THE CALCULUS OF VARIATIONS

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WITH 24 PRINTED FIGURES

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BRAUNSCHWEIG

PRINTED AND PUBLISHED BY FRIEDRICH VIEWEG AND SON

1900

FOREWORD

No new book on the calculus of variations has appeared in Germany or France for more than thirty years now. Of the presentations that have been published in the other branches of the scientific world, none of them seem suited to replace, much less surpass, the commendable, and still quite informative, work of **Moigno** and **Lindelöf**. It might then be timely to attempt a presentation of the calculus of variations that employs the new insights that were achieved in the last thirty years and to carry out the proofs with the kind of rigor that has become necessary in ever -broader circles of mathematicians, mainly under the influence of **Weierstrass**.

I was induced to carry out a detailed study of the calculus of variations mainly because, due to the assignment of lectures at the University of Dorpat, I was rather often called upon to teach the calculus of variations and to utilize the problems that were accessible to that discipline in practical exercises that would be very stimulating to the students. Once I had become familiar with the pedagogical and factual difficulties of the subject, I had the pleasure of undertaking the problem of preparing an entry for the *Encyklopädie der Mathematischen Wissenschaften* on the development of the calculus of variations, which compelled me to carry out a thorough study of its interesting history and literature. Finally, the plan of the work that I thus presented to the mathematical world grew out of a proposal by the firm of **Friedrich Vieweg and Son** that initially pointed in a different direction.

The relationship of my work to the investigations of **Weierstrass**, who did not act merely as a critic, but blazed new trails into positive, creative activities in the calculus of variations, as in other domains, deserves special attention. As is known, his research is not available in the form of a systematic presentation. My most fertile sources were a dissertation by **Zermelo** (Berlin, 1894) and a treatise by **Kobb** (Acta Mathematica, Bd. 16 and 17). Those works are indeed dedicated chiefly to the author's own investigations, but they also include all of the essential ideas of **Weierstrass** that relate to our topic in a modified and generalized form. I have not used anything from the popular lectures that I had prepared and presented to the mathematical society at the University of Berlin, since they were not generally available at the time of printing. That restriction implies no essential handicap. From the rich harvest that the cited treatises, as well as some further dissertations, yield, the ideas of **Weierstrass** can be employed most extensively. In that way, I generally cannot by any means strive to retain the original manner of presentation of the great scholar, since that would not seem appropriate, due to its connection with topics in the calculus of variations that he did not treat.

I have believed that I could count on the approval of the reader in my choice of topics when I avoided vacuous generalities and worked through a number of special problems precisely in each chapter that have a self-explanatory geometric or mechanical meaning, with the exception of the problem of the brachistochrone, which has a venerable history, and they are not merely devised in order to show the fruitfulness of the method. I believed that I could do without detailed historical information and any discussions of questions of priority, since the referring works of **Todhunter** (*History of the calculus of variations*, 1861) and **Pascal** (*Variationsrechnung*, 1899), as well as the aforementioned chapter in the *Encyklopädie*, which will appear soon, make it simple for one to learn about the provenance of the most important theorems. I have therefore included in the bibliography only those works that seemed to me to be best suited to inform the reader of the actual scope of my presentation.

I am happy to mention that I had the pleasure of collaborating with two former auditors in the completion of this work. **Heinrich Karstens** had supported me in a self-sacrificing and sympathetic way during the proofreading of all printed sheets. I worked through part of the manuscript in the final editing with **Erhard Schmidt**. Both men have afforded a worthwhile service to me by suggesting formal and factual changes, and I must express my most heartfelt thanks to them.

December 1899.

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CHAPTER ONE

CONCEPTS OF AND BASIC RULES FOR THE CALCULUS OF VARIATIONS.

____

§ 1. – Differentials and variations. Notations.

In the differential calculus, one regards the dependent variables as well-defined once the values of certain arguments are given. A differential is the increase that a function experiences under small changes in the arguments. By contrast, the calculus of variations treats quantities the depend upon varying functional relationships. The increase that a small change in the relationship will produce is called a *variation*, and we will denote it by *du*, if *u* is the quantity under consideration.

For example, if $y = \varphi(x)$ is the equation of a given curve then the quantity:

$$dy = \varphi(x + dx) - \varphi(x)$$

will be the *differential* of *y* when *dx* is small. By contrast, when one regards the curve as variable and replaces it with the curve that differs from it only slightly:

$$\eta = \varphi(x) + \psi(x) ,$$

the difference that one defines for fixed *x* :

$$\eta - y = \delta y = \psi(x)$$

will be the *variation* of *y*, which will be caused by the change in the dependency relationship between the two variables that that occurs. If one further defines:

$$J = \int_a^b \varphi(x)\, dx$$

then, with the first way of looking at $\varphi(x)$, that quantity will depend upon only *a* and *b*, and its differential will be:

$$dJ = f(b)\, db - f(a)\, da .$$

By contrast, if one regards the function relationship that is denoted by φ as variable then *J* will depend upon its form over the entire interval from *a* to *b*. If the function $\varphi(x)$ takes on the small increase $\psi(x)$ then *J* will increase by the variation:

$$\delta J = \int_a^b \psi(x)\,dx = \int_a^b \delta y(x)\,dx \ .$$

The following notations will be established: Points and systems of values of the available variables might be denoted by the numbers 0, 1, … If x, t, … are any dependent or independent variables then let their values at the points 0, 1, … always be x_0, x_1, …, t_0, t_1, … We shall then employ a substitution symbol by setting:

$$x_0 = x\,|^0 \,, \qquad x_1 - x_0 = x\,\Big|_0^1 \,,$$

and if Φ is a function of x then:

$$\Phi\,(x_1) - \Phi\,(x_0) = \Phi\,(x)\,\Big|_{x_0}^{x_1} = \Phi\,(x)\,\Big|_0^1 \ .$$

In those expressions, the substitution symbol refers to all of the foregoing terms that follow the last substitution or equality sign, or when no such thing precedes, to the entire expression up to the substitution symbol, such that, e.g., one sets:

$$\Phi\,(t) + \Psi\,(t)\,|^0 = \Phi\,(t_0) + \Psi\,(t_0)\,,$$

$$\Theta\,(t)\,|^1 + \Phi\,(t) + \Psi\,(t)\,|^0 = \Theta\,(t_1) + \Phi\,(t_0) + \Psi\,(t_0)\,,$$

If k is a non-negative whole number then we shall let:

$$[u, v, \ldots]_k$$

denote a power series in the bracketed arguments that includes only terms of dimension at least k and converges for all arguments whose absolute values do not exceed a certain positive quantity. If a function of the quantities u, v, … can be represented in the form:

$$f\,(u, v, \ldots) = [u - a, v - b, \ldots]_k\,,$$

i.e., it can be developed into a **Taylor** series, then we shall call it *regular* at the location (a, b, …), as usual. We shall call a curve *regular* in the neighborhood of a location when it defines one of the coordinates as a regular function of the other ones.

We will often use the abbreviated notation:

$$\frac{\partial f\,(u, v, \ldots)}{\partial u} = f_u$$

for partial differential quotients, in which special values of the arguments can be given after the f_u symbol, e.g.:

$$\left(\frac{\partial f(u,v,\ldots)}{\partial u}\right)_{u=a,v=b,\ldots} = f_u\,(a, b, \ldots)\,.$$

§ 2. – Variations of integrals.

Let x, y be the rectangular coordinates of a point in the plane and let $\mathfrak{B}$ be an arc whose endpoints are 0 and 1. Let x and y be continuous functions of t that have continuous first derivatives along it. The derivatives might always be denoted by primes, and the quantities x' and y' might be nowhere-simultaneously-vanishing along the arc $\mathfrak{B}$. With the notational convention that was established, the endpoints will correspond to the arguments t_0, t_1 . Since it is sometimes necessary to replace t with $-t$, one can assume that one will have:

$$t_1 > t_0\,,$$

such that the variable t will increase continually along the arc $\mathfrak{B}$ in the direction from 0 to 1.

Furthermore, let δx, δy be continuous functions of t with continuous first derivatives in the interval from t_0 to t_1 . When the latter quantities run through that interval, the point $(x + \delta x, y + \delta y)$ will describe an arc $\mathfrak{B}^0$ that likewise possesses the continuity properties that are required of the arc $\mathfrak{B}$ and will be regarded as a variation of the latter. The points of both curves are associated with each other uniquely by the values of t. If u is any quantity that is defined at a point of the arc $\mathfrak{B}$ then let $u + \Delta u$ denote the analogously-defined quantity at the corresponds point of the arc $\mathfrak{B}^0$, i.e., the one that is associated with the same value of t. The same notation will apply when u does not depend upon a point, but upon the entire arc $\mathfrak{B}$, e.g., its length. u will also take on the increase Δu when one replaces $\mathfrak{B}$ with $\mathfrak{B}^0$ then. If one regards δx, δy and their first derivatives as small quantities and makes the corresponding approximations then Δu will go to du, and the latter quantity might be called the *variation* of u. Naturally, the fact that it has a well-defined value must be verified in each case.

Obviously, since corresponding points must imply the same value of t :

$$\Delta t = \delta t = 0\,.$$

When $\mathfrak{B}$ goes to $\mathfrak{B}^0$, the quantities x', y' will go to:

$$\frac{d\,(x+\delta x)}{dt}\,, \qquad \frac{d\,(y+\delta y)}{dt}\,.$$

It follows from this that:

$$\Delta x' = \delta x' = \frac{d\,\delta x}{dt}\,, \quad \Delta y' = \delta y' = \frac{d\,\delta y}{dt}\,.$$

We further set:

$$p = \frac{y'}{x'}, \qquad q = \frac{x'}{y'},$$

in which each of those expressions can be considered only where its denominator does not vanish, so:

$$\Delta p = \frac{y' + \delta y'}{x' + \delta x'} - \frac{y'}{x'} = \frac{x'\,\delta y' - y'\,\delta x'}{x'^2} \frac{1}{1 + \frac{\delta x'}{x'}},$$

or when one develops the last fraction in powers of $\delta x'$:

$$\Delta p = \frac{x'\,\delta y' - y'\,\delta x'}{x'^2} \{1 + [\delta x']_1\}.$$

If one neglects the terms here that include $\delta x'$, $\delta y'$ to at least the second power then Δp will go to δp, and one will get:

$$\delta p = \frac{x'\,\delta y' - y'\,\delta x'}{x'^2} = \frac{1}{x'} \frac{d\,\delta y}{dt} - \frac{p}{x'} \frac{d\,\delta x}{dt}.$$

At any location for which x' does not vanish, t can be regarded as a single-valued function of x, and the latter quantity can be introduced as an independent variable. One will then have:

$$\delta p = \frac{d\,\delta y}{dx} - p \frac{d\,\delta x}{dx} = \frac{dx\,d\,\delta y - dy\,d\,\delta x}{dx^2},$$

and analogously, when y' does not vanish:

$$\delta q = \frac{d\,\delta x}{dy} - q \frac{d\,\delta y}{dy}.$$

More generally, let $F(x, y, x', y')$ be a function that is regular in the neighborhood of each system of values (x, y, x', y') that is defined by an element of the curve $\mathfrak{B}$. In that way, it is not excluded that a singularity might appear for $x' = y' = 0$, since that system of values does not occur along the arc $\mathfrak{B}$. From the definition of the Δ symbol, one will then have:

$$\Delta F(x, y, x', y') = F(x + \delta x, y + \delta y, x' + \delta x', y' + \delta y') - F(x, y, x', y'),$$

or since F is regular:

$$\Delta F(x, y, x', y') = F_x\,\delta x + F_y\,\delta y + F_{x'}\,\delta x' + F_{y'}\,\delta y' + [\delta x, \delta y, \delta x', \delta y']_2,$$

with the notation of § **1**. That expression will go to the variation δF when one neglects the last term on the right, such that one will get:

$$\delta F = F_x\,\delta x + F_y\,\delta y + F_{x'}\,\delta x' + F_{y'}\,\delta y' .$$

Finally, one sets:

$$J = \int_{t_0}^{t} F(x, y, x', y')\,dt ,$$

in which, from the remarks above, one can restrict oneself to integrals for which $t - t_0$ is positive. That will immediately imply that:

$$\Delta J = \int_{t_0}^{t} \Delta F\,dt ,$$

and since ΔF goes to δF with the repeatedly-mentioned approximations:

$$\delta J = \int_{t_0}^{t} \delta F\,dt , \qquad \Delta J = \delta J + \int_{t_0}^{t} [\delta x, \delta y, \delta x', \delta y']_2\,dt .$$

When calculating with the expressions for δp, δF, δJ, the same rules of operation will apply when one operates with the δ symbol that are true for the symbol for differentiating with respect to a parameter that is independent of t. In particular, the symbol δ commutes with the ones for differentiation and integration with respect to t.

The most important example of a variation of type considered is provided by the case in which a family of curves that include the curve $\mathfrak{B}$ is defined by the equations:

$$\text{(1)} \qquad \bar{x} = \xi(\tau, a, b, \ldots) , \quad \bar{y} = \eta(\tau, a, b, \ldots) .$$

The latter will be represented by the special equations:

$$x = \xi(\tau, a_0, b_0, \ldots) , \quad y = \eta(\tau, a_0, b_0, \ldots) ,$$

and one lets the functions $\xi(t, a, b, \ldots)$, $\eta(t, a, b, \ldots)$ be regular at the location $(t, a_0, b_0, \ldots)$ when t lies in the interval from t_0 to t_1 . The curve that is represented by the equations (1) can serve as a variation of the arc $\mathfrak{B}$ in the sense that was defined when τ runs from τ_2 to τ_3 , and the quantities:

$$\tau_2 - t_0 = \delta t_0 , \quad \tau_3 - t_1 = \delta t_1 , \quad a - a_0 = \delta a , \quad \ldots$$

have absolute values that are sufficiently small. Namely, if one defines τ to be a function of t by the equation:

$$\begin{vmatrix} \tau - t & t & 1 \\ \delta t_0 & t_0 & 1 \\ \delta t_1 & t_1 & 1 \end{vmatrix} = 0$$

then τ will run through the interval from τ_2 to τ_3 when t assumes all values between t_0 and t_1 . The arc $\mathfrak{B}$ will then be related to the varied arc (1), or:

$$\bar{x} = x + \delta x = \xi(\tau, a_0 + \delta a, \ldots), \qquad \bar{y} = y + \delta y = \eta(\tau, a_0 + \delta a, \ldots)$$

in an invertible, single-valued way. Now since $\tau - t$ is an expression that is linear with respect to δt_0 and δt_1 , its coefficients will be very simple functions of t, so one will have:

$$\delta x = \xi(\tau - t + t, a_0 + \delta a, \ldots) - x(t, a_0, \ldots) = [\delta t_0, \delta t_1, \delta a, \ldots]_1,$$

and likewise:

$$\delta y = [\delta t_0, \delta t_1, \delta a, \ldots]_1,$$

and the coefficients of the power series are regular functions of t between t_0 and t_1 , in any event. The quantities δx, δy will then have the required properties as long as $|\delta t_0|$, $|\delta t_1|$, $|\delta a|$, … are sufficiently-small quantities.

§ 3. – Change of parameter.

By far, the most important of the integrals J are determined from only the curve $\mathfrak{B}$ and the direction of integration along it, but do not depend upon the special choice of the parameter t, i.e., when x and y can also be represented as functions of the new parameter s along the curve $\mathfrak{B}$ and the directions of increasing s and t agree, they will be the ones that yield the equation:

$$\int_{t_0}^{t} F(x, y, x', y')\, dt = \int_{s_0}^{s} F\left(x, y, \frac{dx}{ds}, \frac{dy}{ds}\right) ds,$$

as long as the upper limits are assigned to the same point of the arc $\mathfrak{B}$. When that equation is differentiated, it will yield the result:

$$F(x, y, x', y')\, dt = F\left(x, y, \frac{dx}{ds}, \frac{dy}{ds}\right) ds = F\left(x, y, \frac{x'}{s'}, \frac{y'}{s'}\right) s'\, dt,$$

in which s' can assume an arbitrary positive value at any individual location on the curve $\mathfrak{B}$. Thus, if α is an arbitrary positive constant then one will have the identity:

then one will have the identity:

(2) $$F(x, y, \alpha x', \alpha y') = \alpha F(x, y, x', y') ,$$

and when one differentiates this with respect to x' and y':

(3) $$\begin{aligned} F_{x'}(x, y, \alpha x', \alpha y') &= F_{x'}(x, y, x', y'), \\ F_{y'}(x, y, \alpha x', \alpha y') &= F_{y'}(x, y, x', y'). \end{aligned}$$

Equation (2) is characteristic of functions that are homogeneous of degree one with respect to x', y'. If x' is non-zero then one can set:

$$\alpha = \pm \frac{1}{x'} = \left| \frac{1}{x'} \right|$$

and then get:

$$F(x, y, x', y') = x' F\left(x, y, \pm 1, \pm \frac{y'}{x'} \right) = x' f(x, y, p) .$$

Since dt is positive, the element of the integral J can be written:

$$F(x, y, x', y')\, dt = F(x, y, dx, dy) = dx\, f(x, y, p) ,$$

and it is not, in general, determined by the quantities x, y, p, but initially by one of the systems:

$$x, y, dx, dy ; \quad x, y, p, dx ,$$

since $dx = x'\, dt$ can be positive, as well as negative, because in general two coincident, but oppositely-directed, line elements must differ. The elements of the curve will be regarded as infinitely-small vectors, to some extent. As equation (2) shows, the value $F(x, y, x', y')$ is still not determined by the line element itself, but it will be, e.g., when we impose the relation:

$$x'^2 + y'^2 = 1 ,$$

which can be arranged by a suitable choice of the parameter t at each location. One can then set:

$$x' = \cos \varphi , \quad y' = \sin \varphi .$$

If a half-line rotates from the + x-axis in the sense that would make it rotate through 90° if it were to go to the to the + y-axis then it will have described the angle φ when it points in the same direction as the element considered. The quantity $F(x, y, \cos \varphi, \sin \varphi)$ will then be a continuous function of the position and direction of the element considered and will obviously be regular with respect to x, y, φ when F is regular in the system of values (x, y, x', y') that represents the element considered.

Now one must distinguish two types of homogeneous functions: Either equations (2), (3) will remain valid for negative values of α, as well, or they will not. The first case occurs, e.g., when F is a rational function of the arguments x', y'. The same thing will then be true for f in relation to the argument p. The second case can be found, e.g., under the assumption that:

$$F = \sqrt{x'^2 + y'^2}\,,$$

in which the square root is positive. F will then be an everywhere-single-valued function for real numbers, since the double-valuedness of the root first comes to light in its analytic continuation to the complex number domain. Moreover, F is everywhere regular, with the exception of the location:

$$x' = y' = 0\,.$$

By contrast:

$$f(x, y, p) = \frac{F(x, y, x', y')}{x'} = \sqrt{1 + p^2}$$

is not a single-valued function of p, since the square root has the sign of the quantity x', and one will not have equations (2), (3) for negative α, but:

$$F(x, y, \alpha\, x', \alpha\, y') = -\,\alpha\, F(x, y, x', y')\,,$$
$$F_{x'}(x, y, \alpha\, x', \alpha\, y') = -\,F_{x'}(x, y, x', y')\,.$$

In other cases, those relations are not true, but only a more complicated equation between $F(x, y, \alpha\, x', \alpha\, y')$ and $F(x, y, x', y')$. For example, when:

$$F = y\, x' + \sqrt{x'^2 + y'^2}\,,$$

one will have:

$$F(x, y, \alpha\, x', \alpha\, y') + \alpha\, F(x, y, x', y') = 2\alpha\, y\, x'$$

for $\alpha < 0$.

In the two cases that were distinguished above, the integral:

$$J_{01} = \int_{t_0}^{t_1} F(x, y, x', y')\, dt$$

will exhibit a different behavior when one reverses the direction of integration. In the first case, one has:

$$-F(x, y, -x', -y') = F(x, y, x', y')\,.$$

The differential $dJ = F\,dt$ will then have opposite values for two arc-length elements whose positions coincide but are oppositely directed. Hence, if one sets:

$$-t = s\,, \qquad s_0 = -t_0\,, \qquad s_1 = -t_1\,,$$

such that $s_0 > s_1$, then one will have:

$$J_{10} = \int_{s_1}^{s_0} ds\,F\left(x, y, \frac{dx}{ds}, \frac{dy}{ds}\right) = -\int_{t_1}^{t_0} dt\,F\left(x, y, \frac{dx}{ds}, \frac{dy}{ds}\right) = \int_{t_0}^{t_1} dt\,F\left(x, y, \frac{dx}{ds}, \frac{dy}{ds}\right) = -J_{01}\,.$$

In the second case, this relation does not have to be true. For example, one has:

$$J_{01} = +J_{10}\,; \qquad F(x, y, x', y')\,dt = F(x, y, -x', -y')\;dt$$

for the length integral. The connection between J_{01} and J_{10} can be more complicated for other integrals J that belong to that case. The difference between the two cases is very important for the extremum problems that are accessible by the calculus of variations.

We would also like to establish the meaning of the symbol J that is endowed with two indices when we integrate between points other than 0 and 1, e.g.:

$$J_{23} = \int_{t_2}^{t_3} F(x, y, x', y')\,dt\,.$$

Frequently, the limits of integration will also be denoted by only the symbols of the points between which one integrates, e.g.:

$$J_{23} = \int_2^3 F\,dt\,,$$

since the numerical values of the integration limits will occur only in the examples.

§ 4. – Examples.

From § **2**, one has the formula:

$$\delta J_{01} = \int_0^1 dt\left(F_x\,\delta x + F_y\,\delta y + F_{x'}\,\frac{d\,\delta x}{dt} + F_{y'}\,\frac{d\,\delta y}{dt}\right)\,.$$

If one integrates the last terms by parts then that will give:

(1) $$\delta J_{01} = F_{x'}\,\delta x + F_{y'}\,\delta y\,\Big|^0 + \int_0^1 dt\left(P\,\delta x + Q\,\delta y\right) ,$$

in which one has set:

$$P = F_x - F'_{x'} , \quad Q = F_y - F'_{y'} .$$

Those quantities include x'', y''. In order for no singularities to appear under the integral sign in formula (4) that might jeopardize the integration, we shall assume from now on that x'', y'' are also continuous functions of t along the arc $\mathfrak{B}$.

If one further differentiates equation (2) with respect to α and considers the relations (3) then it will follow that:

$$F = x'\,F_{x'} + y'\,F_{y'} ,$$

so when one differentiates with respect to t :

$$F' = x''\,F_{x'} + y''\,F_{y'} + x'\,F'_{x'} + y'\,F'_{y'} .$$

On the other hand, one obviously has:

$$F' = F_x\,x' + F_y\,y' + F_{x'}\,x'' + F_{y'}\,y'' .$$

If one subtracts that equation from the previous one then that will give the identity:

(5) $$P\,x' + Q\,y' = 0 ,$$

and one can put the expression for δJ_{01} above into the following forms:

(6) $$\begin{aligned}\delta J_{01} &= F_{x'}\,\delta x + F_{y'}\,\delta y\,\Big|^0 + \int_0^1 dt\,Q\left(\delta y - p\,\delta x\right) \\ &= F_{x'}\,\delta x + F_{y'}\,\delta y\,\Big|^0 + \int_0^1 dt\,P\left(\delta x - q\,\delta y\right) .\end{aligned}$$

First example: Determine the variation of arc-length.

The length of the arc $\mathfrak{B}$ will be given by the formula:

$$J_{01} = \int_0^1 dt\,\sqrt{x'^2 + y'^2} ,$$

which is defined with the positive square root. One will then have:

$$\delta J_{01} = \int_0^1 dt\,\delta\sqrt{x'^2+y'^2} = \int_0^1 dt\,\frac{x'\,\delta x' + y'\,\delta y'}{\sqrt{x'^2+y'^2}}$$

$$= \frac{x'\,\delta x' + y'\,\delta y'}{\sqrt{x'^2+y'^2}}\Bigg|_0^1 - \int_0^1 dt\left[\delta x\frac{d}{dt}\left(\frac{x'}{\sqrt{x'^2+y'^2}}\right) + \delta y\frac{d}{dt}\left(\frac{y'}{\sqrt{x'^2+y'^2}}\right)\right],$$

$$P = \frac{d}{dt}\left(\frac{x'}{\sqrt{x'^2+y'^2}}\right), \qquad Q = \frac{d}{dt}\left(\frac{y'}{\sqrt{x'^2+y'^2}}\right).$$

Now if the positive sense of rotation is the one in which a half-line goes from the $+x$-axis to the $+y$-axis by a rotation through 90°, and θ is the angle through which a half-line must be rotated in the positive sense in order to arrive at one of the points that traverse the curve 01 from 0 to 1 upon starting from the $+x$-axis then one will have:

$$\cos\theta = \frac{x'}{\sqrt{x'^2+y'^2}}, \quad \sin\theta = \frac{y'}{\sqrt{x'^2+y'^2}}.$$

With that notation, the formula above can be written:

$$\delta J_{01} = \delta x\cos\theta + \delta y\sin\theta\Big|_0^1 - \int_0^1 d\theta\left[\delta x\cos\left(\theta+\frac{\pi}{2}\right) + \delta y\sin\left(\theta+\frac{\pi}{2}\right)\right].$$

The factor of $d\theta$, which shall be called δn, is the component of the vector that points from the point (x, y) to the point $(x + \delta x, y + \delta y)$ along the normal to the curve that is rotated by 90° in the positive sense from the tangent that points in the direction of increasing t. Under the substitution symbol, one finds the component of that vector with respect to the indicated direction of the tangent. If $d\theta$ has a fixed sign (e.g., a positive one) then the concave side of the curve will point to the defined direction of the normal. One will then have:

$$\delta x\,|^0 = \delta x\,|^1 = \delta y\,|^0 = \delta y\,|^1 = 0,$$

and if δn has a fixed sign then δJ_{01} will have the opposite sign to it. The latter will then be positive or negative according to whether $\mathfrak{B}^0$ lies on the convex or concave side of $\mathfrak{B}$, respectively. It is only when $d\theta = 0$, so the curve $\mathfrak{B}$ will then be a line, that δJ will vanish continually.

Second example: The integral:

$$J = \int y\,dx = \int_0^1 y\,x'\,dt$$

represents the surface that will be swept out by the ordinate of a point that traverses an arc 01 when each element of that surface is taken into account with the sign of $y\,dx$. If one has a closed line along which the direction of increasing t has the same relation to the interior normal that the +y-axis has to the +x-axis then the integral will represent the area, positively-taken, of the surface that it enclosed. Therefore, one will have:

$$\begin{aligned}\delta J &= \int_0^1 \left(x'\,\delta y + y\,\frac{d\,\delta x}{dt}\right) dt \\ &= y\,\delta x\Big|_0^1 + \int_0^1 (x'\,\delta y - y'\,\delta x)\,dt \\ &= y\,\delta x\Big|_0^1 + \int_0^1 (dx\,\delta y - dy\,\delta x),\end{aligned}$$

or when ds is the element of arc-length:

$$\delta J = y\,\delta x\Big|_0^1 + \int_0^1 ds\,\delta n\ ,$$

which can be linked with a discussion that is similar to the one in the previous example. Obviously, we are dealing with the first of the cases in § **3**.

§ 5. – Expressing the integral *J* without using the parameter *t*.

If one would like to express the integral J without the use of the parameter t in terms of the quantities x, y, and their differentials alone then one must start from the equation:

$$F(x, y, x', y') = x'\,f\left(x, y, \frac{y'}{x'}\right) = x'\,f(x, y, p)$$

as the definition of the function f. As we would like to show, it will be regular at any location (x, y, p_0) when the same thing is true for F at the location (x, y, x_0', y_0'), x_0' is non-zero, and one has set:

$$p_0 = \frac{y_0'}{x_0'}\,.$$

If one defines new variables by the equations:

$$x' = r\cos\varphi\,, \qquad y' = r\sin\varphi\,, \qquad x'_0 = r_0\cos\varphi_0\,, \qquad y'_0 = r_0\sin\varphi_0$$

then one can develop:

$$x' - x'_0 = (r - r_0 + r_0)\cos(\varphi - \varphi_0 + \varphi_0) - r_0\cos\varphi_0$$
$$= [r - r_0\,,\ \varphi - \varphi_0]_1\,,$$

and one will get a similar expression for $y' - y'_0$. Now, since, by assumption, an equation will exist:

$$F(x, y, x', y') = [x' - x'_0, y' - y'_0]_0$$

when x, y are assumed to be constant, one will also have:

$$f(x, y, p) = \frac{F(x, y, x', y')}{x' - x'_0 + x'_0} = [x' - x'_0, y' - y'_0]_0 = [r - r_0\,,\ \varphi - \varphi_0]_0\,.$$

That quantity, like p, does not depend upon r, so it will follow that:

$$f(x, y, p) = [\varphi - \varphi_0]_0\,.$$

On the other hand, since p_0 is a finite quantity, one can solve the equation:

$$p - p_0 = \tan(\varphi - \varphi_0 + \varphi_0) - \tan\varphi_0 = \frac{\varphi - \varphi_0}{\cos^2\varphi_0} + [\varphi - \varphi_0]_2$$

for $\varphi - \varphi_0$ and get:

$$\varphi - \varphi_0 = [p - p_0]_1\,.$$

That will then imply that:

$$f(x, y, p) = [p - p_0]_0\,.$$

Now since f depends upon x, y in the same way as F, the assertion is thereby proved.

It follows from this that one can develop:

$$\Delta f(x, y, p) = f_x\,\delta x + f_y\,\delta y + f_p\,\Delta p + [\delta x, \delta y, \Delta p]_2$$

$$= f_x\,\delta x + f_y\,\delta y + f_p\,\delta p + [\delta x, \delta y, \delta x', \delta y']_2$$

at each location along the arc $\mathfrak{B}$ at which x' does not vanish, and that:

$$\delta f = f_x\,\delta x + f_y\,\delta y + f_p\,\delta p$$

is a well-defined quantity. Therefore, when x' does not vanish along the arc 23, one can put the integral J into the form:

$$J_{23} = \int_2^3 (f\,x')\,dt$$

and get:

(7)
$$J_{23} = \int_2^3 \delta(f\,x')\,dt = \int_2^3 (f\,\delta x' + f\,\delta x')\,dt\,.$$

Operating with those formulas and similar ones will be eased by means of the following general notation: If u, v are any continuous, differentiable functions of t between t_0 and t_1 then let the equation:

(8)
$$\int_0^1 u\,dv = \int_{t_0}^{t_1} u\,v'\,dt$$

be the definition of the left-hand side, even when v cannot be introduced as an independent variable for the entire extent of 01. The equation for partial integration will then be:

$$\int_0^1 u\,dv = u\,v\Big|_0^1 - \int_0^1 v\,du\,.$$

With that notation, one will have the formulas:

$$J = \int f\,dx\,, \qquad \delta J = \int (dx\,\delta f + f\,d\,\delta x)\,,$$

the latter of which subsumes the formula (7), even for an arc along which x', and therefore dx, changes sign, and breaks down only at the locations where x' vanishes. f is undefined at those exceptional locations, and one must then call upon the analogously-defined function:

$$\frac{F(x, y, x', y')}{y'} = \bar{f}\left(x, y, \frac{x'}{y'}\right) = \bar{f}(x, y, q)\,.$$

For it, one will have the equations:

$$J = \int \bar{f}\,dy\,, \qquad \delta J = \int (dy\,\delta\bar{f} + \bar{f}\,d\,\delta y)\,,$$

the latter of which will have the more precise meaning of:

$$\delta J = \int \left(y'\,\delta\bar{f} + \bar{f}\,\frac{d\,\delta y}{dt}\right)dt\,,$$

from the definition (8). The expression that is obtained for δJ shows that the symbol δ will commute with d and the symbol for integration when one starts from the definition (8), e.g.:

$$d\,\delta x = \frac{d\,\delta x}{dt}\cdot dt = \delta x'\cdot dt = \delta(x'\,dt) = \delta\,dx\,.$$

By contrast, δ does not commute with the symbol for differentiation with respect to x, since obviously:

$$\delta\left(\frac{dy}{dx}\right) = \delta p = \frac{d\,\delta y}{dx} - p\,\frac{d\,\delta x}{dx}\,.$$

It is only when δx vanishes that one will have:

$$\delta\,\frac{dy}{dx} = \frac{d}{dx}\,\delta y\,.$$

Now in order to introduce f and $\bar{f}$ into the formulas of § **4**, one differentiates the identity:

$$F = x'\,f\left(x, y, \frac{y'}{x'}\right) = y'\,\bar{f}\left(x, y, \frac{x'}{y'}\right)\,.$$

That will then imply that:

$$F_x = y'\,\bar{f}_x\,, \qquad F_y = x'\,f_y\,, \qquad F_{x'} = \bar{f}_q = f - p\,f_p\,, \qquad F_{y'} = f_p = \bar{f} - q\,\bar{f}_q\,,$$

and it will follow from this that:

$$F'_{x'} = y'\,\frac{dF_{x'}}{dy} = y'\,\frac{d\bar{f}_q}{dy}\,, \qquad F'_{y'} = x'\,\frac{dF_{y'}}{dx} = x'\,\frac{df_p}{dx}\,,$$

$$P = y'\left(\bar{f}_x - \frac{d\bar{f}_q}{dy}\right), \qquad Q = x'\left(f_y - \frac{df_p}{dx}\right), \qquad f_y - \frac{df_p}{dx} = -\left(\bar{f}_x - \frac{d\bar{f}_q}{dy}\right),$$

so, from (4), (6):

$$\delta J_{01} = (f - p\,f_p)\,\delta x + f_p\,\delta y\,\Big|_0^1 + \int_0^1 dx\left(f_y - \frac{df_p}{dx}\right)(\delta y - p\,\delta x)$$

$$= \bar{f}_q\,\delta x + (\bar{f} - q\,\bar{f}_q)\,\delta y\,\Big|_0^1 + \int_0^1 dx\left(\bar{f}_x - \frac{d\bar{f}_q}{dy}\right)(\delta x - q\,\delta y)\,.$$

The quantity that appears here:

$$\delta_0 y = \delta y - p\ \delta x$$

has a meaning that is easy to give. Namely, if x' is non-zero and the variations are sufficiently small then both curves $\mathfrak{B}$ and $\mathfrak{B}^0$ will define y as a single-valued function of x that possesses a continuous derivative in the neighborhood of the location considered, such that one can set:

$$y = \varphi(x), \qquad y + \delta y = \Phi(x + \delta x).$$

If one neglects all quantities that vanish in comparison to δx then one will have:

$$\varphi(x + \delta x) = y + p\ \delta x = \Phi(x + \delta x) - \delta y + p\ \delta x,$$

$$\delta_0 y = \delta y - p\ \delta x = \Phi(x + \delta x) - \varphi(x + \delta x).$$

The quantity $\delta_0 y$ then measures the distance between the curves $\mathfrak{B}$ and $\mathfrak{B}^0$ in the direction of the y-axis. One sometimes calls it the *truncated* (*tronquée*) variation of y. If one then sets:

$$\delta_0 u = \delta u - \frac{du}{dx}\delta x,$$

in general, then one will find that:

$$\frac{d\,\delta_0 u}{dx} = \frac{d\,\delta u}{dx} - \frac{du}{dx}\frac{d\,\delta x}{dx} - \frac{d^2 u}{dx^2}\delta x = \delta\left(\frac{du}{dx}\right) - \frac{d^2 u}{dx^2}\delta x,$$

or

$$\delta_0\left(\frac{du}{dx}\right) = \frac{d\,\delta_0 u}{dx}.$$

The symbol δ_0 will then commute with differentiation with respect to x. If x' is non-zero everywhere along the arc $\mathfrak{B}$ then one can set $x = t$, and the operation δ_0 will be identical to δ, since $\delta x = \delta t = 0$.

First example from § 4: One obviously has:

$$J_{01} = \int_0^1 dx\sqrt{1+p^2} = \int_0^1 dy\sqrt{1+q^2}, \quad f = \sqrt{1+p^2},$$

$$\bar{f} = \sqrt{1+q^2},$$

in which the square roots have the signs of the differentials that stand next to them. If we operate on the first formula with the δ symbol according to the rules above then we will get:

$$\begin{aligned}\delta J_{01} &= \int_0^1 (d\,\delta x \cdot \sqrt{1+p^2} + dx\,\delta\sqrt{1+p^2}) \\ &= \int_0^1 \left(d\,\delta x \sqrt{1+p^2} + \frac{p\,\delta p\,dx}{\sqrt{1+p^2}} \right).\end{aligned}$$

Now, from § **2**, one has:

$$\delta p = \frac{d\,\delta y}{dx} - p\frac{d\,\delta x}{dx},$$

so:

$$\delta J_{01} = \int_0^1 \left(\frac{d\,\delta x}{\sqrt{1+p^2}} + \frac{p\,d\,\delta y}{\sqrt{1+p^2}} \right),$$

and when one partially integrates:

$$\begin{aligned}\delta J_{01} &= \left. \frac{\delta x}{\sqrt{1+p^2}} + \frac{\delta y \cdot p}{\sqrt{1+p^2}} \right|_0^1 - \int_0^1 \left[\delta x\, d\left(\frac{1}{\sqrt{1+p^2}} \right) + \delta y\, d\left(\frac{p}{\sqrt{1+p^2}} \right) \right] \\ &= \left. \frac{\delta x}{\sqrt{1+p^2}} + \frac{\delta y \cdot p}{\sqrt{1+p^2}} \right|_0^1 - \int_0^1 (\delta y - p\,\delta x)\, d\left(\frac{p}{\sqrt{1+p^2}} \right),\end{aligned}$$

which naturally agrees with the result that was obtained in § **4**.

§ 6. – The case of many variables.

The general developments that were carried out can be extended with no difficulty to the case in which one considers a simple manifold in a domain with arbitrary-many variables *x*, *y*, *z*, …, *w*, instead of a planar curve. Let such a thing be defined by saying that all of the quantities are set equal to continuous functions of a parameter *t* whose first and second derivatives are likewise continuous. The concept of a variation can be adapted immediately when one assigns the same continuity properties to the quantities $x + dx$, $y + dy$, $z + dz$, … as functions of *t* that one does to the quantities *x*, *y*, *z*, … themselves, and one will have:

$$\delta F(x, y, z, \ldots, w, x', y', \ldots, w') = F_x\,\delta x + F_{x'}\,\delta x' + F_y\,\delta y + F_{y'}\,\delta y' + \cdots + F_{w'}\,\delta w'.$$

If one further sets:

$$P = F_x - F'_{x'}\ ,\quad Q = F_y - F'_{y'}\ ,\ R = P = F_z - F'_{z'}\ , \ldots$$

then formula (4) in § **4** will then take on the extended form:

$$\delta J = \delta\int_0^1 F\,dt = F_{x'}\,\delta x + F_{y'}\,\delta y + \cdots + F_{w'}\,\delta w\Big|_0^1 + \int_0^1 dt\,(P\,\delta x + Q\,\delta y + R\,\delta z + \cdots)\,,$$

when Δ means the increase that occurs when the simple manifold $(x, y, z, \ldots)$ goes to the varied one $(x + \delta x, y + \delta y, z + \delta z, \ldots)$.

The most important case here, as well, is the one in which one can set:

$$F(x, y, z, \ldots, w, x', y', \ldots, w')\,dt = f(x, y, \ldots, \omega, \eta, \zeta, \ldots, \omega)\,dx\,,$$

in which x' is non-zero, and one sets:

$$\eta = \frac{dy}{dx} = \frac{y'}{x'}\,,\qquad \zeta = \frac{dz}{dx} = \frac{z'}{x'}\,,\quad \ldots,\qquad \omega = \frac{dw}{dx} = \frac{w'}{x'}\,.$$

Precisely as in § **5**, one will then get:

$$P\,x' + Q\,y' + \ldots = 0\,,$$

$$F_{x'} = f - \eta\,f_\eta - \zeta\,f_\zeta - \ldots - \omega\,f_\omega\,,\qquad F_{y'} = f_\eta\,,\qquad F_{z'} = f_\zeta\,,\qquad \ldots,$$

$$\delta J = F_{x'}\,\delta x + F_{y'}\,\delta y + \cdots\Big|_0^1 + \int_0^1 dt\,\{Q(\delta y - \eta\,\delta x) + R(\delta z - \zeta\,\delta x) + \cdots\}\ .$$

If one further introduces the notations:

$$\delta_0\,y = \delta y - \eta\,\delta x\,,\qquad \delta_0\,z = \delta z - \zeta\,\delta z\,,\qquad \ldots,\qquad \delta_0\,w = \delta w - \omega\,\delta w$$

then the last equation can be written:

$$\delta J = F_{x'}\,\delta x + F_{y'}\,\delta y + \cdots\Big|_0^1 + \int_0^1 dt\,\{Q\,\delta_0 y + R\,\delta_0 z + \cdots\}\ .$$

Third example: Find the variation of an integral that contains the parameter c :

$$J = \int_0^1 F(x, y, x', y', c)\,dt$$

when one replaces c with $c + \delta c$. One can regard c as a function of t for which:

$$F_{c'} = 0\,, \qquad c' = \frac{d\,\delta c}{dt} = 0\,.$$

If one applies the formula above when one sets $z = c$ then one will get:

$$\delta J = F_{x'}\,\delta x + F_{y'}\,\delta y\,\Big|_0^1 + \int_0^1 dt\,(P\,\delta x + Q\,\delta y) + \delta c\int_0^1 F_c\,dt\,.$$

Fourth example: The arc-length of a space curve 01 has the expression:

$$J = \int_0^1 dt\,\sqrt{x'^2 + y'^2 + z'^2} = \int_0^1 \sqrt{dx^2 + dy^2 + dz^2}\,,$$

in which the square root is taken to be positive. If one operates with the δ symbol as one does with a differential symbol and sets:

$$\sqrt{dx^2 + dy^2 + dz^2} = ds$$

then that will give:

$$\delta J = \int_0^1 \left(\frac{dx}{ds}\,d\,\delta x + \frac{dy}{ds}\,d\,\delta y + \frac{dz}{ds}\,d\,\delta z\right)$$

$$= \frac{dx}{ds}\,d\,\delta x + \frac{dy}{ds}\,d\,\delta y + \frac{dz}{ds}\,d\,\delta z\,\Big|_0^1 - \int_0^1 \left[\delta x\,d\left(\frac{dx}{ds}\right) + \delta y\,d\left(\frac{dy}{ds}\right) + \delta z\,d\left(\frac{dz}{ds}\right)\right].$$

Now, the positive quantity:

$$d\theta = \sqrt{\left(d\,\frac{dx}{ds}\right)^2 + \left(d\,\frac{dy}{ds}\right)^2 + \left(d\,\frac{dz}{ds}\right)^2}$$

is the *contingency angle*, and:

$$\frac{1}{d\theta}\,d\left(\frac{dx}{ds}\right), \qquad \frac{1}{d\theta}\,d\left(\frac{dy}{ds}\right), \qquad \frac{1}{d\theta}\,d\left(\frac{dz}{ds}\right)$$

are the direction cosines of the principal normal with respect to the direction of the center of curvature. Hence, if δn is the component of the vector that points from the point (x, y, z) to the point $(x + \delta x, y + \delta y, z + \delta z)$ along that direction then it will follow that:

$$\delta J = \left.\frac{dx}{ds}\,\delta x + \cdots\right|_0^1 - \int_0^1 d\theta\,\delta n \quad .$$

Therefore, if the original curve and the varied one have the same endpoints and δn has a fixed sign then δJ will have the opposite sign to it. It is only when $d\theta = 0$ everywhere (i.e., the curve is a straight line) that δJ cannot have a different sign in that way.

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CHAPTER TWO

THE SIMPLEST OF THE EXTREMUM PROBLEMS THAT ARE ACCESSIBLE BY THE CALCULUS OF VARIATIONS

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§ 7. – The connection between the extremum and the variation.

Just as the problems of finding the largest or smallest value for a given function gave a strong impetus to the creation of the differential calculus, the calculus of variations owes its creation to an analogous problem of finding a maximum or minimum, or an *extremum*, as we would like to say. Let a quantity whose value is determined by some functional relationship that exists (§ **1**) (e.g., an integral J of the considered form) be extremized by a suitable choice of that connection, so by a curve $\mathfrak{B}$ or its generalization to a higher manifold, in the sense that when it is defined for the curve $\mathfrak{B}$, it will always have a greater value or always have a lesser value than when one replaces the curve $\mathfrak{B}$ with a neighboring one $\mathfrak{B}^0$. In that way, the desired curve can be subjected to restrictions of various kinds. In the case of a maximum, the inequality:

$$J > J + \Delta J, \qquad \Delta J < 0$$

must always be valid, with the notation that was introduced, while the minimum requires that ΔJ must always be positive. The service that the calculus of variations provides is first of all based upon the fact that one can infer the sign of the quantity δJ from that of the quantity ΔJ as soon as the curves $\mathfrak{B}$ and $\mathfrak{B}^0$ deviate from each other sufficiently little. If δJ is non-zero then that will show that in general ΔJ can be positive, as well as negative, so an extremum will be excluded. One then obtains the equation:

$$\delta J = 0$$

as a necessary condition for the extremum, which is entirely similar to how the differential of the function under scrutiny is set to zero in the extremum problem in the differential calculus.

The following general remark will lead to the connection between the signs of δJ and ΔJ : A real power series in any arguments ε_1, ε_2, … that also includes linear terms in them and simultaneously vanishes with all arguments can be positive and negative when the absolute value of the quantities ε remain below an arbitrarily-small positive constant. Namely, if one lets all arguments vanish except for one that occurs linearly, and if, e.g., ε_1 is the non-vanishing argument then the power series will reduce to the form:

$$a\,\varepsilon_1 + [\varepsilon_1]_2 = \varepsilon_1\,(a + [\varepsilon_1]_1)\,,$$

in which a is non-zero. Since $[\varepsilon_1]_1$ decreases indefinitely along with ε_1, the parentheses on the right-hand side will have the same sign as $a\ \varepsilon_1$, so it can become positive, as well as negative.

Furthermore, if the quantities ε, which shall be m in number, are subject to the n equations:

$$(9) \qquad 0 = a_{\mathfrak{a}1}\,\varepsilon_1 + a_{\mathfrak{a}2}\,\varepsilon_2 + \ldots + a_{\mathfrak{a}m}\,\varepsilon_m + [\varepsilon_1, \varepsilon_1, \ldots]_2 \qquad (\mathfrak{a} = 1, 2, \ldots, n),$$

in which $m > n$ and a determinant that is defined by a matrix of the mn quantities a, e.g.:

$$\sum \pm a_{11}\, a_{22} \ldots a_{nn},$$

is non-zero then a power series:

$$\mathfrak{P} = b_1\,\varepsilon_1 + b_2\,\varepsilon_2 + \ldots + b_m\,\varepsilon_m + [\varepsilon_1, \varepsilon_1, \ldots]_2$$

can be expressed in terms of only the quantities $\varepsilon_{n+1}, \varepsilon_{n+2}, \ldots, \varepsilon_m$, and the linear terms will be the same as when the terms of second and higher dimensions are not present. The linear terms in the converted series $\mathfrak{P}$ will then vanish if and only if the equation:

$$a_{\mathfrak{a}1}\,u_1 + a_{\mathfrak{a}2}\,u_2 + \ldots + a_{\mathfrak{a}m}\,u_m = 0$$

always implies the equation:

$$b_1\,u_1 + b_2\,u_2 + \ldots + b_m\,u_m = 0,$$

i.e., when all determinants of order $n + 1$ in the matrix:

$$\begin{matrix} a_{\mathfrak{a}1} & a_{\mathfrak{a}2} & \ldots & a_{\mathfrak{a}m} \\ b_1 & b_2 & \ldots & b_m \end{matrix}$$

vanish. From the result that was just proved, it is only in that case that $\mathfrak{P}$ can have a fixed sign for all quantities ε that are subject to the conditions (9) and lie below a certain positive constant in absolute value. For example, in the case $m = 2$, $n = 1$, $\mathfrak{P}$ can always become positive or negative then for arbitrarily-small values of $|\varepsilon_1|$ and $|\varepsilon_2|$ when the equation:

$$\begin{vmatrix} a_{11} & a_{12} \\ b_1 & b_2 \end{vmatrix} = 0$$

is not valid.

§ 8. – Necessary condition for an extremum.

Let us now pose the problem of drawing a plane curve 01 between the given points 0 and 1 such that when the integral:

$$J = \int_0^1 f(x, y, p)\, dx = \int_0^1 F(x, y, x', y')\, dt$$

is defined along it, in which f is a given function, that integral will be an extremum. We shall approximate the solution to this problem step-wise by first asking: What properties must a curve 01 with the continuity properties that were assumed for the arc $\mathfrak{B}$ in §§ **2** and **4** possess in order for the indicated extremum to be possible? We compare the curve 01 with a curve $\mathfrak{B}^0$ that runs between the same endpoints and has the same continuity properties that runs through the points ($x + \delta x$, $y + \delta y$) and gives the value $J + \Delta J$ to the integral $\int f(x, y, p)\, dx$. From § **2**, we will then have:

$$\Delta J = \delta J + \int_0^1 dt\,[\delta x, \delta y, \delta x', \delta y']_2 \; . \tag{10}$$

If ε means an arbitrarily-small constant then we shall now consider the curve that runs through the points ($x + \varepsilon\, \delta x$, $y + \varepsilon\, \delta y$) and gives the value $J + \Delta_\varepsilon J$ to the integral considered then that curve can also be taken to be $\mathfrak{B}^0$, and we will have:

$$\Delta_\varepsilon J = \varepsilon\, \delta J + [\varepsilon]_2 \; .$$

If δJ is non-zero then, from § **7**, that quantity can be positive, as well as negative, for arbitrarily-small values of $|\varepsilon|$, so there will be curves that are arbitrarily close to the curve 01 with the same continuity properties and endpoints that make the integral J larger, as well as smaller. That will then give:

$$\delta J = 0 \tag{11}$$

as a necessary condition for the extremum, in which δx, δy, $\delta x'$, $\delta y'$ need to lie only within limits for which the power series in formula (10) converges. When applied to the first example in § **4**, that result will show that the shortest line between two points can be nothing but the straight line when they are to have the properties of the arc $\mathfrak{B}$.

Now, in general, let 23 be a segment of the arc $\mathfrak{B}$ and let:

$$\delta x = 0 \; , \qquad \delta y = \varepsilon\, (t - t_1)^3\, (t_3 - t)^3$$

along that segment, but outside of it, one will have:

$$\delta x = \delta y = 0$$

everywhere. δx and δy will be then continuous along the entire interval from t_0 to t_1, along with their derivatives up to second order. The varied path $\mathfrak{B}^0$ will then have the continuity properties of $\mathfrak{B}$. Furthermore, if ε is a sufficiently-small constant then formula (10) will be applicable. From § **4**, since:

$$\delta J_{23} = F_{x'}\,\delta x + F_{y'}\,\delta y\Big|_2^3 + \int_2^3 dt\,Q\,\delta y\,,$$

so since δx and δy vanish at the locations 2, 3, one can set:

$$\delta J_{23} = \int_2^3 dt\,Q\,\delta y = \varepsilon\int_2^3 dt\cdot Q\,(t-t_2)^3(t_3-t)^3\,,$$

and obviously one has the equation:

$$\delta J = \delta J_{01} = \delta J_{23}\,,$$

so equation (11) will then imply that:

$$\int_2^3 dt\cdot Q\,(t-t_2)^3(t_3-t)^3 = 0\,. \tag{12}$$

Now, $Q = F_y - F'_{y'}$ is a continuous function of t along the arc 01 with the assumptions that were introduced. Therefore, if Q does not vanish everywhere then the interval 23 can be chosen in such a way that the quantity Q always possesses a fixed sign and does not vanish in its interior. However, the left-hand side of equation (12) will certainly be non-zero then, since $t - t_2$ and $t_2 - t$ are positive inside of the indicated interval. Equation (12) will then imply that:

$$Q = 0\,.$$

The equation:

$$P = 0$$

will likewise follow from an analogous development, which one gets from the previous identity (5) in § **4** when x' is non-zero. If that is also true for y' then from § **5**, one can replace both equations, which are free of t, with the mutually-equivalent differential equations:

$$f_y - \frac{df_p}{dx} = 0\,, \qquad \bar{f}_x - \frac{d\bar{f}_q}{dy} = 0\,,$$

which are generally of second order.

We call a plane curve that satisfies the equations $P = Q = 0$ an *extremal* of the integral J. It has then been proved that an arc 01 with the properties of the curve $\mathfrak{B}$ can make the integral J take on an extremum in comparison to all sufficiently-close curves with the same endpoints and continuity properties only when 01 is a piece of an extremal. As the expressions for P and Q that were given § **5** will show, the property of a curve that it is an extremal is independent of the choice of parameter t.

