

However, one of the two spheres that have stationary contact with an arbitrary point of the curvature curve will contain three consecutive points of the curvature curve. Thus, one of the two principal tangents of the focal surface at a contact point with the circumscribing ruled surface will cut out three consecutive generators of it; in other words, it is also a principal tangent of the latter.

However, one generally has the theorem: If two surfaces contact along a curve and a principal tangent is common to them at every point of that curve then the curve will be a principal tangent curve.

With that, our assertion is proved.

Now, when one takes the given surface to be a fourth-order surface that contains the imaginary circle at infinity twice (such a surface is the image of a ray system of order and class two that belongs to the linear complex), one will obtain the principal tangent curves of the **Kummer** surface in that way, and that surface will be the focal surface of such a ray system.

The considerations that were contained in the last number were the ones that first led one of us [viz., **Lie** ([5])] to remark that the principal tangent curves of the **Kummer** surface are algebraic curves of order 16. From that, the other of us (viz., **Klein**) found the relationship between those curves and second-degree complexes that belong to the **Kummer** surface and determined their singularities in the way that was set down in the present study ([6]).

([5]) Cf., **Lie**: "Über eine Klasse geometrischer Transformationen," Berichte der Akademie zu Christiania, 1870, or also: "Sur une transformation géométrique," in the Comptes rendus de l'Academie des Sciences of the same year (v. 71, 31 October 1870).

([6]) [As was mentioned in the text above, the foregoing treatise goes back to the collaboration of Lie and myself during our time in Paris, and defines its zenith, so to speak. In the beginning of July, 1870, I got up early one morning and wanted to start directly, when **Lie**, who was still in bed, called me from his room and described to me the connection between the principal tangent curves of a surface and the curvature curves of another surface that he had found in the night in such a way that I did not understand a word. (It was concerned with the line-sphere transformation, but instead of spheres, he operated, semi-intuitively, with rectilinear hyperboloids that went through a fixed real conic.) In any event, he convinced me that the principal tangent curves to the **Kummer** surfaces must be algebraic curves of order 16. Later that morning, while I was visiting the Conservatoire des Arts et Métiers, the thought occurred to me that one must be dealing with those very curves of order 16 that appeared already in treatise II (see, pp. 74) of my "Theorie der Linienkomplexe ersten und zweiten Grades," and I quickly succeeded in deducing the geometric considerations that were given in nos. 1 to 5 of the text above independently of the Lie transformation. When I went to **Lie**'s house at four o'clock in the afternoon, he was gone, and I left him a summary of my results in a letter. I found the figure in no. 6 of the text at the end of July or the beginning of August 1870 during my sojourn in Düsseldorf.

The principal tangent curves of the **Kummer** surface have indeed been investigated many times since then. First, by **Lie** and myself in our treatises in Bd. 5 of Math. Ann. (1871); see the following treatises. Whereas Lie developed them by starting with his line-sphere transformation, I have illuminated the integration process that comes under consideration from some other angles. I refer to these papers more readily for the proof of the main theorem, rather than the one that was developed in nos. 2 and 7 of the present text, since although they are materially correct, perhaps they can seem formally unsatisfactory. What followed were the developments of **Rohn** in his dissertation (Munich, 1878) and in his treatises in Bd. 15, 18 of Math. Ann. (1879, 1881, resp.). The meaning of representing principal tangent curves of the **Kummer** surface by hyperelliptic functions emerged clearly for the first time in them, and its real progress was given for the cases in which one of the two segments that were limited by conic segments in the figure that was given in no. 6 of the text was not valid. One should further confer article III C8 of the Enzyklopädie der Math. Wiss. by **K. Zindler** on algebraic line geometry. K]