A first integral of the differential equation that was obtained can be given immediately in two cases: If f is free of y then one will have:

$$\frac{df_p}{dx} = 0 , \qquad f_p = \text{const.}$$

If f, and therefore F, as well, is free of x then one will have:

$$F'_{x'} = 0 , \qquad F_{x'} = f - p\, f_p = \text{const.}$$

The integral will be employed in most of the examples that were examined up to now. Since it contains only one of the quantities x, y, in addition to the differential quotients, it will lead to a generally finite equation for the extremals by means of a quadrature.

The integration constants shall always be denoted by a, c, … from now on.

§ 9. – Examples. Problems I-V.

Problem I. – Find the shortest line between two given points in the plane.

One has:

$$J = \int \sqrt{x'^2 + y'^2}\, dt .$$

Since F is free of x and y, one will have the first integrals:

$$F_{x'} = \frac{x'}{\sqrt{x'^2 + y'^2}} = \text{const.}, \qquad F_{y'} = \text{const.},$$

so p = const. The extremals are then the straight lines.

Problem II. – Draw a curve between two given points in the plane that generates the smallest area for a surface of revolution when it is rotated around a given axis.

Let the axis of rotation be the x-axis. If ds is an element of arc along the desired line then it will generate an element of the surface of revolution that is bounded by two circles, and its area will be $2\pi y\, ds$. One will then have to minimize the integral:

$$J = \int y\,ds = \int y\sqrt{1+p^2}\,dx = \int y\sqrt{x'^2+y'^2}\,dt\,.$$

The integrand is free of x. Thus, the first integral equation:

$$f - p\,f_p = \text{const.}, \qquad y\sqrt{1+p^2} - \frac{y\,p^2}{\sqrt{1+p^2}} = a$$

will be valid for the extremals, or:

$$\frac{dy}{dx} = p = \sqrt{\left(\frac{y}{a}\right)^2 - 1}\,,$$

$$\frac{dx}{a} = \frac{d\left(\frac{y}{a}\right)}{\sqrt{\left(\frac{y}{a}\right)^2 - 1}} = d\ln\left\{\frac{y}{a} + \sqrt{\left(\frac{y}{a}\right)^2 - 1}\right\},$$

in which the square root has the sign of p. It follows from this that:

$$e^{\frac{x-b}{a}} = \frac{y}{a} + \sqrt{\left(\frac{y}{a}\right)^2 - 1}\,.$$

If one goes to the reciprocal values of both sides of that equation then one will get:

$$e^{-\frac{x-b}{a}} = \frac{y}{a} - \sqrt{\left(\frac{y}{a}\right)^2 - 1}\,, \qquad \text{or} \qquad y = \frac{a}{2}\left(e^{\frac{x-b}{a}} + e^{-\frac{x-b}{a}}\right).$$

Problem III. – Draw a curve of given length that starts from a given point 0 and ends on a fixed line that goes through that point without the position of the endpoint being prescribed and also spans the greatest-possible area with that line.

For the moment, let u, y be rectangular coordinates, let x be the length of the desired curve, as measured from 0, let $y = 0$ be the fixed line, and let l be the prescribed arc-length. If x, y are rectangular coordinates in a second plane then an arc $\mathfrak{B}$ in the first plane will correspond to a similar one $\mathfrak{B}'$ in the second one. The latter is to be chosen such that the integral:

$$J = \int y\,du$$

will become an extremum. Now one obviously has:

$$\left(\frac{du}{dx}\right)^2 = 1 - \left(\frac{dy}{dx}\right)^2 = 1 - p^2, \tag{13}$$

so one can set:

$$J = \int y\sqrt{1-p^2}\,dx .$$

The endpoints of $\mathfrak{B}'$ are given in the second plane, because the point 0 corresponds to the system of values $x = 0$, $y = 0$, while the endpoint of the arc $\mathfrak{B}$ is the system $x = l$, $y = 0$. One must then solve an extremum problem of the type that was considered up to now in the second plane.

Now the integrand in J is free of x in the new form. One will then have the first integral:

$$F_{x'} = f - p\,f_p = \text{const.},$$

and since:

$$f = y\sqrt{1-p^2}\,, \qquad F = y\sqrt{x'^2 - y'^2}\,,$$

that will yield:

$$\frac{y\,x'}{\sqrt{x'^2 - y'^2}} = \frac{y}{\sqrt{1-p^2}} = a .$$

One easily concludes from this that:

$$dx = \frac{a\,dy}{\sqrt{a^2 - y^2}}, \qquad \text{or} \qquad y = a \sin\frac{x+b}{a},$$

i.e., the extremals in the second plane are sinusoids. Furthermore, equation (13) implies that:

$$\left(\frac{du}{dx}\right)^2 = \sin^2\frac{x+b}{a}, \qquad u - c = \pm\, a\cos\frac{x+b}{a}, \qquad y^2 + (u-c)^2 = a^2 .$$

In the first plane, one will then get curves $\mathfrak{B}$ that might possibly yield the desired extremum, which are semi-circles whose centers lie along the fixed line $y = 0$.

Problem IV. – Construct a surface of revolution of given area and largest-possible volume whose surface meets the axis of rotation precisely two times.

If u, y are rectangular coordinates in the meridian plane and $y = 0$ is the axis of rotation then we will have to determine the point 0, 1 along it such that integral:

$$\int_0^1 y\sqrt{dy^2 + du^2}$$

has the prescribed value ω and:

$$J = \int_0^1 y^2\, du$$

is an extremum. We set:

$$x = \int_0 y \sqrt{1 + \left(\frac{dy}{du}\right)^2}\, du\,,$$

in which the arc 01 is integrated up to a variable point. We will then have:

$$dx^2 = y^2 (du^2 + dy^2)\,, \qquad du = \sqrt{-dy^2 + \left(\frac{dx}{y}\right)^2}\,,$$

(14)

$$J = \int y^2 \sqrt{\frac{1}{y^2} - p^2}\, dx\,, \qquad p = \frac{dy}{dx}.$$

If one once more interprets x and y as rectangular coordinates in a second plane then the points 0 and 1 in the first one will correspond to the points:

$$x = y = 0\,, \qquad x = \omega\,, y = 0\,,$$

which are given. One must then find the curve between the given points in the second plane that will make the transformed integral J an extremum. Since the integrand is free of x, one will immediately find the equation for the integral:

$$y\sqrt{1 - p^2 y^2} + \frac{y^3 p^2}{\sqrt{1 - p^2 y^2}} = a = \frac{y}{\sqrt{1 - p^2 y^2}}\,,$$

$$-\frac{dx}{a^2} = \frac{-\dfrac{y}{a}\, d\left(\dfrac{y}{a}\right)}{\sqrt{1 - \left(\dfrac{y}{a}\right)^2}}\,, \qquad b - \frac{x}{a^2} = \sqrt{1 - \left(\frac{y}{a}\right)^2}\,.$$

The extremals in the second plane are then certain ellipses that are easy to characterize. For the ones that go through the point $x = y = 0$, one will get $b^2 = 1$. Now since one obviously has:

$$-2\left(b - \frac{x}{a^2}\right)\frac{dx}{a^2} = -\frac{2y\,dy}{a^2}\,,$$

and y increases with x at the point $x = y = 0$, it will follow that $b = + 1$. Moreover, from (14), one will have:

$$\frac{y^2\,dy^2}{1-\left(\frac{y}{a}\right)^2} = y^2(du^2 + dy^2)\,, \qquad \frac{y\,dy}{\sqrt{1-\left(\frac{y}{a}\right)^2}} = a\,du\,, \qquad \sqrt{1-\left(\frac{y}{a}\right)^2} = \frac{u}{a} + b\,.$$

That equation represents a circle whose center lies along the axis of rotation. The desired body, when it exists and its meridian has the properties of the arc $\mathfrak{B}$, can be only a sphere then.

Problem V. – Construct a body of revolution out of a given quantum of homogeneous matter that attracts according to **Newton**'s law and exerts the greatest-possible attraction on a point at which its surface and axis intersect.

If θ, φ are the angular distance from the North Pole and the geographic longitude, resp., on the Earth then the surface element of a sphere of radius r that is concentric with the Earth is known to be:

$$r^2 \sin\theta\,d\theta\,d\varphi\,,$$

while the volume element is:

$$r^2 \sin\theta\,dr\,d\theta\,d\varphi\,.$$

The volume of an infinitely-thin cone with its vertex at $r = 0$ will then be $\frac{1}{3}r^3 \sin\theta\,d\theta\,d\varphi$. If one then keeps θ constant and integrates over φ from 0 to 2π then one will get $\frac{2}{3}\pi r^3 \sin\theta\,d\theta$ as the volume of the space v that is bounded by two coaxial neighboring cones θ = const. When the density of the volume element is unity, its **Newton**ian attraction at the point $r = 0$ will be $dr \sin\theta\,d\theta\,d\varphi$, so its component along the axis $\theta = 0$ will be $dr \sin\theta \cos\theta\,d\theta\,d\varphi$. The component of the attraction of the space when it is taken in that direction will then be:

$$\int_{\varphi=0}^{2\pi}\int_{r=0}^{r} dr \sin\theta\cos\theta\,d\theta\,d\varphi = 2\pi r \sin\theta\cos\theta\,d\theta\,.$$

Now the meridian of the desired surface of revolution starts from the point $r = 0$ on the axis $\theta = 0$ and let $r = 0$ also be the points at which we wish to make the attraction as great as possible. If a point on the meridian runs from its second point of intersection with the axis to the point $r = 0$ then θ will go from 0 to θ_0. The integral that is defined along the meridian:

$$J = \int_0^{\theta_0} r \sin\theta\cos\theta\,d\theta$$

will then be extremized, while the value:

$$\int_0^{\theta_0} r^3 \sin\theta \, d\theta = \omega$$

will be given. In order to convert this problem into one of a type that is accessible by our methods, we set:

$$\int_0^{\theta_0} r^3 \sin\theta \, d\theta = y , \qquad \cos\theta = x , \qquad \cos\theta_0 = x_0 ,$$

so we will then have:

$$J = -\int_1^{x_0} \sqrt[3]{-\frac{dy}{dx}} \cdot x \, dx ,$$

and we will look for a curve in the xy-plane whose endpoints, which correspond to the values $\theta =$ 0 and $\theta = \theta_0$, have the coordinates:

$$x = 0 , \; y = 0 , \qquad x = x_0 , y = \omega ,$$

so they will be given when a certain condition on θ_0 is added.

As extremals of the problem:

$$\delta \int \sqrt[3]{p} \, x \, dx = 0 ,$$

one will get, with no further analysis, the curves:

$$y = a\,x^{5/2} + b , \qquad \frac{dy}{dx} = \tfrac{5}{2}\, a\, x^{3/2} .$$

The last equation implies that:

$$r = -\sqrt[3]{\frac{dy}{dx}} = -\sqrt[3]{\tfrac{5}{2}\,a}\; x^{1/2} = -\sqrt[3]{\tfrac{5}{2}\,a}\,\sqrt{\cos\theta} ,$$

with which the meridian of the desired surface of revolution that corresponds to the extremals in the xy-plane is expressed in polar coordinates.

§ 10. – Variable endpoints. Transversal position.

A problem that is more general than the ones that were treated up to now is the problem of finding the plane curves 01 that extremize the integral J from among all of the ones whose endpoints have coordinates that are subject to the equations:

$$g_\alpha (x_0, y_0, x_1, y_1) = 0 . \tag{15}$$

If we would not like to revert to the previous problems then the number of those equations must not be greater than three. The most important special case is the one in which x_0 and y_0 are given, but a given equation exists between x_1 and y_1 . Geometrically speaking, that is the problem of drawing the curve from a given point to a given curve that yields an extreme value for J.

Initially, one must extremize J along the curve 01 in comparison to the neighboring curves that have the same endpoints in the case of the most general equations (15), as well. That is because when one establishes the ones whose coordinates satisfy equations (15), the latter will be fulfilled. When the desired curve has the properties of the arc $\mathfrak{B}$, it must necessarily be a piece of an extremal of the integral J then.

However, the equations of constraint will yield a more detailed determination of the arc 01. Let the functions g_α be regular in the neighborhood of the system of values considered x_0, y_0, x_1, y_1, while the quantities δx_0, δy_0, δx_1, δy_1, are subject to the equations:

$$g_\alpha (x_0 + \delta x_0, y_0 + \delta y_0, x_1 + \delta x_1, y_1 + \delta y_1) = 0 . \tag{16}$$

One varies a piece 02 at the beginning of the arc 01 and a piece 31 at its end in the following way: For the former, let:

$$\delta x = \delta x_0 \left(\frac{t - t_2}{t_0 - t_3}\right)^3 , \quad \delta y = \delta y_0 \left(\frac{t - t_2}{t_0 - t_3}\right)^3 ,$$

and for the latter, let:

$$\delta x = \delta x_1 \left(\frac{t - t_2}{t_1 - t_3}\right)^3 , \quad \delta y = \delta y_1 \left(\frac{t - t_2}{t_1 - t_3}\right)^3 ,$$

and for the middle part 23, let:

$$\delta x = \delta y = 0 .$$

The quantities δx, δy that are thus defined for the entire arc 01 are continuous functions of t, along with their first and second order derivatives, such that curve $\mathfrak{B}^0$ that is described by the points ($x + \delta x$, $y + \delta y$) will likewise possess the continuity properties of the arc $\mathfrak{B}$. Now since one further has:

$$\delta x\,|^0 = \delta x_0 , \quad \delta y\,|^0 = \delta y_0 , \quad \delta x\,|^1 = \delta x_1 , \quad \delta y\,|^1 = \delta y_1 ,$$

from (16), the varied arc $\mathfrak{B}^0$ will likewise satisfy the boundary conditions, so it will belong to the curves that should extremize the integral J in comparison to the curve 01. The quantity ΔJ that is defined for the indicated variations δx, δy must then have a fixed sign, no matter what one takes the quantities δx_0 , δy_0 , δx_1 , δy_1 to be, according to the assumptions (16).

Now, from § **4**, one has:

$$\delta J = F_{x'}\,\delta x + F_{y'}\,\delta y\Big|_0^1 + \int_0^1 (P\,\delta x + Q\,\delta y)\,dt\,,$$

so, since the equations:

$$P = Q = 0$$

are true for the extremal 01, one will have:

$$\delta J = F_{x'}\,\delta x + F_{y'}\,\delta y\Big|_0^1\,,$$

and furthermore:

$$\Delta J = \delta J + \int_0^1 [\delta x, \delta y, \delta x', \delta y']_2\,dt\,,$$

so for the values of δx, δy that were introduced:

$$\Delta J = \delta J + F_{x'}\,\delta x + F_{y'}\,\delta y\Big|_0^1 + [\delta x_0, \delta y_0, \delta x_1, \delta y_1]_2\,.$$

However, for arbitrary quantities δx_0 , δy_0 , δx_1 , δy_1 that are subject to equations (16), from § **7**, that expression can possess a fixed sign only when it always follows from the equations:

$$\frac{\partial g_{\mathfrak{a}}}{\partial x_0}u_0 + \frac{\partial g_{\mathfrak{a}}}{\partial y_0}v_0 + \frac{\partial g_{\mathfrak{a}}}{\partial x_1}u_1 + \frac{\partial g_{\mathfrak{a}}}{\partial y_1}v_1 = 0$$

that:

$$-F_{x'}\Big|^0 u_0 - F_{y'}\Big|^0 v_0 + F_{x'}\Big|^1 u_1 + F_{y'}\Big|^1 v_1 = 0\,,$$

or as one usually says, when all of the infinitely-small variations that satisfy the equations:

$$\delta g_{\mathfrak{a}} = 0$$

make the equation:

$$F_{x'}\,\delta x + F_{y'}\,\delta y\Big|_0^1 = 0$$

true. That represents a new necessary condition for an extremum in the present problem.

In particular, if the starting point 0 is prescribed such that $\delta x_0 = \delta y_0 = 0$, and the endpoint is constrained to a given curve:

$$g(x_1, y_1) = 0$$

then the result that was obtained will show that the equations:

$$F_{x'}\,\delta x + F_{y'}\,\delta y\,\Big|^1 = f - p\,f_p\,|^1\;\delta x_1 + f_p\,|^1\;\delta y_1 = 0\,,$$

$$\frac{\partial g}{\partial x_1}\,\delta x_1 + \frac{\partial g}{\partial y_1}\,\delta y_1 = 0$$

are both valid. We would like to introduce the term *transversal* for the special position of the extremal in relation to the curve $g = 0$ that is defined in that way. Above all, if two arc-elements λ, μ that start from a point have the components δx, δy and dx, $p\,dx$ along the coordinate axes, and the equation:

$$F_{x'}\,\delta x + F_{y'}\,\delta y = f - p\,f_p\;\delta x_1 + f_p\;\delta y_1 = 0$$

is true then we will say that μ lies *transversely* to the element λ, and every curve that includes the latter will be intersected by every transversal that includes μ. We can express the result of the developments that were worked through in this terminology. An extremal 01 can extremize the integral J only when among all curves that connect the point 0 to the given point 1 that belongs to the curve $g = 0$, the given curve intersects the extremal transversely at the point 1. That will also be true when one compares the extremal to only the curves that possess the properties of the arc $\mathfrak{B}$.

§ 11. – Examples. Problems I, II, VI, VII.

In Problem I (§ **9**), one had:

$$F_{x'} = \frac{x'}{\sqrt{x'^2 + y'^2}}\,, \qquad F_{y'} = \frac{y'}{\sqrt{x'^2 + y'^2}}\,.$$

Transversal position requires the equation:

$$x'\,\delta x + y'\,\delta y = 0\,, \qquad 1 + p\,\frac{\delta y}{\delta x} = 0\,,$$

so it coincides with the perpendicular position. The shortest line that is drawn from a point to a curve can then be nothing but a normal to the curve if it is to have the continuity properties of the arc $\mathfrak{B}$.

Similar statements are true for Problem II (§ **9**). One will then have:

$$f = y\sqrt{1+p^2}\ , \qquad f - p\, f_p = \frac{y}{\sqrt{1+p^2}}\ , \quad f_p = \frac{y\,p}{\sqrt{1+p^2}}\ ,$$

so the condition for transversal position will be:

$$\delta x + p\ \delta y = 0\ ,$$

just as in Problem I.

Problem VI. – Find the surface of revolution with fixed figure axes that suffers the least resistance when it advances in a fluid in the direction of the axis of rotation when every surface element experiences a normal resistance that is proportional to the area of the element and the square of the normal velocity.

If the axis of rotation is the *x*-axis and *ds* is an element of the meridian of the desired surface then the volume of zone that is bounded by two lateral circles will be $2\pi y\, ds$. Moreover, if θ is the inclination of a normal to the surface with respect to the *x*-axis then one will have:

$$\frac{dy}{dx} = \cos\,\theta\,.$$

$v\cos\,\theta$ is the component of the velocity along that normal, and the normal resistance that the zone thus-defined suffers will be the quantity:

$$v^2 y\, ds\left(\frac{dy}{dx}\right)^2 = y\, ds\ (v\cos\,\theta)^2\ ,$$

while its component along the *x*-axis will be proportional to:

$$v^2 y\, ds\left(\frac{dy}{dx}\right)^2 .$$

For a given value of *v*, one will then be dealing with the problem of extremizing the integral:

$$J = \int y\, ds\left(\frac{dy}{dx}\right)^2 = \int \frac{y\,p^3 dx}{1+p^2}\ .$$

Since the integrand is free of *x*, one will have the integral equation:

(17) $$f - p\, f_p = -\,a\ , \qquad y = \frac{a\,(1+p^3)^2}{2\,p^3}\ .$$

It follows from this that:

(17) [*sic*] $$x = \int \frac{dy}{p} = b + \frac{a}{2}\left(\frac{3}{4p^4} + \frac{1}{p^2} + \ln p\right),$$

and one can regard the equations that are obtained for x and y as the parametric representation of the desired curves. The expression for x shows that p can vanish at either a finite point or an infinite one. If one lets p run through all positive values then one will get a curve that will possess a cusp for $p = \sqrt{3}$, as the equations:

(18) $$\frac{dx}{dp} = \frac{a(1+p^2)(p^2-3)}{2p^5}, \qquad \frac{dy}{dp} = \frac{a(1+p^2)(p^2-3)}{2p^4}$$

show, but y is otherwise defined to be a regular function of x everywhere. Furthermore, since one easily finds from the last equation that:

$$\frac{d^2y}{dx^2} = \frac{2p^5}{a(1+p^2)(p^2-3)},$$

when a, and therefore y, is positive, the concave or convex side of the curve will turn to the x-axis according to whether $p < \sqrt{3}$ or $p > \sqrt{3}$, resp.

Since:

$$f - p\, f_p = \frac{\partial}{\partial p}\left(\frac{y\,p^3}{1+p^2}\right) = \frac{-2p^3 y}{(1+p^2)^2}, \qquad f_p = \frac{y\,p^2(p^2+3)}{(1+p^2)^2},$$

one finds that the condition for the transversal position is the equation:

$$-2p\,\delta x + (p^2+3)\,\delta y = 0.$$

One then sets:

$$p = \tan\varphi, \qquad \frac{\delta y}{\delta x} = \tan\psi,$$

such that ψ means the inclination of the given curve, φ is the inclination of the extremal that intersects it transversally at any point with respect to the x-axis, so one will have:

$$\tan\psi = \frac{2\tan\varphi}{3+\tan^2\varphi} = \frac{\sin 2\varphi}{2+\cos 2\varphi}.$$

Transversal position has a geometric character here that is not as simple as it was in Problems I and II. If a direction lies transversal to a second one then the latter will not generally lie transversal to the former, as was the case in the aforementioned problems.

Since φ depends upon only ψ, it is clear that a line will be intersected at the same angle by all of the extremals that lie transversal to it.

Problem VII. – Represent the shortest line on a given surface in curvilinear coordinates.

Let x, y, z be regular functions of the **Gauss**ian coordinates φ, ψ on the surface. If one sets:

$$E = x_\varphi^2 + y_\varphi^2 + z_\varphi^2, \qquad F = x_\varphi x_\psi + y_\varphi y_\psi + z_\varphi z_\psi, \qquad G = x_\psi^2 + y_\psi^2 + z_\psi^2$$

then two line elements on the surface whose components have the values:

$$dx = x_\varphi\, d\varphi + x_\psi\, d\psi, \quad dy = y_\varphi\, d\varphi + y_\psi\, d\psi, \quad dz = z_\varphi\, d\varphi + z_\psi\, d\psi,$$

$$\delta x = x_\varphi\, \delta\varphi + x_\psi\, \delta\psi, \quad \delta y = y_\varphi\, \delta\varphi + y_\psi\, \delta\psi, \quad \delta z = z_\varphi\, \delta\varphi + z_\psi\, \delta\psi$$

will be perpendicular to each other when:

$$dx\ \delta x + dy\ \delta y + dz\ \delta z = 0$$

or

$$E\, d\varphi\, \delta\varphi + F\,(d\varphi\, \delta\psi + d\psi\, \delta\varphi) + G\, d\psi\, \delta\psi = 0\,.$$

If one further sets:

$$\Phi\,(\varphi,\ \psi,\ d\varphi,\ d\psi) = \sqrt{E\, d\varphi^2 + 2F\, d\varphi\, d\psi + G\, d\psi^2}$$

then the length integral:

$$J = \int \Phi\,(\varphi, \psi, \varphi', \psi')\, dt$$

will be minimized. The condition of transversal position is:

$$\Phi_{\varphi'}\, \delta\varphi + \Phi_{\psi'}\, \delta\psi = 0\,, \qquad (E\, \varphi' + F\, \psi')\, \delta\varphi + (F\, \varphi' + G\, \psi')\, \delta\psi = 0\,,$$

so it will coincide with the condition of perpendicular intersection when one multiplies it by dt. The equations for the extremals – i.e., the geodetic lines – are easy to construct.

§ 12. Integrand that depends upon the integration limits. Problem VIII.

The argument in § **10** also implies necessary conditions for the extremum in the case where the integrand depends upon the limits of integration, such that the integral:

$$J = \int_0^1 F\,(x, y, x', y', x_0, y_0, x_1, y_1)\, dt$$

is to be extremized under the equations of constraint:

(19) $$g_\alpha(x_0, y_0, x_1, y_1) = 0 .$$

In that way, we assume that F, like g_α , is also regular in regard to the last four arguments at the location considered. Obviously, one will then have:

$$J + \Delta J = \int_0^1 F(x+\delta x, y+\delta y, x'+\delta x', y'+\delta y', x_0+\delta x_0, \ldots, y_1+\delta y_1)\, dt .$$

If one next varies in such a way that the endpoints remain fixed then, as in § **10**, one will get the result that the desired curve must be an extremal of the integral J when x_0 and y_0 , x_1 and y_1 are considered to be constant in it. It then follows from the equations of the extremals that:

$$\int_0^1 dt\,(F_x\,\delta x + F_y\,\delta y + F_{x'}\,\delta x' + F_{y''}\,\delta y') = F_{x'}\,\delta x + F_{x'}\,\delta x\Big|_0^1 ,$$

and the expression above for ΔJ will become:

$$\Delta J = F_{x'}\,\delta x + F_{y'}\,\delta y\Big|_0^1 + \delta x_0 \int_0^1 F_{x_0}\,dt + \cdots + \delta y_1 \int_0^1 F_{y_1}\,dt + [\delta x_0, \ldots, \delta y_1]_2 .$$

Should that quantity have a fixed sign for all systems of values δx_0, …, δx_1 that are compatible with equations (19), then the general theorems in § **7** would show that under the assumption that:

$$\frac{\partial g_\alpha}{\partial x_0}\delta x_0 + \frac{\partial g_\alpha}{\partial y_0}\delta y_0 + \frac{\partial g_\alpha}{\partial x_1}\delta x_1 + \frac{\partial g_\alpha}{\partial y_1}\delta y_1 = 0 ,$$

the equation:

$$F_{x'}\,\delta x + F_{y'}\,\delta y\Big|_0^1 + \delta x_0 \int_0^1 \frac{\partial F}{\partial x_0}\,dt + \cdots + \delta y_1 \int_0^1 \frac{\partial F}{\partial y_1}\,dt = 0$$

must be true.

Problem VIII. – Find the brachistochrone in a vertical plane, i.e., the curve that a massive point will traverse in the shortest time when it is constrained to remain on that curve. Let the endpoints of the curve be given or let them move on given curves.

During the motion of a massive point of unit mass along a prescribed curve, which will be assumed to be planar, the equation of *vis viva* is valid in the form:

$$\frac{v^2}{2} = g\,x + \text{const.},$$

when v is the velocity, g is the constant of gravity, and the + x-axis points vertically downwards. If one has the velocity v_0 at the starting point 0 then one will have:

$$\frac{v_0^2}{2} = g\,x_0 + \text{const.}, \qquad \frac{v^2}{2} = g\,(x - x_0) + \frac{v_0^2}{2}$$

or

$$v = \sqrt{2g\,(x-\alpha)}\,, \qquad \alpha = x_0 - \frac{v_0^2}{2\,g}\,,$$

and if ds is the arc-element of the desired curve then the time-of-fall will have the expression:

$$J = \int \frac{ds}{v} = \int \frac{\sqrt{dx^2+dy^2}}{\sqrt{2g\,(x-\alpha)}} = \int \frac{dt\sqrt{x'^2+y'^2}}{\sqrt{2g\,(x-\alpha)}} = \int F\,dt\ .$$

Since F is free of y, one will have:

(20) $$\sqrt{2g}\,F_{y'} = \frac{dy}{ds}\frac{1}{\sqrt{x-\alpha}} = c\,, \qquad p = \frac{dy}{dx} = \sqrt{\frac{x-\alpha}{c^{-2}-(x-\alpha)}}$$

for the extremals of that integral, and the first equation shows that the quantity $c^2(x-\alpha)$ lies between 0 and 1. One can then set:

$$c^2(x-\alpha) = \sin^2\frac{u}{2}\,, \qquad x - \alpha = a\,(1 - \cos u)\,, \qquad 2a\,c^2 = 1\ .$$

Equations (20) will then be fulfilled by the assumption that:

$$dy = a\sin u\sqrt{\frac{1-\cos u}{1+\cos u}}\,du = a\,(1 - \cos u)\,du\,,$$

$$y = b + a\,(u - \sin u)\ .$$

The extremals are then cycloids that are generated by the rolling of a circle of arbitrary radius on the fixed horizontal plane $x = \alpha$.

Now let the problem be specified more precisely by establishing the equations:

$$g\,(x_0, y_0) = 0\,, \qquad h\,(x_1, y_1) = 0\,,$$

i.e., the point on the one curve shall land on another one without the endpoint being prescribed. Let the initial velocity v_0 , and therefore α, be given as a function of x_0, y_0 in any way. Obviously, the natural special case will be:

(21) $$v_0 = 0\,, \qquad \alpha = x_0\,.$$

The general theory then shows that under the assumptions that:

(22) $$\frac{\partial g}{\partial x_0}\delta x_0 + \frac{\partial g}{\partial y_0}\delta y_0 = 0\,, \qquad \frac{\partial h}{\partial x_1}\delta x_1 + \frac{\partial h}{\partial y_1}\delta y_1 = 0\,,$$

the equation:

(23) $$0 = \frac{x'}{\sqrt{x'^2+y'^2}}\frac{\delta x}{\sqrt{x-\alpha}} + \frac{y'}{\sqrt{x'^2+y'^2}}\frac{\delta y}{\sqrt{x-\alpha}}\Bigg|_0^1$$

$$+\ \delta x_0 \int_0^1 \frac{\partial}{\partial x_0}\left(\frac{\sqrt{x'^2+y'^2}}{\sqrt{x-\alpha}}\right)\cdot dt + \delta y_0 \int_0^1 \frac{\partial}{\partial y_0}\left(\frac{\sqrt{x'^2+y'^2}}{\sqrt{x-\alpha}}\right)\cdot dt$$

will be true, since F is free of x_1 and y_1 . If one next sets $\delta x_0 = \delta y_0 = 0$ which is consistent with equations (22), then that will give:

$$x'\,\delta x + y'\,\delta y\,\big|^1 = 0\,,$$

i.e., the curve $h = 0$ on which the endpoint 1 should lie must be intersected perpendicularly by the extremal.

Furthermore, one has:

$$\int_0^1 \frac{\partial}{\partial x_0}\left(\frac{\sqrt{x'^2+y'^2}}{\sqrt{x-\alpha}}\right)\cdot dt = \frac{\partial\alpha}{\partial x_0}\int_0^1 \frac{1}{2}\frac{\sqrt{x'^2+y'^2}\,dt}{\left(\sqrt{x-\alpha}\right)^3} = -\frac{\partial\alpha}{\partial x_0}\int_0^1 \frac{\partial}{\partial x}\left(\frac{\sqrt{x'^2+y'^2}}{\sqrt{x-\alpha}}\right)\cdot dt\,.$$

Now since the equation:

$$P = F_x - F'_{x'} = 0\,, \qquad \text{or} \qquad \frac{\partial F}{\partial x}\,dt = d\left(\frac{\partial F}{\partial x'}\right)$$

is true for the extremals, the expression that was obtained can be written:

$$\frac{\partial\alpha}{\partial x_0}\frac{x'}{\sqrt{x'^2+y'^2}\sqrt{x-\alpha}}\Bigg|_0^1 = \frac{-x'}{\sqrt{x'^2+y'^2}\sqrt{x-\alpha}}\Bigg|_0^1\frac{\partial\alpha}{\partial x_0}\,,$$

and equation (23) will assume the following form when one recalls the corresponding calculation that is performed for the factor of δy_0 :

$$\left.\frac{x'\,\delta x + y'\,\delta y}{\sqrt{x'^2+y'^2}\sqrt{x-\alpha}}\right|_0^1 - \left.\frac{x'}{\sqrt{x'^2+y'^2}\sqrt{x-\alpha}}\right|_0^1 \left(\frac{\partial\alpha}{\partial x_0}\delta x_0 + \frac{\partial\alpha}{\partial y_0}\delta y_0\right) = 0 .$$

Under the special assumption (21), one will then get:

$$\left.\frac{x'}{\sqrt{x'^2+y'^2}\sqrt{x-\alpha}}\right|^1 (\delta x_1 - \delta x_0) + \left.\frac{y'\,\delta y}{\sqrt{x'^2+y'^2}\sqrt{x-\alpha}}\right|_0^1 = 0 .$$

Now the first equation in (20) gives:

$$\left.\frac{y'}{\sqrt{x'^2+y'^2}\sqrt{x-\alpha}}\right|_0^1 = 0 .$$

It will then follow that:

$$\left.\frac{x'}{\sqrt{x'^2+y'^2}\sqrt{x-\alpha}}\right|^1 (\delta x_1 - \delta x_0) + \left.\frac{y'}{\sqrt{x'^2+y'^2}\sqrt{x-\alpha}}\right|^1 (\delta y_1 - \delta y_0) = 0 .$$

If one then sets $\delta x_1 = \delta y_1 = 0$, which would not contradict equations (22), then that will yield:

$$\left.\frac{x'}{\sqrt{x'^2+y'^2}}\right|^1 \delta x_0 + \left.\frac{y'}{\sqrt{x'^2+y'^2}}\right|^1 \delta y_0 = 0 ,$$

i.e., the tangent to the extremal at the point 1 and the tangent to the curve $g = 0$ (i.e., the locus of the starting points) that contacts the point 0 intersect at a right angle.

The assumptions that:

$$\alpha = 0 , \qquad v_0 = \sqrt{2g\,x_0}$$

would then imply that the extremal is perpendicular to both curves $g = 0$, $h = 0$.

§ 13. – Extension to the case of several finite constraint equations. Problem VII.

The development in § **8** can be adapted with no further discussion to the case in which the integral:

$$J = \int f\left(x, y, z, \ldots, \frac{dy}{dx}, \frac{dz}{dx}, \ldots\right) dx = \int F(x, y, z, \ldots, x', y', z', \ldots)\, dt ,$$

whose integrand includes arbitrarily-many (say, $m - 1$) unknown functions of x, should be extremized with a suitable determination of those unknown functions. By analogy with the previous assumptions on the nature of the desired curve, we shall assume here that along that desired simple manifold, all of the quantities $x, y, z, \dots, w$ can be represented as continuous functions of t with continuous first and second derivatives in their domain of definition. The special variation that was used in § **8** will then need only to be written in the form:

$$\delta x = \delta z = \dots = \delta w = 0 , \qquad \delta y = \varepsilon\,(t_3 - t)^3\,(t - t_1)^3 ,$$

and with the notation of § **6**, when one lets $x, y, \dots, w$ enter in place of y, in succession, one will obtain the equations:

$$P = Q = R = \dots = W = 0 .$$

A modification of that argument will become necessary when a number of finite equations of constraint exist between $x, y, z, \dots, w$, say:

$$g_\mathfrak{a}\,(x, y, z, \dots, w) = 0 , \qquad \mathfrak{a} = 1, 2, \dots, n , \tag{24}$$

whose left-hand sides are regular in the neighborhood of all systems of values under consideration. One will then have the n relations:

$$g_\mathfrak{a}\,(x + \delta x, \dots, w + \delta w) = 0 ,$$

(25)

$$\frac{\partial g_\mathfrak{a}}{\partial x}\,\delta x + \cdots + \frac{\partial g_\mathfrak{a}}{\partial w}\,\delta w + [\delta x, \dots, \delta w]_2 = 0$$

for the m variations, about which we would like to assume that n of the variations can be expressed in terms of the remaining $m - n$ as power series. We shall call the former the *dependent* variations and the latter, the *independent* ones. If the latter are continuous functions of t, along with the first two derivatives, and if they contain a constant factor ε then the dependent variations will also have the form $[\varepsilon]_1$. Now when one understands $l_\mathfrak{a}$ to mean arbitrary quantities, one can conclude from equations (25) that:

$$\int_0^1 dt \left[l_\mathfrak{a} \left(\frac{\partial g_\mathfrak{a}}{\partial x}\,\delta x + \cdots + \frac{\partial g_\mathfrak{a}}{\partial w}\,\delta w \right) + [\delta x, , \delta w]_2 \right] = 0 ,$$

so, from § **6** :

$$\Delta J = F_{x'}\,\delta x + \cdots + F_{x'}\,\delta x \Big|_0^1 + \int_0^1 dt \left\{ \delta x \left(P + \sum_\mathfrak{a} l_\mathfrak{a} \frac{\partial g_\mathfrak{a}}{\partial x} \right) + \cdots + \delta w \left(W + \sum_\mathfrak{a} l_\mathfrak{a} \frac{\partial g_\mathfrak{a}}{\partial w} \right) \right\}$$

$$+ \int_0^1 dt\,[\delta x, \delta x', \dots, \delta w, \delta w']_2 .$$

If one determines the quantities l_α such that all dependent variations vanish on the right-hand side of the first integral then we will get n linear equations for the n unknowns whose determinant does not vanish as a result of the assumptions that were made about equations (24). If δx, δy belong to the independent variations then one will vary y as in § **8** when one lets the remaining independent variations vanish. With the given choice of the quantities l_α, one will then get:

$$\Delta J = \varepsilon \int_2^3 dt \left(Q + \sum_\alpha l_\alpha \frac{\partial g_\alpha}{\partial y} \right) (t - t_2)^3 (t_3 - t)^3 + [\varepsilon]_2 \ ,$$

and if the quantities are to have a fixed sign then:

$$Q + \sum_\alpha l_\alpha \frac{\partial g_\alpha}{\partial y} = 0 \ .$$

One will get analogous equations for all of the quantities x, y, …, w whose variations are independent. One has already posed those equations in order to determine the quantities l_α in terms of the remaining ones.

A necessary condition for the quantities x, y, …, w, when coupled by the constraints (24) in the manner considered, to extremize the integral J will then consist of saying that quantities l_α must exist for which the equations:

$$P + \sum_\alpha l_\alpha \frac{\partial g_\alpha}{\partial x} = 0 \ , \qquad Q + \sum_\alpha l_\alpha \frac{\partial g_\alpha}{\partial y} = 0 \ , \qquad \ldots, \qquad W + \sum_\alpha l_\alpha \frac{\partial g_\alpha}{\partial w} = 0$$

are valid.

Problem VII (§ **11**). – Let the equation for the given surface be:

$$g(x, y, z) = 0 \ . \tag{26}$$

The arc-length integral will then be:

$$J = \int_0^1 \sqrt{dx^2 + dy^2 + dz^2} = \int_0^1 ds \ ,$$

and from § **4**, one will have:

$$\Delta J = \delta J + \int_0^1 [\delta x, \delta y, \delta z, \delta x', \delta y', \delta z']_2 \, dt$$

$$= \frac{dx}{ds}\delta x + \frac{dy}{ds}\delta y + \frac{dz}{ds}\delta z\,\Big|_0^1 + \int_0^1 \left\{\delta x\left(\frac{dx}{ds}\right)' + \delta y\left(\frac{dy}{ds}\right)' + \delta z\left(\frac{dz}{ds}\right)' + [\delta x, , \ldots, \delta z']_2\right\} dt\,.$$

On the other hand, equation (26) gives:

$$0 = \int_0^1 \left\{\frac{\partial g}{\partial x}\delta x + \frac{\partial g}{\partial y}\delta y + \frac{\partial g}{\partial z}\delta z + [\delta x, \delta y, \delta z]_2\right\} l\, dt\,,$$

so in general:

$$\Delta J = \frac{dx}{ds}\delta x + \cdots\Big|_0^1 + \int_0^1 dt\left\{\delta x\left[\left(\frac{dx}{ds}\right)' + l\frac{\partial g}{\partial x}\right] + \delta y\left[\left(\frac{dy}{ds}\right)' + l\frac{\partial g}{\partial y}\right] + \delta z\left[\left(\frac{dz}{ds}\right)' + l\frac{\partial g}{\partial z}\right] + [\delta x, , \ldots, \delta z']_2\right\}.$$

If one then assumes that δz is the dependent variation and determines l from the equation:

$$\left(\frac{dz}{ds}\right)' + l\frac{\partial g}{\partial z} = 0$$

then that will yield the further equations:

$$\left(\frac{dx}{ds}\right)' + l\frac{\partial g}{\partial x} = 0\,, \qquad \left(\frac{dy}{ds}\right)' + l\frac{\partial g}{\partial y} = 0\,.$$

If ν is the principle normal to the desired curve and n is the normal to the surface (26) then the first terms on the left-hand sides of those three equations will be proportional to the quantities cos (ν x), cos (ν y), cos (ν z), resp., while the second terms will be proportional to the quantities cos (n x), cos (n y), cos (n z), resp. The lines n and ν will then coincide.

CHAPTER THREE

SUFFICIENT CONDITIONS FOR AN EXTREMUM IN THE SIMPLEST PROBLEM

§ 14. – Fields of extremals.

With the equations:

$$(27) \qquad x = \xi(t, a), \qquad y = \eta(t, a),$$

one can represent a family of extremals of the integrals:

$$J = \int F(x, y, x', y')\, dt = \int f(x, y, p)\, dx.$$

If the system of arguments (t, a) belongs to a region $(\mathfrak{A})$ that is defined by the inequalities:

$$\tau \leq t \leq T, \qquad |a - a_0| \leq \gamma$$

then let the functions ξ, η be regular, let the quantity:

$$x'^2 + y'^2 = \xi_t^2 + \eta_t^2$$

be non-zero, and let the function F be regular at the location (x, y, x', y') that is defined by all elements of the curves (27). In particular, a non-self-intersecting regular piece of a certain extremal $\mathfrak{C}$ will be defined by the equations:

$$x = \xi(t, a_0), \quad y = \eta(t, a_0)$$

when one lets t run from τ to T, and it shall be denoted by $\mathfrak{B}$. Different values of t will then define different points, and every point will then be associated with a uniquely-defined system (t, a_0).

If we introduce the further assumption that the quantity:

$$\Delta = \frac{\partial(\xi, \eta)}{\partial(t, a)} = \xi_t\, \eta_a - \xi_a\, \eta_t$$

is non-zero in the region ($\mathfrak{A}$) then, as we would like to say, the totality of the curve segments (27) that correspond to this region define a *field* of the arc $\mathfrak{B}$, and they will be called the *extremals of the field.* If (t_1, a_1) is any location in the interior of ($\mathfrak{A}$) and one sets:

$$x_1 = \xi(t_1, a_1), \qquad y_1 = \eta(t_1, a_1)$$

then equations (27) can be put into the form:

$$(28) \qquad x - x_1 = [x - x_1, y - y_1]_1, \qquad a - a_1 = [x - x_1, a - a_1]_1,$$

and since $\Delta(t_1, a_1)$ is the determinant of the linear terms that appear on the right-hand side, it can be solved for $t - t_1$ and $a - a_1$. In that way, one will get the expressions:

$$(29) \qquad t - t_1 = [x - x_1, y - y_1]_1, \qquad a - a_1 = [x - x_1, y - y_1]_1.$$

If one sets $a_1 = a_0$, in particular, then, as was pointed out, t_1 will be determined uniquely by the points (x_1, y_1). The same thing will then be true for the developments (28) and their solutions (29). The latter define the quantities t, a as regular functions of the coordinates in the neighborhood of the point (x_1, y_1). Therefore, they will also be regular and single-valued inside of a certain region that includes the arc $\mathfrak{B}$, e.g., the surface $\mathfrak{G}$, which covers a circle of sufficiently-small constant radius ρ when its center traverses the arc $\mathfrak{B}$. If the upper limits of that region ε are the quantities $| t - t_1 |$ and $| a - a_0 |$ that are defined by equations (29) then ε will become infinitely small with ρ, so it can be taken to be less than γ in any event. If one then makes the restricted extension of the field $\gamma = \varepsilon$ and replaces the interval from τ to T with the one from $\tau - \varepsilon$ to $T + \varepsilon$ (whereby the properties of the region ($\mathfrak{A}$) will not be altered when ε is sufficiently small) then the region ($\mathfrak{G}$) will be simply covered by the extremals of the field that are represented by equation (29) precisely. Meanwhile, we will refer to the surface $\mathfrak{G}$ as the field of the arc $\mathfrak{B}$, although, strictly speaking, it can be covered by various fields that are defined by the equations (27).

Let $\mathfrak{C}_0$ be a curve that is regular in the interior of the field and runs through 0, and let its equation be:

$$g(x_0, y_0) = 0.$$

In each of its positions, the point 0 will then lie inside of the field along a certain extremal, and it will then determine a unique pair of values t_0, a, for which the equations:

$$x_0 = \xi(t_0, a), \quad y_0 = \eta(t_0, a)$$

are true, and therefore also:

$$g[\xi(t_0, a), \eta(t_0, a)] = 0.$$

The point of intersection of the curves $\mathfrak{C}$ and $\mathfrak{C}_0$ corresponds to the system of values t_{00}, a_0 such that:

$$g\,[\xi\,(t_{00}, a_0),\ \eta\,(t_{00}, a_0)] = 0\ .$$

Since g, ξ, η are regular functions of their arguments, the previous equation can be written:

$$[t_0 - t_{00}\ ,\ a - a_0]_1 = 0\ ,$$

and that will then imply that in the neighborhood of the pair of values t_{00}, a_0, one will have:

$$t_0 - t_{00} = [a - a_0]_1$$

as long as the derivative of the left-hand side with respect to t_0 does not vanish for $t_0 = t_{00}$, $a - a_0$. However, that derivative is:

$$\frac{\partial}{\partial t} g\,[\xi\,(t,a), \eta\,(t,a)]\Big|^0 = g_x\,\xi_t + g_y\,\eta_t\,\Big|^0 ,$$

so it can vanish only when one of the equations:

$$p = \frac{\eta_t}{\xi_t} = -\frac{g_x}{g_y}\,, \qquad q = \frac{\xi_t}{\eta_t} = -\frac{g_y}{g_x}\,,$$

is valid, i.e., when the curve $\mathfrak{C}_0$ is contacted at the point 0 by the extremal that goes through that point. If we exclude that case then the quantity t_0 will be a regular function of a. Therefore, the integral:

$$u = \int_{t_0}^{t} F(\xi, \eta, \xi_t, \eta_t)\,dt$$

will be an everywhere-regular function of t and a on the field. If we denote the point (t, a) by 1 then, with our notation:

$$u = \int_0^1 F\,dt = \bar{J}_{01}\ .$$

From now on, an overbar on J will suggest that the integral is performed along an extremal of the field.

If one differentiates u then since the lower limit t_0 depends upon a that will give:

$$\frac{\partial u}{\partial t} = F\,(\xi,\ \eta,\ \xi_t,\ \eta_t) = \xi_t\,F_{x'} + \eta_t\,F_{y'}\,,$$

(30)

$$\frac{\partial u}{\partial a} = -F\,\Big|^0 \frac{dt}{da} + \int_{t_0}^{t} \frac{\partial F(\xi, \eta, \xi_t, \eta_t)}{\partial a}\,dt$$

$$= -F\,\Big|^0 \frac{dt_0}{da} + \int_{t_0}^{t} (F_x\,\xi_a + F_y\,\eta_a + F_{x'}\,\xi_{at} + F_{y'}\,\eta_{at})\,dt$$

$$= -F\,\Big|^0 \frac{dt_0}{da} + F_{x'}\,\xi_a + F_{y'}\,\eta_a\,\Big|_0^1 + \int_{t_0}^{t} [(F_x - F'_{x'})\,\xi_a + (F_y - F'_{y'})\,\eta_a]\,dt\,.$$

The integral vanishes here, since one will get an extremal when one sets $x = \xi$, $y = \eta$. One will ultimately get:

(31) $$\frac{\partial u}{\partial a} = -F\,\Big|^0 \frac{dt_0}{da} + F_{x'}\,\xi_a + F_{y'}\,\eta_a\,\Big|_0^1$$

then, or since $F = x'F_{x'} + y'F_{y'}$:

$$\frac{\partial u}{\partial a} = F_{x'}\,\xi_a + F_{y'}\,\eta_a\,\Big|_0^1 - F_{x'}\left(\xi_a + \xi_t\,\frac{dt_0}{da}\right) - F_{y'}\left(\eta_a + \eta_t\,\frac{dt_0}{da}\right)\Bigg|^0 .$$

If one sets:

$$D\,x = \xi_a + \xi_t\,\frac{dt_0}{da}\Bigg|^0 da\,, \qquad D\,y = \eta_a + \eta_t\,\frac{dt_0}{da}\Bigg|^0 da$$

then one of those quantities will be the differential that corresponds to an advance along the curve $\mathfrak{C}_0$, and one will get:

(32) $$\frac{\partial u}{\partial a}\,da = \xi_a + F_{y'}\,\eta_a\,\Big|_0^1 da - F_{x'}\,Dx - F_{y'}\,Dy\,\Big|^0 .$$

Naturally, ξ, η, ξ_t, η_t are the functions F, $F_{x'}$, $F_{y'}$ in all of those formulas.

§ 15. – Generalization of a theorem of Gauss.

Since every function of t and a that is regular in the field can also be represented as a regular function of x and y, the same thing will be true of the quantities:

$$F_{x'}(\xi, \eta, \xi_t, \eta_t)\,, \qquad F_{y'}(\xi, \eta, \xi_t, \eta_t)\,.$$

If they do not both vanish then the equation:

$$F_{x'}(\xi, \eta, \xi_t, \eta_t)\, Dx + F_{y'}(\xi, \eta, \xi_t, \eta_t)\, Dy = 0 \tag{33}$$

will define $Dy : Dx$ or the reciprocal value as a regular function of x and y. One will then get a curve segment $\mathfrak{C}_0$ that is regular at the point considered and cuts the extremals of the field transversally. However, the quantities $F_{x'}$ and $F_{y'}$ will both vanish as a result of the identity:

$$x' F_{x'} + y' F_{y'} = F ,$$

but only where the quantity F also vanishes simultaneously. If the latter is non-zero all along the curve $\mathfrak{C}$, as we would like to assume, then the curve $\mathfrak{C}_0$ can be drawn from an arbitrary point of the arc $\mathfrak{B}$. If one restricts the curve $\mathfrak{C}_0$ and the field, if necessary, then since a and t are regular functions of the coordinates, one can succeed in making every extremal meet the field a single time. Obviously, no contact can take place between the extremals and $\mathfrak{C}_0$ either, since the equation:

$$\xi_t\, Dy - \eta_t\, Dx = 0$$

would be true in that case, so as a result of equation (33), the quantity F would, in turn, have to vanish, which cannot happen along an extremal when the field is sufficiently restricted.

Formula (32) will now become simply:

$$\frac{\partial u}{\partial a} = F_{x'}\, \xi_a + F_{y'}\, \eta_a \Big|^{1} ,$$

and from (30) it will follow from this that:

$$\begin{aligned} du &= F_{x'}(\xi_t\, dt + \xi_a\, da) + F_{y'}(\eta_t\, dt + \eta_a\, da) \\ &= F_{x'}\, dx + F_{y'}\, dy , \end{aligned} \tag{34}$$

in which the function symbols $F_{x'}, F_{y'}$ are thought to replace the arguments ξ, η, ξ_t, η_t, as above.

That equation will also be true under certain assumptions when the curve $\mathfrak{C}_0$ collapses to a point 0 through which all extremals of the field go. One will then have the equations:

$$x_0 = \xi(t_0, a) , \qquad y_0 = \eta(t_0, a) ,$$

in which the value of t_0 can vanish for the various extremals of the field. For example, if it is t_{00} for the curve $\mathfrak{C}$ and the functions ξ, η are regular at the location (t_{00}, a_0), but the quantity $\xi_t^2 + \eta_t^2$ is also non-zero there, then t_0 can be calculated as a regular function of a from one of the last equations, and the equations:

$$\xi_t(t_0, a)\frac{dt_0}{da} + \xi_a(t_0, a) = 0\,, \quad \eta_t(t_0, a)\frac{dt_0}{da} + \eta_a(t_0, a) = 0 \tag{35}$$

will then be true for that t_0 . It obviously follows from this that:

$$\Delta(t_0, a) = 0\,,$$

such that the point 0 itself will no longer belong to the field, and the arc $\mathfrak{B}$ can only get arbitrarily close to it without the extremals that go through it defining the field of the arc $\mathfrak{B}$, in the sense of the definition above. If one now sets:

$$u = \int_{t_0}^{t} F(\xi, \eta, \xi_t, \eta_t)\, dt$$

and assumes that F is also regular in the elements of the extremals that emanate from the point 0 then one will get formula (31), precisely as before, and formulas (35) will, in turn, imply the expression (34) for du. That will still be true in many problems, as special consideration will show, even when the assumptions that were just introduced for the point 0 are invalid, e.g., F and the extremals are singular at that point. Naturally, the integral u must keep its meaning and finite value. The validity of formula (34) will always be assumed to be provable in what follows when we collapse $\mathfrak{C}_0$ to a point.

Since the curve $\mathfrak{C}_0$ can start from any point of the arc $\mathfrak{B}$, the equation:

$$du = F_{x'}\, dx + F_{y'}\, dy\,, \quad u = \text{const.}$$

will define a curve $\mathfrak{C}_1$ with the same character as $\mathfrak{C}_0$ at every location in the field that will intersect the extremals of the field transversally at the variable point 1. In that way, the quantity $\bar{J}_{01}$ will be constant when 0 and 1 belong to the same extremal of the field and the former point traverses the curve $\mathfrak{C}_0$. If one then draws arcs 01 along the extremals of the field that cut the curve $\mathfrak{C}_0$ transversally such that the integral $\bar{J}_{01}$ has the same value then the locus of the point 1 will, in turn, be a regular curve that intersects the extremals of the field transversally. (Cf., Fig. 1, in which, as with most of the later figures, the extremals appear as lines, while their transversal position is suggested by right-angled intersections.)

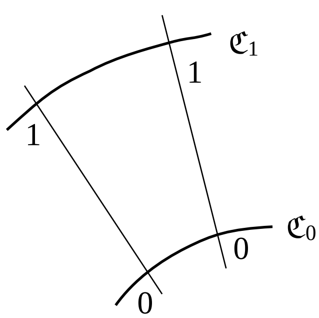

Figure 1.

One recognizes that the result that was obtained is a generalization of the known theorem of **Gauss** that a family of equally-long geodetic arcs that are perpendicular to the line that includes their starting points will also intersect the locus of their endpoints at a right angle. That is because (§ **11**) the transversals are perpendiculars for geodetic lines.

§ 16. – Coordinates that are constant on the extremals and their transversal curves.

Let the integrand $F(x, y, x', y')$ be regular and non-zero (say positive) in either:

a) all elements of the arc $\mathfrak{B}$ that correspond to the direction of increasing t or

b) all line elements that point in arbitrary directions from all of its points.

In case a), the function F preserves those properties for all elements that start from points of the field and are inclined sufficiently little with respect to the distinguished elements of the arc $\mathfrak{B}$, while in case b), it preserves those properties for all elements that start from points of the field, assuming that the field is sufficiently restricted. That is because, from § **3**, one can assume that for every individual line element:

$$x'^2 + y'^2 = 1, \qquad x' = \cos\varphi, \qquad y' = \sin\varphi.$$

φ will then be the inclination of the element with respect to the + x-axis, as measured in the positive sense of rotation (§ **4**), and F will be a regular function of x, y, φ when $F(x, y, x', y')$ is regular in the line element that is characterized by x, y, φ. The asserted statement follows from that, since a function that is regular at one location will preserve that property for a certain neighborhood of that location.

Now, with the narrower assumption, as with the broader one, one will have that:

$$\frac{\partial u}{\partial t} = F(\xi, \eta, \xi_t, \eta_t)$$

is non-zero inside of the field. For every system of values (t, a) that belongs to it, when the curve $\mathfrak{C}_0$ is intersected transversally by the extremals of the field, as before, one can then calculate the quantity t, and therefore x and y, as well, as a regular function of u and a from the equation:

$$u = \int_{t_0}^{t} F(\xi, \eta, \xi_t, \eta_t)\, dt .$$

Since the functional determinant:

$$\frac{\partial(t, a)}{\partial(u, a)} = 1 : \frac{\partial u}{\partial t}$$

is non-zero, the same thing will be true for:

$$(36) \qquad \frac{\partial(x,y)}{\partial(u,a)} = \frac{\partial(\xi,\eta)}{\partial(t,a)}\frac{\partial(t,a)}{\partial(u,a)} = \Delta : \frac{\partial u}{\partial t} = \frac{\Delta}{F(\xi,\eta,\xi_t,\eta_t)} .$$

If one introduces the expressions for x and y in terms of u and a into the function F (x, y, dx, dy) and sets:

$$v = a , \quad \frac{dv}{du} = s$$

then that will give:

$$F(x, y, dx, dy) = G(u, v, du, dv) = g(u, v, s)\, du$$

$$J = \int G(u,v,u',v')\,dt .$$

G and from § **5**, g (u, v, s) as well (as long as du does not vanish), will then be regular in the same line elements as F.

Now, along the extremals of the field, one has:

$$v = a = \text{const.} , \qquad s = 0 ,$$

and since F is positive, u will increase in the direction of increasing t. One will then have:

$$u = \int_0^1 G(u,v,du,0) = \int_0^1 g(u,v,0)\,du ,$$

in which the function g is defined corresponding to a positive value of du. If one differentiates with respect to u and imagines that 0, as the intersection of the curve $\mathfrak{C}_0$ and a line v = const., is independent of u then the identity will follow:

$$1 = g(u, v, 0)$$

or:

$$G(u, v, du, 0) = du$$

for $du > 0$.

One can get a further way of determining the function g from the fact that, from § **15**, all lines u = const. will be intersected transversally in terms of the variables u, v. Namely, the system of values:

$$x, y, x', y'; u, v, u', v',$$

in which the derivatives are taken with respect to an arbitrary parameter, represent the same line element, so one will have the identities:

$$F(x, y, x', y') = G(x, y, x', y') ,$$

$$x' = x_u\, u' + x_v\, v' , \qquad y' = y_u\, u' + y_v\, v' ,$$

$$u' = u_u\, u' + u_v\, v', \qquad v' = v_u\, u' + v_v\, v',$$

and one can differentiate the first of them with respect to x', y'. That will then yield:

$$F_{x'} = G_{u'} \frac{\partial u'}{\partial x'} + G_{v'} \frac{\partial v'}{\partial x'} = u_x\, G_{u'} + v_x\, G_{v'}, \qquad F_{y'} = u_y\, G_{u'} + v_y\, G_{v'}.$$

On the other hand, if one has:

$$Dx = x_u\, Du + x_v\, Dv, \quad Dy = y_u\, Du + y_v\, Dv$$

for the advance in an arbitrary direction then that will imply the identity:

$$F_{x'}\, Dx + F_{y'}\, Dy = \{G_{u'}\,(u_x\, x_u + u_y\, y_u) + G_{v'}\,(v_x\, x_u + v_y\, y_u)\}\; Du + \ldots,$$

or, since one obviously has:

$$\frac{\partial u}{\partial u} = u_x\, x_u + u_y\, y_u = 1, \qquad \frac{\partial v}{\partial u} = v_x\, x_u + v_y\, y_u = 0,$$

along with two analogous equations, that will imply the identity:

$$F_{x'}\, Dx + F_{y'}\, Dy = G_{u'}\, Du + G_{v'}\, Dv.$$

From § **15**, that quantity vanishes when D means the advance along the curve u = const., but the derivatives refer to the extremals of the field. The equation:

$$G_{u'}\, Du + G_{v'}\, Dv = (g - s\, g_s)\, Du + g_s\, Dv = 0$$

must then be fulfilled under the assumption that $s = 0$, $Du = 0$, and that will yield:

$$g_s\,(u, v, 0) = 0.$$

As a result of the value that is found for $g\,(u, v, 0)$, the **Taylor** development of the function g will then take the form:

$$g\,(u, v, s) = 1 + \frac{s^2}{2}\, g_{ss}\,(u, v, \theta s) \tag{37}$$

when θ belongs to the segment from 0 to + 1. That equation is true under assumption a) as long as $|s|$ does not exceed a certain positive quantity, while under assumption b), it will be true for any finite value of s. Obviously, the argument up to now will remain valid when $\mathfrak{C}_0$ degenerates into a point.

Now let 12 (Fig. 2) be any arc $\mathfrak{B}$ that belongs to the field, where, as before, 0 is the intersection point of the extremal $\mathfrak{C}$ that the field contains with $\mathfrak{C}_0$, and 3 is a fixed point of the latter curve that is connected to 2 by a curve $\mathfrak{L}$. It does not leave the field. Let x, y be functions of a parameter τ along it such that the integral:

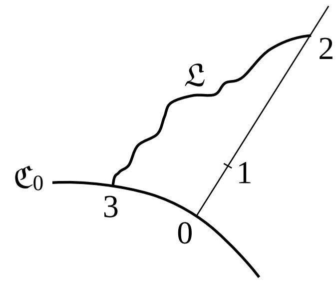

Figure 2.

$$J_{32} = \int_{\tau_3}^{\tau_2} F\left(x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau}\right) d\tau = \int_{\tau_3}^{\tau_2} G\left(u, v, \frac{du}{d\tau}, \frac{dv}{d\tau}\right) d\tau$$

has a certain finite value, and the equation:

$$\text{(38)} \qquad \int_{\tau_3}^{\tau_2} \frac{du}{d\tau} d\tau = u\Big|_3^2 = u\Big|_0^2$$

is true. One will obviously have:

$$\text{(39)} \qquad \bar{J}_{32} - \bar{J}_{02} = \int_{\tau_3}^{\tau_2} \left[G\left(u, v, \frac{du}{d\tau}, \frac{dv}{d\tau}\right) - \frac{du}{d\tau} \right] d\tau$$

then as long as $du / d\tau$ is positive, so from (37) the integrand can be written as:

$$\text{(40)} \qquad \frac{s^2}{2} g_{ss}(u, v, \theta s) \frac{du}{d\tau} .$$

If $du / d\tau$ is negative or zero then the integrand will be positive, like G.

In order to resolve the sign of that integral, one does not need to define the function g, which will often be difficult, but one can start from the linear relations above between x', y', u', v'. A two-fold application of it will yield:

$$\frac{\partial^2 G}{\partial v'^2} = G_{v'v'} = F_{x'x'}\, x_v^2 + 2F_{x'y'}\, x_v\, y_v + F_{y'y'}\, y_v^2 .$$

When one differentiates the identity:

$$F = x' F_{x'} + y' F_{y'}$$

with respect to x' and y', it will follow that:

$$x' F_{x'x'} + y' F_{x'y'} = x' F_{y'x'} + y' F_{y'y'} = 0 .$$

If one imagines that x' and y' do not both vanish then it will follow that one of the quantities:

$$F_{x'x'} : y'^2 , \qquad F_{y'y'} : x'^2$$

has a finite value $F_1 (x, y, x', y')$ for which the equations:

$$F_{x'x'} = F_1 y'^2 , \quad F_{x'y'} = - x' y' F_1 , \qquad F_{y'y'} = x'^2 F_1 , \qquad F_1 \, dt = \frac{f_{pp} \, dx}{x'^4}$$

will be true. If one substitutes that value in the expression for $G_{v'v'}$ then that will give:

$$G_{v'v'} = F_1 (x_v \, y' - y_v \, x')^2 ,$$

and since:

$$G = u' \, g\left(u, v, \frac{v'}{u'} \right) ,$$

it will follow that for non-vanishing u', one will have that:

$$\frac{g_{ss}(u, v, s)}{u'} = F_1 (x_v \, y' - y_v \, x')^2 ,$$

or, from the linear equations that exist between x', y' and u', v':

$$\frac{g_{ss}(u, v, s)}{u'} = F_1 \cdot (x_v \, y_u - x_u \, y_v)^2 u'^2 ,$$

and the expression in parentheses on the right-hand side is non-zero, from (36).

Now for each of the cases a) and b), it was assumed that F_1 had a constant sign and was non-vanishing in just those line elements for which F was regular in each of those cases. An argument similar to the one that was developed at the beginning of this section will show that the given property of F_1 needs to be assumed for only the elements of the arc $\mathfrak{B}$ in case a), and for only the elements that emanate from its points in case b). Along that arc, one has $s = 0$. $| s |$ will not exceed a certain constant along the curve $\mathfrak{L}$, so under the assumption a), the quantities:

$$g_{ss} (u, v, \theta s) , \qquad \int \frac{s^2}{2} g_{ss} (u, v, \theta s) \, du$$

will have the same sign as F_1, and u' will always remain positive. The quantity:

(41) $$G\left(u, v, \frac{du}{d\tau}, \frac{dv}{d\tau}\right) - \frac{du}{d\tau}$$

is always representable in the form (40) and will have the same sign as F_1 in any case. By contrast, as long as $du / d\tau$ ceases to be positive at some locations, the assumption b) must be introduced. The equation:

$$G(u, v, u', v') - u' = \frac{s^2}{2} g_{ss}(u, v, \theta s) u' = \frac{v'^2}{2u'} g_{ss}(u, v, \theta s)$$

will then be true for $u' > 0$. Therefore, if the line element (u, v, u', v') rotates about a fixed point and u' converges to zero from the positive side then the left-hand side of the last equation will approach the limit $G(u, v, 0, v')$, while the right-hand side will have the sign of F_1. The latter must then coincide with the integrands G or F, which we assume to be positive, i.e., F_1 must also be positive. The quantity (41) is positive for positive values of $du / d\tau$ then and vanishes only for $s = 0$. Obviously, it is positive for negative or vanishing values of $du / d\tau$. Therefore, the same thing will be true for the difference $\bar{J}_{32} - \bar{J}_{02}$ as long as equation (38) is valid and the representation of the integral (39) is well-defined.

§ 17. – Strong and weak extrema.

The concept of the extremum must be determined more precisely by being given the totality of all curves that one would like to compare. The general representation of an extremum brings with it the idea that one must next compare neighboring curves. We introduce the following definitions for its neighborhood: If 12 is any planar arc and if the arc $\mathfrak{L}$ lies entirely in the interior of the surface that is covered by a circle with constant radius ρ whose center traverses the arc 12 then we will say that all curves $\mathfrak{L}$ for which the associated quantities ρ do not exceed a certain constant will belong to a *wide* neighborhood of the arc 12. If there is a positive constant ρ_1 with the property that every tangent to the arc $\mathfrak{L}$ defines an angle that is less than ρ_1 with at least one of the arcs 12 whose point of contact is at a distance of less than ρ from the first one then we will say that the arc $\mathfrak{L}$ lies in a *narrow* neighborhood of the arc 12 as long as ρ and ρ_1 remain below certain constants. Obviously, those definitions can be adapted to space curves immediately. Furthermore, if the difference between the integrals J that are formed along both arcs has constant sign as long as $\mathfrak{L}$ lies in a wide neighborhood of the arc 12 then we will say that the arc 12 gives a *strong extremum* for the integral J. By contrast, if that difference has a fixed sign only when $\mathfrak{L}$ lies in a narrow neighborhood of 12 then we will say that a *weak extremum* exists. Obviously, in order to compare the arc 12 with the curve $\mathfrak{L}$, we must appeal to a narrower neighborhood in the latter case than we do in the former.

However, in order to be able to verify that an extremum exists, we must assume certain continuity properties for the curves $\mathfrak{L}$ that essentially stem from the fact that integrals of the form

J are defined along those curves and can be treated with the ordinary rules of operation for the infinitesimal calculus. In order to be able to formulate those properties precisely, we shall generally consider a function $\varphi(\tau)$ that has the following property in an arbitrarily-bounded integral $\mathfrak{J}$: Let it be continuous, and let the quotient:

$$\frac{\varphi(\tau+\varepsilon)-\varphi(\tau)}{\varepsilon}$$

approach a certain finite limit $\varphi'(\tau)$ for every value of τ when ε is positive and decreases indefinitely. In that way, one does not have to exclude the possibility that the quotient:

$$\frac{\varphi(\tau+\varepsilon)-\varphi(\tau)}{-\varepsilon}$$

approaches a different limit or even no limit at all. We call the quantity $\varphi'(\tau)$ the *forward derivative*, as one sometimes does. Just as is true for the ordinary concept of derivative, we have the theorem for it, as well, that the continuous function $\varphi(\tau)$ must be constant in the interval from τ_0 to τ_1 when the forward derivative has the well-defined value of zero over the entire segment. That is because when we set:

$$\psi(\tau) = \varphi(\tau) + \alpha(\tau - \tau_0), \qquad \theta(\tau) = \varphi(\tau) - \alpha(\tau - \tau_0)$$

and understand α to mean a positive constant, $\psi(\tau)$ will have a forward derivative of α and $\theta(\tau)$ will have a forward derivative of $-\alpha$, so the former function will increase from τ_0 to τ_1, while the latter one will decrease, such that when $\tau_0 > \tau_1$, we will have the inequalities:

$$\psi(\tau_1) > \psi(\tau_0), \qquad \theta(\tau_1) < \theta(\tau_0),$$

$$\varphi(\tau_1) + \alpha(\tau_1 - \tau_0) > \varphi(\tau_0), \quad \varphi(\tau_1) - \alpha(\tau_1 - \tau_0) < \varphi(\tau_0),$$

$$\alpha(\tau_1 - \tau_0) > \varphi(\tau_0) - \varphi(\tau_1) > -\alpha(\tau_1 - \tau_0).$$

Since α can be arbitrarily small, the last of them implies that $\varphi(\tau_1) = \varphi(\tau_0)$, with which the assertion is proved.

Except for that special case, the function $\varphi'(\tau)$ has the following property: Let its *variability* in any interval be the upper bound on the absolute value of the differences between any two values of the function that are attained in that interval. The total length of all pieces of the interval $\mathfrak{J}$ inside of which the variability of $\varphi'(\tau)$ exceeds a positive constant can always be made to be infinitely small. According to **Riemann**, the function $\varphi'(\tau)$ will then be integrable, and the integral:

$$\Phi\,(\tau) = \int_{\tau_0}^{\tau_1} \varphi'(\tau)\,d\tau\,,$$

whose integration interval defines a subset of $\mathfrak{J}$, will be a finite, continuous function of τ. We further assume that when ε is positive:

$$\lim_{s\to 0} \varphi'(\tau+\varepsilon) = \varphi'(\tau)\ .$$

The discontinuities in the function $\varphi'(\tau)$ will be restricted in a certain way by that, but it will always still be possible that, e.g., infinitely-many other discontinuities can lie in an arbitrary neighborhood of a discontinuity location. With that assumption, the quantity $\varphi'(\tau)$ will be positive along a finite segment when that is true for a single location. One further finds that for $\varepsilon > 0$:

$$\Phi\,(\tau + e) - \Phi\,(\tau) = \int_{\tau}^{\tau+\varepsilon} \varphi'(\tau)\,d\tau = M\,\varepsilon\,,$$

which means that M lies between the lower and the upper limits of the value that $\varphi'(\tau)$ assumes between τ and $\tau + \varepsilon$. From the previous equation, M will converge to $\varphi'(\tau)$ when ε vanishes. The quantity $\Phi\,(\tau)$ will then have the forward derivative:

$$\Phi'\,(\tau) = \varphi'(\tau)\,,$$

so the difference $\Phi\,(\tau) - \varphi\,(\tau)$ will be constant, and since $\Phi(\tau_0)$ vanishes, that will imply:

$$\Phi\,(\tau) = \varphi\,(\tau) - \varphi\,(\tau_0) = \int_{\tau_0}^{\tau} \varphi'(\tau)\,d\tau\ .$$

Thus, the usual fundamental relation between differentiation and integration will be true when one refers the former operation to the forward derivative.

Now let x and y be functions of the parameter τ along the curve $\mathfrak{L}$ that have the same properties as the function $\varphi\,(\tau)$, so they are integrable and possess forward derivatives, which we will denote by the usual differential symbol. Therefore, the sum:

$$\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2$$

will always lie above a positive constant. Hence, since $\varphi'(\tau)$ does not need to be continuous, it is by no means excluded that the curve might possess a finite or infinite set of corners. However, it does have a well-defined direction at every point that corresponds to the increasing values of τ. If

$\mathfrak{L}$ lies in a sufficiently-bounded narrow or wide neighborhood of the arc $\mathfrak{B}$ according to whether the assumption a) or b) of the previous section was made, resp., then in the former case, every system of values $\left(x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau}\right)$ will deviate arbitrarily little from one that represents an element of the curve $\mathfrak{C}$. In the latter case, F will be regular for arbitrarily-directed line elements that emanate from points of the field. In both cases, that will then be true for every system of values $\left(x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau}\right)$ that is defined by the curve $\mathfrak{C}$, and the variability of the quantity $F\left(x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau}\right)$ that corresponds to a variable interval of the argument τ will become infinitely small with the variability of the quantities $\frac{dx}{d\tau}$, $\frac{dy}{d\tau}$ that exist in the same interval. If we further set:

$$F\left(x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau}\right) = \Theta(\tau)$$

then if $\varepsilon > 0$, we will obviously have:

$$\Theta(\tau) = \lim_{\varepsilon \to 0} \Theta(\tau + \varepsilon),$$

since such an equation is true for $\frac{dx}{d\tau}$ and $\frac{dy}{d\tau}$. $\Theta(\tau)$ will then have all of the properties of $\varphi'(\tau)$ that were assumed above, so it will be integrable and that will yield the equation:

$$\frac{d}{d\tau}\int_{\tau_0}^{\tau} \Theta(\tau)\, d\tau = \frac{d}{d\tau}\int_{\tau_0}^{\tau} F\left(x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau}\right) d\tau = F\left(x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau}\right),$$

which corresponds to the usual rules of integral calculus. The quantity:

$$\frac{d\Phi}{d\tau} = \Phi_x \frac{dx}{d\tau} + \Phi_y \frac{dy}{d\tau}$$

has the same properties as F when $\Phi(x, y)$ is a function of the coordinates that is regular along the curve $\mathfrak{L}$. If one applies those remarks to the quantities u, v, which are regular functions of the coordinates, then it will follow that the quantities:

$$G\left(x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau}\right), \qquad \frac{dx}{d\tau}, \qquad \frac{dy}{d\tau}$$

also have the properties of $\varphi'(\tau)$, and in particular, the difference between the first two quantities along a finite segment will be positive when that is true at a single location. It further follows that equations (38), (39) will be valid under the assumptions that were introduced when the symbol for differentiation with respect to τ always refers to the forward derivative. That is quite natural in the construction of J_{32} , since one integrates in the direction of increasing τ.

On the grounds of that argument, it can now be proved rigorously that the quantity:

$$J_{32} - \bar{J}_{32} = \int_3^2 \left[G\left(u, u, \frac{du}{d\tau}, \frac{dv}{d\tau} \right) - \frac{du}{d\tau} \right] d\tau$$

has the same sign as F_1, and will therefore vanish only when the curves $\mathfrak{L}$ and $\mathfrak{B}$ coincide completely. In the case a), that is clear with no further discussion, since the integral can be replaced with:

$$\frac{s^2}{2} g_{ss}(u, v, \theta\, s) \frac{du}{d\tau} \tag{42}$$

along the entire integration interval, but the quantity $g_{ss}\, du$ does not vanish, by assumption. The integrand is likewise never negative in case b), and if $J_{32} - \bar{J}_{32}$ is to vanish then the quantity $dv / d\tau$ cannot be non-zero if $du / d\tau$ is positive, because in that case, the integrand would be, in turn, representable in the form (42), so it would be positive. However, if $du / d\tau$ is positive at one location then the integrand in the integral above will be, in turn, positive. Now since the latter remains positive along a finite segment when that is true at a single location, as was pointed out above, the integral $J_{32} - \bar{J}_{32}$ can certainly vanish only when v is constant along the curve $\mathfrak{L}$, so the latter will coincide with $\mathfrak{B}$.

With that, the proof of the extremum is complete, regardless of whether one compares the arc 02 with all arcs that have the same endpoints or with all of the ones that go from the curve $\mathfrak{C}_0$ to the point 2. However, if the curve $\mathfrak{C}_0$ collapses to the point 0, such that all extremals of the field go through it, then one must make the special assumption for the curve $\mathfrak{L}$ that it remains inside of the region that is simply covered by the extremals of the field (Fig. 3). One can achieve that, e.g., in such a way that one lets all curves $\mathfrak{L}$ have an arbitrarily-small, constant piece 01 in common with the curve $\mathfrak{C}$ (Fig. 4). One will then have:

$$J_{32} - \bar{J}_{02} = J_{02} - \bar{J}_{02} = \bar{J}_{01} + J_{12} - \bar{J}_{02} = J_{12} - \bar{J}_{12} ,$$

and for the arc 12, the extremum with respect to all arcs with the same endpoints will be assured in the unrestricted sense of the definitions above. That will also be true when the extremals are singular at the point 0, but the assumptions that were expressed for that case in § **15** will then be true. Finally, one easily foresees that the argument will not change when the point 2, as the endpoint of the arc $\mathfrak{L}$, is allowed to vary along a curve $\mathfrak{C}_2$ that intersects the extremals of the field transversally while 0 remains fixed, such that the direction of integration points from the fixed point to the fixed curve, contrary to the assumptions that were made up to now.

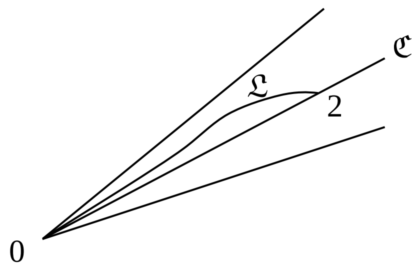

Figure 3.

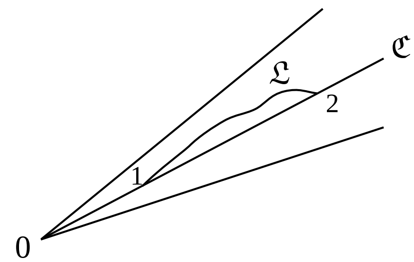

Figure 4.

In § **16**, we restricted the extension of the field inside of which the curves $\mathfrak{L}$ must run in such a way that the assumptions a), b) can be adapted from points or elements of the curve $\mathfrak{C}$ to all extremals of the field, just as we did in § **15** in order to make the curves u = const. remain regular and intersect each extremal of the field only once. When that can be verified by always restricting the field in order to achieve properties in a special problem for a certain field of finite extension, our argument will be true for any curve $\mathfrak{L}$ that runs through that field, and it will then yield more than a mere proof of the extremum, because the latter is already assured when only a field of arbitrarily-small extent has been constructed.

We summarize the most important of the results obtained as follows:

1. **Jacobi**'s condition: Let a nowhere-singular piece of an extremal with the endpoints 0, 2 be surrounded by a field.

2. **Legendre**'s condition: Let the quantities F_1 or $f_{pp}\,dx$ have fixed signs without vanishing, and let F be regular and non-zero in either:

   a. All elements of the arc 02 that run in the direction from 0 to 2 or

   b. All line elements that emanate from the points of it.

The arc 02 will then yield an extremum for the integral J in comparison to all curves $\mathfrak{L}$ with the same endpoint, as well as with all of the ones that connect the curve $\mathfrak{C}_0$ that starts from the point 0 and intersects the extremals of the field transversally with the point 2. The strong or weak extremum is assured according to whether the assumption b) or a) was made, resp. A maximum or a minimum will exist according to whether F_1 is negative or positive, resp.

§ 18. – Examples. Problems I, III,VI, VII.

Problem I (**§ 9**). – Here one has:

$$F\,dt = \sqrt{x'^2 + y'^2}\,dt\,,$$

with the positive square root, so:

$$y'^2 F_1 = \frac{\partial}{\partial x'}\left(\frac{x'}{\sqrt{x'^2 + y'^2}}\right) = \frac{y'^2}{\left(\sqrt{x'^2 + y'^2}\right)^3},$$

$$F_1 = \left(\frac{1}{\sqrt{x'^2 + y'^2}}\right)^3,$$

with a likewise-positive square root. Moreover, one has:

$$F\,dt = f\,dx, \quad f = \sqrt{1+p^2}, \quad f_{pp} = \left(\frac{1}{\sqrt{1+p^2}}\right)^3,$$

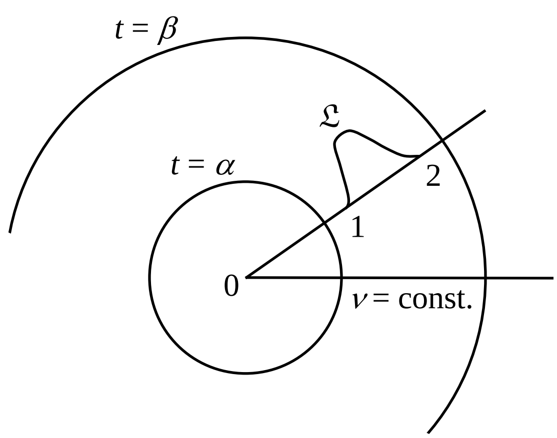

Figure 5.

in which the square root has the sign of dt, such that $f_{pp}\,dx$, as well as F_1, is positive for every arc-element. The **Legendre** condition for a strong minimum is then fulfilled.

In order to investigate whether a straight line segment 12 will yield the shortest line between 1 and 2, we assume (Fig. 5) that $\mathfrak{L}$ is any line 12 with the continuity properties that were given in § **17** that yields the arc integral J_{12} . The length of the segment 12 is $\bar{J}_{12}$. Furthermore, let 0 be a point on the line 12 that does not belong to the line $\mathfrak{L}$ and lies outside of the segment 12. One takes it to be the origin of the rectangular, as well as polar, coordinates when one sets:

$$x = t \cos v, \qquad y = t \sin v,$$

and chooses the constants α, β such that:

$$0 < \alpha < \beta,$$

and chooses the curve $\mathfrak{L}$ to run completely within the annulus between the circles $t = \alpha$, $t = \beta$. That surface, which is covered by the extremals v = const. exactly once, can be considered to be the field of the line segments 12, since the quantity:

$$\frac{\partial(x, y)}{\partial(t, v)} = \begin{vmatrix} \cos v & \sin v \\ -t \sin v & t \cos v \end{vmatrix} = t$$

is non-zero in it. The curves $\mathfrak{C}_0$ of the general theory correspond to the circle t = const., each of which intersects the extremals of the field at a point. u is the length of a line v = const., as measured from the circle $t = \alpha$, so it follows that:

$$u + \alpha = t\,, \qquad F\,dt = \sqrt{dt^2 + t^2 dv^2} = \sqrt{du^2 + (u+\alpha)^2 dv^2} = G\,(u, v, du, dv)\,,$$

$$g\,(u, v, s) = \sqrt{1 + (u+\alpha)^2 s^2}\,,$$

in which the last square root is positive since the expression g will be defined for only the case of $du > 0$. In the difference:

$$J_{12} - \bar{J}_{12} = \int_{\tau_1}^{\tau_2} \left[G\left(u, v, \frac{du}{d\tau}, \frac{dv}{d\tau}\right) - \frac{du}{d\tau} \right] d\tau\,,$$

the integrand will itself be positive as long as $du / d\tau$ negative of vanishes, which characterizes the part of the curve $\mathfrak{L}$ in which the radius t decreases. If $du / d\tau$ is positive then the integrand can be written:

$$\sqrt{1 + (u+\alpha)^2 s^2} - 1\,,$$

so it will likewise be non-negative and vanish only for $s = 0$. If s were non-zero or the integrand were positive at any location then, from § **17**, the same thing would have to be true for a finite segment of the curve $\mathfrak{L}$, so the difference $\bar{J}_{12} - J_{12}$ could vanish only if du were positive everywhere and $s = 0$. Except for that case, one will obviously have:

$$J_{12} > \bar{J}_{12}\,,$$

which proves the existence of a strong minimum, and even more.

Furthermore, let $\mathfrak{C}_1$ be any curve that runs from the point 1 such that x_1, y_1 are regular functions of a parameter v, and let n be the direction of the normal, which has the same relationship to the direction of increasing v that the $+ y$-axis has to the $+ x$-axis. If we set:

$$x = \xi\,(t, v) = x_1 + t\cos(n\,x)\,, \qquad y = \eta\,(t, v) = y_1 + t\cos(n\,y)$$

then v = const will be a normal to the curve $\mathfrak{C}_1$, and we will have:

$$\Delta = \frac{\partial(\xi, \eta)}{\partial(t, v)} = \begin{vmatrix} \cos(n\,x) & \dfrac{dx_1}{dv} + t\,\dfrac{d\cos(n\,x)}{dv} \\ \cos(n\,y) & \dfrac{dy_1}{dv} + t\,\dfrac{d\cos(n\,y)}{dv} \end{vmatrix}\,.$$

If v is the arc-length of the curve $\mathfrak{C}_1$, in particular, then we will have:

$$\cos(n\,x) = -\frac{dy_1}{dv}, \qquad \cos(n\,y) = \frac{dx_1}{dv},$$

$$\Delta = -1 + \left(\frac{dx_1}{dv}\frac{d^2 y_1}{dv^2} - \frac{dx_1}{dv}\frac{d^2 x_1}{dv^2}\right) = -1 + \frac{t}{r},$$

in which r means the radius of the curvature at the point 1, which will be taken to be positive or negative according to whether the center of curvature lies in the direction of n or its opposite, resp.

Now the direction from 1 to the point 2 coincides with n along any normal v = const., such that t_2 will be positive. The straight line segment 12 whose length is t_2 can then be surrounded by a field in which Δ does not vanish when either r is negative or the inequality:

$$0 < t_2 < r$$

is valid, i.e., whenever the segment 12 does not include the center of curvature at the point 1 (Fig. 6). The extremals of the field are the normals v = const., which intersect the curve $\mathfrak{C}_1$ transversally. The general theory then implies that the straight line segment 12 must yield a strong minimum of the distance between $\mathfrak{C}_1$ and the point 2 and is shorter than every line $\mathfrak{L}$ with the continuity properties above that runs through the field and goes from the curve $\mathfrak{C}_1$ to the point 2. The quantity u is the normal distance from the curve $\mathfrak{C}_1$, and one will easily get the formula:

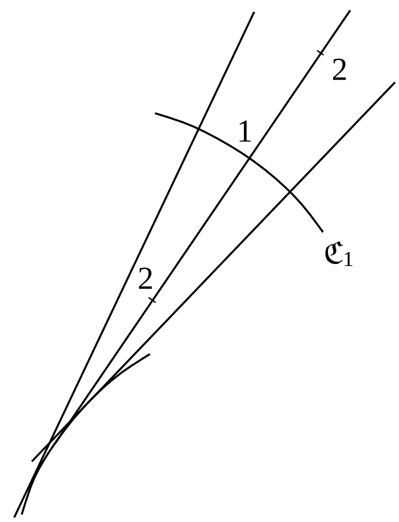

Figure 6.

$$\sqrt{dx^2 + dy^2} = \sqrt{du^2 + (u-r)^2\,dv^2} = G(u, v, du, dv),$$

from which we can draw the same conclusions that we drew for the previous transformation of the arc element.

Problem III (§ 9). – If $\mathfrak{L}$ is any line of length l in the wy-plane that is drawn from the point 0 in the half-plane $y > 0$ to the w-axis, and along which w, y have the properties of $\varphi(\tau)$ (§ **17**) as functions of a parameter τ, then one can verify that the area integral is smaller than the area of the semi-circle of length l. If $dw / d\tau$ is piece-wise negative then we consider the curve $\mathfrak{L}^0$, which is defined by the equations:

$$y = \int_{\tau_0}^{\tau} \frac{dy}{d\tau}\, d\tau\,, \qquad w = \int_{\tau_0}^{\tau} \left|\frac{dw}{d\tau}\right| d\tau\,.$$

When τ runs through the same interval as the curve $\mathfrak{L}$, it will likewise have a length l and the same continuity properties, since $|\,\varphi(\tau)\,|$ possesses the properties that were required of $\varphi(\tau)$ in § **17** in any case, as one easily sees. The surface integral of the curve $\mathfrak{L}^0$ is finite and consists of only positive elements. One obviously has:

$$\int_{\tau_0}^{\tau} y \left|\frac{dw}{d\tau}\right| d\tau \geq \int_{\tau_0}^{\tau} y \frac{dw}{d\tau} d\tau\,,$$

i.e., the area that is bounded by the curve $\mathfrak{L}^0$ and the w-axis is not smaller than the area of the aforementioned semi-circle. In that argument, we now have the advantage that the square root in the integral:

$$J = \int y\sqrt{1-p^2}\, dx = \int y\, du$$

is always taken to be positive. The curve $\mathfrak{L}^0$ corresponds to a curve $\mathfrak{L}_0$ in the xy-plane along which one has:

(43) $$\frac{dx}{d\tau} > 0\,, \qquad -1 \leq \frac{dy}{dx} \leq +1\,, \qquad x > 0\,, \qquad 0 \leq y \leq x$$

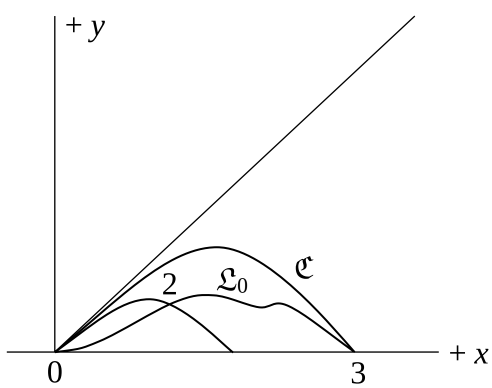

Figure 7.

everywhere, since x is the arc-length of the curve $\mathfrak{L}^0$. It is therefore not at all necessary to verify a strong extremum in the xy-plane, but only the curves that are characterized by those inequalities need to be included in the comparison. The **Legendre** condition for a maximum is fulfilled for all curves $\mathfrak{L}_0$ that come under consideration as long the quantity:

$$f_{pp}\, dx = \frac{-y\, dx}{\left(\sqrt{1-p^2}\right)^3}$$

is negative along it.

The octant in the xy-plane (Fig. 7) in which the curve $\mathfrak{L}_0$ lies according to the last inequality in (43) will now be simply covered by the extremals:

$$y = a \sin\frac{x}{a}\,, \qquad a > 0\,, \qquad p = \cos\frac{x}{a}\,,$$

on each of which one assumes that:

(44) $$0 \leq \frac{x}{a} \leq p\,.$$

That is because when one fixes x, one will have that:

$$\frac{\partial y}{\partial a} = \sin\frac{x}{a} - \frac{x}{a}\cos\frac{x}{a}$$

is always positive, due to the inequality (44) that is true for x. One has $y = 0$ for $a = x / \pi$, while $y = x$ for $a = \infty$. The point (x, y) will then run through all points of the line x = const. that belong to that octant, and when a runs through all positive values greater than x / π, the point (x, y) will go through each point of that line once. One can further set $x = t$, and one will then have:

$$\Delta = \frac{\partial(x,y)}{\partial(t,a)} = \frac{\partial y}{\partial a}\,,$$

and as was remarked before, that is a quantity that will be positive as long as y does not vanish or $y = x$. The points 0, 3 correspond to the values $x = 0$ and $x = a\,\pi$, so every subset of the extremal arc 03 that includes no endpoints will be surrounded by a field. The endpoints themselves require special consideration insofar as $p = \pm 1$ at them, so the function F will then be singular.

If we set:

$$u = \int_0^x y\sqrt{1-p^2}\,dx = a\int_0^x \sin^2\frac{x}{a}\,dx = \frac{a\,x}{2} - \frac{a^2}{2}\sin\frac{x}{a}\cos\frac{x}{a}\,,$$

according to the general theory in the case where the curve $\mathfrak{C}_0$ contracts to the point 0, and in which the positive sign of the square root is employed, then by means of the expressions:

$$F = y\sqrt{x'^2 - y'^2}\,, \qquad F_{x'} = \frac{y\,x'}{\sqrt{x'^2 - y'^2}}\,, \qquad F_{y'} = \frac{-y\,y'}{\sqrt{x'^2 - y'^2}}\,,$$

one can easily verify that:

$$\frac{\partial u}{\partial a} = F_{x'}\frac{\partial x}{\partial a} + F_{y'}\frac{\partial y}{\partial a}\,, \qquad du = F_{x'}\,dx + F_{y'}\,dy\,.$$

Now the differential du can vanish or become negative between 0 and 3 along any curve $\mathfrak{L}_0$, because d refers to the advance along the curve $\mathfrak{L}_0$, while x', y', p refer to the extremal of the field, so since dx is positive and one sets $x = t$ for the extremal, $F_{x'}$ can be regarded as positive, which will imply that:

$$F_{x'} + F_{y'} \frac{dy}{dx} \leq 0 , \qquad 1 + \frac{F_{y'}}{F_{x'}} \frac{dy}{dx} \leq 0 , \qquad p \frac{dy}{dx} \geq 1 ,$$

which can occur only for $p = \pm 1$, so for the points on the x-axis, from the second inequality in (43) and the value of p that was given above. The quantity $du : d\tau$ is always positive between the points 0 and 3 then, such that one can set:

$$dJ_{02} = du \left[1 + \frac{s^2}{2} g_{ss}(u, v, \theta s) \right]$$

everywhere, from which it will follow that:

$$\frac{d\,(J_{02} - u)}{d\tau} < 0 ,$$

with the aforementioned character of the expression $f_{pp}\,dx$.

The transformation of the differential dJ_{02} into the variables u, v will become impossible when:

$$dw = 0 , \qquad dx = \pm\, dy ,$$

since the integrand F will be singular then. Nonetheless, as the original definition:

$$J = \int y \frac{dw}{d\tau} d\tau$$

shows, J_{02} will continue to be a function $\varphi(\tau)$ in the sense of § **17** at such locations. Obviously, $dJ_{02} : d\tau$ will vanish in that way, and one will find that du satisfies:

$$du = dx(F_{x'} \pm F_{y'}) = a\,dx \left(1 \pm \cos \frac{x}{a} \right) ,$$

such that $d\,(u - J_{02})$ will also remain positive. The quantity $u - J_{02}$ will then increase when the point 2 runs from 0 to 3 along the curve $\mathfrak{L}_0$.

The initial value of u requires a closer examination, due to the singularity at the point 0. Clearly, it is initially zero when the inequality:

$$\left(\frac{dy}{d\tau} : \frac{dx}{d\tau} \right) \Bigg|^0 < 1$$

is satisfied. In fact, since the quantity a will become infinite for only $x = y$, by the approximation at the point 0, it will remain below a finite limit along the curve $\mathfrak{L}_0$. The expression for u then shows immediately that u will converge to the limit zero along with x. By contrast, if one has the equations:

$$\left(\frac{dy}{d\tau} : \frac{dx}{d\tau}\right)\Bigg|^0 = 1, \qquad \lim_{\tau\to\tau_0} \frac{y}{x} = 1$$

then the equation that serves as the definition of a, i.e., when one sets:

$$\varphi(z) = \frac{\sin z - z}{z} ,$$

the equation:

$$\frac{y}{x} = 1 + \varphi\left(\frac{x}{a}\right) ,$$

will show that one must have:

$$\varphi\left(\frac{x}{a}\right)\Bigg|^0 = 0 .$$

Now the quantity $\varphi(z)$ will be negative as long as z lies between 0 and π. Since the inequality (44) is valid, it will then follow from this that $\varphi\left(\frac{x}{a}\right)$ will vanish when the equation:

$$\frac{x}{a} = 0$$

is verified, which is then confirmed for the point 0. Obviously, that will also be true when the curve $\mathfrak{L}_0$ has a finite segment in common with the line $y = x$ and 0 means the point at which the line and the curve meet. If one substitutes the value that is obtained for x / a in the equation:

$$u = \frac{a^2}{4}\left\{\frac{2x}{a} - \sin\frac{2x}{a}\right\} = \frac{1}{4}\left\{\frac{(2x)^2}{3!}\cdot\frac{2x}{a} - \frac{(2x)^2}{5!}\left(\frac{2x}{a}\right)^3 + \cdots\right\}$$

then one will see that the quantity u begins with the value 0 at the location 0 as a function of τ in this case, as well. Now since the same thing is true of J_{02}, one will also have:

$$u - J_{02}\,\Big|^{\tau_0} = 0 ,$$

so the difference $u - J_{02}$ will then become positive (since it increases) as soon as the point 2 leaves the line $y = x$ as it runs along the arc $\mathfrak{L}_0$. The limiting value:

$$u \Big|^{3} - J_{03} = \bar{J}_{03} - J_{03}$$

is likewise positive. The lines $\mathfrak{L}$ and $\mathfrak{L}^0$ then enclose a smaller area with the *w*-axis then the semi-circle of the same length.

The theorem that a closed curve always encloses a smaller area than a circle of the same length follows directly from the results obtained when the curve can be divided into two equal pieces by two of its points 1 and 2 in such a way that the straight line segment 12 runs completely in the interior. It can then be identified with the *w*-axis.

Problem VI (§ 11). – One can surround any piece an extremal that does not include a cusp with a field when one, e.g., sets the quantity *a* constant in equations (17) and lets only *b* vary. The extremals of the field will then be curves that are produced from each other by translating parallel to the *x*-axis. When one sets $t = p$, one will then have the equations:

$$\frac{\partial x}{\partial b} = 1 , \qquad \frac{\partial y}{\partial b} = 0 , \qquad \Delta = -\frac{dy}{dp} ,$$

and from (18), Δ can vanish only at a cusp. One further finds that:

$$f_{pp}\, dx = -\frac{2y\, p\,(p^2 - 3)}{(1 + p^2)^3}\, dx .$$

That quantity will also assume opposite values in opposite directions along the arc elements, since the first case of § **3** applies here. The **Legendre** sign condition for the strong extremum is not fulfilled then, but the one for the weak extremum probably is, since $p^2 - 3$ changes its sign only at a cusp. A piece of the extremal that does not contain one will then yield a weak maximum or minimum according to whether it belongs to one or the other of the halves into which the curve is divided by the cusp. If one has $p^2 < 3$ then the desired minimum will occur.

Problem VII (§ 11). – Upon introducing new variables *u*, *v*, one will have a transformation of the form:

$$\sqrt{E\, d\varphi^2 + 2F\, d\varphi\, d\psi + G\, d\psi^2} = \sqrt{E^0\, du^2 + 2F^0\, du\, dv + G^0\, dv^2} = du \sqrt{E^0 + 2F^0\, s + G^0\, s^2} .$$

The equations:

$$g\,(u, v, 0) = 1 , \qquad g_s\,(u, v, 0) = 0$$

will then imply that:

$$E^0 = 1 , \qquad F^0 = 0 .$$

It will then follow from the connection between E, E^0, F, …, and the derivatives of the rectangular coordinates with respect to φ, ψ, u, v that one has the equation:

$$E^0G^0 - F^0F^0 = (EG - F^2)\left(\frac{\partial(\varphi,\psi)}{\partial(u,v)}\right)^2,$$

or when one sets:

$$G^0 = m^0,$$

the equation:

$$\Delta = \frac{\partial(\varphi,\psi)}{\partial(u,v)} = \frac{m}{\sqrt{EG-F^2}}.$$

The special form of the integrand that was considered in § **17** will then be the **Gauss** form of the line element:

$$\sqrt{du^2 + m^2dv^2},$$

and the geodetic lines that are perpendicular to a given line will define a field in the neighborhood of one of them as long as m is finite and non-zero.

The **Legendre** condition for a strong minimum is fulfilled, since:

$$\frac{\partial^2\Phi}{\partial\varphi'^2} = \frac{E}{\Phi} - \frac{(E\varphi' + F\psi')^2}{\Phi^3} = \frac{EG-F^2}{\Phi^3}\psi'^2,$$

with the notation of § **11**, and that quantity is always positive.

§ 19. The Jacobi-Hamilton equation.

The right-hand sides of the equations that are true for u :

$$\frac{\partial u}{\partial x} = F_{x'}, \qquad \frac{\partial u}{\partial y} = F_{y'}$$

depend upon only the ratio $y' : x'$. If one eliminates it then that will give a first-order partial differential equation for u that we would like to refer to as the *Jacobi-Hamilton equation*. If U is any solution of it then one can start with the first of the equations:

(45) $$\frac{\partial U}{\partial x} = F_{x'}, \qquad \frac{\partial U}{\partial y} = F_{y'}$$

and use it to determine the value of $y' : x'$. The second will then be satisfied automatically. When one regards U as given, the first equation can be regarded as an ordinary differential equation between x and y whose integral is:

$$\varphi(x, y, a) = 0 \tag{46}$$

and is represented by a simply-infinite family of curves $\mathfrak{K}$. If one lets D denote the advance along a curve U = const. then one will have:

$$\frac{\partial U}{\partial x} Dx + \frac{\partial U}{\partial y} Dy = 0 , \qquad F_{x'} Dx + F_{y'} Dy = 0 .$$

The curves $\mathfrak{K}$ will then lie transversally to the curves U = const. If one further introduces the values of an arbitrary function:

$$t = \psi(x, y) \tag{47}$$

along a certain curve $\mathfrak{K}$ as a parameter by which the value of x and y are determined then one will have:

$$x' = \frac{\partial x}{\partial t} , \qquad y' = \frac{\partial y}{\partial t} ,$$

in which one thinks of x and y as functions of a and t that are calculated from equations (46) and (47). Obviously, one can also consider t and a, and therefore x' and y', to be functions of x and y on the grounds of just those equations. If the symbol ∂ refers to that way of looking at things, while notating the derivatives by suffixes retains the meaning that it has had up to now, then one will have the equations:

$$\frac{\partial F_{x'}}{\partial y} = \frac{\partial F_{y'}}{\partial x} , \tag{48}$$

$$\begin{aligned} \frac{\partial F_{x'}}{\partial x} &= F_{x'x} + F_{x'x'} \frac{\partial x'}{\partial x} + F_{x'y'} \frac{\partial y'}{\partial x} , \\ \frac{\partial F_{y'}}{\partial x} &= F_{y'x} + F_{y'x'} \frac{\partial x'}{\partial x} + F_{y'y'} \frac{\partial y'}{\partial x} . \end{aligned} \tag{49}$$

As was pointed out in § **16**, it will further follow from the homogeneity of the functions $F_{x'}$, $F_{y'}$ that:

(50) $$F_{x'x'}\,x' + F_{x'y'}\,y' = 0\,, \quad F_{y'x'}\,x' + F_{y'y'}\,y' = 0\,,$$

and on the basis of equations (49), as well as their analogues that are differentiated with respect to *y*, it will follow from the last equation that:

$$x'\,\frac{\partial F_{x'}}{\partial x} + y'\,\frac{\partial F_{y'}}{\partial x} = x'\,F_{x'x} + y'\,F_{y'x} = F_x\,,$$
$$x'\,\frac{\partial F_{x'}}{\partial y} + y'\,\frac{\partial F_{y'}}{\partial y} = F_y\,.$$

However, as a result of equation (48), one can also give the last equations the following form:

$$F_x = \frac{\partial F_{x'}}{\partial x}\,x' + \frac{\partial F_{x'}}{\partial y}\,y' = \frac{dF_{x'}}{dt}\,,$$
$$F_y = \frac{\partial F_{y'}}{\partial x}\,x' + \frac{\partial F_{y'}}{\partial y}\,y' = \frac{dF_{y'}}{dt}\,,$$

which explains the fact that the curves $\mathfrak{K}$ are extremals of the integral *J*. If *V* is any solution of the partial differential equation that arises when one eliminates $x' : y'$ from equations (45) then the curves *V* = const. will be cut transversally by a family of extremals of the integral *J* . Therefore, it is obvious that when one advances along a curve $\mathfrak{K}$:

$$dV = (F_{x'}\,x' + F_{y'}\,y')\,dt = F\,dt\,,$$

one can then set:

$$V = \int F\,dt\,,$$

in which one integrates along an extremal of the family.

If the function *V* includes a constant *c* then the **Jacobi-Hamilton** equation states that *p* can be determined as a function of *x* and *c* such that the identities (45) will be satisfied. If one differentiates it with respect to *c* then one will get:

$$\frac{\partial^2 V}{\partial x\,\partial c} = \frac{\partial F_{x'}}{\partial p}\,\frac{\partial p}{\partial c}\,, \qquad \frac{\partial^2 V}{\partial y\,\partial c} = \frac{\partial F_{y'}}{\partial p}\,\frac{\partial p}{\partial c}\,.$$

Now one can set:

$$F_{x'} = \psi\left(\frac{y'}{x'}\right) = \psi(p)\,,$$

so it will follow that:

$$F_{x'y'} = \frac{1}{x'}\psi'(p) = \frac{1}{x'}\frac{\partial F_{x'}}{\partial p},$$

and a similar relation will be true for $F_{y'}$. One will then get:

$$\frac{\partial^2 V}{\partial x\,\partial c} = x' F_{x'y'}\frac{\partial p}{\partial c}, \qquad \frac{\partial^2 V}{\partial y\,\partial c} = x' F_{y'y'}\frac{\partial p}{\partial c},$$

and since equations (50) are true:

$$x'\frac{\partial^2 V}{\partial x\,\partial c} + y'\frac{\partial^2 V}{\partial y\,\partial c} = 0 .$$

The equation:

$$\frac{\partial V}{\partial c} = b$$

will then be valid along every individual curve $\mathfrak{K}$. If its left-hand side is not constant then it will represent the totality of all extremals, such that it will be known when a function *V* with the given properties has been found.

The generalization of that theory is immediate, and it might be suggested for the case of three variables. Let the function $F(x, y, z, x', y', z')$ be homogeneous with respect to the last three arguments and of dimension one, such that the derivatives $F_{x'}$, $F_{y'}$, $F_{z'}$ depend upon only *x*, *y*, *z*, and the ratios:

$$p = \frac{y'}{x'}, \qquad q = \frac{z'}{x'}.$$

If the equations:

(51) $$\frac{\partial V}{\partial x} = F_{x'}, \qquad \frac{\partial V}{\partial y} = F_{y'}, \qquad \frac{\partial V}{\partial z} = F_{z'}$$

are true for a function *V*, as well as the **Jacobi-Hamilton** differential equation that results from them by eliminating *p* and *q*, then when one understands the differentiation ∂ to mean the same thing as before, one will have:

$$\frac{\partial F_{x'}}{\partial y} = \frac{\partial F_{y'}}{\partial x}, \qquad \frac{\partial F_{y'}}{\partial z} = \frac{\partial F_{z'}}{\partial y}, \qquad \frac{\partial F_{z'}}{\partial x} = \frac{\partial F_{x'}}{\partial z}.$$

It follows from this that:

$$\frac{\partial F_{x'}}{\partial x}x' + \frac{\partial F_{y'}}{\partial x}y' + \frac{\partial F_{z'}}{\partial x}z' = \frac{\partial F_{x'}}{\partial x}x' + \frac{\partial F_{x'}}{\partial y}y' + \frac{\partial F_{z'}}{\partial z}z',$$

along with two analogous equations. When combined with the equations that correspond to what was cited in (49) and (50), they will imply that:

$$F_x - \frac{dF_{x'}}{dt} = 0, \qquad F_y - \frac{dF_{y'}}{dt} = 0, \qquad F_z - \frac{dF_{z'}}{dt} = 0.$$

If one then regards two of the equations (51) as ordinary differential equations in x, y, z then the curves that represent their integrals will be extremals of the integral $\int F\,dt$. If V includes two constants c_1, c_2 then equations (51) will yield:

$$\frac{\partial^2 V}{\partial x\,\partial c_1} = \frac{\partial F_{x'}}{\partial p}\frac{\partial p}{\partial c_1} + \frac{\partial F_{x'}}{\partial q}\frac{\partial q}{\partial c_1} = x'\left(F_{y'x'}\frac{\partial p}{\partial c_1} + F_{z'x'}\frac{\partial q}{\partial c_1}\right),$$

along with two similar equations, from which the analogues of equations (50) will yield:

$$x'\frac{\partial^2 V}{\partial x\,\partial c_\alpha} + y'\frac{\partial^2 V}{\partial y\,\partial c_\alpha} + z'\frac{\partial^2 V}{\partial z\,\partial c_\alpha} = 0, \qquad \alpha = 1, 2.$$

The equations:

$$\frac{\partial V}{\partial c_1} = b_1, \qquad \frac{\partial V}{\partial c_2} = b_2$$

will then represent the extremals of the integral $\int F\,dt$. If they contain $n + 1$ quantities x, y, z, … then V must be endowed with n arbitrary constants if it is to be able to serve as a representation of all extremals.

Example. – If x is time, and y, z, … are the independent parameters of a system of masses, and one sets:

$$p = \frac{dy}{dx}, \qquad q = \frac{dz}{dx}, \qquad \ldots$$

then **Hamilton**'s principle will be mostly an equation of the form:

$$\delta\int H(x, y, z, \ldots, p, q, \ldots)\,dx = 0,$$

which must be integrated between two given systems of values (x, y, z, …). According to **Helmholtz**, H is called the *kinetic potential*. If one would like to put the integral into the form:

$$\int F\, dt$$

then one must set:

$$p = \frac{y'}{x'}, \qquad q = \frac{z'}{x'}, \qquad F = x' H(x, \ldots, p, q, \ldots),$$

such that:

$$F_{x'} = H - p\frac{\partial H}{\partial p} - q\frac{\partial H}{\partial q} - \ldots, \qquad F_{y'} = \frac{\partial H}{\partial p}, \qquad F_{z'} = \frac{\partial H}{\partial q}, \ldots$$

The **Jacobi-Hamilton** equation then arises by eliminating p, q, … from the equations:

$$\frac{\partial V}{\partial x} = H - p\frac{\partial H}{\partial p} - q\frac{\partial H}{\partial q} - \ldots, \qquad \frac{\partial V}{\partial y} = \frac{\partial H}{\partial p}, \ldots \tag{52}$$

If time x does not enter into H explicitly then the equation:

$$F_{x'} = \text{const.}, \qquad p\frac{\partial H}{\partial p} + q\frac{\partial H}{\partial q} + \ldots - H = -h \tag{53}$$

will be true for the extremals, and h will be the generalized constant of the *vis viva*. Now since the quantities p, q, … also refer to the extremals in equations (52), one will have:

$$\frac{\partial V}{\partial x} = h, \qquad V = h\,x + W,$$

in which W does not include time explicitly and it satisfies the equations:

$$\frac{\partial W}{\partial y} = \frac{\partial H}{\partial p}, \quad \frac{\partial W}{\partial z} = \frac{\partial H}{\partial q}, \quad \ldots$$

If the number of quantities x, y, … is $n + 1$, such that the system has n degrees of freedom, and if $n - 1$ constants $c_1, c_2, \ldots, c_{n-1}$ are included in W then the motions of the system will be represented by the following equations:

$$\frac{\partial V}{\partial c_1} = \frac{\partial W}{\partial c_1} = b_1, \qquad \frac{\partial V}{\partial c_2} = \frac{\partial W}{\partial c_2} = b_2, \qquad \ldots, \qquad \frac{\partial V}{\partial h} = x + \frac{\partial W}{\partial h} = x_0,$$

in which arbitrary constants appear on the right-hand side.

That argument is independent of the form of the function H. In particular if one has:

$$-H = T + U,$$

and if T is a quadratic form in the quantities p, q, …, but U is free of them, then since one now has:

$$2T = p\frac{\partial T}{\partial p} + q\frac{\partial T}{\partial q} + \ldots,$$

equation (53) will go to the usual equation for *vis viva*:

$$T = U + h .$$

§ 20. – The Weierstrass method.

As in § **14**, whose assumptions are the only ones that shall now be established, let $\mathfrak{C}_0$ be an arbitrary regular curve (Fig. 8) that $\mathfrak{C}$ starts from, meets the extremals of the field at a non-vanishing angle, and goes through the point 0. Let 12 be any piece of the curve $\mathfrak{C}$ that does not include the point 0. Let the points 1 and 2 be connected by a further curve $\mathfrak{L}$ that belongs to the field, and along which the point 3 runs from 1 to 2. That curve has the properties of the one in § **17** that was similarly denoted, such that x_3, y_3 are functions of a parameter τ that belongs to the $\varphi(\tau)$ that was examined there. The parameter τ increases in the direction from 1 to 2. A certain extremal of the field goes through each position of the point 3 whose parameters a, just like the argument t of the point 3 in § **14** that corresponds to it, are regular functions of x_3, y_3 . That extremal specifies a point 0 on the curve $\mathfrak{C}_0$ whose coordinates, just like the argument t_0 in § **14** that belongs to it, are regular functions of a, and for a sufficient restriction of the field, the curve segment $\mathfrak{C}_0$ will be determined uniquely. If one then sets:

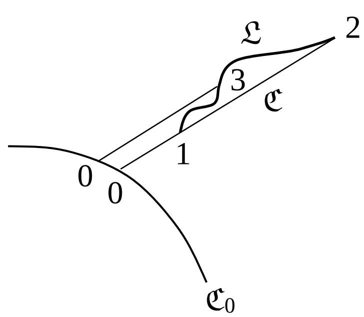

Figure 8.

$$\omega(a) = F(\xi, \eta, \xi_t, \eta_t)\,\frac{dt_0}{da} + F_{x'}(\xi, \ldots, \eta_t)\,\xi_a + F_{y'}(\xi, \ldots, \eta_t)\,\eta_a\Big|^{0}$$

then that expression will be a regular function of a. When one, in turn, sets:

$$u = \bar{J}_{03} ,$$

formula (31) will say that:

$$\frac{\partial u}{\partial a} = \frac{\partial \bar{J}_{03}}{\partial a} = -\,\omega(a) + F_{x'}(\xi, \ldots, \eta_t)\,\xi_a + F_{y'}(\xi, \ldots, \eta_t)\,\eta_a\Big|^{3} ,$$

and when the point 3 is thought to move along a certain extremal 03, that will immediately imply that:

$$\frac{\partial u}{\partial t} = F(\xi, \eta, \xi_t, \eta_t)\Big|^3.$$

Now a, t, like x_3, y_3, are functions of τ that belong to the ones in § **17** that were denoted by $\varphi(\tau)$. Therefore, if any differentiation with respect to τ always refers to forward derivatives then the last two equations will imply that:

$$\frac{d\bar{J}_{03}}{d\tau} = -\,\omega(a)\,\frac{da}{d\tau} + F_{x'}(\xi, \ldots, \eta_t)\,\frac{dx}{d\tau} + F_{y'}(\xi, \ldots, \eta_t)\,\frac{dy}{d\tau}\Bigg|^3.$$

Furthermore, if the integral J_{32} that is defined along the curve $\mathfrak{L}$, which likewise has the character of $\varphi(\tau)$ as a function of τ, satisfies the equation:

$$\frac{d\bar{J}_{32}}{d\tau} = -\,F\left(x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau}\right)\Bigg|^3$$

then when one replaces ξ_t, η_t with x', y', one will get:

$$\frac{d(\bar{J}_{03} + J_{32})}{d\tau} = -\,\omega(a)\,\frac{da}{d\tau} + F_{x'}(x, y, x', y')\,\frac{dx}{d\tau} + F_{y'}(x, y, x', y')\,\frac{dy}{d\tau} - F\left(x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau}\right).$$

The sum of the last three terms is called $\mathcal{E}$, or more precisely:

$$\mathcal{E}\left(x, y, x', y', \frac{dx}{d\tau}, \frac{dy}{d\tau}\right).$$

Now if the point 3 lies at 1 in its initial position and at 2 in its final position then the corresponding values of the sum $\bar{J}_{03} + J_{32}$ will be:

$$\bar{J}_{01} + J_{12}, \qquad \bar{J}_{02} = \bar{J}_{01} + \bar{J}_{12},$$

and it then follows that the difference between them is:

$$\bar{J}_{12} - J_{12} = \int_{\tau_1}^{\tau_2} \frac{d(\bar{J}_{03} + J_{32})}{d\tau}\,d\tau = -\int_{\tau_1}^{\tau_2} \omega(a)\,\frac{da}{d\tau}\,d\tau + \int_{\tau_1}^{\tau_2} \mathcal{E}\,d\tau.$$

However, the first summand on the right vanishes, because when one sets:

$$\int \omega(a)\,da = \zeta(a),$$

ζ will be a regular function for $a = a_0$ in any event, and one will have:

$$\int_{\tau_1}^{\tau_2} \omega(a)\frac{da}{d\tau}\,d\tau = \int_{\tau_1}^{\tau_2} \frac{d\zeta(a)}{d\tau}\,d\tau = \zeta(a)\Big|_1^2 .$$

However, that quantity has the value 0, since a has the same value a_0 at 1 and 2. It will then follow that:

$$\bar{J}_{12} - J_{12} = -\int_{\tau_1}^{\tau_2} \mathcal{E}\left(x, y, x', y', \frac{dx}{d\tau}, \frac{dy}{d\tau}\right) d\tau, \tag{54}$$

and the left-hand side will have a fixed sign when that is true of $\mathcal{E}$.

If one would like to test the extremal arc 02 directly in the case where all extremals of the field run through the point 0 then one only has to replace the curve $\mathfrak{C}_0$ with the fixed point 0. The formula:

$$d\bar{J}_{03} = du = F_{x'}\,dx + F_{y'}\,dy, \tag{55}$$

which we must necessarily prove by some special consideration, will imply immediately that:

$$\frac{d(\bar{J}_{03} + J_{32})}{d\tau} = \mathcal{E}, \qquad J_{02} - \bar{J}_{02} = -\int_{\tau_0}^{\tau_2} \mathcal{E}\,d\tau, \tag{56}$$

only as long as the curve $\mathfrak{L}$ that is under consideration remains inside the surface that the extremals that start from the point 0 cover simply.

From § **15**, the simple formula (55) for *du* will also remain valid when the curve $\mathfrak{C}_0$ is intersected transversally by the extremals of the field (Fig. 9). $\omega(a)$ will vanish identically then. We shall make use of that fact in order to derive a criterion for the extremum with one variable limit by means of the expression $\mathcal{E}$, as well. The point 4 lies on the curve $\mathfrak{C}_0$, while 2 lies on the curve $\mathfrak{C}$, as before. Let both of them be connected by a curve that remains in the field and runs from the point 3 to the point 2 in the direction of 4. The initial and final values of the sum $\bar{J}_{03} + J_{32}$ will then be the quantities:

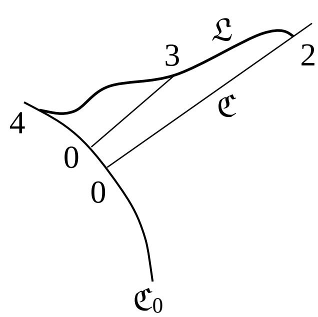

Figure 9.

$$J_{42}\,, \qquad \bar{J}_{02}\,,$$

and one will have, as before:

$$\frac{d\bar{J}_{03}}{d\tau} = F_{x'}\frac{dx}{d\tau} + F_{y'}\frac{dy}{d\tau}\,, \qquad \frac{d\bar{J}_{32}}{d\tau} = -F\left(x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau}\right),$$

so

(57) $$J_{42} - \bar{J}_{02} = -\int_{\tau_4}^{\tau_2} \mathcal{E}\, d\tau\,.$$

§ 21. – The geometric interpretation of $\mathcal{E}$.

The quantity $\mathcal{E}$, by whose examination the question of the existence of an extremum can be resolved, obviously depends upon the point 2 = (x, y) and the two line elements that start from it, which are determined along the extremal of the field and the curve $\mathfrak{L}$ in the sense of increasing parameters t and τ, resp. If both quantities x' and $dx / d\tau$ are non-zero then one can set:

$$Dx = \frac{dx}{d\tau}d\tau\,, \qquad Dy = \frac{dy}{d\tau}d\tau = \bar{p}\,Dx\,, \qquad dy = p\,dx\,,$$

$$F(x, y, x', y')\,dt = f(x, y, p)\,dx\,,$$

$$F\left(x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau}\right) d\tau = f(x, y, \bar{p})\,Dx\,,$$

and that will give:

(58) $$\mathcal{E}\,d\tau = \{f(x, y, p) - (\bar{p} - p)\,f_p(x, y, p) - f(x, y, \bar{p})\}\,Dx\,.$$

Indeed, the quantities x' and y' that characterize the former line element enter into only $F_{x'}$ and $F_{y'}$, which are expressions that depend upon only $x' : y'$. However, one cannot conclude from this that $\mathcal{E}$ will keep the same value when one replaces the element of the extremal with its opposite, because when F is also a single-valued function of x' and y', $F_{x'}$ and $F_{y'}$ do not, however, need to be single-valued functions of $x' : y'$ (§ **3**). Therefore, the quantities:

$$\mathcal{E}\left(x, y, x', y', \frac{dx}{d\tau}, \frac{dy}{d\tau}\right), \qquad \mathcal{E}\left(x, y, -x', -y', \frac{dx}{d\tau}, \frac{dy}{d\tau}\right)$$

do not need to be equal, any more than the quantities:

$$\mathcal{E}\left(x, y, x', y', \frac{dx}{d\tau}, \frac{dy}{d\tau}\right), \qquad \mathcal{E}\left(x, y, x', y', -\frac{dx}{d\tau}, -\frac{dy}{d\tau}\right)$$

need to be opposite (see the example below).

The two line elements upon which $\mathcal{E}$ depends will coincide if and only if a positive quantity α exists for which the equations:

$$x' = \alpha\frac{dx}{d\tau}, \qquad y' = \alpha\frac{dy}{d\tau} \tag{59}$$

are satisfied. In that case, one will have:

$$F_{x'}(x, y, x', y') = F_{x'}\left(x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau}\right),$$

$$F_{y}(x, y, x', y') = F_{y'}\left(x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau}\right).$$

Now since one always has:

$$\frac{dx}{d\tau}F_{x'}\left(x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau}\right) + \frac{dy}{d\tau}F_{y'}\left(x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau}\right) - F\left(x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau}\right) = 0,$$

it will follow that:

$$\mathcal{E} = 0.$$

We then say that $\mathcal{E}$ vanishes in an *ordinary way*. If $\mathcal{E}$ assumes the value zero without equation (59) being true for a positive value of α then we shall call the vanishing *extraordinary*.

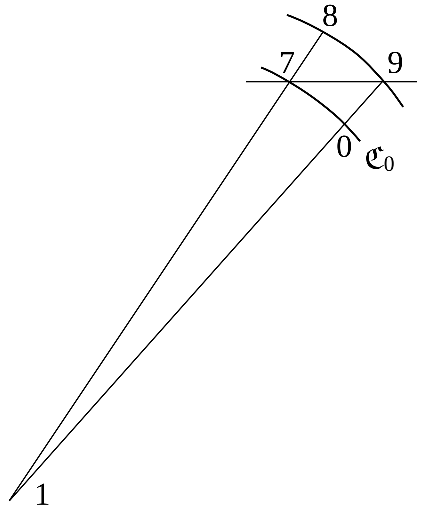

Figure 10.

Geometrically, we can characterize the meaning of the quantity $\mathcal{E}$ when we assume, as we will prove later (§ **30**), that every sufficiently-small piece of an extremal can be surrounded by a field, in general. Let 78 and 79 be the line elements (Fig. 10) that belong to the system of values (x, y, dx, dy), (x, y, Dx, Dy). We then draw an extremal in the direction 78 and surround it with a field in the neighborhood of the point 7, whose extremals might intersect the curve $\mathfrak{C}^0$ that goes through the point 7, as well as a curve 89, transversally. The former will be cut by the extremal of the field that goes through 9 at the point 0. $\bar{J}_{09}$ or $\bar{J}_{78}$ can then be regarded as the differential of the quantity u that is measured from the curve $\mathfrak{C}^0$ in the sense of § **15**, and we will have from formula (34) that:

$$\bar{J}_{09} = \bar{J}_{78} = F_{x'}(x, y, dx, dy)\, Dx + F_{y'}(x, y, dx, dy)\, Dy\,.$$

If the extremals 78 and 09 intersect at a point 1 that lies on the opposite side of 7 to 8 then, from §**15**, one will have:

$$\bar{J}_{17} = \bar{J}_{10}\,, \qquad \mathcal{E}\, d\tau = \bar{J}_{78} - \bar{J}_{79} = \bar{J}_{19} - \bar{J}_{17} - \bar{J}_{79}\,.$$

One sees from this that the quantity $\mathcal{E}\, d\tau$ is independent of the coordinates, which is also clear from the identity that was developed in § **16**:

$$F_{x'}(x, y, dx, dy)\, Dx + F_{y'}(x, y, dx, dy)\, Dy$$
$$= G_{u'}(u, v, du, dv)\, Du + G_{v'}(u, v, du, dv)\, Dv\,.$$

In particular, one can then imagine rotating the coordinate system in such a way that the differential dx has the same value for two elements 78 and 79, i.e., both elements point to the same side of a parallel to the y-axis. One will then have that:

$$p = \frac{dy}{dx}\,, \qquad \bar{p} = \frac{dy}{d\tau} : \frac{dx}{d\tau} = Dy : Dx$$

are finite quantities, and the elements will be defined by the systems of values (x, y, p, dx), (x, y, $\bar{p}$, dx). Now if F is regular for all arc elements that start from the point (x, y) then, from § **5**, the same thing will be true of:

$$f(x, y, p) = \frac{F}{x'}$$

for all elements that fall within the concave angle that is defined by the directions 78 and 79, since x' is non-zero for it. f will then be regular for the interval from p to $\bar{p}$, and when θ is a proper fraction, from (58), one can set:

(60)
$$f(x, y, \bar{p}) = f(x, y, p) + (\bar{p} - p)\, f_p(x, y, p) + \tfrac{1}{2}(\bar{p} - p)^2 f_{pp}[x, y, p + \theta(\bar{p} - p)] ,$$
$$\mathcal{E}\, d\tau = -\tfrac{1}{2}(\bar{p} - p)^2 f_{pp}[x, y, p + \theta(\bar{p} - p)]\, dx .$$

The quantity $\mathcal{E}$ has the fixed sign of $-F_1$ and will vanish in an ordinary way only when the **Legendre** condition for a strong extremum is fulfilled, because the product:

$$f_{pp}[x, y, p + \theta(\bar{p} - p)]\, dx$$

is identical to the quantity $F_1(x, y, dx, dy)\, dx^4$, which belongs to the arc element that goes through the point (x, y) and whose components are:

$$dx , \qquad [p + \theta(\bar{p} - p)]\, dx = dy ,$$

with which the assertion is proved. However, when only the **Legendre** condition for a weak extremum is assumed to be fulfilled, $\mathcal{E}$ will have the two indicated properties only as long as the two determining elements 78, 79 deviate sufficiently little in position and direction from an element of the curve $\mathfrak{C}$ that is taken in the direction of integration. Conversely, one cannot deduce the sign of the quantity $\mathcal{E}$ from that of the quantity F_1 in the same way.

§ 22. – Ordinary vanishing of $\mathcal{E}$ along a curve.

In general, when t and a refer to the point 3 that runs along the curve $\mathfrak{L}$, one will have:

$$x = \xi(t, a) , \qquad y = \eta(t, a) ,$$

$$\frac{dx}{d\tau} = \xi_t \frac{dt}{d\tau} + \xi_a \frac{da}{d\tau} , \qquad \frac{dy}{d\tau} = \eta_t \frac{dt}{d\tau} + \eta_a \frac{da}{d\tau} ,$$

$$x' = \xi_t , \qquad y' = \eta_t .$$

Now when the quantity $\mathcal{E}$ vanishes ordinarily at every point along the entire curve $\mathfrak{L}$, that will imply equations (59), and therefore:

$$\begin{vmatrix} x' & \dfrac{dx}{d\tau} \\ y' & \dfrac{dy}{d\tau} \end{vmatrix} = \begin{vmatrix} \xi_t & \xi_a \dfrac{da}{d\tau} \\ \eta_t & \eta_a \dfrac{da}{d\tau} \end{vmatrix} = \Delta \frac{da}{d\tau} = 0 .$$

Now since Δ is non-zero, it follows that:

$$\frac{da}{d\tau} = 0$$

for a finite interval of the argument τ. It follows from this that since a has the same properties as the function $\varphi(\tau)$ that was defined in § **17** as a function of τ, a must be constant along the entire curve $\mathfrak{L}$ and equal to its initial value a_0 .

If the curve $\mathfrak{L}$ does not coincide with $\mathfrak{C}$ completely then an extraordinary vanishing of the quantity $\mathcal{E}$ will be excluded, either completely or by restricting the curve $\mathfrak{L}$, and for an ordinary vanishing along the curve $\mathfrak{L}$, no change of sign will take place, so the right-hand sides of equations (54), (56), (57) in § **20** will be non-zero and have a fixed sign. The extremals 12, 02 will then, in fact, yield an extremum for the integral J relative to the curve $\mathfrak{L}$ that was restricted in the given way, and indeed it will be a maximum or a minimum according to whether $\mathcal{E}$ is positive or negative, resp.

In particular, if the curve $\mathfrak{L}$ lies in a narrow neighborhood of the arc 12 of the curve $\mathfrak{C}$ that is defined by ρ and ρ_1 as in § **17** then the angle that the tangent to the latter that is drawn in the sense of increasing t defines with a similar tangent to an extremal of the field whose contact point is at a distance of less than ρ from the former is below a limit ρ_2 that increases with ρ with no restrictions. Now for any tangent to the curve $\mathfrak{L}$ there is a corresponding one to the arc 12 whose contact point is also at a distance from it of less than ρ and defines an angle with it that does not exceed ρ_1 . The angle between the tangent to the curve $\mathfrak{L}$ and the extremal that goes through its contact point will then be less than $\rho_1 + \rho_2$ and will become arbitrarily small for a sufficient bounding of the neighborhood. In order to ensure the weak extremum, $\mathcal{E}$ then needs to have a fixed sign and vanish in an ordinary way only for the case in which the two line elements that its values determine are inclined arbitrarily little with respect to each other and to an element of the curve $\mathfrak{C}$ that is taken in the direction of integration. That convention can enter in place of the one that relates to F_1 in case a) of § **17**, from which it will follow as in § **21**.

In order to further conclude a strong extremum from equations (54), (56), (57) of § **20**, first of all, the integrand must be regular for an arbitrarily-directed line element of the field, but in such a way that the quantity $\mathcal{E}(x, y, x', y', Dx, Dy)$ will have the properties that were indicated above when

(Dx, Dy) is an arbitrary line element that starts from the point (x, y), but x', y' refer to the extremal of the field that goes through that point. In the criteria for a strong extremum that were presented before (§ **17**), the **Legendre** condition in case b) can be replaced with the less-demanding **Weierstrass** condition that the quantity possesses a fixed sign at all points of a certain region that includes the arc 02 and will vanish only when the line elements (x', y') and (Dx, Dy) coincide.

In both cases a) and b), the requirement that was used essentially in § **16** and § **17** that the integrand F must not vanish can be further dropped, since the present development is obviously independent of that assumption.

§ 23. – Examples. Problems I, II, IV.

Problem I (§ **9**). – Here, one sets (with the positive square roots):

$$F = \sqrt{x'^2 + y'^2}\ ,$$

$$\begin{aligned}\mathcal{E} &= \frac{x'}{\sqrt{x'^2+y'^2}}\frac{dx}{d\tau} + \frac{y'}{\sqrt{x'^2+y'^2}}\frac{dy}{d\tau} - \sqrt{\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2} \\ &= \sqrt{\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2}\,\{\cos(ds, Ds) - 1\}\ ,\end{aligned}$$

in which ds, Ds are the arc-elements that correspond to positive differentials dt, $d\tau$. The quantity $\mathcal{E}$ is nowhere-positive then and vanishes only when ds and Ds coincide. Formula (58) of § **21** will give the same thing when one sets:

$$F\,dt = \sqrt{1+p^2}\,dx\ , \qquad F\left(x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau}\right)d\tau = \sqrt{1+\bar{p}^2}\,Dx\ ,$$

in which the square roots must have the same signs as the differentials that are next to them. One then gets:

$$\begin{aligned}\mathcal{E}\,d\tau &= \left\{\sqrt{1+p^2} + \frac{(\bar{p}-p)\,p}{\sqrt{1+p^2}} - \sqrt{1+\bar{p}^2}\right\}Dx \\ &= \sqrt{1+\bar{p}^2}\,Dx\left\{\frac{\bar{p}\,p-1}{\sqrt{1+p^2}\sqrt{1+\bar{p}^2}} - 1\right\},\end{aligned}$$

and this expression is nowhere-positive, as well, since:

$$\sqrt{1+\bar{p}^2}\,Dx$$

is positive, but the bracketed expression next to it is negative or zero as a result of the inequality:

$$(1+p\,\bar{p})^2 \leq (1+p^2)(1+\bar{p}^2)\ .$$

Geometrically, from § **21**, that result means that in a triangle 179 in which the side 79 is small compared to the others, one will have the inequality:

(61)
$$\bar{J}_{19} - \bar{J}_{17} - \bar{J}_{79} < 0\ ,$$

i.e., the sum of the sides 17, 79 is greater than 19. The proof of the existence of an extremum that was carried out by means of the expression $\mathcal{E}$ can be interpreted in a very geometrically-elementary way in this problem. If 1, 4, 5 are three points (Fig. 11) on a line, and 7, 9 are two positions of the point 2 that runs from 4 to 5 along an arbitrary curve 45 then from (61), when one denotes the lengths of straight line segments by $\bar{J}$, one will have:

$$\bar{J}_{17} + J_{75} > \bar{J}_{19} + J_{75},$$

so the quantity $\bar{J}_{12} + J_{25}$ must decrease and have a greater value at the beginning of the motion of the point 2 than it has at the end.

Problem II (§ 9). – The quantity $\mathcal{E}$ differs from the one that was obtained Problem I only by the positive factor *y*, so the **Weierstrass** condition for a strong maximum is fulfilled.

If we introduce the hyperbolic functions and set:

$$\cosh z = \frac{e^z + e^{-z}}{2}, \qquad \sinh z = \frac{e^z - e^{-z}}{2}, \qquad \tanh z = \frac{1}{\coth z} = \frac{\sinh z}{\cosh z},$$

for which, the following equations will be true:

$$\cosh^2 z - \sinh^2 z = 1\ , \qquad d\cosh z = \sinh z\,dz\ , \quad d\sinh z = \cosh z\,dz\ ,$$

$$d\tanh z = \frac{dz}{\cosh^2 z}, \quad d\coth z = \frac{-dz}{\sinh^2 z},$$

then the following equations will be true for the extremals:

(62)
$$y = a\cosh\frac{x-b}{a}, \qquad p = \sinh\frac{x-b}{a}.$$

In particular, we direct our attention to those extremals that intersect the given curve $\mathfrak{C}_0$ transversally, i.e., at right angles (§ **11**). Let the equation of that curve be:

$$y_0 = f(x_0) .$$

If the constants of the extremals that intersect perpendicularly are then subject to the equations:

$$y_0 = a \cosh \frac{x_0 - b}{a}, \qquad 1 + p_0 f'(x_0) = 0 ,$$

or when one sets:

$$u = \frac{x - b}{a}, \qquad u_0 = \frac{x_0 - b}{a},$$

one will have:

$$(63) \qquad f(x_0) = a \cosh u_0 , \qquad 1 + f'(x_0) \sinh u_0 = 0 .$$

If one differentiates this with respect to x, x_0, a, b, along with the first equation in (62), then that will give:

$$\begin{aligned} dy &= \sinh u \, dx + 0 \cdot dx_0 + (\cosh u - u \sinh u) \, da - \sinh u \, db \\ &= [\sinh u_0 - f(x_0)] \, dx_0 + (\cosh u_0 - u \sinh u_0) \, da - \sinh u_0 \, db , \end{aligned}$$

(64)

$$0 = \left[\frac{f'(x_0)}{a} \cosh u_0 + f''(x_0) \sinh u_0 \right] dx_0 - \frac{f'(x_0)}{a} u_0 \cosh u_0 \, da - \frac{f'(x_0)}{a} u_0 \cosh u_0 \, db .$$

The determinant of the terms in the last two equations that are endowed with da, db is:

$$M = \frac{f'(x_0) \cosh u_0}{a} \begin{vmatrix} \cosh u_0 - u_0 \sinh u_0 & -\sinh u_0 \\ -u_0 & -1 \end{vmatrix} = \frac{f'(x_0) \cosh^2 u_0}{-a} ,$$

so it will be non-zero when the curve $\mathfrak{C}_0$ does not run parallel to the x-axis at the location considered, as one would like to assume. One can then think of a and b as being regular functions of x_0 that are determined by the equations (63) and obtain an expression for y in terms of x and x_0. Furthermore, since y is a single-valued function of x along any catenary, one can set $t = x$. For the family of extremals that intersect $\mathfrak{C}_0$ perpendicularly, one will then get:

$$\Delta = \frac{\partial(x, y,)}{\partial(x, x_0)} = \frac{\partial y}{\partial x_0} , \qquad dy = p \, dx + \Delta \, dx_0 .$$

However, when one eliminates da and db from equations (64) by an elementary calculation, one will have:

$$M\,dy = M\sinh u\,dx + dx_0\,(\dots)\,,$$

in which the factor of dx_0 is the determinant of the coefficients of dx_0, da, db in the cited equations. One will then have:

$$M\,\Delta = \begin{vmatrix} 0 & \cosh u - u\sinh u & -\sinh u \\ \sinh u_0 - f'(x_0) & \cosh u_0 - u_0\sinh u_0 & -\sinh u_0 \\ \dfrac{f'(x_0)}{a}\cosh u_0 + f''(x_0)\sinh u_0 & -\dfrac{f'(x_0)}{a}u_0\cosh u_0 & -\dfrac{f'(x_0)}{a}\cosh u_0 \end{vmatrix}.$$

As long as that expression does not vanish, the extremal arc 02 that cuts the line $\mathfrak{C}_0$ perpendicularly at the point 0 will yield a strong maximum for the surface of revolution in comparison to all curves that connect to $\mathfrak{C}_0$ at the point 2.

If one further considers the second equation in (63) and sets:

$$\frac{f''(x_0)}{\left(\sqrt{1+[f'(x_0)]^2}\right)^3} = \frac{f''(x_0)}{(1+\sinh^{-2}u_0)^{3/2}} = f''(x_0)\tanh^3 u_0 = \frac{1}{r}\,,$$

then $|\,r\,|$ will be the radius of curvature of the curve $\mathfrak{C}_0$, and one can write:

$$M\,\Delta$$
$$= -\frac{\cosh^3 u_0}{r\sinh^2 u_0}\begin{vmatrix} \cosh u - u\sinh u & \sinh u \\ \cosh u_0 - u_0\sinh u_0 & \sinh u_0 \end{vmatrix} + \frac{\cosh^2 u_0}{y_0\sinh u_0}\begin{vmatrix} 0 & \cosh u - u\sinh u & \sinh u \\ \cosh u_0\coth u_0 & \cosh u_0 - u_0\sinh u_0 & \sinh u_0 \\ -1 & u_0 & 1 \end{vmatrix}.$$

If $\mathfrak{C}_0$ goes to an infinitely-small circle then one will have $r = 0$, and the equation $\Delta = 0$ will demand that:

$$\begin{vmatrix} \cosh u - u\sinh u & \sinh u \\ \cosh u_0 - u_0\sinh u_0 & \sinh u_0 \end{vmatrix} = 0\,, \qquad \coth u - u = \coth u_0 - u_0\,,$$

or

$$\frac{y}{p} - x = \frac{y_0}{p_0} - x_0\,.$$

That equation expresses the fact that the tangents to the extremal at the points 0 and (x, y) intersect on the x-axis. Therefore, as long as that does not happen for any two tangents to the curve 02, the extremals that go through the point 0 will define a field of that arc 12, which is a subset of 02. Now since the points 0 and 1 can get arbitrarily close to each other, it will follow that every arc of the extremal, no two of whose tangents can intersect on the x-axis, will yield an extremum for fixed endpoints.

In the general case of a finite curvature for the curve $\mathfrak{C}_0$, the equation $\Delta = 0$ will have the form:

$$A\,(\cosh u - u \sinh u) + B \sinh u = 0\,,$$

in which A and B depend upon only u_0, y_0, and $1 : r$, and are linear in the latter quantity. If A is non-zero then one can write the equation:

$$\coth u - u + \frac{B}{A} = 0\,. \tag{65}$$

The left-hand side then has the derivative:

$$-\frac{1}{\sinh^2 u} - 1\,,$$

which then decreases with increasing values of u. Now one easily finds that:

$$\coth(\mp\infty) = \mp 1\,, \qquad \coth(\pm 0) = \pm\infty\,,$$

so equation (65) will have a single finite root when:

$$|B| > |A|\,,$$

and in the contrary case, it will have no root. If one further fixes u_0 and y_0, i.e., one fixes the point 0 and the direction of the curve $\mathfrak{C}_0$, then one can add to the curvature of that curve the requirement that $B : A$ must preserve a given value. By a suitable choice of r, one can always succeed in making the extremal that is perpendicular to the curve $\mathfrak{C}_0$ cease to exist at a given point 2 then, so one can be certain that the curve that connects to the point 2 will yield a minimum in comparison to the curve $\mathfrak{C}_0$.

Problem IV (§ 9). – If $\mathfrak{L}$ is an arbitrary line in the wy-plane that once more descends from the point $w = y = 0$ to the line y in the half-plane $y > 0$ and has the continuity properties of the line that was similarly denoted in § **17**, and if the parameter τ has the value τ_0 at the coordinate origin then when the symbol u in § **9** is replaced with w, one will have:

$$x = \int_{\tau_0}^{\tau} y \sqrt{\left(\frac{dw}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2}\, d\tau\,, \tag{66}$$

with the positive square root, so one certainly has:

(67) $$x \geq \int_{\tau_0}^{\tau} y \frac{dy}{d\tau} d\tau \;, \qquad x \geq \frac{y^2}{2} \;.$$

The equality sign is true only when one always has:

$$\frac{dw}{d\tau} = 0$$

between τ_0 and τ. The corresponding curve $\mathfrak{L}_1$ in the *xy*-plane then lies in the first quadrant ($x > 0$, $y > 0$) and in the interior (i.e., the concave side) of the parabola:

(68) $$y^2 = 2x \;.$$

The interior of that parabola will be simply covered by the extremals that start from the point 0 ($x = y = 0$):

(69) $$1 - \frac{x}{a^2} = \sqrt{1 - \left(\frac{y}{a}\right)^2} \;, \qquad \frac{x^2}{a^2} = 2x - y^2 \;,$$

since those equations will yield a single positive value for a^2 when x, y are given and $2x - y^2$ is positive. If a piece of the curve $\mathfrak{L}$ first goes parallel (*ein Stück weit parallel*) to the *y*-axis then the curve $\mathfrak{L}_1$ will have a corresponding piece 01 in common with the parabola (68) that yields the value zero for J. Therefore, if 3 is the endpoint of the curve $\mathfrak{L}_1$, so $y_2 = 0$, then one will have:

(70) $$J_{03} = J_{13}$$

along it. If the curve $\mathfrak{L}_1$ leaves the parabola at some point then it cannot go back to it again, since $dw : d\tau$ will then be at least piece-wise non-zero, so the equality sign can no longer be valid in the relation (67).

The ellipses (69) define a field inside of the half of the parabola (68) for which $y > 0$, because here one has:

$$x = \xi = t \;, \qquad y = \eta = \sqrt{2t - \left(\frac{t}{a}\right)^2} \;,$$

from which one will get:

$$\Delta = \frac{\partial(\xi, \eta)}{\partial(t, a)} = \frac{\partial y}{\partial a} = \frac{t^2}{a^2}\left(2t - \frac{t^2}{a^2}\right)^{-1/2} ,$$

which is a value that is obviously non-zero. One further finds that when a and the square root are positive, and 2 is any point of the region considered:

$$u = \bar{J}_{02} = \int_0^x y\sqrt{1-y^2p^2}\,dx = \int_0^x \frac{y^2}{a}\,dx = \frac{x^2}{a} - \frac{x^3}{3a^3}, \tag{71}$$

and when one expresses dy in terms of dx and da by means of equation (69) and refers x', y', p to the extremals of the field, one has the equations:

$$F_{x'} = a, \qquad F_{y'} = -\,a\,y^2p, \qquad du = F_{x'}\,dx + F_{y'}\,dy.$$

It follows from this that when the point 2 runs from 1 to 3 along the curve $\mathfrak{L}_1$, one will have:

$$d\,(\bar{J}_{02} + J_{23}) = \mathcal{E}\,dt,$$

and from formula (58), when Dx and $\bar{p}$ refer to the line $\mathfrak{L}_1$, and p refers to the extremal, one will have:

$$\mathcal{E}\,dt = y\left\{\frac{1-y^2p\,\bar{p}}{\sqrt{1-y^2p^2}} - \sqrt{1-y^2\,\bar{p}^2}\right\}Dx,$$

in which the square root in the denominator is positive. Now for any direction that comes under consideration, the quantity:

$$y\frac{dy}{dx} = \frac{dy}{\sqrt{dy^2+dw^2}}$$

will lie between the limits ± 1 and will attain those limits only for $y = 0$, even on the extremals of the field. Therefore, as long as y does not vanish, $y^2p\,\bar{p}$ will be a proper fraction and the quantity $1 - y^2p\,\bar{p}$ will be negative. Since y and Dx are positive, the same thing will obviously be true of $\mathcal{E}\,d\tau$ when $\sqrt{1-y^2\bar{p}^2}$ is negative. If that quantity has a positive value then it will follow from the general inequality:

$$(1 - \alpha\,\beta)^2 \geq (1-\alpha^2)(1-\beta^2)$$

that $\mathcal{E}\,d\tau$ is never negative and vanishes only when $p = \bar{p}$, i.e., only in an ordinary way, since the differential dx that relates to the extremal is positive, just like Dx. If the equation:

$$1 - y^2 \bar{p}^2 = 0$$

exists then the integrand of the integral J will indeed be singular when expressed in terms of x, y. However, the original definition:

$$J = \int y^2 \frac{dw}{d\tau} d\tau = \int y \sqrt{1 - y^2 \bar{p}^2} \frac{dx}{d\tau} d\tau$$

shows that J_{23} will nonetheless remain a function $\varphi(\tau)$ in the sense of § **17**. The quantity dJ_{23} will vanish in that case. du has the expression:

$$du = a\,(1 - y^2 p\,\bar{p})\,dx\,,$$

which then remains positive. Therefore, when $\mathcal{E}$ does not vanish in an ordinary way everywhere along the curve $\mathfrak{L}_1$, i.e., when that curve does not coincide with one of the extremals (69), the quantity $\bar{J}_{02} + J_{23}$ will increase continually as long as the point 2 lies between 1 and 3. Of course, our argument will no longer be valid at those points. If the point 2 approaches the positions 1 and 3 then that quantity will converge to the limiting values J_{13} and $\bar{J}_{03}$, resp., because when the point 2 approaches a well-defined location on the parabola, formulas (69), (71) will imply that:

$$\lim \frac{x}{a} = 0\,, \qquad \lim u = \lim \bar{J}_{02} = 0\,.$$

From the behavior of the quantity $\mathcal{E}$, with consideration given to the relation (70) when $\mathfrak{L}_1$ is not one of the curves (69), one will get:

$$\bar{J}_{03} > J_{13}\,, \qquad \bar{J}_{03} > J_{013}\,,$$

i.e., the sphere has a larger volume than the body with the same surface area that is generated by rotating the curve $\mathfrak{L}$.

§ 24. – Sufficient conditions for an extremum.

If one goes along the curve $\mathfrak{C}$ starting from the point 0 in the direction of increasing t then let 6 be the first point at which the quantity Δ vanishes after one has left 0. We assume that this happens before the functions ξ and η lose their assumed properties along the curve $\mathfrak{C}$, such that 6 will still be a regular point of that curve in whose neighborhood a region $\mathfrak{G}'$ for the quantities t, a will be defined by the inequalities:

$$|t - t_6| < \varepsilon, \quad |a - a_0| < \varepsilon,$$

inside of which the functions ξ, η, $F(\xi, \eta, \xi_t, \eta_t)$ remain regular and the quantity $\xi_t^2 + \eta_t^2$ is non-zero. The corresponding points (x, y), which are determined by the equations:

$$x = \xi(t, a), \quad y = \eta(t, a),$$

no longer belong to the field of a part of the curve $\mathfrak{C}$ since they generally cover certain regions of the plane doubly. If the curve $\mathfrak{C}_0$, which is intersected transversally by the extremals of the field, begins at the point 0, as before, then we will call 6 the *extremal focal point* of the curve $\mathfrak{C}_0$ on the extremal $\mathfrak{C}$. If the curve $\mathfrak{C}_0$ contracts to the point 0, through which all extremals of the field then go, then 0 and 6 will be called *conjugate points*, or more precisely, *extremal conjugate points*. For example, according to **23**, the conjugate points in Problem II are characterized by the fact that their tangents intersect along the x-axis. The extremal focal point of an arbitrary curve $\mathfrak{C}_0$ will be defined by equation (65), which is easily discussed.

Some important properties of the conjugate and focal points are based upon the fact that the quantity Δ satisfies a linear differential equation that will be determined by the extremal $\mathfrak{C}$ alone. The geometric interpretation of the quantity Δ leads to that equation, which can be seen from the following consideration. Let:

$$x = \xi(t, a), \quad y = \eta(t, a),$$

(72)

$$\bar{x} = \xi(\tau, a + \delta a), \quad \bar{y} = \eta(t, a + \delta a)$$

be two neighboring extremals of the field and lay the point $(\bar{x}, \bar{y})$ on the normal to the first curve at the point (x, y). The direction cosines of that line are:

$$X = \frac{-y'}{\sqrt{x'^2 + y'^2}}, \qquad Y = \frac{x'}{\sqrt{x'^2 + y'^2}}.$$

If one takes the square roots to be positive then those quantities will relate to the direction that lies in the direction of increasing t in the same way that the $+ y$-axis relates to the $+ x$-axis. The equation of the normal is:

$$(\bar{x} - x)\, x' + (\bar{y} - y)\, y' = 0.$$

One can write equations (72) as:

$$\bar{x} - x - \{\xi_t(\tau - t) + \xi_a\, \delta a + [\tau - t, \delta a]_2\} = 0,$$
$$\bar{y} - y - \{\eta_t(\tau - t) + \eta_a\, \delta a + [\tau - t, \delta a]_2\} = 0,$$

in which the quantities ξ_t, ξ_a, … are taken with respect to the system of arguments t, a. The functional determinant of the left-hand sides of those three equations with respect to $\bar{x}$, $\bar{y}$, τ is:

$$\begin{vmatrix} x' & y' & 0 \\ 1 & 0 & -\xi_t \\ 0 & 1 & -\eta_t \end{vmatrix} = x'^2 + y'^2$$

when they are set to x, y, t, respectively, so it is non-zero. One can then develop the quantities $\bar{x} - x$, $\bar{y} - y$, $\tau - t$ in δa, and the linear terms will be the same as when the equations contain those three differences in only a strictly-linear way. It then follows that:

$$\bar{x} - x = \frac{-\eta_t\,\Delta\,\delta a}{\xi_t^2 + \eta_t^2} + [\delta a]_2\,, \qquad \bar{y} - y = \frac{\xi_t\,\Delta\,\delta a}{\xi_t^2 + \eta_t^2} + [\delta a]_2\,,$$

or when one sets:

(73) $$\omega = \frac{\Delta\,\delta a}{\sqrt{\xi_t^2 + \eta_t^2}}\,,$$

one will have:

$$\bar{x} - x = X\,\omega + [\delta a]_2\,, \qquad \bar{y} - y = Y\,\omega + [\delta a]_2\,.$$

ω is then the normal distance between the two extremals when δa is infinitely-small, which is taken to be positive or negative according to whether the direction to the point $(\bar{x}, \bar{y})$ does or does not coincide with the normal direction that was defined above, respectively.

Now the point $(\bar{x}, \bar{y})$ that is defined by equations (72) will describe an extremal for an arbitrarily-established value of δa. If the equations of one of them are:

$$P = 0, \quad Q = 0,$$

as in § **4**, one can substitute $\bar{x}, \bar{y}$ for x, y and differentiate with respect to δa. If one then sets $\delta a = 0$ then, e.g., the first equation will give:

$$\frac{\partial P}{\partial x}\frac{\partial \bar{x}}{\partial\,\delta a} + \frac{\partial P}{\partial x'}\frac{\partial \bar{x}'}{\partial\,\delta a} + \frac{\partial P}{\partial x''}\frac{\partial \bar{x}''}{\partial\,\delta a} + \frac{\partial P}{\partial y}\frac{\partial \bar{y}}{\partial\,\delta a} + \cdots = 0\,.$$

That result can also be expressed in the following way: One will have the equations:

$$\delta P = 0\,, \qquad \delta Q = 0$$

when one sets:

(74) $$\delta x = \omega\,X\,, \qquad \delta y = \omega\,Y\,.$$

As we would like to say, the latter equations define a *normal variation.* Namely, since one obviously has the equations:

$$\delta a \frac{\partial \bar{x}}{\partial\,\delta a} = \omega\,X\,, \qquad \delta a \frac{\partial \bar{y}}{\partial\,\delta a} = \omega\,Y\,,$$

up to the term $[\delta a]_2$, one can consider the operations:

$$\delta, \quad \delta a \frac{\partial}{\partial\,\delta a}$$

to be identical under the assumption (74). In order to perform the suggested calculations, we start from the identities:

$$P = F_x - \frac{dF_{x'}}{dt}\,, \qquad \delta P = \delta F_x - \frac{d\,\delta F_{x'}}{dt}\,.$$

The last one easily implies that:

$$\delta P = \left(F_{xx} - \frac{dF_{xx'}}{dt}\right)\delta x + \left(F_{xy} - \frac{dF_{x'y}}{dt}\right)\delta y - \frac{d}{dt}(F_{x'x'}\,\delta x' + F_{x'y'}\,\delta y') + (F_{x'x'} - F_{x'y'})\,\delta y'\,,$$

and one will find analogously that:

$$\delta Q = \left(F_{yx} - \frac{dF_{xy'}}{dt}\right)\delta x + \left(F_{yy} - \frac{dF_{yy'}}{dt}\right)\delta y - \frac{d}{dt}(F_{y'x'}\,\delta x' + F_{y'y'}\,\delta y') + (F_{yx'} - F_{xy'})\,\delta x'\,.$$

One now introduces the quantity F_1 that was defined in § **16** and considers the fact that for the normal variation (74) one will have:

$$y'\,\delta x' - x'\,\delta y' = -\,\omega'\sqrt{x'^2 + y'^2}\,.$$

If one then sets:

$$F^1 = F_1(x'^2 + y'^2)\,,$$

$$F_2 = \left(F_{xx} - \frac{dF_{xx'}}{dt}\right)X^2 + \left(2F_{xy} - \frac{dF_{x'y}}{dt} - \frac{dF_{y'x}}{dt}\right)X\,Y + \left(F_y - \frac{dF_{yy'}}{dt}\right)Y^2 + (F_{xy'} - F_{yx'})(X\,Y' - Y\,X')$$

then that will give:

$$X\,\delta P + Y\,\delta Q = F_2\,\omega - X\frac{d}{dt}(F^1 X\,\omega') - Y\frac{d}{dt}(F^1 Y\,\omega')\,,$$

with the values of δQ and δQ that were written down. When one considers the identities:

$$X^2 + Y^2 = 1 , \qquad X X' + Y Y' = 0 ,$$

it will follow that:

(75) $$X\,dP + Y\,dQ = F_2\,\omega - \frac{d}{dt}\left(F^1 \frac{d\omega}{dt}\right) ,$$

and that is purely an identity under the assumption (74) and with arbitrary values of ω that we shall employ later on.

If one takes ω to have the value (73) then that will yield:

$$X\,dP + Y\,dQ = 0, \qquad F_2\,\omega - \frac{d}{dt}\left(F^1 \frac{d\omega}{dt}\right) = 0 .$$

Therefore, when:

$$x = \xi(t, a) , \quad y = \eta(t, a)$$

along the piece of the extremal that is being considered (say, along the arc 06), the quantity F_1 will be non-zero, so ω will satisfy a linear differential equation whose coefficients will be regular functions of t from t_0 to t_1 when one writes them in the form:

$$\omega'' + M\,\omega' + N\,\omega = 0 .$$

When the quantity ω does not vanish identically, its derivative with respect to t cannot vanish either, and as a result of equation (73), the same will be true of Δ, such that one will have:

$$\Delta t\,(t_6, a_0) \neq 0 ,$$

in any event.

§ 25. – Envelope of extremals near conjugate and focal points.

The inequality that was obtained shows that one can solve the equation:

(76) $$\Delta(t, a) = 0 ,$$

which is satisfied by the system of values t_6 , a_0 , for t in the neighborhood of those values. If one writes that equation in the form:

$$\Delta_t(t_6 , a_0)\,(t - t_6) + \Delta_a(t_6 , a_0)\,(a - a_0) + [a - a_0 , t - t_0]_2 = 0$$

then it is clear that one will get from it that:

$$t - t_6 = -\left.\frac{\Delta_a}{\Delta_t}\right|^6 (a - a_0) + [a - a_0]_2 .$$

If one substitutes that value in the developments of ξ and η for a neighborhood of the location 6 then a curve will be defined by the equations:

$$(77) \qquad \begin{aligned} x &= \xi(t, a) = x_6 + [a - a_0]_1 , \\ y &= \eta(t, a) = y_6 + [a - a_0]_1 \end{aligned}$$

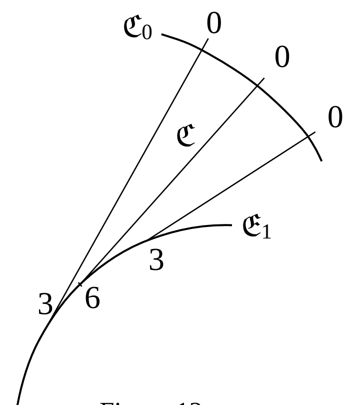

Figure 12.

that includes the point 6 and is denoted by $\mathfrak{C}$ (Fig. 12). It is regular at all points that correspond to sufficiently-small, but non-vanishing, values of $| a - a_0 |$. It can exhibit a cusp at the point 6 itself or also, in particular, contract to that point completely. The latter happens in the case where the power series $[a - a_0]_1$ in equations (77) vanishes identically. The increments in the coordinates while advancing along that curve have the following expressions:

$$Dx = \frac{dx}{da}\,da = \left(\xi_a + \xi_t \frac{dt}{da}\right) da = \frac{\xi_t\,\Delta_a - \xi_a\,\Delta_t}{-\Delta_t}\,da ,$$

$$Dy = \frac{dy}{da}\,da = \left(\eta_a + \eta_t \frac{dt}{da}\right) da = \frac{\eta_t\,\Delta_a - \eta_a\,\Delta_t}{-\Delta_t}\,da .$$

Now since ξ_t and η_t do not vanish simultaneously, equation (76) can be written in one of the forms:

$$\eta_a = \frac{\eta_t\,\xi_a}{\xi_t} , \qquad \xi_a = \frac{\xi_t\,\eta_a}{\eta_t} ,$$

and that will imply one of the two equations:

$$(78) \qquad \begin{aligned} Dy &= \frac{\eta_t}{\xi_t}\left(\xi_a + \xi_t \frac{dt}{da}\right) da = \frac{\eta_t}{\xi_t}\,Dx , \\ Dx &= \frac{\xi_t}{\eta_t}\left(\eta_a + \eta_t \frac{dt}{da}\right) da = \frac{\xi_t}{\eta_t}\,Dy . \end{aligned}$$

At each of its points that are sufficiently close to 6, the curve $\mathfrak{C}$ will then contact the extremal of the field that belongs to the same value of a as the point considered. The curve will then be the envelope of the extremals of the field, and the existence of such a thing will be verified under the assumptions that were introduced.

One now approaches the point 6 along the curve $\mathfrak{E}$ that belongs to $a = a_0$ when one increases a by da and assumes that:

(79) $$(a - a_0)\, da < 0 .$$

If ξ_t does not vanish at the point 6, moreover, and:

(80) $$\xi_t \left.\frac{dx}{da}\right|^6 (a - a_0) < 0 , \quad \left|\left|\frac{dx}{da}\right|\right|^6 > 0$$

then under the assumption (79), one will have:

$$\xi_t \left.\frac{dx}{da}\right|^6 da > 0 .$$

Therefore, from the sign on the quantity $a - a_0$ that is determined from the relation (80), the quantities ξ_t and Dx will have the same signs, and the advance along the curve $\mathfrak{E}$ that points away from the point 6 will coincide with the direction of increasing t along the contacting extremal. We shall call the half of the curve $\mathfrak{E}$ for which the inequalities (80) are true $\mathfrak{E}_1$. From (78), we will have the two equations:

(81) $$Dx = \alpha \cdot \xi_t\, dt = \alpha\, dx ,$$
$$Dy = \alpha \cdot \eta_t\, dt = \alpha\, dy$$

along it, in which α is positive. The existence of such an arc $\mathfrak{E}_1$ will be doubtful only in the case:

$$\left.\frac{dx}{da}\right|^6 = 0 , \qquad \xi_t\, \Delta_a - \xi_a\, \Delta_t \,\Big|^6 = 0 ,$$

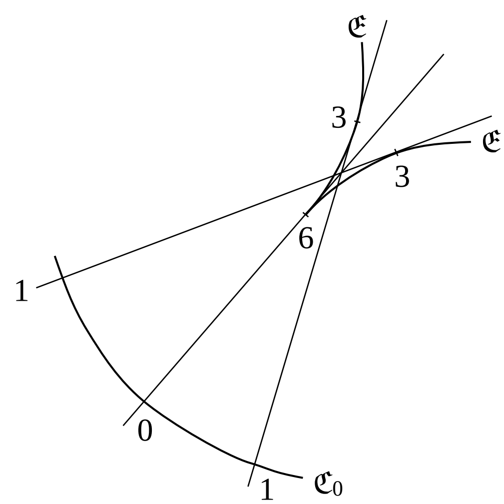

Figure 13.

in which we also have the equation:

$$\Delta_t \left.\frac{dy}{da}\right|^6 = \eta_t\, \Delta_a - \eta_a\, \Delta_t \,\Big|^6 = \eta_t \left(\Delta_a - \frac{\Delta_t\, \xi_a}{\xi_t}\right)\Bigg|^6 = 0 .$$

The curve $\mathfrak{E}$ will possibly have a cusp at the point 6 then (Fig. 13). It is only when both branches at the point 6 advance in the direction that corresponds to increasing values of *t* along the curve $\mathfrak{E}$ that the arc $\mathfrak{E}_1$ will exist. We shall refer to that as the *exceptional* case.

As before, let 0 be a point on the curve $\mathfrak{C}_0$ and let 3 be a point that lies with it along the same extremal of the field to which the arguments *a* and *t* refer. The quantity:

$$u = \bar{J}_{03} = \int_0^3 F(\xi, \eta, \xi_t, \eta_t)\, dt$$

is not only in the field then, as we know, but is also a regular function of *a* and *t* in the region $\mathfrak{G}'$ in which the quantities $\xi, \eta, F(\xi, \eta, \xi_t, \eta_t)$ remain regular. The equations:

$$\frac{\partial u}{\partial a} = F_{x'}(\xi, \eta, \xi_t, \eta_t)\, \xi_a + F_{y'}(\xi, \eta, \xi_t, \eta_t)\, \eta_a\,, \qquad \frac{\partial u}{\partial t} = F(\xi, \eta, \xi_t, \eta_t)\,,$$

which are first derived for the field, are therefore also true in the region $\mathfrak{G}'$, since their right-hand sides, as well as their left, remain regular functions of *a* and *t* in it. In particular, let 3 be a point of the curve $\mathfrak{E}$. When one advances along it, one will have:

$$\begin{aligned} du &= F_{x'}(\xi, \ldots, \eta_t)\,(\xi_a\, da + \xi_t\, dt) + F_{y'}(\xi, \ldots, \eta_t)\,(\eta_a\, da + \eta_t\, dt) \\ &= F_{x'}(\xi, \ldots, \eta_t)\, Dx + F_{y'}(\xi, \ldots, \eta_t)\, Dy\,. \end{aligned}$$

On the other hand, if one defines the integral:

$$J_{36} = \int_3^6 F(x, y, Dx, Dy)\ ,$$

such that the symbol *D* corresponds to the motion along the curve $\mathfrak{E}$ that points away from the point 6. One will then have:

$$d\bar{J}_{36} = -F(x, y, Dx, Dy) = -F(\xi, \eta, Dx, Dy)\,.$$

That quantity will be opposite to the one that is obtained for *du* when one can apply the homogeneity equations:

$$\begin{aligned} F_{x'}(\xi, \eta, \xi_t, \eta_t) &= F_{x'}(\xi, \eta, Dx, Dy)\,, \\ F_{y'}(\xi, \eta, \xi_t, \eta_t) &= F_{y'}(\xi, \eta, Dx, Dy)\,, \end{aligned}$$

so when equations (81) are valid and α is positive, such that the point 3 belongs to the arc $\mathfrak{E}_1$. In that case, one has the equations:

$$F_{x'}(\xi, \ldots, \eta_t)\,Dx + F_{y'}(\xi, \ldots, \eta_t)\,Dy = F_{x'}(\xi, \eta, Dx, Dy)\,Dx + F_{y'}(\xi, \eta, Dx, Dy)\,Dy$$
$$= F(\xi, \eta, Dx, Dy)\,,$$

$$du + d\bar{J}_{36} = d(\bar{J}_{03} + \bar{J}_{36}) = 0\,.$$

Now since the equation:

$$\lim\,(\bar{J}_{03} + \bar{J}_{36}) = \bar{J}_{06}$$

will obviously be valid when the distance between the points 3 and 6 becomes infinitely small, and therefore the extremals 03 and $\mathfrak{C}$ will coincide, it will generally follow that:

$$\bar{J}_{03} + J_{36} = \bar{J}_{06}\,. \tag{82}$$

If the curve $\mathfrak{E}$ contracts to the point 6 then one will obviously get:

$$J_{36} = 0\,.$$

The values of $\bar{J}_{06}$ will then be equal on the various extremals of the field.

Now the curve 036, which consists of the extremal arc 03 and the piece 36 that belongs to the curve $\mathfrak{E}_1$, can be regarded as a narrow neighborhood of the arc 06, because as long as the distance 36 is sufficiently small, the tangents to the arc 36 will differ arbitrarily little from the one at the point 6, and the tangent to the extremal 03, which goes continuously along the arc 06, will differ arbitrarily little from the tangent to the latter. The arc 036 will then belong to the curve $\mathfrak{L}$ to which one must compare the arc 06 if it is to also make the integral J become only a weak extremum in comparison to the other lines that connect the curve $\mathfrak{C}_0$ to the point 6. The difference:

$$\bar{J}_{06} - J_{036} = \bar{J}_{06} - (\bar{J}_{03} + \bar{J}_{36})$$

must then have a fixed sign without vanishing, while, from (82), it must have the value zero. The extremal arc 06 does not yield an extremum for the integral J then, not even a weak one. Rather, the extremum ceases to exist at the point 6 in the way that was indicated by equation (82). Therefore, if one assumes the existence of the field, the following result has been proved:

If the regular curve $\mathfrak{C}_0$ is intersected transversally by the extremal $\mathfrak{C}$ at the point 0 and the latter runs through the point 5 then the arc will cease to give a weak extremum for the integral J to the lines that go from the point 5 to the curve $\mathfrak{C}_0$ as soon as the point 5 moves to the focal point of the curve $\mathfrak{C}_0$. The same thing will be true when the curve $\mathfrak{C}_0$ contracts to a point 0 for the extremum over the curves that connect the fixed points 0 and 5 as soon as the point 5 goes over to the point that is conjugate to 0. An exception will exist only when the envelope of the curves of the field (which always exists) possesses a cusp of a special type at the focal point. In that case, our argument will show that in an arbitrary neighborhood of the extremal segment that is delimited by two conjugate points, there will be other ones that have the same starting point that the extremal has, and that will happen in such a way that it can no longer yield an extremum. For example, in Fig. 13, one obviously has:

$$\bar{J}_{13} = \bar{J}_{06} + \bar{J}_{63} .$$

§ 26. – Examples. Infinitesimal transformations in fields.

Example: For the rectilinear elastic oscillation of a material point, **Hamilton**'s principle will lead to the integral:

$$J = \int dx\,(p^2 - y^2)\,,$$

whose extremal will be determined by the simple equation:

$$\frac{dp}{dx} + y = 0 .$$

The extremals:

$$y = a \sin x$$

all go through the fixed point 0 ($x = y = 0$), to which we shall imagine that the curve $\mathfrak{C}_0$ contracts. If we set $x = t$ then we will get:

$$\Delta = \frac{\partial(x, a \sin x)}{\partial(x, a)} = \sin x .$$

For the point 6 that is conjugate to the point 0, we will then have $x = \pi$, so x and $a \sin x$ are obviously regular functions in its vicinity. The curve $\mathfrak{C}$ contracts to the point 6 since all extremals of the family in question go through it. The integrals J that are obtained along them have the same value:

$$J_{06} = \int_0^\pi a^2\,dx\,(\cos^2 x - \sin^2 x) = \left.\frac{a^2 \sin 2x}{2}\right|_0^\pi = 0 .$$

The case occurs for a great circle on the sphere as the shortest lines. Every point is conjugate to its diametrical opposite one. Along the geodetic lines that start from an umbilic point of a three-axis ellipsoid, the starting point will be conjugate to the diametrical opposite one, and the two geodetic arcs that connect them will have equal length.

The simplest example of the general equation (82) is provided by the basic property of the evolute that its arcs measure the difference between the endpoints of the corresponding radii of curvature of the evolvent. One will get that theorem when one applies equation (82) to Problem I and considers the evolvent to be the curve $\mathfrak{C}_0$.

Problem VI (§ 11). – A similarity transformation whose center lies along the axis and has the abscissa x_0 will be defined by the equations:

$$\bar{y} = \alpha y , \qquad \bar{x} - x_0 = \alpha (x - x_0) , \quad \text{or} \qquad \bar{x} = \alpha x - (\alpha - 1) x_0 .$$

If one has then represented an extremal by the equations:

$$y = \frac{a(1+p^2)^2}{2p^3}, \qquad x = b + \frac{a}{2}\left(\frac{3}{4p^4} + \frac{1}{p^2} + \ln p\right)$$

then the point $(\bar{x}, \bar{y})$ will describe an extremal in any event. An arbitrary tangent to one of them will have the equation:

$$Y - y = p (X - x)$$

when X, Y are the running coordinates, so it will cut the x-axis at a point T whose abscissa is:

$$X = x - \frac{y}{p}.$$

Now since one then has:

$$\frac{dX}{dp} = \frac{y}{p^2},$$

and the values $p = 0$, $p = +\infty$ yield the corresponding values $-\infty$, $+\infty$ for X, the point T will traverse the x-axis precisely once in the positive sense when a, and therefore y, is positive, while the contact point of the tangent considered (§ **11**) will then run through the entire curve when p increases through all positive values.

Now let a line $\mathfrak{C}_0$ (Fig. 14) be inclined at an acute angle ψ with respect to the $+ x$-axis. It will then be intersected transversally by an extremal $\mathfrak{C}$ at the point 0 when the equation:

$$\tan \psi = \frac{2p}{3+p^2} = \frac{p}{1+\frac{1}{2}(1+p^2)}$$

is true for it, which obviously implies that:

$$\tan \psi < p . \tag{83}$$

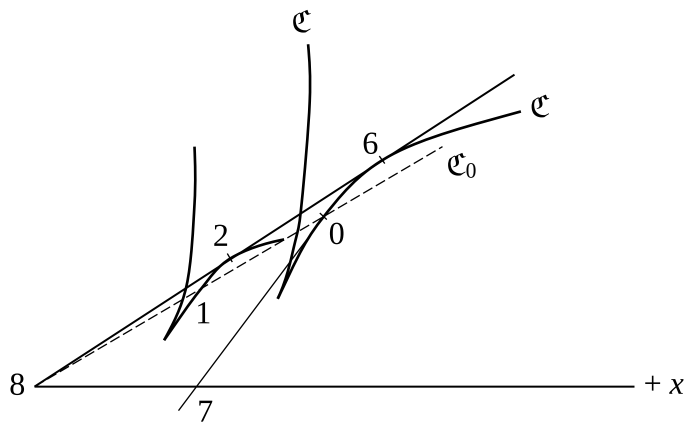

Figure 14.

Any given value of tan ψ is associated with two real values of p whose product will be 3 as long as the inequality:

$$\tan \psi < \frac{1}{\sqrt{3}}, \qquad \text{so} \qquad \psi < 30^{\circ}$$

is valid. Either of those values that we employ to construct $\mathfrak{C}$ will then be less than $\sqrt{3}$ and belong to a point of the branch of the extremal whose segment yields a minimum for the resistance (§ **18**). Furthermore, if 7 is the intersection point of the tangent to the extremal at the point 0 with the x-axis and 8 is the intersection point of it with the line $\mathfrak{C}_0$ then, from (83), the point 7 will lie to the right of 8, i.e., on the positive side. Therefore, if a point on the extremal that was constructed runs from 0 in the direction of increasing x, so decreasing p, then the point T will move from the position 7 towards $-\infty$, and thus arrive at the position 8 in between. Let the contact point of the tangent to the curve $\mathfrak{C}$ that goes through 8 be 6.

The point 8 now defines the center of a family of similarity transformations. They move the curve $\mathfrak{C}$ in a family of extremals that intersect the line $\mathfrak{C}_0$ transversally and contact the tangent 68. The latter then represents the envelope $\mathfrak{C}$ of the general theory, where 6 is the focal point of the line $\mathfrak{C}_0$ along the curve $\mathfrak{C}$, and $\mathfrak{E}_1$ is the segment 68. If 2 is any point of the segment and if the extremal that contacts it intersects the family of lines $\mathfrak{C}_0$ that was constructed at the point 1 then the equation:

$$\bar{J}_{12} + J_{26} = \bar{J}_{06}$$

will be true for the resistance integral, in which J_{26} is defined along the line 68. If one lets the point 5 run along the curve $\mathfrak{C}$ then the arc 05 will give a weak minimum for the resistance compared to the curve that is drawn from the point 5 to the line $\mathfrak{C}_0$ as long as the point 5 belongs to the arc 06. If it shifts to the position 6 then the equation above will show how the minimum property can already be lost for the arc 06.

The method that was applied to Problem VI in order to construct and delimit a field can be adapted to any problem in which the extremals of a family can go to each other under a known continuous group of transformations. The boundary of the field lies where the so-called trajectories of the transformation contact the extremals of the family.

§ 27. – Surrounding regular systems of solutions.

The fact that extremals can be surrounded by fields, at least piece-wise, is mostly quite obvious in the individual problems. However, in order to be able to obtain general theorems about the situation, we shall give a function-theoretic analysis of a general character that we shall also make use of later on.

For the differential equations:

(84) $$\frac{dy}{dx} = f(x, y, z, \ldots), \quad \frac{dz}{dx} = g(x, y, z, \ldots), \quad \ldots,$$

which are n in number, like the number of unknowns $y, z, \ldots$, let the right-hand sides be regular in the neighborhood of the location:

(85) $$x = x_0, \qquad y = Y_0, \qquad z = Z_0, \qquad \ldots$$

It is known that there will then exist a system of integrals whose terms will assume the arbitrarily-prescribed values:

$$y = y_0, \qquad z = z_0, \qquad \ldots$$

as long as the quantities:

$$|y_0 - Y_0|, \qquad |z_0 - Z_0|, \qquad \ldots$$

lie below a certain limit. One will get series developments for those integrals when one differentiates equations (84) with respect to x and represents everything on the right-hand sides in terms of functions of $x, y, z, \ldots$ alone while eliminating the differential quotients. If one sets $x, y, z, \ldots$ equal to the values $x_0, y_0, z_0, \ldots$, resp., in the expressions thus-obtained, which are likewise regular at the location (85), then one will get the coefficients of the **Taylor** series developments of those well-defined solutions $y, z, \ldots$ in powers of $x - x_0$. As was shown in the proof of the existence of the integral, the absolute values of those coefficients are smaller than the absolute values of the corresponding ones for a certain convergent power series in the argument $x - x_0$ that is independent of the particular choice of the quantities $y_0, z_0, \ldots$ If one regards the latter as variable then the series that are obtained for $y, z, \ldots$ will converge uniformly in a certain region. Their values for a certain argument x will then be representable as power series in $y_0 - Y_0, z_0 - Z_0, \ldots$ Obviously one can set:

$$y = y_0 + (x - x_0)\,[x - x_0, y_0 - Y_0, z_0 - Z_0, \ldots]_0,$$
$$z = z_0 + (x - x_0)\,[x - x_0, y_0 - Y_0, z_0 - Z_0, \ldots]_0, \quad \ldots,$$

such that the functional determinant:

$$\frac{\partial(y, z, \ldots)}{\partial(y_0, z_0, \ldots)}$$

will have the value 1 for $x = x_0$. The values of $y, z, \ldots$ are then mutually-independent as functions of the quantities $y_0, z_0, \ldots$ for any chosen argument x.

Now let the quantities $Y, Z, \ldots$ be defined to be regular functions of x that satisfy the equations (84) in a real interval $\mathfrak{J}$ that is bounded below by the value x_0 . Let:

$$Y = Y_0 , \qquad Z = Z_0 , \qquad \ldots$$

for them at $x = x_0$ while:

$$Y = Y_1 , \qquad Z = Z_1 , \qquad \ldots$$

for $x = x_1$, and every system of values $(x, Y, Z, \ldots)$ in the interval $\mathfrak{J}$ will belong to the ones in whose neighborhood the functions $f, g, \ldots$ are regular. As a result of the argument that was developed, there will then be positive values δ, ε with the property that a system of integral that is regular in the interval:

$$x_0 \leq x \leq x_0 + \varepsilon \tag{86}$$

will exist that assumes the prescribed values $y_1, z_1, \ldots$ for $x = x_1$ when x_1 means any location in that interval and the inequalities:

$$| y_1 - Y_1 | < \delta, \qquad | z_1 - Z_1 | < \delta, \qquad \ldots$$

are satisfied. The values of $y, z, \ldots$ at any well-defined location in the interval (86) are then regular, mutually-independent functions of $y_1, z_1, \ldots$ in the neighborhood of the system of values $(Y_1, Z_1, \ldots)$. Furthermore, let γ be the upper limit of all possible values of ε. The quantity:

$$\bar{x} = x_0 + \gamma$$

cannot lie in the interior of the interval $\mathfrak{J}$ then, because if that were true then one could determine positive values $\bar{\varepsilon}$, $\bar{\delta}$ such that the quantity $\bar{\delta}$ would have the same meaning for the interval:

$$-\bar{\varepsilon} < x - \bar{x} < \bar{\varepsilon} \tag{87}$$

that δ has for the region (86). However, since the latter can get arbitrarily close to the point $\bar{x}$, there will be locations x_1 in it that likewise belong to the region (87). Therefore, if δ_1 is the lesser of the quantities $\delta, \bar{\delta}$ and one assumes that:

$$| y_1 - Y_1 | < \delta_1 , \qquad | z_1 - Z_1 | < \delta_1 , \qquad \ldots$$

then a system of integrals will be defined by each system of values $y_1, z_1, \ldots$ that satisfy those inequalities that is regular in both regions (86), (87), so in the entire region:

$$x_0 \leq x \leq \bar{x} + \bar{\varepsilon} ,$$

and is so arranged that its values at a certain location will be regular, mutually-independent functions of the quantities $y_1, z_1, \ldots$ Therefore, if one sets:

$$\varepsilon_1 = \gamma + \bar{\varepsilon} , \qquad \bar{x} + \bar{\varepsilon} = x_0 + \varepsilon_1$$

then the quantity δ_1 will have the same meaning for the regions:

$$0 \leq x - x_0 < \varepsilon_1$$

that δ has for the region (86). However, that contradicts the concept of an upper bound, since $\varepsilon_1 > \gamma$. The upper bound of all values $x_0 + \varepsilon$ then lies outside the interval $\mathfrak{J}$ or at its limit. The interval (86) can then be identified with any other one that reaches arbitrarily close to x_0 at the limit of the interval $\mathfrak{J}$.

With that, it has been shown that the integral manifold $(x, Y, Z, \ldots)$ can be surrounded by neighboring ones in the following way: There will be integrals:

$$y = \Phi(x, a, b, \ldots, k) , \qquad z = \Psi(x, a, b, \ldots, k) , \qquad \ldots,$$

which includes n constants $a, b, \ldots, k$, and for a certain special system of values:

$$a = A , \qquad b = B , \qquad \ldots, \qquad k = K ,$$

they will go to the integrals $Y, Z, \ldots$, which will be regular functions of their $n + 1$ arguments, moreover, as long as x belongs to the interior of the interval $\mathfrak{J}$, but the differences:

$$| a - A | , \qquad | b - B | , \qquad \ldots, \qquad | k - K |$$

are sufficiently small. Under those assumptions, the functional determinant:

$$\frac{\partial(\Phi, \Psi, \ldots)}{\partial(a, b, \ldots)}$$

will be non-zero.

§ 28. – The second variation.

If the integrand in the integral:

$$J + \Delta J = \int F\,(x + \delta x, y + \delta y, x' + \delta x', y' + \delta y')\,dt$$

is developed in a **Taylor** series then the integral of the double sum of all terms that are quadratic in the variations will be:

$$\int dt\,\{F_{xx}\,\delta x^2 + 2F_{xy}\,\delta x\,\delta y + \cdots + F_{x'x'}\,\delta x'^2 + \cdots\}$$

That expression arises from the variation:

$$\delta J = \int (F_x\,\delta x + F_y\,\delta y + F_{x'}\,\delta x' + F_{y'}\,\delta y')\;dt$$

when one applies the operation δ according to the rules of differentiation and regards t, δx, $\delta x'$, δy, $\delta y'$ as constants under that operation. One accordingly denotes that expression by $\delta^2 J$ and calls it the *second variation* of J. If the variations δx, δy include a constant factor ε then one will obviously have:

(88) $$\Delta J = \delta J + \tfrac{1}{2}\delta^2 J + [\varepsilon]_3\,,$$

and $\delta^2 J$ will then include the factor ε^2. Now one can set:

$$\delta J = F_{x'}\,\delta x + F_{y'}\,\delta y + \int dt\,(P\,\delta x + Q\,\delta y)\,,$$

so it will follow that:

$$\delta^2 J = \delta F_{x'}\,\delta x + \delta F_{y'}\,\delta y + \int dt\,(\delta P\,\delta x + \delta Q\,\delta y)\,,$$

since the symbol δ can be switched with integration over t, which is a formula that is very easy to verify by calculation. If one has:

$$\delta x\,|^0 = \delta x\,|^1 = \delta y\,|^0 = \delta y\,|^1 = 0\,,$$

in particular, in this case then one will get:

$$\delta^2 J_{01} = \int_0^1 dt\,(\delta P\,\delta x + \delta Q\,\delta y)\,.$$

That formula can be applied to a normal variation, which is defined by the equations:

$$\delta x = \omega X , \qquad \delta y = \omega Y ,$$

in which X, Y have the same meaning that they had in § **24** and ω vanishes at the locations 0 and 1. From the identity that was derived there, one then has:

(89) $$\delta^2 J_{01} = \int_0^1 \omega(X\,\delta P + Y\,\delta Q)\,dt = \int_0^1 \omega\left(F_2\,\omega - \frac{d\,(F^1\omega')}{dt}\right) dt ,$$

or after a partial integration:

(90) $$\delta^2 J_{01} = \int_0^1 dt\,(F^1\,\omega'^2 + F_2\,\omega^2) .$$

Now let 0 and 1, in turn, be conjugate points. The arc 02 reaches beyond the latter one. From § **24**, the equation:

(91) $$F_2\omega - \frac{d\,(F^1\omega')}{dt} = 0$$

will then have an integral that vanishes at the points 0 and 6 that will change sign at the point 6. Let the arc 02 be delimited in such a way that 0 and 6 are the only zeroes of the integral ω that belong to it. If ε is a sufficiently-small constant then and F^1 is non-zero along the entire arc 02 then, from § **27**, the equation:

$$F_2 - \varepsilon)\,w - \frac{d\,(F^1 w')}{dt} = 0$$

will have an integral that is regular along the segment 02, like ω, satisfies the equation:

$$w\,|^0 = \omega\,|^0 = 0 ,$$

and likewise possesses a single zero 7 between 0 and 2 that coincides with 6 when $\varepsilon = 0$. From the general formula (89), with the normal variation:

$$\delta x = X\,w\,\alpha , \quad \delta y = Y\,w\,\alpha ,$$

in which α is constant, one will have the formula:

$$\delta^2 J_{07} = \alpha^2 \int_0^7 w\left(F_2\,w - \frac{d(F^1 w')}{dt}\right) dt = \alpha^2\,\varepsilon \int_0^7 w^2\,dt .$$

Therefore, if one sets:

$$\delta x = \delta y = 0$$

in the interval 72 then since the point 7 belongs to the interior of the interval 02 for a sufficiently-small ε, one will have:

$$\delta^2 J_{02} = \delta^2 J_{07} = \alpha^2 \varepsilon \int_0^7 w^2\, dt\,,$$

and since the varied curve is an extremal, from (88), one will have:

$$\delta J_{02} = 0\,, \qquad \Delta J_{02} = \tfrac{1}{2}\delta^2 J_{07} + [\alpha]_3 = \tfrac{1}{2}\alpha^2 \varepsilon \int_0^7 w^2\, dt + [\alpha]_3\,.$$

As long as α is sufficiently small, that quantity will obviously have the same sign as ε, so it can be positive, as well as negative, such that arc 02 can certainly yield no extremum of the integral J.

That argument, whose basic ideas go back **Erdmann** and **Weierstrass**, differs from the ones that were given in §§ **24**, **25** by the fact that, in general, the latter ideas already show that an extremum will not exist for an arc that is delimited by two conjugate points, but it excludes an exceptional case. The argument that was just developed also extends to the exceptional case, but always demands that the arc considered must reach beyond the point that is conjugate to its starting point.

In the event that the arc 01 does not include a pair of conjugate points and F^1 does not vanish along it, formula (90) will tell us that the sign of the second variation agrees with those of the quantities F_1 or F^1. Namely, when u is a regular function t from t_0 to t_1, one will obviously have:

$$\int_0^1 (u'\,\omega^2 + 2u\,\omega\,\omega')\, dt = \int_0^1 \frac{d}{dt}(u\,\omega^2)\, dt = u\,\omega^2\Big|_0^1 = 0\,,$$

or

$$\delta^2 J_{01} = \int_0^1 dt\,[F^1\,\omega'^2 + 2u\,\omega\,\omega' + (F_2 + u')\,\omega^2]\,, \tag{92}$$

and in that expression, the integrand will have the same sign as F_1 when one can determine u from the equation:

$$(F_2 + u')\,F^2 - u_2 = 0\,. \tag{93}$$

One will then have simply:

$$\delta^2 J_{01} = \int_0^1 F^1 \left(\omega' + \frac{u\,\omega}{F^1}\right) dt\,. \tag{94}$$

However, equation (93), like any equation of the form:

$$u' = L\,u^2 + M\,u + N\,,$$

can be converted into a second-order linear differential equation. When one sets:

$$u = -\frac{1}{L}\frac{\omega'}{\omega},$$

one will get:

$$\omega'' - \left(M + \frac{L'}{L}\right)\omega' + L\,N\,\omega = 0 .$$

One gets to equation (91), which is satisfied by the quantity $\Delta\,(t, a) : \sqrt{\xi_t^2 + \eta_t^2}$ precisely from equation (93) in that way. With the assumption that was introduced, that will be non-zero along the arc 01, so it will yield a quantity u that is regular in the integration interval, namely:

$$u = -\frac{F^1\sqrt{\xi_t^2 + \eta_t^2}}{\Delta(t,a)}\frac{d}{dt}\left(\frac{\Delta(t,a)}{\sqrt{\xi_t^2 + \eta_t^2}}\right).$$

Therefore, when one knows a family of extremals, one can exhibit the formula (94) explicitly for an arc with the assumed character and a normal variation.

One will get a proof of the existence of the extremum with respect to all curves that emerge from $\mathfrak{C}$ by a normal variation and belong to a narrow neighborhood when one replaces equation (93), in which one understands ε to mean a sufficiently-small constant, with the following one:

$$(F_2 + u')\,F^1 - u^2 = \varepsilon^2 .$$

From § **27**, that will have a regular integral when ε is sufficiently small, just like the one in that section. If one introduces that formula into the formula (92) then the quadratic form under the integral sign will be well-defined, and the formula:

$$\Delta J_{01} = \tfrac{1}{2}\delta^2 J_{01} + \int_0^1 [\omega, \omega']_3 \, dt$$

will easily imply that the extremum actually occurs.

§ 29. – Applying § 27 to the equations of extremals.

If 02 is any piece of a curve that is regular in the neighborhood of its points (§ **1**) then x and y can always be represented as regular, single-valued functions of a parameter t along it, e.g., the arc-length, as measured 0 to 2. That will be defined by the equations:

$$\frac{dt}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,, \qquad \frac{dt}{dy} = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,,$$

in which each of the square roots have the same sign as the differentials in their denominators when one advances in the direction 02. Namely, if the quantity $dy : dx$ is finite at the location 1, so y is a regular function of x, then the same thing will be true of dt / dx, and that quantity will be non-zero:

$$\frac{dt}{dx} = \left.\frac{dt}{dx}\right|^1 + [x - x_1]_1\,, \qquad t - t_1 = \left.\frac{dt}{dx}\right|^1 (x - x_1) + [x - x_1]_2\,,$$

but the second equation will imply:

$$x - x_1 = [t - t_1]_1\,,$$

such that x and y are regular function of t at the location 1. Naturally, one will get the same result from a similar development when $dx : dy$ is finite, and therefore x can be represented as a regular function of y. The convention on the sign of the square roots implies that dt will always be positive when one advances in the direction 02, such that different points of the arc will belong to different values of t.

Now if the arc 02 defines a piece of an extremal and it defines only the system of values (x, y, x', y') in whose neighborhood the integrand F is regular then one will have the equations:

$$\text{(95)} \qquad \begin{aligned} F_x - F_{x'x'}\,x' - F_{x'y}\,y' - F_{x'x'}\,x'' - F_{x'y'}\,y'' &= 0\,, \\ x'\,x'' + y'\,y'' &= 0\,. \end{aligned}$$

$$\text{(96)} \qquad \begin{aligned} F_y - F_{y'x'}\,x' - F_{y'y}\,y' - F_{y'x'}\,x'' - F_{y'y'}\,y'' &= 0\,, \\ x'\,x'' + y'\,y'' &= 0\,. \end{aligned}$$

x'', y'' can be calculated as functions of x, y, x', y' from each of them, and will take the form of fractions whose denominators are regular functions of t along the arc 02 and whose numerators are the determinants:

$$\begin{vmatrix} F_{x'x'} & F_{x'y'} \\ x' & y' \end{vmatrix}\,, \qquad \begin{vmatrix} F_{y'x'} & F_{y'y'} \\ x' & y' \end{vmatrix}\,,$$

and when one considers the identities that were derived in § **16**, they can be put into the following forms:

$$F_1\,y'(x'^2 + y'^2) = F_1\,y' = \frac{\bar{f}_{qq}}{y'^2}\,, \qquad F_1\,x'(x'^2 + y'^2) = F_1\,x' = \frac{f_{qq}}{x'^2}\,.$$

If one once more introduces the restriction that was valid in § **17** that the quantity F_1 must be non-zero for any line element of the arc 02, by which, it is possible that certain singular solutions of equations (95), (96) will be excluded, then one will get expressions for x'' and y'' when one uses one or the other system (95), (96) as a basis that are regular in the neighborhood of any system of values (x, y, x', y') that is defined by an element of the arc 02. If one writes:

$$\frac{dx'}{dt} = M(x, y, x', y'), \qquad \frac{dy'}{dt} = N(x, y, x', y')$$

for those expressions and adds the equations:

$$\frac{dx}{dt} = x', \qquad \frac{dy}{dt} = y',$$

whose right-hand sides are always regular, then one will get a system with the character that was considered in § **27** in which x is replaced with t, and the interval $\mathfrak{I}$ reaches from t_0 or 0 to t_2 . There will then be a system of integrals of the form:

$$x = X(t, a, b, \alpha, \beta), \quad y = Y(t, a, b, \alpha, \beta), \tag{97}$$

in which the functions X, Y will be regular as long as t belongs to the indicated interval, and the differences:

$$|a - x_0|, \qquad |b - y_0|, \qquad |\alpha - x'_0|, \qquad |\beta - y'_0|$$

are sufficiently small. One will then have the identities:

$$\begin{aligned} X(0, a, b, \alpha, \beta) &= a, \quad X_t(0, a, b, \alpha, \beta) = \alpha, \\ Y(0, a, b, \alpha, \beta) &= b, \quad Y_t(0, a, b, \alpha, \beta) = \beta. \end{aligned} \tag{98}$$

If one then introduces the condition:

$$\alpha^2 + \beta^2 = 1$$

then from the second equation in (95), one will generally have:

$$X_t^2 + Y_t^2 = 1,$$

and the quantity t will also represent the arc-length, as measured from the point (a, b), on all curves:

$$x = X, \qquad y = Y.$$

§ 30. – A field whose extremals intersect a given curve transversally.

If we now pose the equation:

$$\Gamma(a, b) = 0, \tag{99}$$

which will represent a curve $\mathfrak{C}_0$ that is regular at the location $a = x_0$, $b = y_0$ when a and b are coordinates and along which Γ_a and Γ_b do not vanish simultaneously anywhere, then equations (97) will define a family of extremals that starts from the points of the curve $\mathfrak{C}_0$ with the value t = 0. They will cut that curve transversally when the further assumption is introduced that:

$$\Lambda(a, b, \alpha, \beta) = \Gamma_b F_{x'}(a, b, \alpha, \beta) - \Gamma_a F_{y'}(a, b, \alpha, \beta) = 0. \tag{100}$$

Obviously, one has:

$$\begin{aligned}\Lambda_\alpha &= \Gamma_b F_{x'x'}(a, b, \alpha, \beta) - \Gamma_a F_{y'x'}(a, b, \alpha, \beta) \\ &= \beta F_1(a, b, \alpha, \beta)(\alpha\Gamma_a + \beta\Gamma_b),\end{aligned}$$

$$\Lambda_\beta = -\alpha F_1(a, b, \alpha, \beta)(\alpha\Gamma_a + \beta\Gamma_b).$$

The functional determinant:

$$\frac{\partial(\Lambda, \alpha^2 + \beta^2)}{\partial(\alpha, \beta)} = 2F_1(a, b, \alpha, \beta)(\alpha\Gamma_a + \beta\Gamma_b)$$

will then be non-zero when that is true for the expression $\alpha\Gamma_a + \beta\Gamma_b$, so when the quantity:

$$x_0'\,\Gamma_a(x_0, y_0) + y_0'\,\Gamma_b(x_0, y_0)$$

does not vanish either, i.e., when the curve $\mathfrak{C}_0$ does not contact the arc 02 at the point 0. That can happen only when the equations:

$$x_0'\,\Gamma_a(x_0, y_0) + y_0'\,\Gamma_b(x_0, y_0) = 0,$$

$$F_{y'}(x_0, y_0, x_0', y_0')\,\Gamma_a(x_0, y_0) - F_{x'}(x_0, y_0, x_0', y_0')\,\Gamma_b(x_0, y_0) = 0$$

exist together, from which the equation:

$$F(x_0, y_0, x_0', y_0') = 0$$

would follow. If we exclude that possibility, as in § **15**, then the equations:

(101) $$\alpha^2 + \beta^2 = 1\ ,\quad \Lambda\,(a, b, \alpha, \beta) = 0$$

will yield expressions for α and β that are regular in the arguments a, b. Since equation (99) defines one of the quantities a, b as a regular function of the other one, moreover (e.g., b as a function of a), when the inequality:

$$\Gamma_b\,(x_0, y_0) \neq 0$$

is true, on the basis of equations (99), (100), (101), all four of the constants that appear in X and Y can be expressed in terms of only a, such that the equations of the family of extremals will assume the form:

$$x = X = \xi\,(t, a)\ , \qquad y = Y = \eta\,(t, a)\ ,$$

and the functions ξ and η will be regular at the location $(0, x_0)$. In that way, one will obviously have:

$$\xi_t = X_t\ , \qquad \xi_a = X_a + X_b\,\frac{db}{da} + X_\alpha\,\frac{d\alpha}{da} + X_\beta\,\frac{d\beta}{da}\ ,$$

and those equations will remain valid when one replaces ξ and η with X and Y.

If we now define the functional determinant:

$$2D = \frac{\partial(X, Y, \alpha^2 + \beta^2, \Gamma, \Lambda)}{\partial(t, a, b, \alpha, \beta)}$$

$$= \begin{vmatrix} X_t & X_a & X_b & X_\alpha & X_\beta \\ Y_t & Y_a & Y_b & Y_\alpha & Y_\beta \\ 0 & 0 & 0 & 2\alpha & 2\beta \\ 0 & \Gamma_a & \Gamma_b & 0 & 0 \\ 0 & \Lambda_a & \Lambda_b & \Lambda_\alpha & \Lambda_\beta \end{vmatrix}$$

and consider that as a result, equations (98) can be posed:

$$X = a + \alpha\,t + [t]_2\ , \qquad Y = b + \beta\,t + [t]_2\ ,$$

which will give:

$$X_a = 1\ , \qquad X_b = X_\alpha = X_\beta = 0\ , \qquad X_t = \alpha\ ,$$

$$Y_b = 1\ , \qquad Y_a = Y_\alpha = Y_\beta = 0\ , \qquad Y_t = \beta$$

for $t = 0$, then based upon the values Λ_α, Λ_β above, we will have:

$$D\Big|^{t=0} = \begin{vmatrix} \alpha & 1 & 0 & 0 & 0 \\ \beta & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \alpha & \beta \\ 0 & \Gamma_a & \Gamma_b & 0 & 0 \\ 0 & \Lambda_a & \Lambda_b & \beta & -\alpha \end{vmatrix} F_1\,(\alpha\,\Gamma_a + \beta\,\Gamma_b)$$

$$= -\,F_1\,(\alpha\,\Gamma_a + \beta\,\Gamma_b)^2,$$

which is a value that is obviously non-zero. If we now multiply the third, fourth, and fifth columns in D by $\dfrac{db}{da}, \dfrac{d\alpha}{da}, \dfrac{d\beta}{da}$, resp., and add them to the second one then that will give:

$$D = \begin{vmatrix} \xi_t & \xi_a \\ \eta_t & \eta_a \end{vmatrix} \begin{vmatrix} 0 & \alpha & \beta \\ \Gamma_b & 0 & 0 \\ \Lambda_b & \Lambda_\alpha & \Lambda_\beta \end{vmatrix}$$

$$= \begin{vmatrix} \xi_t & \xi_a \\ \eta_t & \eta_a \end{vmatrix} \Gamma_b\, F_1\,(\alpha\,\Gamma_a + \beta\,\Gamma_b)\,.$$

Therefore, the quantity:

$$\Delta = \frac{\partial(\xi,\eta)}{\partial(t,a)}$$

is also non-zero in the neighborhood of the location 0. The extremals that are constructed to intersect the curve $\mathfrak{C}_0$ transversally will then define a field for the arc 02 or a finite part of it that begins at 0 when one does not consider them beyond a certain limit, precisely in the sense of § **14**.

When one defines the quantity D for the curve 02, it will include only the first and second derivatives of Γ for the system of values $a = x_0$, $b = y_0$. One easily concludes from this that any two curves $\mathfrak{C}_0$ that osculate at the point 0 will have the same focal points on the extremal 02. If the curve $\mathfrak{C}_0$ contracts to a point 0 then a and b will no longer be freely-varying, and X, Y can then be dropped from the function symbols. As a result of equations (98), the determinant:

$$D_0\,(t) = \begin{vmatrix} X_t & X_\alpha & X_\beta \\ Y_t & Y_\alpha & Y_\beta \\ 0 & \alpha & \beta \end{vmatrix} = \frac{\partial(X, Y, \alpha^2+\beta^2)}{\partial(t,\alpha,\beta)}$$

can be written:

$$D_0\,(t) = \begin{vmatrix} \alpha+[t]_1 & t+[t]_2 & [t]_2 \\ \beta+[t]_1 & [t]_2 & t+[t]_2 \\ 0 & \alpha & \beta \end{vmatrix}$$

$$= \begin{vmatrix} \alpha & t & 0 \\ \beta & 0 & t \\ 0 & \alpha & \beta \end{vmatrix} + [t]_2 = - t + [t]_2 ,$$

so it can indeed vanish at the point 0 itself, but not in a neighborhood of it. Now if, e.g., y_0' is non-zero, and β is therefore regular as a function of α at the location x_0', and we set:

$$X(t, \alpha, \sqrt{1-\alpha^2}) = \xi(t, \alpha) ,$$
$$Y(t, \alpha, \sqrt{1-\alpha^2}) = \eta(t, \alpha) ,$$

then ξ and η will be regular at the location $(0, x_0')$, and we will have:

$$\alpha + \beta \frac{d\beta}{d\alpha} = 0 ,$$
$$\xi_\alpha = X_\alpha + X_\beta \frac{d\beta}{d\alpha}, \qquad \eta_\alpha = Y_\alpha + Y_\beta \frac{d\beta}{d\alpha} .$$

It then easily follows from this that:

$$D_0(t) = \frac{\partial(\xi, \eta)}{\partial(t, a)} = \beta \Delta .$$

Δ is also non-zero in the neighborhood of the point 0 then, and when 1 is a point on the arc 02 that is sufficiently close to 0, the arc 12 can be surrounded by a field whose extremals all go through the point 0, at least in the neighborhood of the point 1.

§ 31. – The Jacobi condition in its Weierstrass and Hesse forms.

If we restrict the four constants that appear in X and Y to only the relations:

$$\alpha^2 + \beta^2 = 1 , \qquad \Gamma(a, b) = 0$$

then we can express them in terms of two constants, say a and:

$$c = \frac{\beta}{\alpha} ,$$

when the quantities:

$$x_0' , \qquad \Gamma(x_0, y_0)$$

are non-zero. If we get:

(102) $$x = \mathfrak{x}\,(t, a, c)\,, \qquad y = \mathfrak{y}\,(t, a, c)$$

then $\mathfrak{x}$ and $\mathfrak{y}$ will be regular at the location:

$$t = 0\,, \qquad a = x_0\,, \qquad c = \frac{y_0'}{x_0'}$$

and have the forms:

$$\mathfrak{x}\,(t, a, c) = a + \frac{t}{\sqrt{1+c^2}} + [t]_2\,,$$

$$\mathfrak{y}\,(t, a, c) = b + \frac{c\,t}{\sqrt{1+c^2}} + [t]_2\,,$$

in which the square root has the same sign in both cases. That will give:

$$\frac{\partial(\mathfrak{x},\mathfrak{y})}{\partial(t,a)} = \begin{vmatrix} \dfrac{1}{\sqrt{1+c^2}} & 1 \\ \dfrac{c}{\sqrt{1+c^2}} & \dfrac{db}{da} \end{vmatrix} = \frac{\dfrac{db}{da} - c}{\sqrt{1+c^2}}$$

for $t = 0$. That quantity will be non-zero when the quantity $\left| c - \dfrac{y_0'}{x_0'} \right|$ remains sufficiently small since the extremal $\mathfrak{C}$ and the curve $\Gamma = 0$ do not contact, by assumption. One can then represent the set of all extremals that differ sufficiently little from a well-defined one (say $\mathfrak{C}$) in the neighborhood of a well-defined point 0 that belongs to the latter in the form (102), in which when one keeps $a = a_0$, $c = c_0$ for $\mathfrak{C}$ itself and associates the point 0 with the parameter t_{00} , the functions $\mathfrak{x}$, $\mathfrak{y}$ will be regular at the location (t_{00} , a_0 , c_0), and at least one of the quantities:

$$\theta_1\,(t, a, c) = \frac{\partial(\mathfrak{x},\mathfrak{y})}{\partial(t,a)}, \qquad \theta_2\,(t, a, c) = \frac{\partial(\mathfrak{x},\mathfrak{y})}{\partial(t,c)}$$

is non-zero. However, the functions $\mathfrak{x}$ and $\mathfrak{y}$ do not need to have the special form that was given above, which served as only a concrete example of the stated assertion.

If one now directs one's attention to the extremals that go through the point 0, in particular, then one will have:

$$x_0 = \mathfrak{x}\,(t_0\,, a, c)\,, \qquad y_0 = \mathfrak{y}\,(t_0\,, a, c)$$

for them. One can solve those equations for the quantities θ as a result of the assumption that was made in regard to t_0 and a or t_0 and c. If one has, say:

$$\theta_2\,(t_{00}\,,\,a_0\,,\,c_0) \neq 0\,,$$

then one will get expressions of the form $[a - a_0]_1$ for $t - t_0$ and $c - c_0$, and the values that are defined by them:

$$\mathfrak{x}\,(t, a, c) = \xi\,(t, a)\,, \qquad \mathfrak{y}\,(t, a, c) = \eta\,(t, a)$$

will be regular at the location:

$$t = t_{00}\,, \qquad\qquad a = a_0\,.$$

Since one further has:

$$\xi_t = \mathfrak{x}_t\,, \qquad \xi_a = \mathfrak{x}_a + \mathfrak{x}_c\,\frac{dc}{da}\,, \qquad \eta_t = \mathfrak{y}_t\,, \qquad \eta_a = \mathfrak{y}_a + \mathfrak{y}_c\,\frac{dc}{da}\,,$$

$$\mathfrak{x}_t\,dt + \mathfrak{x}_a\,da + \mathfrak{x}_c\,dc\,\Big|^0 = 0\,, \qquad \mathfrak{y}_t\,dt + \mathfrak{y}_a\,da + \mathfrak{y}_c\,dc\,\Big|^0 = 0\,,$$

and the latter equations yield:

$$\frac{dc}{da} = -\,\frac{\theta_1(t_0, a, c)}{\theta_2(t_0, a, c)}\,,$$

one will then get:

$$\Delta = \frac{\partial(\xi,\eta)}{\partial(t,a)} = \theta_1\,(t\,,\,a,\,c) + \theta_2\,(t,\,a,\,c)\,\frac{dc}{da} = \frac{D(t_0,t)}{\theta_2(t_0,a,c)}\,,$$

in which one sets:

$$D\,(t_0,\,t) = \begin{vmatrix} \theta_1(t,a,c) & \theta_2(t,a,c) \\ \theta_1(t_0,a,c) & \theta_2(t_0,a,c) \end{vmatrix} = -\,D\,(t,\,t_0)\,.$$

That expression, which was considered by **Weierstrass**, is also the determinant of the coefficients of dt, dt_0, da, dc in the linear forms dx_0, dy_0, dx, dy . Namely, one easily verifies the identity:

$$-\,D\,(t_0,\,t) = \begin{vmatrix} \mathfrak{x}_t(t) & 0 & \mathfrak{x}_a(t) & \mathfrak{x}_c(t_0) \\ \mathfrak{y}_t(t) & 0 & \mathfrak{y}_a(t) & \mathfrak{y}_c(t_0) \\ 0 & \mathfrak{x}_t(t_0) & \mathfrak{x}_a(t_0) & \mathfrak{x}_c(t_0) \\ 0 & \mathfrak{y}_t(t_0) & \mathfrak{y}_a(t_0) & \mathfrak{y}_c(t_0) \end{vmatrix}.$$

The expression for Δ shows that the **Jacobi** condition for the extremum will be fulfilled for the arc 12 along any extremal when the points 0, 1, 2 follow in succession in the direction of increasing t, and the equation:

$$D\,(t_0,\,t) = 0$$

will possess no other roots than $t = t_0$ in the interval from t_0 to t_1 . That form of the **Jacobi** condition employs the point 0 that lies outside the arc 12 under scrutiny, which is a defect that can be remedied in the following way: If t and t_0 lie in the neighborhood of a fixed value t_1 then one can develop:

$$D(t_0, t) = [t - t_1, t - t_0]_1$$

(103)

$$= [(t - t_0) + (t_0 - t_1), t_0 - t_1]_1 = \sum_{a=1}^{\infty} f_a(t_0)(t - t_0)^a .$$

The coefficients in this series initially appear to be functions of t_0 and t_1, but from the latter argument, they must be independent, just like $D(t_0, t)$. Furthermore, since θ_1 and θ_2 can be regarded as special cases of the quantity Δ that do not vanish, along with their derivatives with respect to t, from § **24**, the same thing will then be true for $D(t, t_0)$ when that expression does not vanish identically, and it will follow that:

$$| f_1(t_0) | > 0 .$$

If one then restricts the quantity t_0 to a certain interval:

(104) $$| t_0 - t_1 | < \varepsilon$$

then the quantity $| f_1(t_0) |$ will lie above a positive limit γ. As a result of the original development (103), there will be further positive constants g, ρ such that with the assumption that:

(105) $$| t_0 - t_1 | < \rho , \quad | t - t_1 | < \rho ,$$

the inequality:

$$\frac{1}{\mathfrak{a}!}\left|\frac{\partial^a D(t_0, t)}{\partial t^a}\right| < g\,\rho^{-a}$$

will be valid, and since the left-hand side goes to $| f_a(t_0) |$ for $t = t_a$, if one assumes the first inequality in (105) then:

$$| f_a(t_0) | < g\,\rho^{-a} ,$$

$$\left|\sum_{a=2}^{\infty} f_a(t_0)(t - t_0)^{a-1}\right| < \frac{g}{\rho}\left|\sum_{a=2}^{\infty}\left(\frac{t - t_0}{\rho}\right)^{a-1}\right| .$$

There will then be a positive constant δ that depends upon only g, ρ, γ, but not t_0, with the property that under the assumption that:

$$| t - t_0 | < \delta ,$$

along with the assumptions in (104), (105), the inequality:

$$\left| f_1(t_0) + \sum_{\mathfrak{a}=2}^{\infty} f_{\mathfrak{a}}(t_0)(t-t_0)^{\mathfrak{a}-1} \right| > 0$$

will be valid, so the equation:

$$D\,(t_0, t) = 0$$

will possess no other root between $t_0 - \delta$ and $t_0 + \delta$ than $t = t_0$.

t_0 is now assumed to be close enough to t_1 that not only the relations (104), (105) are fulfilled, but also the inequalities:

$$t_0 < t_1 < t_0 + \delta\,, \qquad t_0 + \delta > t_1 + \delta_0\,, \tag{106}$$

when δ_0 is a constant that lies between 0 and δ. Assume further that the equation:

$$D\,(t_1, t) = 0 \tag{107}$$

has only the root $t = t_1$ in the interval from t_1 to t_2 , so it will have none at all in the interval from $t_1 + \delta$ to t_2 . Therefore, as long as $|\, t_1 - t_0 \,|$ is sufficiently small, the latter will also be true for the equation:

$$D\,(t_0, t) = 0\,,$$

which possesses only the one root t_0 between $t_0 - \delta$ and $t_0 + \delta$, so from (106), between $t_0 - \delta$ and $t_0 + \delta_0$, as well. It will also be the single root of the latter equation for the entire interval from t_0 to t_2 then. With that, it has been shown that the **Jacobi** condition is fulfilled in the form that was given above under the assumption that was introduced into equation (107), i.e., that they extremals that go through a suitably-chosen point 0 define a field of the arc 12. When one is dealing with the extremum for the arcs 12 with fixed endpoints, the **Jacobi** condition can also be replaced with the requirement that equation (107) possesses no other root than $t = t_1$ along the segment from t_1 to t_2.

If one can set $x = t$, in particular, then one will get:

$$y = \mathfrak{y}\,(x, a, c)\,, \qquad \theta_1 = \frac{\partial y}{\partial a}\,, \qquad \theta_2 = \frac{\partial y}{\partial c}\,,$$

$$D\,(t_0, t) = \Delta\,(t_0, t) = \begin{vmatrix} \dfrac{\partial y}{\partial a} & \dfrac{\partial y}{\partial b} \\ \left|\dfrac{\partial y}{\partial a}\right|^0 & \left|\dfrac{\partial y}{\partial c}\right|^0 \end{vmatrix}\,.$$

If one sets that expression equal to zero then one will get the criterion for the conjugate points in the form that **Hesse** gave to it.

CHAPTER FOUR

THE SIMPLEST RELATIVE EXTREMUM

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§ 32. – The general isoperimetric problem.

The isoperimetric problem, in the broader sense of the term, requires one to extremize an integral:

$$J = \int f(x, y, p)\, dx = \int F(x, y, x', y')\, dt$$

when another integral of the same form:

$$K = \int g(x, y, p)\, dx = \int G(x, y, x', y')\, dt$$

has a prescribed value. That is, among all curves that give a prescribed value to the integral K, find the one along which J takes on an extreme value. Such an extremum is called a *relative* one, while the extrema that were considered up to now are, by contrast, *absolute* ones. One gets the isoperimetric problem, in the narrower sense, when J is the area integral and K is arc-length.

Now in order to derive necessary conditions for that extremum, let $\mathfrak{B}$ be a curve segment with the properties that were required in §§ **2** and **4** for the similarly-denoted arc. Let the functions F and G be regular for the system of arguments that is defined for an element of that arc. The points 0, 1, 2, 3 might follow each other in the direction of increasing values of t along it. We then vary the two arcs 01 and 23 according to the method that was employed in § **8**.

Let ε, η, $\bar{\varepsilon}$, $\bar{\eta}$ denote constants and set:

$$T = (t - t_0)^3\, (t_1 - t)^3, \quad U = (t - t_2)^3\, (t_3 - t)^3 .$$

Then let:

$$\delta x = \varepsilon\, T, \qquad \delta y = \eta\, T$$

for the arc 01 and:

$$\delta x = \bar{\varepsilon}\, U, \qquad \delta y = \bar{\eta}\, U$$

for the arc 23, but:

$$\delta x = \delta y = 0$$

everywhere outside the segments 01 and 23. Since the first and second derivatives of the quantities δx, δy vanish at the locations 0, 1, 2, 3, the point $(x + \delta x, y + \delta y)$ will describe a curve with the continuity properties that were assumed for $\mathfrak{B}$. Obviously, with the notation of the first two sections, one has the equations:

(1) $$\Delta J = \Delta J_{01} + \Delta J_{23}\,, \qquad \Delta K = \Delta K_{01} + \Delta K_{23}\,.$$

Hence, in order for the integral K to have the same value along the varied curve that it has along the original one, one must fulfill the equation:

$$\Delta K = \Delta K_{01} + \Delta K_{23} = 0\,.$$

If we further set:

$$P = F_x - F'_{x'}\,, \quad Q = F_y - F'_{y'}\,, \quad R = G_x - G'_{x'}\,, \quad S = G_y - G'_{y'}$$

then, from § **4**, when the variations vanish at the limits of integration vanish:

$$\Delta J = \delta J + \int dt\,[\delta x, \delta y, \delta x', \delta y']_2\,,$$
$$\delta J = \int dt\,(P\,\delta x + Q\,\delta y)\,,$$

and analogous formulas will be true for the integral K. On the grounds of formulas (1), that will give:

$$\Delta J = \varepsilon \int_0^1 PT\,dt + \eta \int_0^1 QT\,dt + \bar{\varepsilon} \int_2^3 PU\,dt + \bar{\eta} \int_2^3 QU\,dt + [\varepsilon, \eta, \bar{\varepsilon}, \bar{\eta}]_2\,,$$

$$\Delta K = \varepsilon \int_0^1 RT\,dt + \eta \int_0^1 ST\,dt + \bar{\varepsilon} \int_2^3 RU\,dt + \bar{\eta} \int_2^3 SU\,dt + [\varepsilon, \eta, \bar{\varepsilon}, \bar{\eta}]_2\,,$$

with the given special variations. Now if the curve $\mathfrak{B}$ is to yield a relative extremum of the integral J in the sense that was defined then ΔJ would need to have to have a fixed sign for all varied curves for which ΔK vanishes. Since the constants ε, η, $\bar{\varepsilon}$, $\bar{\eta}$ are freely available, from § **7**, it will follow from this that the second-order determinants that are defined by any two columns of the matrix:

$$\begin{matrix} \int_0^1 RT\,dt\,, & \int_0^1 ST\,dt\,, & \int_2^3 RU\,dt\,, & \int_2^3 SU\,dt\,, \\ \int_0^1 PT\,dt\,, & \int_0^1 QT\,dt\,, & \int_2^3 PU\,dt\,, & \int_2^3 QU\,dt \end{matrix}$$

must vanish. Since the quantity T is positive along the segment 01, and U is likewise positive along the segment 23, one can put that matrix into the following form:

$$R_m \int_0^1 T\,dt\,, \qquad S_m \int_0^1 T\,dt\,, \qquad R_\mu \int_2^3 U\,dt\,, \qquad S_\mu \int_2^3 U\,dt\,,$$

$$P_m \int_0^1 T\,dt\,, \qquad Q_m \int_0^1 T\,dt\,, \qquad P_\mu \int_2^3 U\,dt\,, \qquad Q_\mu \int_2^3 U\,dt\,,$$

in which the index m means that t takes on a value in the interval from t_0 to t_1, and μ has the same meaning for the interval from t_2 to t_3 . The integrals that appear here are all positive. All second-order determinants that are defined by the columns of the system:

$$(2) \qquad \begin{matrix} R_m, & S_m, & R_\mu, & S_\mu, \\ P_m, & Q_m, & P_\mu, & Q_\mu \end{matrix}$$

will also vanish then.

We now imagine that P, Q, R, S are continuous functions of t under the assumed properties of the arc $\mathfrak{B}$. Thus, if the segments 01 and 23 are shrunk without limit while one fixes the points 0 and 2 then the quantities with the indices m and μ will approach certain values:

$$\begin{gathered} \lim P_m = P_0\,, \quad \lim Q_m = Q_0\,, \quad \lim P_\mu = P_2\,, \quad \lim Q_\mu = Q_2\,, \\ \lim R_m = R_0\,, \quad \lim S_m = S_0\,, \quad \lim R_\mu = R_2\,, \quad \lim S_\mu = S_2\,, \end{gathered}$$

and the same thing will be true of the second-order determinants that are defined by those quantities that was true of the quantities (2). In particular, we will have the equations:

$$P_0\,S_2 - R_0\,Q_2 = 0\,, \qquad Q_0\,S_2 - S_0\,Q_2 = 0\,.$$

From now on, we introduce the assumption that the curve $\mathfrak{B}$ is not an extremal of either of the two integrals J and K, such that neither the quantities R and S nor the quantities P and Q will vanish everywhere along them. We can then choose the point 2 such that one of the quantities P_2, Q_2 (say, Q_2) is non-zero, and the last equation will give:

$$P_0\,\frac{S_2}{Q_2} - R_0 \;=\; Q_0\,\frac{S_2}{Q_2} - S_0 \;=\; 0\,.$$

If we had $S_2 = 0$ then it would follow from those equations that with an arbitrary position of the point 0:

$$R_0 = S_0 = 0\,,$$

so the curve $\mathfrak{B}$ would be an extremum of the integral K, which would contradict the assumption. There would then be a quantity:

$$\lambda = -\,\frac{Q_2}{S_2}$$

that is finite, non-zero, and independent of the position of the point 0, and has the property that the equations:

$$P_0 + \lambda R_0 = Q_0 + \lambda S_0 = 0$$

will be satisfied. The same consequence will follow from the equations:

$$P_0 R_2 - R_0 P_2 = 0 , \qquad Q_0 R_2 - S_0 P_2 = 0$$

when P_2 does not vanish. The curve $\mathfrak{B}$ then satisfies the differential equations:

$$F_x - F'_{x'} + \lambda (G_x - G'_{x'}) = 0 , \qquad F_y - F'_{y'} + \lambda (G_y - G'_{y'}) = 0 ,$$

or with the notation:

$$H = F + \lambda G ,$$

the equations:

$$H_x - H'_{x'} = 0 , \qquad H_y - H'_{y'} = 0 ,$$

in which λ means a finite, non-zero constant. We shall call any curve that satisfies those equations for an arbitrary, but non-vanishing, value of λ an *extremal* for the given problem of the relative extremum. Since we can divide the equations by λ, it is clear that the totality of all extremals will remain the same when the roles of the integrals J and K are switched.

The result of our investigation can now be expressed by saying that the curve that yields the desired relative extremum under the assumed continuity properties is nothing but a piece of an extremal when it is not, perhaps, an extremal in the sense of an absolute extremum for one of the integrals J, K. We shall overlook the latter exceptional case in the general theory.

§ 33. – The variable end-point problem. Transversal position.

Let the integral J be extremized, in particular, by a curve 01 whose endpoints are not given, but are restricted only insofar as any sort of relations:

$$g_{\mathfrak{b}} (x_0, y_0, x_1, y_1) = 0$$

can be prescribed between the coordinates x_0, y_0, x_1, y_1 , whose number cannot be greater than four. A curve 01 that solves that problem and has the properties that we have assumed to pertain to the arc $\mathfrak{B}$ must then be a piece of an extremal, except for the exceptional case that was referred to at the conclusion of § **32**, because otherwise, from § **32**, we could vary it without displacing its endpoints in such a way that J could increase, as well as decrease, for unvaried values of K. Hence, should the endpoints also be varied then the equations:

$$g_{\mathfrak{b}} (x_0 + \delta x_0 , y_0 + \delta y_0 , x_1 + \delta x_1 , y_1 + \delta y_1) = 0 \tag{3}$$

would have to be true. In particular, if we set:

$$\delta x = \delta x_0 \left(\frac{t-t_2}{t_0-t_2}\right)^3, \quad \delta y = \delta y_0 \left(\frac{t-t_2}{t_0-t_2}\right)^3 + \varepsilon\,(t_2-t)^3\,(t-t_0)^3$$

along an arc 02 of the curve 01, and:

$$\delta x = \delta x_1 \left(\frac{t-t_3}{t_1-t_3}\right)^3, \quad \delta y = \delta y_1 \left(\frac{t-t_3}{t_1-t_3}\right)^3$$

along an arc 31 that is separate from it, but:

$$\delta x = \delta y = 0$$

in between, then the variations along the entire curve 01 will be continuous functions of t, along with their first derivatives.

Now, one has, in general:

$$\Delta K_{01} = G_{x'}\,\delta x + G_{x'}\,\delta x\Big|_0^1 + \int_0^1 dt\,\{R\,\delta x + S\,\delta y + [\delta x, \delta y, \delta x', \delta y']_2\}\ ,$$

so with the given special variations:

$$\Delta K_{01} = [\delta x, \delta y, \delta x', \delta y']_1 + \varepsilon \int_0^2 S\,(t_2-t)^3(t-t_0)^3\,dt + [\varepsilon, \delta x, \delta y, \delta x', \delta y']_2\ .$$

The factor of ε is non-zero here, since 01 is a regular curve along which the quantity S has only distinct zeroes when it does not vanish everywhere. In the latter case, since the curve considered should not be an extremal of the integral K, the quantity R will not vanish everywhere along the curve 01, and one needs only to switch x and y throughout the entire argument. Thus, if S is non-zero, at least with the exception of isolated locations, then one can choose the interval 02 in such a way that S will remain non-zero in its interior. One can then use the equation:

$$\Delta K_{01} = 0 \tag{4}$$

to calculate:

$$\varepsilon = [\delta x_0\ ,\ \delta y_0\ ,\ \delta x_1\ ,\ \delta y_1]_1\ .$$

On the other hand, with the assumption (4), one will have:

$$\Delta J_{01} = \Delta\,(J_{01} + \lambda\,K_{01})\ .$$

One will get analogous formulas for ΔK_{01} when one considers the equations of the extremals:

$$\Delta J_{01} = H_{x'}\,\delta x + H_{y'}\,\delta y\Big|_0^1 + \int_0^1 dt\,[\delta x, \delta y, \delta x', \delta y']_2 \; .$$

If the desired extremum is to be provided by the curve 01 then that quantity must have a fixed sign for all values of $\delta x_0, \ldots, \delta y_1$ that satisfy equations (3). From § **7**, that requires that the equation:

$$H_{x'}\,\delta x + H_{y'}\,\delta y\Big|_0^1 = 0$$

must be a consequence of the linear relations:

$$\frac{\partial g_b}{\partial x_0}\,\delta x_0 + \frac{\partial g_b}{\partial y_0}\,\delta y_0 + \frac{\partial g_b}{\partial x_1}\,\delta x_1 + \frac{\partial g_b}{\partial y_1}\,\delta y_1 = 0 \; .$$

In particular, if x_0, y_0 are given, and should the point 1 lie along the curve:

$$h\,(x, y) = 0$$

such that one always has:

$$\delta x_0 = \delta y_0 = 0$$

then the equation:

$$H_{x'}\,\delta x + H_{y'}\,\delta y\Big|^1 = 0$$

would have to be true under the assumption that:

$$h_x\,\delta x + h_y\,\delta y\Big|^1 = 0 \; .$$

We will then say that the extremal 01 cuts the curve $h = 0$ *transversally*. The fact that this can occur is a necessary condition for the extremal 01 to yield an extremum for the integral J from among all of the lines that are drawn from the point 0 to the curve $h = 0$ that imply the prescribed value for K_{01} .

§ 34. – Examples. Problems IX, X.

Problem IX. – Find the curve of given length that maximizes the area integral.

Obviously, one has (§ **4**):

$$J = \int y\,dx = \int y\,x'\,dt\,, \qquad K = \int dt\sqrt{x'^2+y'^2}\,, \quad H = y\,x' + \lambda\sqrt{x'^2+y'^2}\,, \qquad H_x = 0\,,$$

with the positive square roots, so that will first imply the following equation for the extremals:

$$(5) \qquad H_{x'} = y + \frac{\lambda\,x'}{\sqrt{x'^2+y'^2}} = a\,, \quad \frac{dx}{dy} = \frac{a-y}{\lambda\sqrt{1-\left(\frac{a-y}{\lambda}\right)^2}}\,,$$

and then:

$$\frac{x-b}{\lambda} = \sqrt{1-\left(\frac{a-y}{\lambda}\right)^2}\,, \qquad (y-a)^2 + (x-b)^2 = \lambda^2$$

upon integration. The totality of all extremals will then be identical to that of all circles in the plane, and the square of the isoperimetric constant will be equal to the radius. If the endpoints 0, 1 are prescribed then the integral J will be equal to the area between the curve 01 and the straight line 01, plus the constant area of the trapezoid that is defined by the ordinates of the straight line segment. Thus, if the first area is to be a maximum then the desired curve with the properties of the arc $\mathfrak{B}$ can be nothing but a circular arc. The transversal position will be defined by the equation:

$$H_{x'}\,Dx + H_{y'}\,Dy = \left(y + \frac{\lambda\,x'}{\sqrt{x'^2+y'^2}}\right)Dx + \frac{\lambda\,y'}{\sqrt{x'^2+y'^2}}\,Dy = 0\,.$$

Along the x-axis ($y = 0$), it goes over to the special form:

$$x'\,Dx + y'\,Dy = 0\,.$$

Therefore, should the desired curve begin at the given point 0 and end at a point on the x-axis that is not prescribed then the surface that is bounded by that axis, the ordinate of the point 0, and the desired curve will not be a maximum for a given length of the latter in any other case than when the curve is a circular arc whose center lies along the x-axis. (Cf., Problem III.)

If the desired curve is closed then, from § **4**, the integral J will represent the area that it enclosed when one integrates around the curve precisely once. One will then have the conditions:

$$x_0 - x_1 = y_0 - y_1 = 0\,, \qquad \delta x_0 - \delta x_1 = \delta y_0 - \delta y_1 = 0\,,$$

as a result of which, the condition:

$$H_{x'}\,\delta x + H_{y'}\,\delta y\Big|_0^1 = 0$$

will be fulfilled by itself when the curve has the properties of the arc $\mathfrak{B}$ along its entire circuit. Under that assumption, it can then yield an extreme area for a given length only when it is a circle.

Should that area pertain to the surface swept out, not by the ordinate, but by the desired curve and a given curve $\mathfrak{H}$ that encloses it:

$$y = h(x)$$

then one would need to set:

$$J = \int [y - h(x)]\, x'\, dt .$$

One would then have:

$$H = [y - h(x)]\, x'\, dt + \lambda \sqrt{x'^2 + y'^2} ,$$

so the quantities H_y, $H_{y'}$, and $H_x - H'_{x'}$ would then be the same as they were in the previous case, such that the extremals would also be circles now. The condition for transversal intersection is:

$$\left[y - h(x) + \frac{\lambda\, x'}{\sqrt{x'^2 + y'^2}} \right] Dx + \frac{\lambda\, y'}{\sqrt{x'^2 + y'^2}}\, Dy = 0 .$$

The curve $\mathfrak{H}$ will then be perpendicular to the extremals that it intersects transversally. This problem will be most interesting geometrically when the endpoint 1, or also both endpoints 0 and 1, are not prescribed points on the curve $\mathfrak{H}$. The solution, if it exists, will then be a circle that is normal to the curve $\mathfrak{H}$ at one or two points.

The sign of the quantity λ is determined from the first equation in (5). Namely, if the direction of increasing t has the same relationship to the outward-pointing radius of the circle that the $+y$-axis has to the $+x$-axis then since $|\lambda|$ is always the radius of the circle, one will have:

$$\frac{x'}{\sqrt{x'^2 + y'^2}} = - \frac{y - a}{|\lambda|} ,$$

so $\lambda = |\lambda|$. When the direction of increasing t has the same relationship to the radius that the $+x$-axis has to the $+y$-axis, that will give the opposite sign.

Problem X. – Draw the shortest-possible line on a given surface that encloses a region of given surface area with a given curve, or also by itself when it is closed. (The curve of shortest circumference.)

Let x and y be isometric, curvilinear coordinates on the surface, i.e., let the position of a point be determined uniquely by those quantities in a certain region that includes the point $x = y = 0$. Thus, the line element will have the form:

$$ds = \sqrt{M\,(dx^2 + dy^2)}\,.$$

We shall call the segment of a line x = const. that is drawn from a point (x, y) to the line $y = 0$ its *ordinate*. $M\,dx\,dy$ will then be the area of a rectangular surface element whose sides are element of the lines x = const. and y = const. Therefore, if:

$$y = h\,(x)$$

is the equation of any curve then the area of the surface that is bounded by two of its neighboring ordinates will be:

$$dx \cdot \int_0^y dy\,M = dx\,[N\,(x, y) - N\,(x, 0)]\,,$$

in which N is a function of x and y that satisfies the equation:

$$\frac{\partial N}{\partial y} = M\,.$$

The total area that is covered by the ordinates of the curve $y = h\,(x)$ when one focusses on the arc 01 will then be:

$$\int_0^1 dx\,\{N\,[x, h\,(x)] - N\,[x, 0]\}\,.$$

If (x, y) then means a point on another curve 01 then the area that lies between both of them will be:

$$K = \int_0^1 dx\,\{N\,[x, y] - N\,[x, h\,(x)]\}\,.$$

The integral to be minimized is:

$$J = \int_0^1 dx\sqrt{M\,(1 + p^2)}\,.$$

One will then have:

$$H = \sqrt{M\,(x'^2 + y'^2)} + \lambda\{N[x, y] - N[x, h(x)]\}\,x'\,,$$

and the one equation of the extremals will be:

$$H_y - H'_{y'} = \frac{M_y\sqrt{x'^2 + y'^2}}{2\sqrt{M}} + \lambda M\;x' - \frac{d}{dt}\left(\frac{y'\sqrt{M}}{\sqrt{x'^2 + y'^2}}\right) = 0\,,$$

or when one defines the angle θ by the equations:

$$\cos\,\theta = \frac{x'}{\sqrt{x'^2 + y'^2}}\,, \quad \sin\,\theta = \frac{y'}{\sqrt{x'^2 + y'^2}}\,,$$

one will have:

$$\lambda\, M\, dx = d\,(\sqrt{M}\,\sin\theta) - \frac{ds}{\sqrt{M}}\frac{\partial\sqrt{M}}{\partial y}\,.$$

Now, one obviously has:

$$\frac{ds}{\sqrt{M}} = \sqrt{dx^2 + dy^2} = dx\,\cos\,\theta + dy\,\sin\,\theta\,.$$

Hence, the penultimate equation will imply that:

$$\begin{aligned}\lambda\, M\, dx &= \left(\frac{\partial\sqrt{M}}{\partial x}dx + \frac{\partial\sqrt{M}}{\partial y}dy\right)\sin\theta + \sqrt{M}\; d\sin\theta - \frac{\partial\sqrt{M}}{\partial y}(dx\cos\theta + dy\sin\theta) \\ &= \left(\frac{\partial\sqrt{M}}{\partial x}\sin\theta - \frac{\partial\sqrt{M}}{\partial y}\cos\theta\right)dx + \sqrt{M}\,\cos\theta\, d\theta\,,\end{aligned}$$

and since one further has:

$$\cos\,\theta = \sqrt{M}\,\frac{dx}{ds}\,, \qquad \sin\,\theta = \sqrt{M}\,\frac{dy}{ds}\,,$$

one will get:

$$\lambda\, M = \frac{\partial\sqrt{M}}{\partial x}\sin\theta - \frac{\partial\sqrt{M}}{\partial y}\cos\theta + M\,\frac{d\theta}{ds}\,.$$

If one now considers any family of extremals that belong to the same value of λ and covers part of the surface simply then θ can be regarded as a function of x and y, and one will have:

$$\frac{d\theta}{ds} = \frac{\partial\theta}{\partial x}\frac{dx}{ds} + \frac{\partial\theta}{\partial y}\frac{dy}{ds} = \frac{1}{\sqrt{M}}\left(\cos\theta\frac{\partial\theta}{\partial x} + \sin\theta\frac{\partial\theta}{\partial y}\right) = \frac{1}{\sqrt{M}}\left(-\frac{\partial\cos\theta}{\partial x} + \frac{\partial\sin\theta}{\partial y}\right).$$

When that value is substituted in the previous equation, one will get:

$$\lambda\, M = -\,\frac{\partial(\sqrt{M}\,\cos\theta)}{\partial x} + \frac{\partial(\sqrt{M}\,\sin\theta)}{\partial y}\,.$$

Based upon a general formula of **Bonnet**, it is clear from this that λ is the geodetic curvature of the curve in question. The curves of shortest circumference will then have constant geodetic curvature. Transversal position also coincides with being perpendicular to the curve $y = h(x)$ here, as well.

The fact that a sufficiently-bounded piece of an extremal can be classified in a family of the indicated type follows from § **30**, since the extremals that belong to a fixed value of λ can be regarded as extremals of the integral $J + \lambda K$, in the sense of the absolute extremum.

§ 35. – Doubly-infinite family of extremal arcs.

The extremals of an isoperimetric problem generally depend upon three parameters, namely, the quantity λ and the two integration constants that the integration of the differential equation of the problem introduce. Therefore, a two-fold manifold of extremals will go through a fixed point 0, in general. Such a thing will be represented by the equations:

$$x = \xi(t, a, b), \qquad y = \eta(t, a, b),$$

which is very easy to do in many individual cases, and we shall introduce the following assumption on the right-hand sides: If t lies within a certain interval $\mathfrak{T}$ then equations:

$$x = \xi(t, a_0, b_0), \qquad y = \eta(t, a_0, b_0)$$

will imply a nowhere-singular piece of a certain extremal $\mathfrak{C}$ that contains the point 0. Let the functions $\xi(t, a, b)$, $\eta(t, a, b)$ be regular as long as the system of values (t, a, b) belongs to an arbitrarily-bounded region $(\mathfrak{A})$ that contains all systems of values for which the quantities $|a - a_0|$, $|b - b_0|$ do not exceed certain limits, but t belongs to the interval $\mathfrak{T}$. In that region $(\mathfrak{A})$, let at least one of the quantities ξ_t, η_t be always non-zero, and suppose that the three functional determinants:

$$\frac{\partial(\xi,\eta)}{\partial(t,a)}, \qquad \frac{\partial(\xi,\eta)}{\partial(t,b)}, \qquad \frac{\partial(\xi,\eta)}{\partial(a,b)} \tag{6}$$

never vanish simultaneously. Thus, the first two of them cannot vanish simultaneously either, since, e.g., when ξ_t is non-zero at the location in question, it will follow from the equations:

$$\xi_t\,\eta_a - \eta_t\,\xi_a = \xi_t\,\eta_b - \eta_t\,\xi_b = 0$$

that

$$\eta_a = \frac{\eta_t\,\xi_a}{\xi_t}, \qquad \eta_b = \frac{\eta_t\,\xi_b}{\xi_t}, \qquad \eta_a\,\xi_b - \eta_b\,\xi_a = 0,$$

such that all three quantities in (6) would have to vanish, which was excluded. Finally, different extremals that go through the fixed point 0 might belong to different systems of values a, b in the region ($\mathfrak{A}$), whose integration and isoperimetric constants can be regarded as functions of *a* and *b*. We shall likewise denote the set of all those extremal arcs by ($\mathfrak{A}$).

The quantity t_0 can have different values on the different extremals of that family. If it is equal to t_{00} for the curve $\mathfrak{C}$ then one can use at least one of the equations:

$$x_0 = \xi(t_0, a, b), \qquad y_0 = \eta(t_0, a, b)$$

to derive an expression for t_0 that has the form:

$$t_0 - t_{00} = [a - a_0, b - b_0]_1,$$

since the derivative with respect to t_0 on the right-hand side of at least one of those equations will not vanish for $a = a_0$, $b = b_0$. Furthermore, if 1 is a point of the curve that is not equal to 0 and belongs to the region ($\mathfrak{A}$) then one will have, with the assumptions that were introduced:

$$x_1 = \xi(t_1, a_0, b_0), \qquad y_1 = \eta(t_1, a_0, b_0),$$

$$x - x_1 = [t - t_1, a - a_0, b - b_0]_1,$$
$$y - y_1 = [t - t_1, a - a_0, b - b_0]_1.$$

Therefore, when, e.g., the first of the determinants (6) is non-zero for $t = t_1$, $a = a_0$, $b = b_0$, one will get expressions of the form:

$$[x - x_1, y - y_1, b - b_0]_1$$

upon solving those equations for $a - a_0$ and $t - t_1$. A certain neighborhood of the point 1 will then be covered by the extremals of the family ($\mathfrak{A}$), and indeed, infinitely many times, in general.

Now, every arc-element of any extremal of the family ($\mathfrak{A}$) will imply a system of values (x, y, x', y') in whose neighborhood the functions F and G will be regular. One can then define the integral:

$$\bar{J}_{03} = \int_0^3 F\,dt = \int_{t_0}^{t_3} F[\xi(t,a,b), \ldots, \eta_t(t,a,b)]\,dt$$

along any of those curves that run through the point 3. It is a regular function of t_3, a, b, and since the lower limit t_0 is a function of a and b, one will obviously have:

$$\frac{\partial \bar{J}_{03}}{\partial a} = -F\,\Big|^0 \frac{\partial t_0}{\partial a} + \int_{t_0}^{t_3} \frac{\partial(\xi, \ldots, \eta_t)}{\partial a}\,dt$$

$$= -F\,\Big|^0 \frac{\partial t_0}{\partial a} + \int_{t_0}^{t_3} dt\,\{F_x\,\xi_a + F_y\,\eta_a + F_{x'}\,\xi_{at} + F_{y'}\,\eta_{at}\}\,,$$

or when one partially integrates and does not write out the arguments of the system ξ, η, ξ_t, η_t:

(7) $$\frac{\partial \bar{J}_{03}}{\partial a} = -\,F\,\Big|^0 \frac{\partial t_0}{\partial a} + F_x\,\xi_a + F_y\,\eta_a\,\Big|_0^3 + \int_{t_0}^{t_3} dt\,\{\xi_a(F_x - F'_{x'}) + \eta_a(F_y - F'_{y'})\}\ .$$

In this formula, *a* can obviously be replaced with *b*, and independently of that, *J*, *F* can be replaced with *K*, *G*. The same thing will be true for the formula:

(8) $$\frac{\partial \bar{J}_{03}}{\partial a} = F\,\Big|^3 = \xi_t\,F_{x'} + \eta_t\,F_{y'}\,\Big|^3\ .$$

Now, the definition of the quantity t_0 implies that:

$$\xi_t\,\Big|^0 \frac{\partial t_0}{\partial a} + \xi_a\,\Big|^0 = \eta_t\,\Big|^0 \frac{\partial t_0}{\partial a} + \eta_a\,\Big|^0 = \xi_t\,\Big|^0 \frac{\partial t_0}{\partial b} + \xi_b\,\Big|^0 = \ldots = 0\ .$$

On the other hand, one has the identity:

$$H\,(\xi,\,\ldots,\,\eta_t) = \xi_t\,H_{x'}\,(\xi,\ldots,\eta_t) + \eta_t\,H_{y'}\,(\xi,\ldots,\eta_t)\,.$$

It then follows that:

(9) $$H\,\Big|^0 \frac{\partial t_0}{\partial a} + H_{x'}\,\xi_a + H_{y'}\,\eta_a\Big|^0 = H\,\Big|^0 \frac{\partial t_0}{\partial b} + H_{x'}\,\xi_b + H_{y'}\,\eta_b\Big|^0 = 0\ .$$

If one then combines the formula (7) with the analogously-constructed one for *K*, and if λ is the isoperimetric constant of the extremal that runs through 3 then the terms in the expression:

$$\frac{\partial \bar{J}_{03}}{\partial a} + \lambda\,\frac{\partial \bar{K}_{03}}{\partial a} = -H\,\Big|^0 \frac{\partial t_0}{\partial a} + H_{x'}\,\xi_a + H_{y'}\,\eta_a\Big|_0^3 + \int_0^3 \cdots$$

that relate to the point 0 will vanish, and likewise, the integral, due to the equations for the extremals, such that:

(10) $$\frac{\partial \bar{J}_{03}}{\partial a} + \lambda\,\frac{\partial \bar{K}_{03}}{\partial a} = H_{x'}\,\xi_a + H_{y'}\,\eta_a\Big|^3,$$

in which *a* can naturally be replaced with *b*. If *a*, *b*, and t_3 are functions of *t* then it will follow from that and formula (3) that:

$$\frac{\partial \bar{J}_{03}}{\partial a} + \lambda \frac{\partial \bar{K}_{03}}{\partial a} = H_{x'}\left(\xi_a \frac{da}{d\tau} + \xi_b \frac{db}{d\tau} \right) + H_{y'}\left(\eta_a \frac{da}{d\tau} + \eta_b \frac{db}{d\tau} \right) + H \frac{dt}{d\tau} \Bigg|^3 ,$$

or since the identity:

$$H = \xi_t H_{x'} + \eta_t H_{y'}$$

is true:

(11) $$\frac{\partial \bar{J}_{03}}{\partial a} + \lambda \frac{\partial \bar{K}_{03}}{\partial a} = H_{x'} \frac{dx}{d\tau} + H_{y'} \frac{dy}{d\tau} \Bigg|^3 .$$

§ 36. – Concept of a field. Weierstrass construction.

We refer *t* to the point 3 and set:

$$\bar{K}_{03} = \omega (t, a, b) ,$$

and introduce the new assumption that the functional determinant:

$$\Delta = \frac{\partial (\xi, \eta, \omega)}{\partial (t, a, b)} = \begin{vmatrix} \xi_t & \eta_t & \omega_t \\ \xi_a & \eta_a & \omega_a \\ \xi_b & \eta_b & \omega_b \end{vmatrix}$$

does not vanish in the region ($\mathfrak{A}$), except at the point 0. That includes the assumption that was made above in regard to the determinants (6), and we now say that the extremals ($\mathfrak{A}$) define a *field* if each arc 12 belongs to the curve $\mathfrak{C}$ inside of the region ($\mathfrak{A}$). The point 1 can lie arbitrarily close to the position 0 without however attaining it. The property of defining a field will be preserved when one switches the roles of the integrals *J* and *K*. Namely, if one multiplies the first and second columns in the determinant Δ by $H_{x'}$ and $H_{y''}$, resp., and adds them to the third one, multiplied by – λ, then on the basis of the identity:

$$H_{x'} \xi_t + H_{y'} \eta_t = \frac{\partial \bar{J}_{03}}{\partial t} + \lambda\, \omega_t$$

and equations (8), (10), that will give:

$$-\lambda\, \Delta = \begin{vmatrix} \xi_t & \eta_t & \dfrac{\partial \bar{J}_{03}}{\partial t} \\ \xi_a & \eta_a & \dfrac{\partial \bar{J}_{03}}{\partial a} \\ \xi_b & \eta_b & \dfrac{\partial \bar{J}_{03}}{\partial b} \end{vmatrix} .$$

Now since λ is non-zero, the same thing will be true for the determinant on the right-hand side in the region ($\mathfrak{A}$). Since that will emerge from Δ when one replaces K with J, the assertion is proved.

Now let the points 1 and 2 be connected by a curve $\mathfrak{L}$ that traverses part of the plane that is covered by the extremals ($\mathfrak{A}$), and possesses the same continuity properties as the one that was denoted similarly in § **17**. Let F and G be regular in the system of values $\left(x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau}\right)$ that is defined by the elements of $\mathfrak{L}$. Furthermore, let:

$$\bar{K}_{12} = K_{12}\,, \tag{12}$$

when one integrates the left-hand side over the curve $\mathfrak{C}$, but the integral without the overbar refers to $\mathfrak{L}$, as it always will from now on. The parameter τ, which x and y are functions of, increases along the latter curve in the direction from 1 to 2. Let 3 be one of its points that belongs to the interior of the field ($\mathfrak{A}$) and corresponds to the values $a = a_3$, $b = b_3$, and let it be connected with 0 by the extremal 03. One will then have the following equations in the neighborhood of the point 3:

$$\begin{aligned}
x - x_3 &= \xi(t, a, b) - \xi(t_3, a_3, b_3) &&= [t - t_3, a - a_3, b - b_3]_1\,,\\
y - y_3 &= \eta(t, a, b) - \eta(t_3, a_3, b_3) &&= [t - t_3, a - a_3, b - b_3]_1\,,\\
\omega - \omega_3 &= \omega(t, a, b) - \omega(t_3, a_3, b_3) &&= [t - t_3, a - a_3, b - b_3]_1\,,
\end{aligned}$$

and the coefficients of the linear terms on the right-hand side have the determinant $\Delta\,|^3$. One can then calculate the quantities $t - t_3$, $a - a_3$, $b - b_3$ from those three equations in the form:

$$[x - x_3, y - y_3, \omega - \omega_3]_1\,, \tag{13}$$

and those series have a non-vanishing domain of convergence. The system of values (t, a, b) thus-obtained, like (t_3, a_3, b_3), belongs to the interior of the region ($\mathfrak{A}$) as long as the quantities $|x - x_3|$, $|y - y_3|$, $|\omega - \omega_3|$ do not exceed certain limits. Thus, if the point 4 lies sufficiently close to 3 (Fig. 15) and α is a sufficiently small, prescribed value then one can connect the points 0 and 4 by an extremal of the field such that the integral $\bar{K}_{04}$ will have the value $\omega_3 + \alpha$ along it. In particular, if one sets:

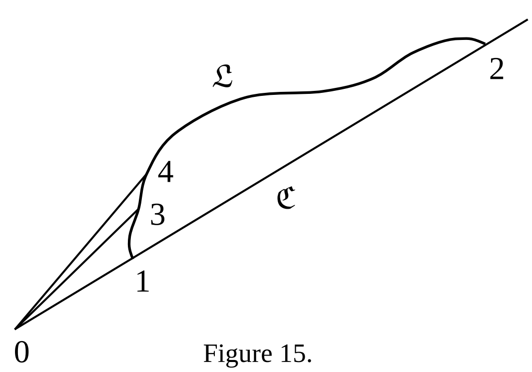

Figure 15.

$$\alpha = K_{34}$$

when one defines the integral along the curve $\mathfrak{L}$ then all arguments of the expressions (13), and therefore the values of a and b that one obtains, will be functions of τ of the type that was denoted by $\varphi(\tau)$ in § **17**. One will then have the relation:

$$\bar{K}_{04} = \bar{K}_{03} + K_{03}$$

for the two extremals 03 and 04, and it will follow from this that:

$$\bar{K}_{03} + K_{32} = \bar{K}_{03} + K_{42} \ .$$

If we then let the point 3 go to a neighboring position and connect it with the point 0 with the defined extremal 03 then the quantity $\bar{K}_{03} + K_{32}$ will remain constant. We can let the point 3 run through the entire path $\mathfrak{L}$ in that way, and if $\mathfrak{C}$ is the starting point and end point of the extremal 03 that was constructed then we will say that the **Weierstrass** construction is possible. The quantities a, b, which belong to the extremal 03, will then be functions of τ in the entire interval from τ_1 to τ_2, and they will have the properties of the quantities $\varphi(\tau)$ in § **17**. Their initial and final values are a_0, b_0, while the constant quantities $\bar{K}_{03} + K_{32}$ will assume the values:

$$\bar{K}_{01} + K_{12}, \qquad \bar{K}_{02} = \bar{K}_{01} + K_{12}$$

for $\tau = \tau_1$ and $\tau = \tau_2$, which are equal, from (12). For special problems, the **Weierstrass** construction can often be seen to be possible. We will develop a general condition for it to be carried out in § **38**.

Now in order to get closer to the question of the extremum of the J, we assume that the **Weierstrass** construction is possible and define the quantity:

$$W = \bar{J}_{03} + J_{32} \,,$$

such that the first integral will be defined along the extremal 03, while the second one will be defined along the curve $\mathfrak{L}$. That quantity is a function $\varphi(\tau)$ in the sense of § **17**, and since the extremal 03 ultimately goes to the position $\mathfrak{C}$, one will have:

$$W\Big|^{\tau_1} = \bar{J}_{01} + J_{12} \ , \qquad W\Big|^{\tau_2} = \bar{J}_{01} + \bar{J}_{12} = \bar{J}_{02} \ .$$

Therefore, if it so happens that the sign of the quantity dW is fixed then the difference $\bar{J}_{12} - J_{12}$ will have the same sign. However, the quantity dW is given explicitly by the formulas of § **35**. Namely, since $\bar{K}_{03} + K_{32}$ is constant, so:

$$\frac{d(\bar{K}_{03}+K_{32})}{d\tau} = 0$$

one will have:

$$\frac{dW}{d\tau} = \frac{d\bar{J}_{03}}{d\tau}+\lambda\frac{d\bar{K}_{03}}{d\tau}+\frac{dJ_{32}}{d\tau}+\lambda\frac{dK_{32}}{d\tau},$$

in which λ is the isoperimetric constant of the extremal 03. With the use of formula (11), and the obviously-correct equations:

$$\frac{dJ_{32}}{d\tau} = - F\left(x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau}\right), \qquad \frac{dK_{32}}{d\tau} = - G\left(x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau}\right),$$

that will imply:

(14)
$$\frac{dW}{d\tau} = H_{x'}(x, y, \xi_t, \eta_t)\frac{dx}{d\tau}+H_{y'}(x, y, \xi_t, \eta_t)\frac{dy}{d\tau}-H\left(x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau}\right).$$

That expression is identical to the previous one that was denoted by:

$$\frac{1}{d\tau}\mathcal{E}(x, y, x', y', Dx, Dy) = \mathcal{E}\left(x, y, x', y', \frac{dx}{d\tau}, \frac{dy}{d\tau}\right),$$

in which F is replaced by H. The properties of $\mathcal{E}$ that were derived above (§ **21**) can then be employed, and especially the distinction between ordinary and extraordinary vanishing. If $\mathcal{E}$ always has a constant sign and does not vanish along any of the curves $\mathfrak{L}$ that are under consideration then the desired extremum of the integral J will actually be provided by the arc 12.

When $\mathcal{E}$ vanishes ordinarily everywhere along a finite part of the curve $\mathfrak{L}$, one will have:

$$\frac{dx}{d\tau} = m\,\xi_t, \quad \frac{dy}{d\tau} = m\,\eta_t, \quad m > 0$$

along it. However, since the equations:

$$x = \xi, \quad y = \eta$$

can be differentiated with respect to τ, that will give:

$$\xi_t\frac{dt}{d\tau}+\xi_a\frac{da}{d\tau}+\xi_b\frac{db}{d\tau} = m\,\xi_t,$$

(15)

$$\eta_t \frac{dt}{d\tau} + \eta_a \frac{da}{d\tau} + \eta_b \frac{db}{d\tau} = m\ \eta_t\,.$$

Furthermore, one obviously has:

$$\frac{d\omega}{d\tau} = \omega_t \frac{dt}{d\tau} + \omega_a \frac{da}{d\tau} + \omega_b \frac{db}{d\tau} = m\ \omega_t\,.$$

However, with what was done in (15), that equation can be regarded as homogeneous and linear in the quantities:

$$\frac{da}{d\tau}, \quad \frac{db}{d\tau}, \quad \frac{dt}{d\tau} - m\,.$$

Now, since the determinant of the coefficients is identical to Δ, and thus non-zero, the three equations can be true simultaneously only when:

$$\frac{da}{d\tau} = \frac{db}{d\tau} = 0\,.$$

However, since a and b are determined from this as functions $\varphi(\tau)$ in the sense of § **17** by the **Weierstrass** construction, when 3 and 4 are any two points of the arc in question, it will follow that:

$$a\Big|_4^3 = b\Big|_4^3 = 0\,.$$

The arc will then be a piece of an extremal in the field. If $\mathcal{E}$ vanishes along the entire curve $\mathfrak{L}$ in the ordinary way then since a_0, b_0 are the final values of a, b, $\mathfrak{L}$ will coincide with $\mathfrak{C}$. If that is not the case, and if one has an extraordinary vanishing of $\mathcal{E}$, as well as a sign change that is excluded either unconditionally or by restricting the comparison curve $\mathfrak{L}$ then the difference:

$$W\Big|_{\tau_1}^{\tau_2} = \int_{\tau_1}^{\tau_2} \mathcal{E}\,dt = \bar{J}_{12} - J_{12}$$

will be non-zero and have the same sign as $\mathcal{E}$, which guarantees the extremum.

In total, there are then three sufficient conditions for the occurrence of an extremum:

1. The existence of a field, or the **Jacobi** condition.

2. The **Weierstrass** sign condition, i.e., the fixed sign of the quantity $\mathcal{E}$, in the given sense.

3. The possibility of making the **Weierstrass** construction inside of the field.

If those conditions are fulfilled then the extremal $\mathfrak{C}$ will produce an extremum for the integral J_{12} in comparison to the curves 12 or $\mathfrak{L}$ that run through the field and which will imply a prescribed value for K_{12} . The extremum will be strong or weak according to whether the sign of $\mathcal{E}$ is or is not guaranteed everywhere, resp., when $\mathfrak{L}$ lies in a narrow neighborhood of the curve $\mathfrak{C}$.

In that way, the extremum will not be provided by the arc that begins at 0, but by the arc 12, whose starting point 1 can be moved arbitrarily close to the position 0.

Moreover, if one replaces the function F with H in the general examination of § **21** then one will see that the quantity $\mathcal{E}$ that is considered here will possess a fixed sign as long as the same thing is true of the quantity:

$$H_1 = \frac{1}{y'^2} H_{x'x'} = \frac{1}{x'^2} H_{y'y'} = \frac{1}{x'^2} (F_{y'y'} + \lambda G_{y'y'}) ,$$

or also of the product $(f_{pp} + \lambda\, g_{pp})\, dx$, when x' does not vanish. The **Weierstrass** sign condition can then be replaced by the **Legendre** condition of § **17** when one lets $F + \lambda\, G$ enter in place of F in it.

If the conditions 1) and 2) are fulfilled then they will remain fulfilled for the problem that one gets when one switches the roles that the integrals J, K had up to now. For the condition 1), that will follow immediately from the remarks above that the property that certain extremals define a field will remain preserved under that exchange. Analogous statements will be true for the condition 2), as a result of the fact that H and $\mathcal{E}$ can be replaced by:

$$G + \frac{1}{\lambda} F , \qquad \frac{1}{\lambda} \mathcal{E} ,$$

resp. Obviously, since λ is non-zero, the latter quantity will have a fixed sign when that is true of $\mathcal{E}$, in which, of course, the sign does not need to be that of the quantity $\mathcal{E}$. The extremum will then have the same character in the new problem that it had in the old one or the opposite one according to whether λ is positive or negative, resp. With that, the simplest case of a reciprocity theorem that was presented by **Mayer** has been proved.

§ 37. – Problem IX with fixed endpoints.

Problem IX (§ **34**). – The extremals are circles of radius $\pm\lambda$. The ones that go through a point 0 – e.g., the coordinate origin – can be represented in the following way: Let a, b be the coordinates of the center m, and let the radius be r (Fig. 16). One can then set:

$$(16) \qquad x = a + r \cos \theta , \qquad y = b + r \sin \theta , \qquad r^2 = a^2 + b^2 ,$$

and θ is the angle that the radius must describe when it rotates in the positive sense upon starting from the position mx that is parallel to the $+x$-axis in order to bring its endpoint to the point 4 on the circle. As usual, positive is the sense of rotation that that would take the $+x$-axis to the $+y$-axis by a rotation through 90°. Furthermore, if φ is the angle by which the ray $0m$ deviates from the $+x$-axis, as measured in the positive sense, then one will have:

$$a = r \cos \varphi, \quad b = r \sin \varphi, \tag{17}$$

and the radius $0m$ will then be inclined by $\varphi + \pi$ with respect to mx. If one then sets:

$$t = \pi + \varphi - \theta, \qquad \theta = \pi + \varphi - t$$

then $r\,t$ will be the arc-length of 04, measured in the negative sense from the point 0, and from (16), (17), one will have:

$$K_{03} = \omega = t\sqrt{a^2 + b^2}, \qquad \xi(t, a, b) = a - a \cos t - b \sin t, \qquad \xi(t, a, b) = a - a \cos t - b \sin t.$$

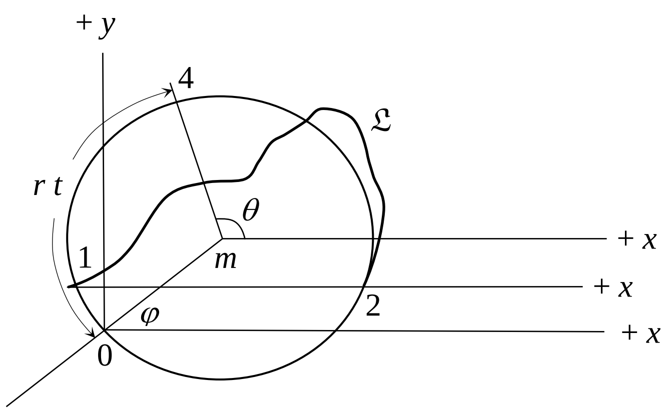

Figure 16.

Those functions are everywhere regular, so they will remain between two positive limits along r, which are completely arbitrary, moreover. Furthermore, one has:

$$\Delta = \begin{vmatrix} 1 - \cos t & \sin t & at/r \\ -\sin t & 1 - \cos t & bt/r \\ a \sin t - b \cos t & a \cos t + b \sin t & r \end{vmatrix},$$

and when one expresses a and b in terms of other variables α, β:

$$\frac{\partial(\xi, \eta, \omega)}{\partial(t, \alpha, \beta)} = \frac{\partial(\xi, \eta, \omega)}{\partial(t, a, b)} \cdot \frac{\partial(t, a, b)}{\partial(t, \alpha, \beta)} = \Delta \, \frac{\partial(a, b)}{\partial(\alpha, \beta)}. \tag{18}$$

In particular, if a, b are the rectangular coordinates of the point m in a new system then the factor next to Δ will have the value ± 1. Now, when $b = 0$, in particular, one will have $a^2 = r^2$, and that will make:

$$\Delta = \begin{vmatrix} 1-\cos t & \sin t & t \\ -\sin t & 1-\cos t & 0 \\ \sin t & \cos t & 1 \end{vmatrix}$$

$$= r\,(2 - 2\cos t - t\sin t) = 4r\sin\frac{t}{2}\left(\sin\frac{t}{2} - \frac{t}{2}\cos\frac{t}{2}\right).$$

However, if the point 4 traverses the circle in question then t will move in the interval:

$$0 \le t < 2\pi, \tag{19}$$

so the factor in the expression for Δ will be non-zero, except at the point 0. If b does not vanish then one replaces the coordinate system (a, b) with a new one (α, β) whose abscissa axis goes through m. From formula (18), Δ will then likewise remain non-zero. With that, it is shown that all circles that go through the point 0 and are subject to an inequality:

$$\varepsilon^2 < a^2 + b^2 < \gamma^2, \tag{20}$$

when ε, γ mean positive constants, will define a field in the sense of § **36** as long as one restricts t by the inequality (19), i.e., none of the circles are thought of as being traversed more than once.

Since:

$$F = y\,x', \qquad G = \sqrt{x'^2 + y'^2}, \qquad H = y\,x' + \lambda\sqrt{x'^2 + y'^2}$$

for this problem, and the square roots are positive, one will have:

$$\mathcal{E} = \left(y + \frac{\lambda\,x'}{\sqrt{x'^2 + y'^2}}\right)Dx + \frac{\lambda\,y'}{\sqrt{x'^2 + y'^2}}\,Dy - (y\,Dx + \lambda\sqrt{Dx^2 + Dy^2})$$

$$= \lambda\sqrt{Dx^2 + Dy^2}\left\{\frac{x'}{\sqrt{x'^2 + y'^2}}\frac{Dx}{\sqrt{Dx^2 + Dy^2}} + \frac{y'}{\sqrt{x'^2 + y'^2}}\frac{Dy}{\sqrt{Dx^2 + Dy^2}} - 1\right\}.$$

The bracketed quantity is never positive (§ **23**, Problem 1), and $\mathcal{E}$ vanishes only when:

$$\frac{x'}{\sqrt{x'^2 + y'^2}} = \frac{Dx}{\sqrt{Dx^2 + Dy^2}}, \qquad \frac{y'}{\sqrt{x'^2 + y'^2}} = \frac{Dy}{\sqrt{Dx^2 + Dy^2}},$$

i.e., only in an ordinary way. Furthermore, since the direction of increasing t has the same relationship to the outward-pointing radius that the $+x$-axis has to the $+y$-axis, from § **34**, λ will be negative, so $\mathcal{E}$ will be positive, and the desired extremum will be a maximum.

Now let the points 12 be connected by an arc of a circle. The point 0 lies on that circle outside of that arc. The abscissa axis will be laid through it and point parallel to the line 12, while the y-axis is laid through it such that y is positive along that line. The direction of increasing t that was defined above will then go along the arc $\mathfrak{C}$ from 1 to the point 2, and the integral:

$$\bar{J}_{12} = \int_1^2 y\, x'\, dt$$

will be the positive area between $\mathfrak{C}$ and the straight line segment 12, plus the right angle that is covered by the ordinates of the latter. Furthermore, let $\mathfrak{L}$ be an arbitrary curve that connects 1 and 2 and has the same continuity properties as in § **17** and the same length as $\mathfrak{C}$ and does not go into a circle of radius 2ε and center 0, which can always be arranged by a suitable choice of that point. All circles that go through 0 and a point 3 that belongs to the curve $\mathfrak{L}$ will then have a radius that is greater than ε. If one restricts oneself to ones whose radius does not exceed a certain limit γ then they will all belong to the field (20).

Now, if the point 3 runs through the arc $\mathfrak{L}$ then one can connect it with 0 in each of its positions with a circular arc of length:

$$\bar{\bar{K}}_{01} + K_{12} = \text{arc } 01 + \text{arc } 13 ,$$

in which the first arc is measured on the circle 012, while the second one is measured on the curve $\mathfrak{L}$. That quantity is greater than the rectilinear distance 03, so one will get a finite radius for the desired arc, and since it obviously varies continuously with 3, a finite upper limit γ. Of the two circular arcs with the desired behavior that lie symmetrically to the line 03, we shall always take the one that emerges continuously from the initial position $\mathfrak{C}$. Since the radius remains finite, $\mathfrak{C}$ will also be the final position of the arc 03, and that proves that the **Weierstrass** construction is possible inside of the field (20) in full rigor. Therefore, the general theory of § **36** implies that:

$$\bar{J}_{12} > J_{12} ,$$

i.e., the segment that is bounded by the circular arc $\mathfrak{C}$ and the straight line segment 12 is greater than the one that is defined by the curve $\mathfrak{L}$. Obviously that proves more than the maximum property.

§ 38. – Condition for the possibility of the Weierstrass condition. Problem XI

The **Weierstrass** construction can be implemented when one imposes a restriction on the curve $\mathfrak{L}$ that seems natural in many problems and consists of saying that when the integral K is defined along curves that start from 0 and go through neighboring points to the curves $\mathfrak{L}$ and $\mathfrak{C}$, it will deviate only slightly from the integrals along those curves. In order to establish that more precisely, we assume that an arbitrary curve that starts from the point 0 runs through the point 6, that K_{06} is the integral that one obtains along that curve, and that z is the third of the rectangular space coordinates. If we then set:

$$x = x_6 , \qquad y = y_6 , \qquad z = K_{06} ,$$

in general, then the point (x, y, z) will define a certain space curve that likewise starts from the point 0. Let the points in space that are associated with any point 1, …, 6 in that way be denoted by $1^0, \ldots, 6^0$, resp. Any extremal of the field will then correspond to a curve:

$$x = \xi(t, a, b) , \qquad y = \eta(t, a, b) , \qquad z = \omega(t, a, b) .$$

For $a = a_0$, $b = b_0$, one will get a space curve $\mathfrak{C}^0$ that corresponds to the curve $\mathfrak{C}$. The line that is composed of the extremal arc 01 and the curve $\mathfrak{L}$ corresponds to a space curve that coincides with $\mathfrak{C}^0$ up to the point 1^0 whose coordinates are:

$$x = x_1 , \qquad y = y_1 , \qquad z = \bar{K}_{01} = \omega(t_1, a_0, b_0) ,$$

but then splits apart from it. The point 3^0 has the coordinates:

$$x = x_3 , \qquad y = y_3 , \qquad z = \bar{K}_{01} + K_{13} . \tag{21}$$

The new convention now consists of saying that only those curves $\mathfrak{L}$ can be compared to $\mathfrak{C}$ in regard to the value of the integral J whose associated space curve $\mathfrak{L}^0$ belongs to a certain neighborhood (§ **17**) of the curve $\mathfrak{C}^0$.

A point 5 on the arc 01 corresponds to the point 5^0 on the latter for which:

$$x = x_5 , \qquad y = y_5 , \qquad z = \bar{K}_{05} = \omega(t_5, a_0, b_0) = z_5 .$$

The equations:

$$\begin{aligned} x - x_5 &= \xi(t, a, b) - x_5 = [t - t_5 , a - a_0 , b - b_0]_1 , \\ y - y_5 &= \eta(t, a, b) - y_5 = [t - t_5 , a - a_0 , b - b_0]_1 , \\ z - z_5 &= \zeta(t, a, b) - z_5 = [t - t_5 , a - a_0 , b - b_0]_1 , \end{aligned} \tag{22}$$

in which the linear terms on the right-hand side have the determinant $\Delta\,(t_5, a_0, b_0)$, can then be fulfilled in such a way that one substitutes certain well-defined series of the form:

$$[x - x_5\,,\, y - y_5\,,\, z - z_5]_1$$

for $t - t_5$, $a - a_0$, $b - b_0$. Those expressions define t, a, b as functions of x, y, z that are regular at every location 5^0 and will therefore be determined regularly and uniquely when the curve $\mathfrak{C}$ does not intersect itself (so different point 5^0 also belong to different values of t_5) inside of a certain region $\mathfrak{G}^0$ that surrounds the arc $1^0\,2^0$, such that they will be fulfilled precisely and simply by the extremals that corresponds to the expressions a, b that were defined above. If the point in question falls along the curve $\mathfrak{C}^0$ itself then a and b will go to a_0 and b_0 . One can represent the region $\mathfrak{G}^0$ as the space that a ball of constant radius will sweep out when its center traverses the arc $1^0\,2^0$.

Now, if the curve $\mathfrak{L}^0$ runs completely within the region $\mathfrak{G}^0$ then every point 3^0 will lie along a certain extremal of the field, and one will have the equations:

$$x_3 = \xi\,(t_3\,, a, b)\,, \qquad y_3 = \eta\,(t_3\,, a, b)\,, \qquad z_3 = \omega\,(t_3\,, a, b)\,.$$

According to the third equation in (21), it will follow from this that:

$$\omega\,(t_3\,, a, b) = \bar{K}_{03} = \bar{K}_{01} + K_{13}\,,$$

so

$$\bar{K}_{03} + K_{32} = \bar{K}_{01} + K_{12} = \bar{K}_{01} + \bar{K}_{12} = \bar{K}_{02}\,.$$

The projection of the constructed curve 03^0 onto the xy-plane is then precisely the desired extremal 03 in the **Weierstrass** construction, and from the remarks that were connected with equations (22), it will go to $\mathfrak{C}$ when the point 3^0 moves to one of the positions 1^0 and 2^0. With the newly-introduced restriction on the curve $\mathfrak{L}$, the **Weierstrass** construction will always be possible then, and one of the sufficient conditions for the extremum that was given in § **36** will drop out.

Example. Problem XI. – A massive homogeneous string of given length and endpoints is laid in a vertical plane in such a way that its center of mass lies as deep as possible (equilibrium figure, catenary).

If the $+y$-axis is direction of gravity, l is the length of the string 01, and the mass per unit length is unity then the ordinate of the center of gravity will be:

$$\frac{1}{l}\int_0^1 y\sqrt{x'^2 + y'^2}\,dt\,.$$

Therefore, it is the integral:

$$J = \int_0^1 y\sqrt{x'^2 + y'^2}\,dt$$

that must be maximized for a prescribed value of the arc-length integral:

$$K = \int_0^1 \sqrt{x'^2 + y'^2}\,dt\ ,$$

and the first variation of the integral:

$$J + \lambda\,K = \int (y+\lambda)\sqrt{x'^2 + y'^2}\,dt$$

must be set equal to zero. One gets the equation for the extremals from that (§ **9**, Problem II):

$$y + \lambda = \frac{a}{2}\left\{e^{\frac{x-b}{a}} + e^{\frac{-x+b}{a}}\right\} = a\cosh\frac{x-b}{a}\ .$$

The right-hand side has the sign of the quantity a for $x = \pm\,\infty$, so the curve will be convex downwards for $a < 0$. $y + \lambda$ will always be negative then, and the quantity:

$$\mathcal{E} = (y+\lambda)\sqrt{Dx^2 + Dy^2}\left\{\frac{x'\,Dx + y'\,Dy}{\sqrt{x'^2 + y'^2}\sqrt{Dx^2 + Dy^2}} - 1\right\}$$

will be positive and will be equal to zero only when the system of values (x', y') and (Dx, Dy) represent the same direction. The **Weierstrass** sign condition for a strong maximum is then fulfilled.

Further calculations are simplified by means of the easily-verified formulas:

$$\begin{aligned}\cosh(u+v) &= \cosh u\ \cosh v + \sinh u\ \sinh v,\\ \sinh(u+v) &= \sinh u\ \cosh v + \sinh u\ \cosh v.\end{aligned}\tag{23}$$

One has the relation:

$$y_0 + \lambda = a\cosh\frac{x_0 - b}{a}$$

for the constants of the extremal that goes through the fixed point 0, and the general equation of that family of curves can then be written:

$$y - y_0 = a\left\{\cosh\frac{x-b}{a} - \cosh\frac{x_0 - b}{a}\right\}.$$

The right-hand side of that equation is regular as long as a is non-zero, and one can set:

$$x = \xi(t, a, b) = t, \qquad y = \eta(t, a, b) = y_0 + a\left\{\cosh\frac{x-b}{a} - \cosh\frac{x_0-b}{a}\right\}.$$

Now since:

$$\xi_t = 1, \qquad \xi_a = \xi_b = 0,$$

one has:

$$\Delta(t, a, b) = \frac{\partial(\xi,\eta,\omega)}{\partial(t,a,b)} = \frac{\partial(\eta,\omega)}{\partial(a,b)}.$$

Furthermore, ω is the arc-length that one measures from the point 0, so:

$$\omega = \int_0 \sqrt{1+y'^2}\,dt = \int_0 \sqrt{1+\sinh^2\frac{t-b}{a}}\,dt = \int_0 \cosh\frac{t-b}{a}\,dt = a\left\{\sinh\frac{t-b}{a} - \sinh\frac{t_0-b}{a}\right\}.$$

When one sets:

$$\frac{t-b}{a} = u, \qquad \frac{t_0-b}{a} = u_0,$$

on the grounds of the formulas for the differentials of the hyperbolic function (§ **23**), one will have:

$$-\Delta = \begin{vmatrix} \cosh u - \cosh u_0 - (u\sinh u - u_0\sinh u_0) & \sinh u - \sinh u_0 \\ \sinh u - \sinh u_0 - (u\cosh u - u_0\cosh u_0) & \cosh u - \cosh u_0 \end{vmatrix}.$$

If one further differentiates that and employs equations (23) then it will follow that:

$$-\frac{d\Delta}{du} = \begin{vmatrix} -u\cosh u & \sinh u - \sinh u_0 \\ -u\sinh u & \cosh u - \cosh u_0 \end{vmatrix} \begin{vmatrix} \cosh u - \cosh u_0 - (u\sinh u - u_0\sinh u_0) & \sinh u - \sinh u_0 \\ \sinh u - \sinh u_0 - (u\cosh u - u_0\cosh u_0) & \cosh u - \cosh u_0 \end{vmatrix}$$

$$= (u - u_0)\cosh(u - u_0) - \sinh(u - u_0).$$

It will then follow that:

$$-\frac{d^2\Delta}{du^2} = (u - u_0)\sinh(u - u_0).$$

Since the quantities $u - u_0$ and $\sinh(u - u_0)$ have the same sign, one will always have:

$$\frac{d^2\Delta}{du^2} \le 0.$$

Furthermore, since the quantities Δ and $d\Delta : du$ vanish for $u = u_0$, the latter will always have the sign of $u - u_0$, and the former will be negative as long as $u \neq u_0$. If one starts from the point 0 on the catenary and moves in the direction of increasing x then:

$$u - u_0 = \frac{x - x_0}{a} ,$$

and just as a is negative, Δ will also be negative. Therefore, the sufficient conditions for a strong minimum with the given restriction on the curve $\mathfrak{L}$ will be fulfilled for an arbitrarily long piece of the convex-downward catenary. From a remark that was made in § **36**, that will imply a maximum for any arc 12 whose starting point lies arbitrarily-close to 0. Now since the position of the latter is not subject to any restriction, the arc 12 can be regarded as an arbitrary part of the curve.

The restriction on the curve $\mathfrak{L}$ that was introduced here seems natural, because for any small displacement of a string, the individual material points that are defined by a certain string length, as measured from the point 0, will be displaced only slightly. The curves $\mathfrak{L}$ and $\mathfrak{C}$ will then be related to each other in a single-valued way such that corresponding points will have a small separation distance and will yield the same value of the integral K.

§ 39. – Sufficient conditions for an extremum with one moving endpoint.

The general developments of the last section will remain valid with minor modifications when one assumes that the extremals:

(24) $$x = \xi(t, a, b) , \qquad y = \eta(t, a, b) ,$$

to which $\mathfrak{C}$ belongs, do not go through a fixed point 0 but all intersect a fixed curve $\mathfrak{C}_0$ transversally. One will then get a criterion for a piece of the curve $\mathfrak{C}$ to yield an extremum for the integral from among all curves that connect a fixed point to the curve $\mathfrak{C}_0$ and give a prescribed value to the integral K.

Let:

$$\Gamma(x, y) = 0$$

be the equation of the curve $\mathfrak{C}_0$. The latter will cut $\mathfrak{C}$ and the curves of the family (24) that deviate from it sufficiently little at the variable point 0 at a non-vanishing angle, and let it be regular at that point. One will then have the equation:

(25) $$\Gamma[\xi(t_0, a, b), \eta(t_0, a, b)] = 0 ,$$

which can be solved for t_0 , since the inequality:

$$\Gamma_x\,\xi_t + \Gamma_y\,\eta_t \neq 0$$

will be true for the point 0 in question. When t_{00} is the parameter that belongs to the curve $\mathfrak{C}$ at its point of intersection with $\mathfrak{C}_0$, one will then get:

$$t_0 - t_{00} = [a - a_0\,,\, b - b_0]_1\,.$$

Since the extremals of the family (24) further intersect the curve $\mathfrak{C}_0$ transversally, under the assumption that:

$$\Gamma_x\,Dx + \Gamma_y\,Dy\,|^0 = 0\,,$$

one will always get the equation:

$$H_{x'}\,Dx + H_{y'}\,Dy\,\Big|^0 = 0\,.$$

It will then follow that:

(26) $$\Lambda = H_{x'}\,\Gamma_y - H_{y'}\,\Gamma_x\,\Big|^0 = 0\,,$$

in which one takes the arguments:

(27) $$x = \xi\,, \qquad y = \eta\,, \qquad x' = \xi_t\,, \qquad y' = \eta_t\,.$$

If one then differentiates equation (25) with respect to a and b then that will give:

$$\Gamma_x\left(\xi_t\,\frac{\partial t_0}{\partial a} + \xi_a\right) + \Gamma_y\left(\eta_t\,\frac{\partial t_0}{\partial a} + \eta_a\right)\Bigg|^0 = 0\,,$$

$$\Gamma_x\left(\xi_t\,\frac{\partial t_0}{\partial b} + \xi_b\right) + \Gamma_y\left(\eta_t\,\frac{\partial t_0}{\partial b} + \eta_b\right)\Bigg|^0 = 0\,.$$

One can replace the quantities Γ_x and Γ_y with $H_{x'}$ and $H_{y'}$ in those equations. They will then be precisely identical to the formulas (9). If one then defines the integrals:

$$\bar{J}_{03} = \int_0^3 F\,dt = \int_{t_0}^{t} F(\xi, \eta, \xi_t, \eta_t)\,dt\,, \qquad \bar{K}_{03} = \int_{t_0}^{t} G(\xi, \ldots, \eta_t)\,dt$$

along any extremal of the family that belong belongs to the variable point 3 then equation (7), which is independent of the connection between t_0, a, b, will still be true, and equations (10), (11) will then follow, precisely as they did in § **35**, on the basis of formula (9).

As before, we will now set:

$$\bar{K}_{03} = \omega(t, a, b)$$

and assume that the determinant:

$$\Delta = \frac{\partial(\xi, \eta, \omega)}{\partial(t, a, b)}$$

does not vanish along the arc 02 that belongs to the curve $\mathfrak{C}$, except at the point 0. The same thing will be true for all arcs of the extremals of the family that begin at the curve $\mathfrak{C}_0$ and deviate from 02 sufficiently little. The totality of all arcs for which Δ remains non-zero is called the *field* of the arc 02. Δ itself vanishes everywhere along the curve $\mathfrak{C}_0$, as is easily inferred from the formulas (7), (9). A certain neighborhood of the arc 02 will then be covered infinitely-many times by the extremals of the field, in general, as an argument that was made in § **35** will also show now. Inside of that neighborhood, a point 1 that belongs to the curve $\mathfrak{C}$ will be connected with 2 by a line $\mathfrak{L}$ for which the equation:

$$K_{12} = \bar{K}_{02} \tag{28}$$

is satisfied, whose right-hand side is integrated along the curve $\mathfrak{C}$, while the integral without the overbar might also always refer to the curve $\mathfrak{L}$ now. It will run through the point 3. One will then have a construction that is analogous to the **Weierstrass** one that consists of saying that an extremal of the field that varies with 3 is constructed for which the quantity $\bar{K}_{03} + K_{32}$ remains constant and which will go to the curve $\mathfrak{C}$ when the points 3 and 2 coincide. The initial and final values of those quantities are obviously the two sides of equation (28) then. However, the proof that was given in § **38** for the possibility of making the **Weierstrass** construction cannot be adapted to the present problem with no further assumptions, since Δ vanishes along the curve $\mathfrak{C}_0$, as was pointed out. That fact does not affect the argument in § **36**, by which $\mathcal{E}$ can vanish everywhere in an ordinary way only along an extremal. Since the quantity $\bar{K}_{03} + K_{32}$ is now independent of τ, and furthermore, equations (7), (10), (11) in § **35** are also true now, as was shown, that will imply, as in § **36**, that:

$$\frac{dW}{d\tau} = \frac{d(\bar{J}_{03} + J_{32})}{d\tau} = H_{x'}\frac{dx}{d\tau} + H_{y'}\frac{dy}{d\tau} - H\left(x, y, \frac{dx}{d\tau}, \frac{dy}{d\tau}\right) = \mathcal{E}\left(x, y, x', y', \frac{dx}{d\tau}, \frac{dy}{d\tau}\right).$$

The initial and final values of the quantity W are different from the previous ones. Obviously, one must now set:

$$W\Big|^{\tau_1} = J_{12}, \qquad W\Big|^{\tau_2} = \bar{J}_{02}.$$

The difference $\bar{J}_{02} - J_{12}$ then has the sign of the quantity $\mathcal{E}$.

If the extremals of the field intersect the regular curve $\mathfrak{C}_0$ transversally without contacting it then the sufficient conditions for an extremum that were given in § **36** will have the same meaning for the extremum from among all lines $\mathfrak{L}$ that connects the curve $\mathfrak{C}_0$ to a fixed point.

Example. – Problem IX will be specialized in the following way: Draw an arbitrary fixed curve $\mathfrak{F}$ from a point 2 to a circular line $\mathfrak{C}_0$. Connect the point 2 with the circle $\mathfrak{C}_0$ with a second curve of given length that encloses the greatest possible area with the circle and the curve $\mathfrak{F}$.

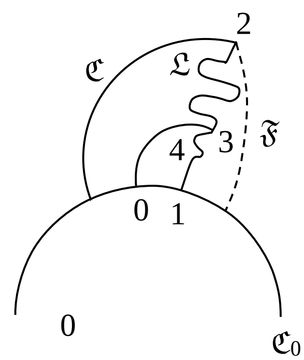

Figure 17.

The extremals are circles (Fig. 17) and, from § **34**, they lie transverse to the circle $\mathfrak{C}_0$ when they intersect it perpendicularly. Now, since any point 3 can be connected with the circle $\mathfrak{C}_0$ by a circular arc 03 of given length that is orthogonal to it as long as that length exceeds the shortest line segment that connects the point and $\mathfrak{C}_0$, the **Weierstrass** construction will always be possible along any arc $\mathfrak{L}$ or 12 of the prescribed length, such that the arc 03 is equal to the arc 13, as measured along the curve $\mathfrak{L}$. In that way, the circular arc 03 can be chosen such that a point that moves along the circle $\mathfrak{C}_0$ in a certain direction will go towards the concave side of the arc 03 at each location 0. However, that construction can be performed only for a piece 42 of the arc $\mathfrak{L}$ when it begins with the straight line segment 14 that is normal to the circle $\mathfrak{C}_0$, but leaves the line 14 at the point 4. The segment 14 will then be the limit of the arc 03 when one lets 3 and 4 coalesce, such that one can set:

$$\bar{J}_{03} + J_{32}\Big|^{\tau_4} = J_{12}\,, \qquad \bar{K}_{03} + K_{32}\Big|^{\tau_4} = K_{12}\,. \tag{29}$$

Furthermore, since the limiting position of the arc 03 will be the arc $\mathfrak{C}$ or 02 to which $\mathfrak{L}$ is compared when the point 3 moves into the position 2, the equation:

$$\lim\ \bar{J}_{03} = \bar{J}_{02}$$

will be valid, so from (29):

$$\bar{J}_{03} + J_{32}\Big|_{\tau_4}^{\tau_2} = \int_{\tau_4}^{\tau_2} \mathcal{E}\,dt = \bar{J}_{02} - J_{12}\,,$$

from which the extremum will follow as long as the **Jacobi** condition is fulfilled, since $\mathcal{E}$ has a fixed sign.

In order to recognize the circumstances under which that will be the case, we assume that the equation of the circle $\mathfrak{C}_0$ is:

$$x_0^2 + y_0^2 = r^2 .$$

If b is the arc-length that increases in the direction $\mathfrak{R}$ then the coordinates of the center of an arc 03 of radius | a | will be:

$$\alpha = x_0 + a\frac{dx_0}{db}, \qquad \beta = y_0 + a\frac{dy_0}{db}.$$

From the meaning of $\mathfrak{R}$, a is positive. From the manner of representation that was employed in § **37**, when one replaces the coordinate origin with the point 0, the circular arc 03 can be represented by the following equations:

$$x - x_0 = (\alpha - x_0) - (\alpha - x_0)\cos t - (\beta - y_0)\sin t ,$$
$$y - y_0 = (\beta - y_0) + (\alpha - x_0)\sin t - (\beta - y_0)\cos t ,$$

or when one sets:

$$\xi(t, a, b) = x_0 + a\frac{dx_0}{db} - a\frac{dx_0}{db}\cos t - a\frac{dy_0}{db}\sin t ,$$

$$\eta(t, a, b) = y_0 + a\frac{dx_0}{db} + a\frac{dx_0}{db}\sin t - a\frac{dy_0}{db}\cos t ,$$

by the equations:

$$x = \xi(t, a, b), \qquad y = \eta(t, a, b).$$

Obviously, $t = 0$ at the point 0. One will then have:

$$\bar{K}_{03} = a\,t = \omega(t, a, b),$$

$$\Delta = \frac{\partial(\xi, \eta, \omega)}{\partial(t, a, b)}$$

$$= \begin{vmatrix} a\frac{dx_0}{db}\sin t - a\frac{dy_0}{db}\cos t & a\frac{dx_0}{db}\cos t + a\frac{dy_0}{db}\sin t & a \\ \frac{dx_0}{db}(1-\cos t) - \frac{dy_0}{db}\sin t & \frac{dy_0}{db}(1-\cos t) + \frac{dx_0}{db}\sin t & t \\ \frac{dx_0}{db} + a\frac{d^2x_0}{db^2}(1-\cos t) - a\frac{d^2y_0}{db^2}\sin t & \frac{dy_0}{db} + a\frac{d^2y_0}{db^2}(1-\cos t) + a\frac{d^2x_0}{db^2}\sin t & 0 \end{vmatrix}.$$

Now when the equations:

(30) $$\frac{dy_0}{db} = 0, \qquad \frac{dx_0}{db} = 1$$

are true for the arc 03 in question, the identity:

$$\frac{dy_0}{db}\frac{d^2y_0}{db^2}+\frac{dx_0}{db}\frac{d^2x_0}{db^2} = 0$$

will give the immediate consequence:

$$\frac{d^2x_0}{db^2} = 0 ,$$

and one will then get:

$$\Delta = -\,a\,(\sin t - t\cos t) + a^2\frac{d^2y_0}{db^2}\begin{vmatrix} \sin t & \cos t & 1 \\ 1-\cos t & \sin t & t \\ -\sin t & 1-\cos t & 0 \end{vmatrix} ,$$

and with the relation:

$$\varphi(t) = \sin t - t\cos t ,$$

that will give:

$$\Delta = -\,a\,\varphi(t) + 4a^2\frac{d^2y_0}{db^2}\sin\frac{t}{2}\,\varphi\!\left(\frac{t}{2}\right) .$$

Furthermore, one obviously has the equations:

$$x_0 = r\cos\frac{b}{r}, \qquad y_0 = r\sin\frac{b}{r}$$

along the circle $\mathfrak{C}_0$, and when one makes the assumption:

$$b = \frac{3r\pi}{2} ,$$

for the point 0, the equations (30) will be satisfied, and therefore also the following ones:

$$\frac{d^2y_0}{db^2} = -\frac{1}{r}\sin\frac{b}{r} = \frac{1}{r} ,$$

$$\Delta = -\frac{a}{r}\left\{r\,\varphi(t) - 4a\frac{t}{2}\,\varphi\!\left(\frac{t}{2}\right)\right\} .$$

Therefore, as long as the circular arc 03, as measured in units of the radius a, does not attain the smallest root of the equation:

$$r\,\varphi(t) - 4\,a\sin\frac{t}{2}\,\varphi\left(\frac{t}{2}\right) = 0\,,$$

the arc 02 will actually yield an extremum of the area that is enclosed by that arc, the circle $\mathfrak{C}_0$, and a fixed line that connects the latter to 2.

One easily sees that this result will remain valid when $\mathfrak{C}_0$ is an arbitrary regular curve and r is its radius of curvature at the point at which that curve intersects the circular arc 03 orthogonally.

Problem XI (§ 38). – Investigate the stability of a massive string when one endpoint is fixed and the other one moves along a given curve.

We begin with the following general remark: Let a two-fold infinite family of extremals that go through a point 0 or cut the curve $\mathfrak{C}_0$ transversally be given by the equations:

$$x = \xi(t, a, b, c)\,, \qquad y = \eta(t, a, b, c)\,,$$

$$\bar{K}_{03} = \omega = \omega(t, a, b, c)\,, \qquad g(a, b, c) = 0\,,$$

and let g_c be non-zero. The quantity Δ will then have the following form:

$$\Delta = \frac{\partial(\xi,\eta,\zeta)}{\partial(t,a,b)} = \begin{vmatrix} \xi_t & \xi_a + \xi_c\dfrac{\partial c}{\partial a} & \xi_b + \xi_c\dfrac{\partial c}{\partial b} \\ \eta_t & \eta_a + \eta_c\dfrac{\partial c}{\partial a} & \eta_b + \eta_c\dfrac{\partial c}{\partial b} \\ \omega_t & \omega_a + \omega_c\dfrac{\partial c}{\partial a} & \omega_b + \omega_c\dfrac{\partial c}{\partial b} \end{vmatrix},$$

and since the equations:

$$g_a + g_c\frac{\partial c}{\partial a} = g_b + g_c\frac{\partial c}{\partial b} = 0$$

are obviously true, it will follow that:

$$\Delta g_c^2 = \begin{vmatrix} \xi_t & \xi_a\,g_c - \xi_c\,g_a & \xi_b\,g_c - \xi_c\,g_b \\ \eta_t & \eta_a\,g_c - \eta_c\,g_a & \eta_b\,g_c - \eta_c\,g_b \\ \omega_t & \omega_a\,g_c - \omega_c\,g_a & \omega_b\,g_c - \omega_c\,g_b \end{vmatrix} = \begin{vmatrix} \xi_t & \xi_a & \xi_b & \xi_c \\ \eta_t & \eta_a & \eta_b & \eta_c \\ \omega_t & \omega_a & \omega_b & \omega_c \\ 0 & g_a & g_b & g_c \end{vmatrix},$$

such that $g_c\,\Delta$ will be equal to the fourth-order determinant on the right in that.

The given curve $\mathfrak{C}_0$ has the equation:

$$y_0 = f(x_0)\,.$$

Let it not be horizontal at the location considered, so $f'(x_0)$ will be non-zero. The extremals, for which one has:

$$y + \lambda = a \cosh \frac{x-b}{a}, \qquad p = \sinh \frac{x-b}{a},$$

must be intersected perpendicularly by $\mathfrak{C}_0$ at the point 0, when 0 is the moving endpoint of the string. One will then have the equations:

$$y_0 + \lambda = a \cosh \frac{x-b}{a},$$

(31)

$$1 + f'(x_0)\, p_0 = 1 + f'(x_0) \sinh \frac{x_0 - b}{a} = 0,$$

and when one sets – $x = t$ and lets ω denote the length that is measured from the point 0 in the sense of decreasing x, the total family of transversally-intersecting extremals be defined by the following equations:

$$x = -t, \qquad y = f(x_0) + a\left(\cosh \frac{x-b}{a} - \cosh \frac{x_0 - b}{a}\right),$$

$$\omega = -a\left(\sinh \frac{x-b}{a} - \sinh \frac{x_0 - b}{a}\right), \qquad 0 = 1 + f'(x_0) \sinh \frac{x_0 - b}{a}.$$

The quantity – Δ, when multiplied by the analogue of the quantity that was denoted by g_c above, will then be the functional determinant of the right-hand side of the last three equations with respect to a, b, x_0 . If one sets:

$$\frac{x-b}{a} = u, \qquad \frac{x_0 - b}{a} = u_0$$

then one will immediately find on the basis of the addition formulas for the functions sinh and cosh that:

$$\Delta = -\Phi(u) + \frac{1}{a} \coth^2 u_0 \, \Phi'(u) \left[f''(x_0) \sinh u_0 - \frac{1}{a} \coth u_0 \right]^{-1},$$

in which one has set:

$$\Phi(u) = 2 - 2 \cosh (u - u_0) + (u - u_0) \sinh (u - u_0).$$

The following equations are true for that function:

$$\Phi(u_0) = \Phi'(u_0) = 0, \quad \Phi''(u_0) = (u - u_0) \sinh (u - u_0),$$

$$\Phi(u)\,\Phi''(u) - [\Phi'(u)]^2 = [\Phi(u)]^2\,\frac{d}{du}\left[\frac{\Phi'(u)}{\Phi(u)}\right] = -\,[u - u_0 - \sinh(u - u_0)]^2\,,$$

$$\frac{\Phi'(u)}{\Phi(u)} = \frac{u - u_0 - \tanh(u - u_0)}{(u - u_0)\tanh(u - u_0) - 2 + [2 : \cosh(u - u_0)]}\,,$$

from which it will immediately follow that the quantities $\Phi(u)$ and $\Phi'(u)$ will always be positive for $u > u_0$, but the quotient $\Phi'(u) : \Phi(u)$ will continually decrease in the interval from $+\infty$ to $+1$ when one starts from u_0 and lets it become positively infinite.

If one now focusses upon only those equilibrium positions for which the **Legendre** condition for the maximum of the center of gravity ordinate is fulfilled then the extremals in question must be convex downwards, and therefore, since $+y$ is the direction of gravity, $a < 0$. The equations:

$$\text{(32)} \qquad \Delta = 0\,, \qquad \frac{\Phi'(u)}{\Phi(u)} = \tanh u_0\left[\frac{a\,f''(x_0)\sinh^2 u_0}{\cosh u_0} - 1\right]$$

will then have no root above the value u_0 when $\sinh u_0$, and therefore $\tanh u_0$, as well as $f''(x_0)$, is positive. However, since the equation can be written:

$$\cosh u_0\left[\coth u_0\,\frac{\Phi'(u)}{\Phi(u)} + 1\right] = a\,f''(x_0)\sinh^2 u_0\,,$$

the same thing will be true when $\sinh u_0$ is negative, so $f''(x_0)$ will not be positive. That is because one will then have:

$$\coth u_0 < 1\,,$$

so the left-hand side of the previous equation will certainly be negative for $u > u_0$, while the right-hand side will positive or zero. Now, the quantities $\sinh u_0$ and $f''(x_0)$ will have opposite signs, from (31). Hence, $f'(x_0)$ and $f''(x_0)$ will have opposite signs in the two cases considered, such that the direction of decreasing x (or since a is negative, increasing u) will refer to the convex side of the curve C_0. The **Jacobi** condition for the stability of equilibrium will therefore always be fulfilled when the string goes to the convex side of the curve $\mathfrak{C}_0$ at the point 0 or the latter has zero curvature at the point 0, i.e., it is a straight line. When the **Weierstrass** construction is possible in one of those cases, all arcs 03 will lie inside of the field, and the stability will be guaranteed for an arbitrarily-long piece of the curve.

If $f''(x_0)$ and $f'(x_0)$ have the same sign, such that the string (for which $u > u_0$) goes to the concave side of the curve $\mathfrak{C}_0$, then equation (32) will have a single root $\bar{u}$ for $u > u_0$ as long as the inequality:

$$\tanh u_0 \left[\frac{a\, f''(x_0) \sinh^2 u_0}{\cosh u_0} - 1 \right] > 1$$

is satisfied. If one then regards the direction of the curve $\mathfrak{C}_0$ and the form of the string, i.e., the quantities a and u_0, as fixed, and lets the curvature of the curve $\mathfrak{C}_0$ vary then a root $\bar{u}$ will appear as long as the absolute value of the curvature has attained a certain limit. If it increases beyond all bounds then $\bar{u}$ will continually decrease and approach the limit u_0 , i.e., the piece of the string for which the **Jacobi** condition is fulfilled will become infinitely small.

In order to discuss the **Weierstrass** construction, we shall now look for a catenary that contacts the fixed line T that is neither horizontal nor vertical at the point 0 and contains the point 1, in addition, which does not lie vertically to it, like 0. One can write the equation for the line T:

$$y - y_0 = (x - x_0) \sinh u_0 .$$

The value u_0 will then be determined uniquely since the function sinh increasingly runs through all values from $-\infty$ to $+\infty$ for an increasing argument. The set of all catenaries that contact the line T at the point 0 will be represented by the equations:

$$\text{(33)} \qquad y - y_0 = a\,(\cosh u - \cosh u_0) , \qquad x = x_0 + a\,(u - u_0) ,$$

in which u is a parameter. If we regard a as varying and x as fixed then we will have:

$$\frac{\partial y}{\partial a} = \cosh u + a \sinh u \frac{\partial y}{\partial a} - \cosh u_0 = \cosh u - (u - u_0) \sinh u - \cosh u_0 .$$

The derivative of the right-hand side with respect to u is $(u_0 - u) \cosh u$, so it will have the sign of the difference, while the quantity itself will vanish for $u = u_0$ and will then be negative as long as u is finite and remains different from u_0 . Now, for infinitely-small values of a, one will get:

$$\cosh u = +\infty , \qquad y = \pm\infty ,$$

according to whether a is positive or negative, resp. Furthermore, since the first equation in (33) can be written as:

$$y - y_0 = (x - x_0) \sinh u_0 + (x - x_0)\, [u - u_0]_1 ,$$

in which the coefficients of the last series are independent of a, it will follow that for $a = \pm\infty$, one has the equation:

$$y = y_0 + (x - x_0) \sinh u_0 .$$

The quantity y then decreases from that value down to $-\infty$, and then likewise from $+\infty$ back down to that value, when one lets a increase from $-\infty$ to $+\infty$ and establishes a fixed value for x (say, x_1).

When $x_1 - x_0$ is non-zero, one will then get $y = y_1$ for a single, well-defined value of a, with which the desired curve has been found and proves to be uniquely determined. It will be convex downwards ($a < 0$) when the point (x, y) lies above the line T, i.e., when:

$$y_1 < y_0 + (x_1 - x_0) \sinh u_0 .$$

We shall now restrict ourselves to the case $f'(x_0) > 0$, in which the catenary proceeds in the direction of decreasing x and then drops below the curve $\mathfrak{C}_0$. If the point 3 on the curve $\mathfrak{L}$ then approaches the position 1 (Fig. 18), i.e., its point of intersection with $\mathfrak{C}_0$, then the distance 13 or K_{12} that is measured along the curve $\mathfrak{L}$ will become infinitely small. If one connects the point 3 to a fixed point 4 on the curve $\mathfrak{C}_0$ that lies below the point 1 by an extremal that is at right angles to the curve $\mathfrak{C}_0$ then it will be convex downwards since as long as the arc 13 is sufficiently small, the point 3 will lie above the normal at the point 4, and the length of the extremal arc 34, which we will call $\mathfrak{K}_{34}$, will remain above a positive limit when K_{13} decreases indefinitely. It follows from this that one can isolate a part $\mathfrak{L}_1$ of the curve $\mathfrak{L}$ that begins at 1, and for which the inequality:

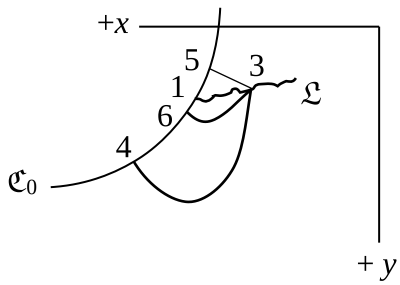

Figure 18.

(34) $$K_{12} < \mathfrak{K}_{34}$$

is valid, as long as 3 is a point of the arc $\mathfrak{L}_1$. Furthermore, if 35 is the normal to the curve $\mathfrak{C}_0$ that goes through 3 and one lets the point 6 go from 4 to 5 along the curve $\mathfrak{C}_0$ then it will never lie vertically above or below 3, and its normal will always run through the point 3. Therefore, an upward-convex extremal 36 that intersects the curve $\mathfrak{C}_0$ perpendicularly can always be constructed whose length $\mathfrak{K}_{36}$ will satisfy the inequality:

$$\mathfrak{K}_{36} > \mathfrak{K}_{35} ,$$

when $\mathfrak{K}_{35}$ is the rectilinear distance 35, and $\mathfrak{K}_{36} - \mathfrak{K}_{35}$ can be arbitrarily small. Now since one obviously has:

$$K_{13} \geq \mathfrak{K}_{35} ,$$

the inequality (34) will imply that:

$$K_{13} = \mathfrak{K}_{36}$$

for at least one position of the point 6, which completes the **Weierstrass** construction for the arc $\mathfrak{L}_1$. For the rest of the curve $\mathfrak{L}$, the possibility of making that construction will follow from § **38** when one varies in the manner that was given there, because the quantity Δ will no longer vanish then as long as the **Jacobi** condition for the catenary 02 is fulfilled. The case in which the curve $\mathfrak{L}$ begins with a straight line segment can be resolved just as easily as it was in Problem IX above.

§ 40. – Conjugate points. Extremal focal points.

The following development will be valid regardless of whether the extremals of the field have a common fixed point 0 or intersect a fixed curve $\mathfrak{C}_0$ transversally at the variable point 0.

If one moves along the arc $\mathfrak{C}$ upon starting from the point 0 that lies along it then the quantity Δ must vanish for the first time at a point 1 for an argument $t = t_1$ such that the functions ξ, η, ω are regular at the location (t_1, a_0, b_0) and the quantities ξ_t, η_t do not both vanish. In the first of the cases that were distinguished above, the point 1 will then be called *conjugate* to the point 0, and in the second, it will be called the *extremal focal point* of the curve $\mathfrak{C}_0$ for the present problem, and one will have the characteristic property:

$$\Delta\,(t_1, a_0, b_0) = 0 \,. \tag{35}$$

The development in § **36** that was connected with equations (10), (11) will then show one how to exhibit a curve 01 along which the **Weierstrass** construction is possible, but the quantity $\mathcal{E}$ vanishes everywhere, such that the equations:

$$\bar{J}_{01} = J_{01} \,, \qquad \bar{K}_{01} = K_{01} \tag{36}$$

will be satisfied, and the extremal will end there. To that end, t will next be defined as a function of a and b by the equation:

$$\Delta\,(t, a, b) = 0 \,. \tag{37}$$

If we assume that:

$$\Delta\,(t_1, a_0, b_0) \neq 0$$

then when one recalls equation (35), that will give an expression for t :

$$t = t_1 + [a - a_0, b - b_0]_1 \,.$$

We further assume that the differential equation:

$$\begin{vmatrix} \xi_t\,dt + \xi_a\,da + \xi_b\,db & \eta_t\,dt + \eta_a\,da + \eta_b\,db \\ \xi_t & \eta_t \end{vmatrix} = 0$$

or

$$(38) \qquad (\xi_t\, \eta_a - \xi_a\, \eta_t)\, da + (\xi_t\, \eta_b - \xi_b\, \eta_t)\, db = 0$$

is satisfied, and replace t with the expression above. The factors of da and db will then be power series in the arguments $a - a_0$, $b - b_0$ which do not both vanish for $a = a_0$, $b = b_0$ when we introduce the assumption that the second-order determinants in the matrix:

$$(39) \qquad \begin{matrix} \xi_t & \xi_a & \xi_b \\ \eta_t & \eta_a & \eta_b \end{matrix}$$

do not all vanish for the system of values $t = t_1$, $a = a_0$, $b = b_0$. Namely, if those two factors have the value zero then the same thing would follow for the third determinant of the matrix, since ξ_t and η_t do not both vanish. If a_0 and b_0 are supposed to be associated values, equation (38) will then give a relation that takes one of the forms:

$$b - b_0 = [a - a_0]_1 , \qquad a - a_0 = [b - b_0]_1 .$$

If, e.g., the first equation is true, and one substitutes the values that were obtained for b and t in the equations:

$$(40) \qquad x = \xi\,(t, a, b), \qquad y = \eta\,(t, a, b)$$

then x and y will take the form of expressions $[a - a_0]_0$, and one will have:

$$x = x_1 , \qquad x = y_1$$

for $a = a_0$ in particular. Equations (40) will then represent a curve $\mathfrak{C}$ that includes the point 1 and is regular in the neighborhood of that point, except for the point itself. Should the point 1 be a cusp then one would have the following equations for it:

$$\xi_t\, dt + \xi_a\, da + \xi_b\, db = 0 , \qquad \eta_t\, dt + \eta_a\, da + \eta_b\, db = 0 .$$

Now since the equation:

$$\Delta_t\, dt + \Delta_a\, da + \Delta_b\, db = 0$$

is true, and one of the quantities db / da and da / db is non-zero, it would follow that:

$$(41) \qquad \frac{\partial(\xi, \eta, \Delta)}{\partial(t, a, b)} = 0 .$$

If that equation is not true for $t = t_1$, $a = a_0$, $b = b_0$ then the curve $\mathfrak{C}$ will not have a cusp at the point 1.

Now, since ξ_t and η_t do not vanish at the same time, it will follow from equation (38) that one can set:

$$\begin{aligned} \xi_t\,dt + \xi_a\,da + \xi_b\,db &= m\,\xi_t\,dt, \\ \eta_t\,dt + \eta_a\,da + \eta_b\,db &= m\,\eta_t\,dt. \end{aligned} \tag{42}$$

The curve $\mathfrak{C}$ will then be contacted at every point of the extremal (40) that belongs to the system of values (a, b), and the envelope will be a certain simply-infinite family of those curves.

On the basis of equations (37), and with the assumption that was introduced in regard to the system (39), equations (42) will imply that:

$$\omega_t\,dt + \omega_a\,da + \omega_b\,db = m\,\omega_t\,dt\,,$$

or since one can obviously set:

$$\omega_t = G\,(\xi,\ \eta,\ \xi_t,\ \eta_t)\,,$$

that:

$$\omega_t\,dt + \omega_a\,da + \omega_b\,db = m\,G\,(\xi,\ \eta,\ \xi_t,\ \eta_t)\,dt\,.$$

Now when $m\,dt$ is positive, due to the homogeneity of the function G and equations (42), one can conclude that:

$$\omega_t\,dt + \omega_a\,da + \omega_b\,db = G\,(\xi,\ \eta,\ \xi_t\,dt + \xi_a\,da + \xi_b\,db,\ \eta_t\,dt + \eta_a\,da + \eta_b\,db)\,.$$

The differentials dt, da, db all refer to an advance along the curve $\mathfrak{C}$. Moreover, if one denotes the increment that corresponds to that advance by D then one can write the last equation and equations (42) as:

$$Dx = m\,\omega_t\,dt\,, \qquad Dy = m\,\eta_t\,dt\,, \qquad D\omega = G\,(\xi,\ \eta,\ Dx,\ Dy)\,.$$

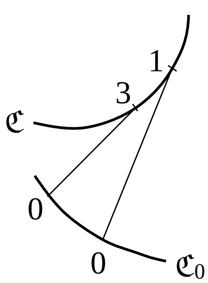

Figure 19.

When the integral sign K without an overbar refers to the curve $\mathfrak{C}$ and 3 is a point on it (Fig. 19), it will then follow from the equation:

$$D\,K_{31} = -\,G\,(\xi,\ \eta,\ Dx,\ Dy)$$

for $m\,dt > 0$ that:

$$D\omega + D\,K_{31} = D(\bar{K}_{03} + K_{31}) = 0\,, \tag{43}$$

in which the overbarred K refers to the extremal 03 that is defined by the system of values (a, b) that belongs to the point 3. Those extremals then imply the **Weierstrass** construction for the curve $\mathfrak{C}$, and the quantity:

$$\mathcal{E}\,(x, y, x', y', Dx, Dy) = \mathcal{E}\,(x, y, \xi_t, \eta_t, Dx, Dy)$$

will then vanish in an ordinary way when $m\,dt$ is positive.

In order to be able to infer equations (36) from this, the symbol D must mean the advance along the curve $\mathfrak{C}$ in the direction of the point 1. That direction might or might not coincide with the direction of increasing t along the extremal that contacts the point 3. The latter case will always occur when the curve $\mathfrak{C}$ has a cusp at the point 1 from which both branches of the curve $\mathfrak{C}$ start in the direction of increasing t. If both branches have opposite directions or if no cusp is present then the direction of the curve $\mathfrak{C}$ towards the point will coincide with the direction of increasing t along the extremal 03 along at least one of the two halves into which the curve $\mathfrak{C}$ splits in the neighborhood of the point 1. That half will be denoted by $\mathfrak{C}_1$. The latter direction will be defined by the pair of values:

$$x' = \xi_t , \qquad y' = \eta_t .$$

Along the half $\mathfrak{C}_1$, Dx then differs from ξ_t and Dy from η_t by a positive factor, such that $m\,dt$ is positive, and the quantity that is denoted by $\mathcal{E}$ will then vanish. Such an arc $\mathfrak{C}_1$ is then present in any case when equation (41) is not true, and therefore excludes a cusp.

Now let 3 be any point on the arc $\mathfrak{C}_1$ that lies arbitrarily-close to 1. The extremal 03 that was constructed above, along with the arc 31 of the curve $\mathfrak{C}_1$, will then define a connected path with continuously-varying tangents that deviates arbitrarily-little from the arc 01 that belongs to the curve $\mathfrak{C}$, and indeed in such a way that the tangents will also differ only slightly at neighboring points of both paths. In that way, one will have the equation:

$$d(\bar{J}_{03} + J_{31}) = \mathcal{E}(x, y, x', y', Dx, Dy) = 0 ,$$

because the expressions for the partial differential quotients of $\bar{J}_{03}$ and J_{31} that were derived above will also remain valid in the neighborhood of the location (t_1, a_0, b_0), since the indicated integrals are also regular functions of t, a, b here. It follows from the equation that was obtained and equation (43), which is characteristic of the **Weierstrass** construction, that:

$$\bar{J}_{03} + J_{31} = \bar{J}_{01} , \qquad \bar{K}_{03} + K_{31} = \bar{K}_{01} .$$

The arc 01 of the curve $\mathfrak{C}$ will no longer yield an extremum of the integral J in comparison to all curves that give the same value to the integral K. Rather, the weak extremum will already break down at the point 1 due to the given properties of the path 031.

Example. Problem IX will be specialized as follows: Draw a line of given length from a given point on one leg of an angle on the convex side to the other leg that will enclose the greatest-possible area.

Let the second leg be the line $y = 0$. The area to be extremized differs from the integral $\int y\,x'\,dt$ only by the constant triangular area that the altitude that is dropped from the given point to the second leg will cut out of the angle. The extremals that cut the line $y = 0$ transversally are perpendicular to it (§ **34**) and are circles, so they can be represented by the equations:

$$x = a + b\cos t, \qquad y = b\sin t\,. \tag{44}$$

One always has $t = 0$ for the point 0, such that:

$$\bar{K}_{03} = \omega(t, a, b) = b\,t\,.$$

That will then give:

$$\Delta = \begin{vmatrix} -b\sin t & b\cos t & b \\ 1 & 0 & 0 \\ \cos t & \sin t & t \end{vmatrix} = b\,(\sin t - t\cos t)\,.$$

The extremal focal point of the line $y = 0$ along any circle that is orthogonal to it will then be defined by the smallest root of the equation:

$$\tan t = t\,,$$

which will be denoted by t_1 and has the approximate value:

$$t_1 = \frac{257.5}{360}\cdot 2\pi\,,$$

i.e., it corresponds to the angle 257.5°. Equation (38), which provides the connection that defines the curve $\mathfrak{C}$, will become simply:

$$b\,db + b\cos t_1\,da = 0\,, \qquad db + \cos t_1\,da = 0\,.$$

Such a relation between a and b selects a singly-infinite family of circles from the doubly-family (44) that contact two fixed lines. If α is the concave angle between them then:

$$\sin\frac{\alpha}{2} = -\cos t_1 = \cos(t_1 - \pi)\,, \qquad \frac{\alpha}{2} = \frac{3\pi}{2} - t_1 = 12.5^{\circ}.$$

The breakdown of the extremum in this family can be exhibited by elementary means (Fig. 20). Namely, if 1, 3 are two points that lie on the same leg of the angle α, and 3 is closer to the vertex than 1, and if 0_0, 0 are the points of intersection of the circles of the family that go through 1, 3 with the bisector of the angle α, moreover, and if the arcs 10_0, 30 have the central angle t_1 then the arcs 10_0 and the combined line 13_0 will have equal length. If the altitude 14 is dropped from 1 to the bisector of the angle α then the horizontally-lined figure 10_041 and the vertically-lined one 13041 will have equal area, such that the arcs 10_0 will certainly not provide an extremum for the area with a given arc-length. The arc $\mathfrak{C}_1$ is the line segment 13.

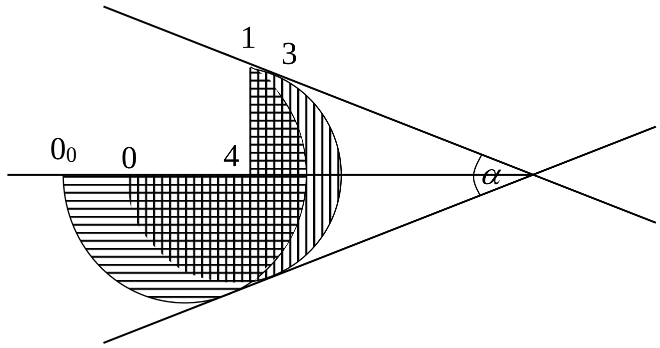

Figure 20.

§ 41. – Existence of a field whose extremals intersect a given curve transversally.

Let the point 0 be chosen along the extremal $\mathfrak{C}$ such that ξ_t does not vanish. The differential equation of the extremals can then be written in the form:

$$f_y + \lambda\, g_y - \frac{d\,(f_p + \lambda\, g_p)}{dx} = 0\,,$$

and that will imply an expression for dp / dx that is regular in x, y, p when the inequality:

$$f_{pp} + \lambda\, g_{pp}\,|^0 \neq 0 \tag{45}$$

is satisfied. Under that assumption, the general theorem in § **27** will then say that the differential equation will possess an integral of the form:

$$y = \Phi\,(x, a, b, \lambda) \tag{46}$$

that leads to the curve for $a = a_0$, $b = b_0$, $\lambda = \lambda_0$ and is regular at the location (x_{00}, a_0, b_0, λ_0), if the abscissa of the point 0 is denoted by x_{00} . If one takes a and b to be the values of y and p for x_{00}, resp., in particular, then:

$$\Phi\,(x, a, b, \lambda) = a + b\,(x - x_{00}) + [x - x_{00}]_2\,,$$

and for $x = x_{00}$, one has:

(47) $$\Phi_b = 0\,, \qquad \Phi_{bx} = 1\,.$$

We shall now attempt to select a two-fold manifold of extremals from the three-fold manifold that is given by (46) that intersect the regular curve $\mathfrak{C}_0$ that starts from the point 0 and has the equation:

$$y_0 = \gamma(x_0)\,.$$

That will happen when the following equations are fulfilled:

(48) $$\begin{aligned} M &= \Phi(x_0, a, b, \lambda) - \gamma(x_0) = 0\,, \\ N &= f + \lambda\, g - p\,(f_p + \lambda\, g_p) + (f_p + \lambda\, g_p)\,\gamma'(x)\Big|^{x=x_0} = 0\,. \end{aligned}$$

In that way, the values Φ and Φ_x are substituted for y and p in the second equation. The left-hand sides of those equations are regular at the location $x_0 = x_{00}$, $a = a_0$, $b = b_0$, $\lambda = \lambda_0$. It is easy to see that one can express two of the quantities x, a, b, λ as functions of the remaining ones. Namely, if one defines:

$$\frac{\partial(M, N)}{\partial(x_0, b)} = \begin{vmatrix} \Phi_x - \gamma'(x) & \Phi_b \\ \dfrac{\partial N}{\partial x} + \Phi_x \dfrac{\partial N}{\partial y} + \Phi_{xx} \dfrac{\partial N}{\partial p} & \dfrac{\partial N}{\partial y}\Phi_b + \dfrac{\partial N}{\partial p}\Phi_{px} \end{vmatrix}^{x=x_0}$$

and observes that obviously:

$$\frac{\partial N}{\partial p} = [\gamma'(x) - \Phi_x](f_{pp} + \lambda\, g_{pp})\Big|^{x=x_0},$$

then one will get from equations (47) that for $x = x_{00}$:

$$\frac{\partial(M, N)}{\partial(x_0, b)} = -\,[\gamma'(x) - \Phi_x]^2 (f_{pp} + \lambda\, g_{pp})\Big|^{x=x_{00}},$$

and that expression will be non-zero at the location (x_{00}, a_0, b_0, λ_0) when:

$$\Phi_x\,(x_{00}, a_0, b_0, \lambda_0) - \gamma'(x_{00}) \neq 0\,,$$

i.e., when the curves $\mathfrak{C}$ and $\mathfrak{C}_0$ do not contact each other. Equations (48) will then give expressions of the form:

$$[a - a_0,\ \lambda - \lambda_0]_1$$

for $b - b_0$ and $x_0 - x_{00}$. If one replaces $b - b_0$ with the former in Φ then one will get an expression:

(49) $$y = \varphi\,(x, a, \lambda)$$

that is regular at the location (x_{00}, a_0, λ_0) and defines a double infinitude of extremals that intersect the curve $\mathfrak{C}_0$ transversally. If the curve $\mathfrak{C}_0$ contracts to the point 0 then that will give the same result (i.e., an equation for the extremals that go through it once) when one solves the equation:

$$\Phi\,(x_{00}, a, b, \lambda) - y_{00} = 0 \tag{50}$$

for b.

In each of those cases, a connection between a, b, λ is represented by an equation:

$$\Omega\,(a, b, \lambda) = 0\,, \tag{51}$$

for which:

$$\Omega_b\,(a_0, b_0, \lambda_0) \neq 0\,, \tag{52}$$

and the left-hand side is regular in the neighborhood of the location (a_0, b_0, λ_0). In that way, one will obviously get:

$$\Omega_a + \Omega_b\,\frac{\partial b}{\partial a} = 0\,, \qquad \Omega_\lambda + \Omega_b\,\frac{\partial b}{\partial \lambda} = 0\,,$$

$$\varphi_a\,(x, a, \lambda) = \Phi_a + \Phi_b\,\frac{\partial b}{\partial a}\,, \qquad \varphi_\lambda\,(x, a, \lambda) = \Phi_\lambda + \Phi_b\,\frac{\partial b}{\partial \lambda}\,,$$

and analogous equations will be true for every quantity that first takes the form of a function of a, b, λ, and goes to one in a and λ alone.

If one sets, e.g.:

$$\Theta\,(x, a, b, \lambda) = \int_{x_0}^{x} g\,(x, \Phi, \Phi_x)\,dx\,,$$

and if that quantity goes to $\omega\,(x, a, \lambda)$ under the assumption (51) then since one can set $t = x$, it will have the same meaning for the extremals (49) that it did before for the quantity $\omega\,(t, a, b)$, and one will get the following expression for Δ when one replaces b with λ :

$$\Delta = \frac{\partial(\xi, \eta, \omega)}{\partial(x, a, \lambda)} = \frac{\partial(\varphi, \omega)}{\partial(a, \lambda)} = \begin{vmatrix} \Phi_a + \Phi_b\,\dfrac{\partial b}{\partial a} & \Phi_\lambda + \Phi_b\,\dfrac{\partial b}{\partial \lambda} \\ \Theta_a + \Theta_b\,\dfrac{\partial b}{\partial a} & \Theta_\lambda + \Theta_b\,\dfrac{\partial b}{\partial \lambda} \end{vmatrix} = \frac{1}{\Omega_b} \begin{vmatrix} \Phi_a & \Phi_b & \Phi_\lambda \\ \Omega_a & \Omega_b & \Omega_\lambda \\ \Theta_a & \Theta_b & \Theta_\lambda \end{vmatrix} = \frac{D}{\Omega_b}\,.$$

If one succeeds in proving that this determinant does not vanish identically then one will have shown that the extremals $x = \varphi\,(x, a, \lambda)$ define a field in the sense of § **39**, at least in a neighborhood of the point 0.

In order to investigate the last determinant that was obtained, we shall start from the fact that x_0 is defined as a regular function of a, b, λ . One will then have the equation:

$$\Theta_a = -g\frac{\partial x_0}{\partial a}\bigg|^0 + \int_{x_0}^{x}\frac{\partial g}{\partial a}dx = -g\frac{\partial x_0}{\partial a}\bigg|^0 + g_p\,\Phi_a\Big|_{x_0}^{x} + \int_{x_0}^{x}\left(g_y - \frac{dg_p}{dx}\right)\Phi_a\,dx\ ,$$

along with similar ones in which a is replaced with b or λ. It will then follow that:

$$D = \frac{\partial(\Phi,\Omega,\Theta)}{\partial(a,b,\lambda)} = \begin{vmatrix} \Phi_a & \Phi_b & \Phi_\lambda \\ \Omega_a & \Omega_b & \Omega_\lambda \\ A + \int_{x_0}^{x}\left(g_y - \frac{dg_p}{dx}\right)\Phi_a\,dx & B + \int_{x_0}^{x}\left(g_y - \frac{dg_p}{dx}\right)\Phi_b\,dx & C + \int_{x_0}^{x}\left(g_y - \frac{dg_p}{dx}\right)\Phi_\lambda\,dx \end{vmatrix},$$

in which the quantities A, B, C are independent of x. That is because the terms in the expressions Θ_a , Θ_b, Θ_λ that appear before the integral sign and depend upon x will drop out when one multiplies the first row in the determinant by g_p and subtracts it from the last one. Now, when that expression for D vanishes identically, the second-order determinants in the matrix:

$$\text{(53)} \qquad \begin{Vmatrix} \Omega_a & \Omega_b & \Omega_\lambda \\ A + \int_{x_0}^{x}\cdots & B + \int_{x_0}^{x}\cdots & C + \int_{x_0}^{x}\cdots \end{Vmatrix}$$

can likewise vanish identically. Their differential quotients with respect to x will be the determinants of the matrix:

$$\begin{Vmatrix} \Omega_a & \Omega_b & \Omega_\lambda \\ \left(g_y - \frac{dg_p}{dx}\right)\Phi_a & \left(g_y - \frac{dg_p}{dx}\right)\Phi_b & \left(g_y - \frac{dg_p}{dx}\right)\Phi_\lambda \end{Vmatrix}.$$

Now since the quantity:

$$R = g_y - \frac{dg_p}{dx}$$

does not vanish along the $\mathfrak{C}$, as an extremal in the problem of the relative extremum, the determinants of the matrix:

$$\begin{Vmatrix} \Omega_a & \Omega_b & \Omega_\lambda \\ \Phi_a & \Phi_b & \Phi_\lambda \end{Vmatrix}$$

will also vanish, e.g.:

$$\Omega_a \, \Phi_b - \Omega_b \, \Phi_a = 0 \,, \qquad \Omega_\lambda \, \Phi_b - \Omega_b \, \Phi_\lambda = 0 \,. \tag{54}$$

However, the first of those equations is obviously impossible, since one would have:

$$\Phi_a = 1 \,, \qquad \Phi_b = 0 \,, \qquad \Omega_b \neq 0 \tag{55}$$

for $x = x_{00}$, $a = a_0$, $b = b_0$, $\lambda = \lambda_0$. The second will also be impossible, since Φ_b begins with the first power of $x - x_{00}$, while Φ_λ begins with a higher power. The quantity Φ_λ cannot vanish identically, because otherwise the expression Φ would be independent of λ, so the equation:

$$f_y - \frac{df_p}{dx} = 0$$

would also be satisfied, and its left-hand side, like R, is non-zero for the curve $\mathfrak{C}$.

In particular, it follows from this analysis that neither of the determinants:

$$\begin{vmatrix} \Omega_a & \Omega_b \\ A + \int_{x_0}^{x} R \, \Phi_a \, dx & B + \int_{x_0}^{x} R \, \Phi_b \, dx \end{vmatrix} ,$$

$$\begin{vmatrix} \Omega_\lambda & \Omega_b \\ C + \int_{x_0}^{x} R \, \Phi_\lambda \, dx & B + \int_{x_0}^{x} R \, \Phi_b \, dx \end{vmatrix} \tag{56}$$

will vanish identically, since their derivatives are the left-hand sides of equations (54), multiplied by the non-zero quantity R. However, from a general theorem from the theory of determinants, the adjoints of two parallel rows in a vanishing determinant will be proportional. If one applies that theorem to the first and third row in the determinant D then that will imply that the last two determinants are proportional to the left-hand sides of equations (54). Now since one neither has that one of those four quantities vanishes identically, nor that the quantity R vanishes identically, that fact will imply that the logarithmic derivatives of the quantities (56) with respect to x are equal, so the quantities themselves will be related by:

$$\begin{vmatrix} \Omega_a & \Omega_b \\ A + \int_{x_0}^{x} R\Phi_a\,dx & B + \int_{x_0}^{x} R\Phi_b\,dx \end{vmatrix} = M \begin{vmatrix} \Omega_\lambda & \Omega_b \\ C + \int_{x_0}^{x} R\Phi_\lambda\,dx & B + \int_{x_0}^{x} R\Phi_b\,dx \end{vmatrix},$$

when M refers to a finite, non-zero quantity that is independent of x. It then follows from this upon differentiation, while recalling the non-vanishing value of the quantity R, that:

$$\Omega_a\,\Phi_b - \Omega_b\,\Phi_a = M\,(\Omega_\lambda\,\Phi_b - \Omega_b\,\Phi_\lambda)\;.$$

If we set $x - x_{00}$ in this then the left-hand side will reduce to $-\Omega_b$ and the right-hand side to zero, since the relations (55) are valid. The assumption that D vanishes identically then leads to a contradiction.

Therefore, on the grounds of the relationship between D and Δ, one can prove the following: If we add the inequality (45) to the previous assumptions in regard to the extremal $\mathfrak{C}$ then the arc that begins at the point 0 can always be surrounded by a field whose extremals either all go through the point 0 or intersect a given regular curve $\mathfrak{C}_0$ transversally that intersects the curve $\mathfrak{C}$ transversally at the point 0 but does not contact it. Thus, for a sufficiently-small extremal arc, the **Jacobi** condition for an extremum will be fulfilled, regardless of whether one or two endpoints of the desired points are fixed.

§ 42. – The Jacobi condition in the Mayer and Weierstrass forms.

If the point 0 is fixed and the equation $\Omega = 0$ is replaced by the special relation (50) accordingly then one will have:

$$\Omega_a = \Phi_a\,(x_0\,,a,b,\lambda)\,,\qquad \Omega_b = \Phi_b\,(x_0\,,a,b,\lambda)\,,\qquad \Omega_\lambda = \Phi_\lambda\,(x_0\,,a,b,\lambda)\,,$$

and then get:

$$D = \frac{\partial\{\Phi(x,a,b,\lambda), \Phi(x_0,a,b,\lambda), \int_{x_0}^{x} g[x, \Phi(x,a,b,\lambda), \Phi_x(x,a,b,\lambda)]\,dx}{\partial(a,b,\lambda)}\;.$$

Now since that expression has the value:

$$\Delta\,\Phi_b\,(x_0\,,a,b,\lambda)\,,$$

the quantity:

$$\Delta\,(x_0, x) = D\,\Big|^{a=a_0,\,b=b_0,\,\lambda=\lambda_0}$$

will vanish when the point 0 is conjugate to (x, y). **Mayer**'s way of characterizing conjugate points is thus derived with that. One will obtain the analogous formula of **Weierstrass** under the assumption that the extremals can be represented in the form:

$$x = X(t, a, b, \lambda), \qquad y = Y(t, a, b, \lambda). \tag{57}$$

One will then have the identity:

$$Y = \Phi(X, a, b, \lambda),$$

and the relations that follow from it:

$$Y_t = \Phi_x(X, a, b, \lambda)\, X_t, \qquad Y_a = \Phi_x X_a + \Phi_a, \qquad Y_b = \Phi_x X_b + \Phi_b, \qquad Y_\lambda = \Phi_x X_\lambda + \Phi_\lambda,$$

$$X_t\,\Phi_a = X_t Y_a - X_a Y_t = \theta_1(t), \qquad X_t\,\Phi_b = X_t Y_b - X_b Y_t = \theta_2(t), \qquad X_t\,\Phi_\lambda = X_t Y_\lambda - X_\lambda Y_t = \theta_3(t).$$

If the point 0 belongs to the argument $t = t_0$ then that will give:

$$D\, X_t(t, a, b, \lambda)\, X_t(t_0, a, b, \lambda) = \begin{vmatrix} \theta_1(t) & \theta_2(t) & \theta_3(t) \\ \theta_1(t_0) & \theta_2(t_0) & \theta_3(t_0) \\ \dfrac{\partial K}{\partial a} & \dfrac{\partial K}{\partial b} & \dfrac{\partial K}{\partial \lambda} \end{vmatrix},$$

in which one has set:

$$K = \int_{x_0}^{x} g\, dx = \int_{x_0}^{x} G(X, Y, X_t, Y_t)\, dx.$$

Now since x_0 does not depend upon a, b, λ, one will have the equation:

$$\frac{\partial K}{\partial a} = \Phi_a\, g_p \Big|_{x_0}^{x} + \int_{x_0}^{x} \Phi_a \left(g_y - \frac{dg_p}{dx} \right) dx,$$

along with similar formulas in which a has been replaced with b and λ. If one then adds the first row in the determinant D, times a suitable factor, to the last one then that will give:

$$\int_{x_0}^{x} \Phi_a \left(g_y - \frac{dg_p}{dx} \right) dx, \quad \int_{x_0}^{x} \Phi_b \left(g_y - \frac{dg_p}{dx} \right) dx, \quad \int_{x_0}^{x} \Phi_\lambda \left(g_y - \frac{dg_p}{dx} \right) dx.$$

From § **5**, when one introduces:

$$G(x, y, x', y') = x'\, g\left(x, y, \frac{y'}{x'} \right),$$

one will further have:

$$x'\left(g_y - \frac{dg_p}{dx}\right) = G_y - G'_y = -\frac{x'}{y'}(G_x - G'_{x'}) \ .$$

It will follow from this that:

$$\Phi_a\left(g_y - \frac{dg_p}{dx}\right)dx = \left(Y_a - X_a\frac{Y_t}{X_t}\right)(G_y - G'_{y'})\,dt = [X_a\,(G_x - G'_{x'}) + Y_a\,(G_y - G'_{y'})]\,dt \,.$$

If one then sets:

$$\Theta_1\,(t_0\,,\,t) = \int_{t_0}^{t} dt\,[X_a\,(G_x - G'_{x'}) + Y_a\,(G_y - G'_{y'})] \ ,$$

and if Θ_2 and Θ_3 can be inferred from that when one replaces a with b and λ, resp., then one will get the **Weierstrass** determinant:

$$D\,X_t\,(t, a, b, \lambda)\,X_t\,(t_0, a, b, \lambda) = \begin{vmatrix} \theta_1(t) & \theta_2(t) & \theta_3(t) \\ \theta_1(t_0) & \theta_2(t_0) & \theta_3(t_0) \\ \Theta_1(t_0,t) & \Theta_2(t_0,t) & \Theta_3(t_0,t) \end{vmatrix} = D\,(t_0, t) \,.$$

In that, one should observe that D is defined only under the assumption that y is defined to be a regular function of x by way of the curve $\mathfrak{C}$ everywhere in the interval $t_0 \ldots t$. By contrast, the right-hand side is meaningful for any arc $\mathfrak{C}$ along which x and y are regular functions of t and the quantity $H_1(x, y, x', y')$ does not vanish, since, from § **27**, such a thing can always be embedded in a family of curves, as it was represented in equations (57).

If $D\,(t_0, t)$ does not vanish identically then at least one of the quantities $\theta_1\,(t_0)$, $\theta_2\,(t_0)$, $\theta_3\,(t_0)$ must be non-zero. Let it be, e.g., the last one. t_0 and λ can be calculated as regular functions of a and b from the equations:

$$x_0 = X\,(t, a, b, \lambda), \qquad y_0 = Y\,(t_0, a, b, \lambda) \,.$$

If one substitutes those expressions for t_0 and λ then one will get:

$$X = \xi\,(t, a, b) \,, \qquad Y = \eta\,(t, a, b) \,.$$

The following equations will be valid:

$$X_t\,\frac{\partial t_0}{\partial a} + X_\lambda\,\frac{\partial \lambda}{\partial a} + X_a\,\Big|^{t_0} = Y_t\,\frac{\partial t_0}{\partial a} + Y_\lambda\,\frac{\partial \lambda}{\partial a} + Y_a\,\Big|^{t_0} = 0 \,,$$

$$X_t\,\frac{\partial t_0}{\partial b} + X_\lambda\,\frac{\partial \lambda}{\partial b} + X_b\,\Big|^{t_0} = Y_t\,\frac{\partial t_0}{\partial b} + Y_\lambda\,\frac{\partial \lambda}{\partial b} + Y_b\,\Big|^{t_0} = 0 \,,$$

$$\xi_t = X_t\,, \qquad \eta_t = Y_t\,, \qquad X_a + X_\lambda \frac{\partial\lambda}{\partial a} = \xi_a\,, \qquad Y_a + Y_\lambda \frac{\partial\lambda}{\partial a} = \eta_a\,,$$

$$X_b + X_\lambda \frac{\partial\lambda}{\partial b} = \xi_b\,, \qquad Y_b + Y_\lambda \frac{\partial\lambda}{\partial b} = \eta_b\,.$$

It follows easily from the first four equations that:

$$\theta_1\,(t_0) + \theta_3\,(t_0)\frac{\partial\lambda}{\partial a} = \theta_2\,(t_0) + \theta_3\,(t_0)\frac{\partial\lambda}{\partial b} = 0\,,$$

and from the last four, that:

$$\theta_1\,(t_0) + \theta_3\,(t_0)\frac{\partial\lambda}{\partial a} = \xi_t\,\eta_a - \xi_a\,\eta_t\,, \qquad \theta_2\,(t_0) + \theta_3\,(t_0)\frac{\partial\lambda}{\partial b} = \xi_t\,\eta_b - \xi_b\,\eta_t\,,$$

and the definition of the quantities Θ will give:

$$\Theta_1\,(t_0, t) + \frac{\partial\lambda}{\partial a}\,\Theta_3\,(t_0, t) = \int_{t_0}^{t} dt\,\{(G_x - G'_{x'})\,\xi_a + (G_y - G'_{y'})\,\eta_a\} = A\,,$$

$$\Theta_2\,(t_0, t) + \frac{\partial\lambda}{\partial b}\,\Theta_3\,(t_0, t) = \int_{t_0}^{t} dt\,\{(G_x - G'_{x'})\,\xi_b + (G_y - G'_{y'})\,\eta_b\} = B\,.$$

If one then adds the third row in the determinant $D\,(t_0, t)$, when multiplied by $\partial\lambda : \partial a$ and $\partial\lambda : \partial b$, to the first and second ones, resp., then one will get:

$$D\,(t_0, t) = -\theta_3(t_0)\begin{vmatrix} \xi_t\,\eta_a - \xi_a\,\eta_t & \xi_t\,\eta_b - \xi_b\,\eta_t \\ A & B \end{vmatrix}.$$

Now, since, from § **35**:

$$\omega_a = G_{x'}\,\xi_a + G_{y'}\,\eta_a + A, \qquad \omega_b = G_{x'}\,\xi_b + G_{y'}\,\eta_b + B, \qquad \omega_t = G = \xi_t\,G_{x'} + \eta_t G_{y'}\,,$$

it will ultimately follow that:

$$D\,(t_0, t) = -\theta_3(t_0)\frac{\partial(\xi,\eta,\omega)}{\partial(t,a,b)} = -\,\theta_3\,(t_0)\,\Delta\,.$$

Under the assumption that was introduced, Δ and $D\,(t_0, t)$ will vanish only simultaneously, and the equation:

$$D\,(t_0, t) = 0$$

will be a necessary and sufficient condition for the points that belong to the arguments t_0 and t to be conjugate as long as its left-hand side does not vanish identically. Therefore, if the quantity $D(t_0, t)$ is everywhere non-zero along an arc 02, except for the point 0 itself, then that will yield an extremum of the desired type for any arc 12 whose starting point differs from 0 arbitrarily little.

On the basis of the argument that was presented at the conclusion of § **31**, it will follow from this that the **Jacobi** condition for the extremum for the arc 12 with fixed endpoints can also be formulated in general as follows: The equation:

$$D(t_1, t) = 0$$

possesses no roots besides $t = t_1$ along the segment 12. Namely, that consideration can be applied with no alterations to the quantity that is now denoted by $D(t_0, t)$ as long as the inequality:

$$\left.\frac{\partial D(t_0, t)}{\partial t}\right|^{t=t_0} \neq 0$$

is satisfied. Even when the quantity:

$$d_0(t_0) = \left.\frac{\partial^{\mathfrak{a}} D(t_0, t)}{\partial t^{\mathfrak{a}}}\right|^{t=t_0}$$

does not vanish identically for $\mathfrak{a} = 1, 2, \ldots, n$, but $d_{n+1}(t_1)$ is non-zero, the argument in § **31** will need to be modified only insofar as the quantity $d_{n+1}(t_0)$ must enter in place of $f_1(t_0)$. It is only when $d_{n+1}(t_0)$ does not vanish identically, but $d_{n+1}(t_1)$ has the value zero, that it will become doubtful whether the stated consideration is applicable.

CHAPTER FIVE

DISCONTINUOUS SOLUTIONS

§ 43. – Necessary conditions for a curve with a corner to yield an absolute extremum.

The developments in the foregoing chapter do not always give a solution to the original extremum problem of drawing a curve between two given points that yields an absolute or relative extremum to the integral J. It was only shown that when the desired curve was supposed to have certain continuity properties, it could be nothing but a regular segment of an extremal. Conditions for the extremum were then derived under the assumption that the two given endpoints of the desired curve were already connected by a singularity-free arc of an extremal. If that assumption is not fulfilled (which already happens in simple problems) then the desired extremum cannot be provided by a curve with the assumed continuity properties, and one will then arrive at the question of whether a curve that is endowed with singularities (e.g., corners) might not solve the problem that was posed. If one can succeed in answering that question by allowing certain types of singularities then one will indeed still not have an infallible method for determining the desired extremum, but at least the scope of the cases in which the problem can be solved will have been extended.

In particular, we ask when the integral:

$$J = \int F\,dt$$

will become an absolute extremum on a curve that consists of a finite number of pieces that come together at corners and each possess the properties that were previously required of the entire curve, i.e., along each segment, let x and y be continuous functions of a parameter t and have continuous first and second derivatives. The considerations of Chapter Two will remain valid for each individual piece, since one can leave the corners unvaried and restrict the variation to the interior of the piece. Let the points 0 and 2 of the arcs that meet at the corner 1 be taken to be close enough to the corner that the arcs 01 and 12 include no conjugate pairs (Fig. 21). The points 0 and 2 can then be connected to any point 3 that is sufficiently distant from 1 by extremals 03 and 32, and from § **2**, they can be considered to be variations of the arcs 01 and 12. The broken line 012 can then be replaced with 032 as a segment of the desired curve, since the variational formulas in § **2** will remain valid for each of the pieces 01 and 02. If one now sets:

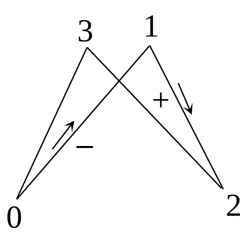

Figure 21.

$$\delta x_1 = x_3 - x_1\,, \qquad \delta y_1 = y_3 - y_1\,,$$

and if one attaches the – or + sign to any quantity according to whether it is defined for the arc 01 in the direction of 1 or for the arc 12 in the direction of 2, resp., then one will have the formulas:

$$\Delta \bar{J}_{01} = \left. F_{x'}^{-}\,\delta x + F_{y'}^{-}\,\delta y \right|^{1} + [\delta x_1, \delta y_1]_2 \;,$$

$$\Delta \bar{J}_{12} = -\left. F_{x'}^{+}\,\delta x - F_{y'}^{+}\,\delta y \right|^{1} + [\delta x_1, \delta y_1]_2 \;,$$

$$F_{x'}^{-} = F_{x'}(x, y, x'_{-}, y'_{-}) \,, \qquad F_{x'}^{+} = F_{x'}(x, y, x'_{+}, y'_{+}) \,,$$

so when one adds them:

$$\Delta \bar{J}_{012} = \left. (F_{x'}^{-} - F_{x'}^{+})\,\delta x + (F_{y'}^{-} - F_{y'}^{+})\,\delta y \right|^{1} + [\delta x_1, \delta y_1]_2 \;.$$

Those quantities must have a constant sign if the curve, which belongs to the broken line 012, is to yield an extremum for the integral J. Therefore, if the corner point 3 is freely-available in the neighborhood of the position 1, so the quantities δx_1, δy_1 are mutually-independent, then (§ **7**) the coefficients of the linear terms will vanish, and one will get **Erdmann**'s theorem:

(1) $$F_{x'}^{-} = F_{x'}^{+} \,, \qquad F_{y'}^{-} = F_{y'}^{+} \;.$$

By contrast, if the corner point is connected with the curve:

(2) $$h(x_1, x_2) = 0$$

from the outset, such that the equation:

$$0 = h_x(x_1, x_2)\,\delta x_1 + h_y(x_1, x_2)\,\delta y_1 + [\delta x_1, \delta y_1]_2$$

exists, then, from § **7**, $\Delta \bar{J}_{012}$ can have an unvarying sign only when:

$$\begin{vmatrix} F_{x'}^{-} - F_{x'}^{+} & F_{y'}^{-} - F_{y'}^{+} \\ h_x(x_1, y_1) & h_y(x_1, y_1) \end{vmatrix} = 0 \,,$$

i.e., when the equation:

(3) $$F_{x'}^{-}\,\delta x_1 + F_{y'}^{-}\,\delta y_1 = F_{x'}^{+}\,\delta x_1 + F_{y'}^{+}\,\delta y_1$$

is valid for an infinitely-small displacement of the point 1 along the curve (2).

If one denotes the segment 13 by ds and assumes that t means the arc-length along each of the extremals 01 and 12, which always increases in the direction from 0 to 1 or 2, then one can introduce three angles σ, θ_+, θ_- that satisfy the following equations:

$$\delta x_1 = \delta s \cos \sigma, \qquad \delta y_1 = \delta s \sin \sigma,$$

$$x'_- = \cos \theta_-, \quad y'_- = \sin \theta_-, \quad x'_+ = \cos \theta_+, \quad y'_+ = \sin \theta_+.$$

One further defines the quantities u, v by the equations:

(4) $$\delta x_1 = x'_+ u + x'_- v, \qquad \delta y_1 = y'_+ u + y'_- v.$$

That will easily give:

$$u = \frac{\delta s \sin(\sigma - \theta_-)}{\sin(\theta_+ - \theta_-)}, \quad v = \frac{\delta s \sin(\sigma - \theta_+)}{\sin(\theta_+ - \theta_+)}$$

then, and those quantities are proportional to the altitudes that can be dropped from the point 3 to the extremals 01 and 12. In that way, equations (4) will imply that:

$$\begin{aligned}(F_{x'}^- - F_{x'}^+)\,\delta x_1 + (F_{y'}^- - F_{y'}^+)\,\delta y_1 & \\ &= u\,(x'_+ F_{x'}^- + y'_+ F_{y'}^- - F^+) - v\,(x'_- F_{x'}^+ + y'_- F_{y'}^+ - F^-) \\ &= u\,\mathcal{E}(x_1, y_1, x'_-, y'_-, x'_+, y'_+) - v\,\mathcal{E}(x_1, y_1, x'_+, y'_+, x'_-, y'_-),\end{aligned}$$

and from the above, that expression must vanish for every allowable displacement of the corner point.

First example: The principle of least action in the **Euler-Jacobi** form for the central motion of a free point in the plane. Let the repulsion be, e.g., proportional to the distance. If one sets:

$$r = \sqrt{x^2 + y^2}$$

then the potential will be $c\, r$, and the cited principle will say that the trajectories must minimize the integral:

$$\int \sqrt{cr + h}\,\sqrt{dx^2 + dy^2},$$

in which h means the constant of the *vis viva*. If one assumes that $h = 0$, in particular, then the radii that emanate from the center of force will be included in the trajectories that are thus defined, and along which the moving point can approach the center asymptotically, and one will have to set:

$$F = \sqrt{x^2 + y^2}\,\sqrt{x'^2 + y'^2}.$$

When r is non-zero, equations (1) will give:

$$\frac{x'_-}{\sqrt{x'^2_- + y'^2_-}} = \frac{x'_+}{\sqrt{x'^2_+ + y'^2_+}} , \qquad \frac{y'_-}{\sqrt{x'^2_- + y'^2_-}} = \frac{y'_+}{\sqrt{x'^2_+ + y'^2_+}} ,$$

i.e., the directions +, – coincide. A corner can then appear only at the force center ($r = 0$). Equations (1) are fulfilled for any two radii that emanate from it. Now since the radii are the only trajectories that go through the force center under the assumption that $h = 0$, a broken line can yield an extremum only when it consists of two straight line segments that meet at the center.

One can pose similar considerations when the force is proportional to an arbitrary power of the distance. One will then have to minimize the integral:

$$\int (x^2 + y^2)^n \sqrt{dx^2 + dy^2} .$$

Second example. Problem VII (§ 11). – The shortest line on the surface whose arc-length is given by:

$$ds^2 = E\,dx^2 + 2F\,dx\,dy + G\,dy^2$$

cannot exhibit a freely-available corner anywhere as long as $EG - F^2$ is non-zero, because equations (1) would then require that:

$$E\left(\frac{dx}{ds}\right)_- + F\left(\frac{dy}{ds}\right)_- = E\left(\frac{dx}{ds}\right)_+ + F\left(\frac{dy}{ds}\right)_+ ,$$

$$F\left(\frac{dx}{ds}\right)_- + G\left(\frac{dy}{ds}\right)_- = F\left(\frac{dx}{ds}\right)_+ + G\left(\frac{dy}{ds}\right)_+ ,$$

from which it would follow, with the given assumption, that:

$$\left(\frac{dx}{ds}\right)_- - \left(\frac{dx}{ds}\right)_+ = \left(\frac{dy}{ds}\right)_- - \left(\frac{dy}{ds}\right)_+ = 0 .$$

However, a corner can appear when one demands that the points 0 and 2 are connected by the shortest line that has a point 1 that is not prescribed in common with a given curve. If D denotes the increase when one advances along that curve and Ds denotes its arc-length element then equation (3) will imply that the quantity:

$$\left[E\left(\frac{dx}{ds}\right)_- + F\left(\frac{dy}{ds}\right)_-\right]Dx + \left[F\left(\frac{dx}{ds}\right)_- + G\left(\frac{dy}{ds}\right)_-\right]Dy$$

must preserve its value when one replaces the index – with + . If ds_+, ds_- are the elements of the geodetic curves 01, 12 then that quantity will be Ds cos (Ds, ds_-) . One will then get the condition for the extremum:

$$\cos (Ds, ds_-) = \cos (Ds, ds_+) .$$

The directions for the geodetic lines that are denoted by the signs +, – will then be symmetric to the direction of the given curve, so the two arcs 01, 12 will lie on the same side of that curve and define equal acute angles with it.

§ 44. – The desired curve is restricted to a bounded region and includes pieces of the boundary.

In the reasoning of § **8**, it was essential that the quantity ε was assigned positive and negative values, so the desired curve was compared to varied curves that deviated from it on one side, as well as the other. That will be excluded when the comparison curves are restricted to a certain region $\mathfrak{G}$ that is bounded by some barriers, as we would like to assume. The stated argument will no longer be valid then, and the desired curve will not need to satisfy the differential equation of the extremals. We must then examine, in particular, when a line that consists partly of extremal arcs and partly coincides with the barrier can extremize the integral J. The fact that the parts of the arc that do not coincide with the barrier must be extremals obviously follows from the considerations in § **43** that showed that the desired curve consisted of extremal arcs between the corners.

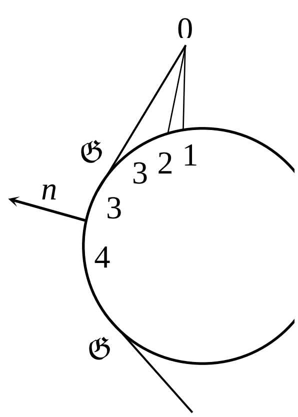

Figure 22.

An extremal meets the barrier at the point 1 (Fig. 22). Let the point 0 be separated from 1 on the former so little that the arc 01 can be surrounded by a field whose extremals go through the point 0. That field also belongs to the part of the barrier that lies in a neighborhood of the point 1 – say, up to the point 2 – and let the broken line 012 be a part of the combined curve that is to be examined for the occurrence of an extremum. Everything that was proved for the curve $\mathfrak{L}$ in § **20** can then be applied to the arc of the barrier that belongs to the field. If x and y are functions $\varphi(\tau)$ in the sense of § **17** along the barrier, τ increases in the direction 12, and the point 3 passes the position 1 while traversing the barrier then one will have:

$$\frac{d(\bar{J}_{03} + J_{32})}{d\tau} = \mathcal{E}\left(x, y, x', y', \frac{dx}{d\tau}, \frac{dy}{d\tau} \right)\Bigg|^{3},$$

in which the quantities x', y' refer to the direction of the extremal 01 that goes from 0 to 1. Therefore, when that quantity $\mathcal{E}$ is non-zero, the sum $\bar{J}_{03} + J_{32}$ can become greater than $\bar{J}_{01} + J_{12}$, as well as smaller for arbitrarily-small distances between the points 1 and 3, and the latter quantity will certainly not define an extremum of the integral J. Should it be provided by the broken line 012 then the equation:

$$\mathcal{E}\left(x, y, x', y', \frac{dx}{d\tau}, \frac{dy}{d\tau}\right)\Big|^{1} = 0$$

must exist, i.e., either the extremals must have contact with the barrier, and indeed in such a way that the direction of advance on 012 does not vary continuously at the point 1, or the quantity $\mathcal{E}$ must vanish in an ordinary way.

One will get a further necessary condition for an extremum when one varies a piece 34 of the barrier that belongs to the desired curve, and indeed in such a way that the endpoints are fixed and the variations of the coordinates have the same properties as in § **2**. If one sets:

$$\frac{F_x - F'_{x'}}{y'} = -\frac{F_y - F'_{y'}}{x'} = M$$

then that will give:

$$\delta J = \int M(y'\,\delta x - x'\,\delta y)\,dt\,,$$
$$\Delta J = \delta J + \int [\delta x, \delta y, \delta x', \delta y']_2\,dt\,.$$

In that way, any point $(x + \delta x, y + \delta y)$ must belong to the region $\mathfrak{G}$ to which the desired curve is constrained. If one lets n denote the normal to the barrier that points to the interior of $\mathfrak{G}$ and lets ε denote a positive constant then the requirements that were expressed above will be fulfilled by the assumption that:

$$\delta x = \varepsilon \cos(n\,x)\,(t - t_3)^3\,(t_4 - t)^3,$$
$$\delta y = \varepsilon \cos(n\,y)\,(t - t_3)^3\,(t_4 - t)^3,$$

and one will get:

$$\delta J = \varepsilon \int [y' \cos(n\,x) - x' \cos(n\,y)]\,M\,(t - t_3)^3 (t_4 - t)^3\,dt\,,$$
$$\Delta J = \delta J + [\varepsilon]_2\,.$$

Therefore, should ΔJ possess a fixed sign, then the same thing would have to be true of the integral $\delta J : \varepsilon$ when it does not vanish. Now since the points 3 and 4 can be brought arbitrarily close to each other, the quantity:

$$[y' \cos(n\,x) - x' \cos(n\,y)]\,M$$

must have the required sign when it does not vanish (i.e., negative in the case of a maximum and positive in the case of a minimum). Now, the determinant:

$$\begin{vmatrix} \cos(n\,x) & \cos(n\,y) \\ dx & dy \end{vmatrix}$$

is positive or negative according to whether the direction n has the same relationship to that of the element (dx, dy), i.e., the direction of integration, that the $+x$-axis has to the $+y$-axis or is opposite to it, resp., which means the same thing as saying according to whether the region $\mathfrak{G}$ is encircled by the path of integration in the negative or positive sense, resp. In the first case, so when a maximum is to occur:

$$M \leq 0 ,$$

and if a minimum is desired then

$$M \geq 0 .$$

In the second case, those inequalities will have the opposite sense.

First example: Draw the shortest line between two points in the plane that does not go beyond the barrier of a given region $\mathfrak{G}$.

The problem becomes interesting only when the line that connects the two points intersects the barrier, so it will no longer be the desired shortest connecting line, and one must bring combinations of straight line segments and pieces of the barrier under consideration. Now since:

$$F = \sqrt{x'^2 + y'^2} , \qquad M = \frac{1}{x'} \frac{d}{dt}\left(\frac{y'}{\sqrt{x'^2 + y'^2}} \right) ,$$

$\mathcal{E}$ cannot vanish in an extraordinary way, so the straight line segments must have contact with the barrier where they meet it. If one further lets ds denote an element of the barrier in the direction of integration then one will have:

$$M = \frac{1}{x'} \frac{d \cos(ds, y)}{dt} .$$

Therefore, if $x' > 0$, the coordinate axes have the usual the orientation, and the first of the cases above is the one that occurs (i.e., the region $\mathfrak{G}$ lies to the right of a person who advances in the direction ds) then cos (ds, y) must increase in order for a minimum to be possible, i.e., the angle (ds, y) must decrease, ds must rotate in the positive sense to the left, and the concave side of the barrier must lie on that same side of ds. Hence, the region must be convex towards the interior of the bounded region, insofar as it defines part of a shortest connecting line. The same thing will be true for the other possible orientations of the figure.

One likewise easily finds for a surface whose line element is represented in isometric coordinates by means of the expression for the geodetic curvature in § **34** that the geodetically-convex side of a barrier to which a shortest connecting line belongs must turn to the interior of the encircled region, i.e., the projection of the radius of curvature of the barrier onto the tangent plane to the surface must point to the outside of the region.

Second example: In Problem II (§ **9**), it is clearly appropriate to look for the desired curve only in the region $y \geq 0$ and to regard the axis of rotation $y = 0$ as the barrier. In general, the latter will not be cut by extremals. However, since one sets:

$$F = y\sqrt{x'^2 + y'^2}\ ,$$

the equations of the extremals:

$$F_{x'} = \frac{d}{dt}\frac{y\,x'}{\sqrt{x'^2+y'^2}} = 0\ , \qquad F_y - F_{y'} = \sqrt{x'^2+y'^2} - \frac{d}{dt}\left(\frac{y\,y'}{\sqrt{x'^2+y'^2}}\right) = 0$$

will be fulfilled by the assumption that:

$$x' = 0\ ,$$

such that the lines x = const. can be regarded as extremals. The quantity $\mathcal{E}$ vanishes on the y-axis for an arbitrary direction of the arc-length element that determines it, so in an extraordinary way, since it has the factor y. Furthermore, for $y = 0$, $x' > 0$, one has the equation:

$$M = -\frac{1}{x'}(F_y - F'_{y'}) = -1\ . \tag{5}$$

If one now integrates along the x-axis in the direction of increasing x then the region $\mathfrak{G}$ will be encircled in the positive sense, and the second of the cases that were distinguished will occur. The condition of the minimum that related to M is then fulfilled. One can then consider the discontinuous solution to the problem to be the broken line that consists of a piece of the axis of rotation and two positive ordinates of arbitrary length that are erected at its endpoints, i.e., the necessary conditions for a minimum that were considered are fulfilled by that line-path. Two points of the half-plane $y > 0$ can always be connected by a broken line with the given character, but by no means can they always be connected with an arc of an extremal. It is clear from this that the solution of the extremum problem will require the introduction of broken lines.

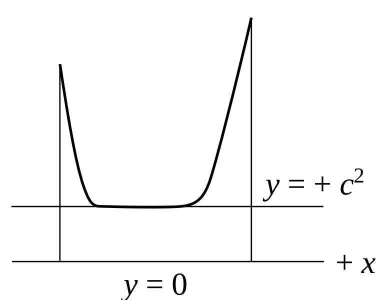

Figure 23.

If one introduces a line $y = c^2$ (Fig. 23) as a barrier and if $y - c^2$ is positive for a given point then equation (5) will be true for the barrier when it is traverses in the direction of increasing x, while the quantity $\mathcal{E}$ can vanish only in an ordinary way. The desired line must now be composed of pieces of the lines $y = c^2$ and contact arcs of catenaries that do not degenerate into the lines x = const.

§ 45. – Relative extremum for a curve that subdivides into different pieces.

As in Chapter Four, let the integral:

$$J = \int F\,dt$$

be extremized for a prescribed value of the integral:

$$K = \int G\,dt\ .$$

Along an extremal $\mathfrak{E}$ that goes through the point 1, let y be representable as a regular function of x in a neighborhood of it, and let the quantity:

$$F_{y'y'} + \lambda G_{y'y'} = x'^2 H_1$$

be non-zero. From § **27**, the set of all extremals in the neighborhood of the point 1 and the curve $\mathfrak{E}$ can be represented in the form:

$$y = \Phi\,(x, a, b, \lambda)\ ,$$

and the function Φ is regular in the neighborhood of the system of values $(x_1, a^0, b^0, \lambda^0)$ that is defined by the curve $\mathfrak{C}$. In particular, we consider all extremals that go through 1 and a point 0 that belongs to the curve $\mathfrak{C}$ such that the equations:

$$(6) \qquad y_1 = \Phi\,(x_1, a, b, \lambda)\ , \qquad y_0 = \Phi\,(x_0, a, b, \lambda)$$

will define the quantities a and b as functions of λ as long as the determinant:

$$\Psi(x_0, x_1, a^0, b^0, \lambda^0) = \begin{vmatrix} \Phi_a\,(x_1, a^0, b^0, \lambda^0) & \Phi_b\,(x_1, a^0, b^0, \lambda^0) \\ \Phi_a\,(x_0, a^0, b^0, \lambda^0) & \Phi_b\,(x_0, a^0, b^0, \lambda^0) \end{vmatrix}$$

is non-zero. Now, since one can obviously set:

$$\Phi\,(x, a, b, \lambda) = y_1 + a + b\,(x - x_1) + [x - x_1]_2\ ,$$

it will follow that:

$$\Phi_a\,(x, a, b, \lambda) = 1 + [x - x_1]_2\ ,$$
$$\Phi_b\,(x, a, b, \lambda) = x - x_1 + [x - x_1]_2\ .$$

That determinant will then take on the form:

$$\begin{vmatrix} 1 & 0 \\ 1+[x_0-x_1]_2 & x_0-x_1+[x_0-x_1]_2 \end{vmatrix},$$

and will be non-zero as long as the quantity $| x_0 - x_1 |$ is sufficiently small. Equations (6) will then give:

$$0 = \Phi_a(x_1, a, b, \lambda)\frac{da}{d\lambda} + \Phi_b(x_1, a, b, \lambda)\frac{db}{d\lambda} + \Phi_\lambda(x_1, a, b, \lambda) \;,$$
$$0 = \Phi_a(x_0, a, b, \lambda)\frac{da}{d\lambda} + \Phi_b(x_0, a, b, \lambda)\frac{db}{d\lambda} + \Phi_\lambda(x_0, a, b, \lambda) \;.$$

In addition, one will obviously have to regard $\bar{K}_{01}$ as a function of λ for which the equation exists:

$$\frac{d\bar{K}_{01}}{d\lambda} = \frac{\partial\bar{K}_{01}}{\partial a}\frac{da}{db} + \frac{\partial\bar{K}_{01}}{\partial b}\frac{db}{d\lambda} + \frac{\partial\bar{K}_{01}}{\partial\lambda},$$

when the partial derivatives are defined under the assumption that x_0 and x_1 are fixed, but not y_0 and y_1 . It will then follow from this that:

$$0 = \begin{vmatrix} \Phi_a(x_0, a, b, \lambda) & \Phi_b(x_0, a, b, \lambda) & \Phi_\lambda(x_0, a, b, \lambda) \\ \Phi_a(x_1, a, b, \lambda) & \Phi_b(x_1, a, b, \lambda) & \Phi_\lambda(x_1, a, b, \lambda) \\ \dfrac{\partial\bar{K}_{01}}{\partial a} & \dfrac{\partial\bar{K}_{01}}{\partial b} & \dfrac{\partial\bar{K}_{01}}{\partial\lambda} - \dfrac{d\bar{K}_{01}}{d\lambda} \end{vmatrix}$$

or

$$\Delta\,(x_0, x_1) = \frac{d\bar{K}_{01}}{d\lambda}\cdot\Psi(x_0, a, b, \lambda).$$

From § **42**, the left-hand side of that equation will be non-zero as long as the distance between the points 0 and 1 is sufficiently small and one sets:

$$(7) \qquad a = a^0, \qquad b = b^0, \qquad \lambda = \lambda^0.$$

The quantity Ψ is finite then, and it will follow that:

$$(8) \qquad 0 \neq \left.\frac{d\bar{K}_{01}}{d\lambda}\right|^{a=a^0,\, b=b^0,\, \lambda=\lambda^0}.$$

Now one obviously has:

$$\frac{d\bar{K}_{01}}{d\lambda} = \frac{d}{d\lambda}\int_{x_0}^{x_1} g\,(x, y, p)\,dx$$

$$= \int_{x_0}^{x_1}\left\{g_y\left(\Phi_a\frac{da}{d\lambda}+\Phi_b\frac{db}{d\lambda}+\Phi_\lambda\right)+g_p\,\frac{d}{dx}\left(\Phi_a\frac{da}{d\lambda}+\Phi_b\frac{db}{d\lambda}+\Phi_\lambda\right)\right\}dx$$

$$= g_p\left(\Phi_a\frac{da}{d\lambda}+\Phi_b\frac{db}{d\lambda}+\Phi_\lambda\right)\Bigg|_{x_0}^{x_1}+\int_{x_0}^{x_1}\left(g_y-\frac{dg_p}{dx}\right)\left(\Phi_a\frac{da}{d\lambda}+\Phi_b\frac{db}{d\lambda}+\Phi_\lambda\right)dx$$

$$= \int_{x_0}^{x_1}\left(g_y-\frac{dg_p}{dx}\right)\left(\Phi_a\frac{da}{d\lambda}+\Phi_b\frac{db}{d\lambda}+\Phi_\lambda\right)dx\,,$$

and likewise:

$$\frac{d\bar{J}_{01}}{d\lambda} = \int_{x_0}^{x_1}\left(g_y-\frac{dg_p}{dx}\right)\left(\Phi_a\frac{da}{d\lambda}+\Phi_b\frac{db}{d\lambda}+\Phi_\lambda\right)dx\ .$$

Moreover, since an equation:

$$f_y-\frac{df_p}{dx}+\lambda\left(g_y-\frac{dg_p}{dx}\right) = 0$$

exists for any extremal, that will give:

$$(9)\qquad -\frac{d\bar{J}_{01}}{d\lambda} = \lambda\frac{d\bar{K}_{01}}{d\lambda}\,,$$

and since we assume that λ is non-zero, the left-hand side of that equation will also be non-zero under the assumption (7). Obviously, the following developments are therefore true:

$$\bar{J}_{01}(\lambda) = \bar{J}_{01}(\lambda^0)+(\lambda-\lambda^0)\left(\frac{d\bar{J}_{01}}{d\lambda}\right)_{\lambda=\lambda^0}+[\lambda-\lambda^0]_2\ ,$$

$$\bar{K}_{01}(\lambda) = \bar{K}_{01}(\lambda^0)+(\lambda-\lambda^0)\left(\frac{d\bar{K}_{01}}{d\lambda}\right)_{\lambda=\lambda^0}+[\lambda-\lambda^0]_2\ .$$

The entire argument is repeated for a second extremal $\mathfrak{C}_1$ along which the points 2 and 3 lie. We denote the value of λ that belongs to it by λ_+^0. For the sake of symmetry, we set λ equal to λ_- in the formulas up to now, and above all, the attachment of the symbols –, + shall suggest that a quantity refers to the curve $\mathfrak{C}$ or $\mathfrak{C}_1$, resp. One will then get the equations:

$$\bar{J}_{23}(\lambda_+) = \bar{J}_{23}(\lambda_+^0) + (\lambda_+ - \lambda_+^0)\left(\frac{d\bar{J}_{23}}{d\lambda_+}\right)_{\lambda_+ = \lambda_+^0} + [\lambda_+ - \lambda_+^0]_2 ,$$

$$\bar{K}_{23}(\lambda_+) = \bar{K}_{23}(\lambda_+^0) + (\lambda_+ - \lambda_+^0)\left(\frac{d\bar{K}_{23}}{d\lambda_+}\right)_{\lambda_+ = \lambda_+^0} + [\lambda_+ - \lambda_+^0]_2 .$$

It will now be assumed that the curves $\mathfrak{C}$ and $\mathfrak{C}_1$ are parts of a piecewise-linear path that is composed in any way and which extremizes the integral J for a prescribed value of K without having the continuity properties of the curve $\mathfrak{B}$ that was considered in § **2**. One then varies it in such a way that one replaces the arcs 23 and 01 of the curves $\mathfrak{C}_1$ and $\mathfrak{C}$, resp., with other extremal arcs with the same endpoints for which the notations that were used up to now might still be valid. Since the integral K should have the same value along the varied line that it had along the original, one will have:

$$\bar{K}_{01}(\lambda_-) + \bar{K}_{23}(\lambda_+) = \bar{K}_{01}(\lambda_-^0) + \bar{K}_{23}(\lambda_+^0) ,$$

so the quantities λ_- , λ_+ will be coupled by the following relation:

$$0 = (\lambda_- - \lambda_-^0)\left(\frac{d\bar{K}_{01}}{d\lambda_-}\right)_{\lambda_- = \lambda_-^0} + (\lambda_+ - \lambda_+^0)\left(\frac{d\bar{K}_{23}}{d\lambda_+}\right)_{\lambda_+ = \lambda_+^0} + [\lambda_- - \lambda_-^0, \lambda_+ - \lambda_+^0]_2 .$$

The extremum will then demand that the quantity:

$$\bar{J}_{01}(\lambda_-) + \bar{J}_{23}(\lambda_+) - \bar{J}_{01}(\lambda_-^0) - \bar{J}_{23}(\lambda_+^0)$$

must have a fixed sign. Since it can be represented in the form:

$$(\lambda_- - \lambda_-^0)\left(\frac{d\bar{J}_{01}}{d\lambda_-}\right)_{\lambda_- = \lambda_-^0} + (\lambda_+ - \lambda_+^0)\left(\frac{d\bar{J}_{23}}{d\lambda_+}\right)_{\lambda_+ = \lambda_+^0} + [\lambda_+ - \lambda_+^0, \lambda_- - \lambda_-^0]_2 ,$$

the general theorem in § **7** will give that:

$$0 = \begin{vmatrix} \dfrac{d\bar{J}_{01}}{d\lambda_-} & \dfrac{d\bar{J}_{23}}{d\lambda_+} \\ \dfrac{d\bar{K}_{01}}{d\lambda_-} & \dfrac{d\bar{K}_{23}}{d\lambda_+} \end{vmatrix}$$

for $\lambda_- = \lambda_-^0$, $\lambda_+ = \lambda_+^0$. From the relations (8), (9), and the analogous ones that pertain to the curve 23, it will then follow from this that:

$$0 = \frac{d\bar{K}_{01}}{d\lambda_-}\frac{d\bar{K}_{23}}{d\lambda_+}\begin{vmatrix} \lambda_-^0 & \lambda_+^0 \\ 1 & 1 \end{vmatrix}, \qquad \lambda_-^0 = \lambda_+^0 .$$

That equation expresses the simplest case of the theorem that **Mayer** presented in regard to the conservation of the isoperimetric constant, which has the same value for $\mathfrak{C}$ and $\mathfrak{C}_1$, from the result that was obtained.

Example: Draw a line of given length between two points 0 and 2 that goes through a given point 1 and encloses the greatest-possible area with the line 02.

If the desired curve has no discontinuity between 0 and 1, as well as between 1 and 2, then the arcs 01 and 12 must be extremals – so circles – on the grounds of the reasoning that implied the corresponding result in § **43**. From the theorem that was obtained, the desired line will necessarily consist of two circular arcs 01, 02 of equal radius. If one lets them vary then one will easily see that the total length of the two arcs can assume any value between certain limits.

§ 46. – Extending the result in § 43 to a relative extremum.

The extremals $\mathfrak{C}$ and $\mathfrak{C}_1$ might now have the point 1 in common. Assume that the points 0 and 2 are close enough to the point 1 that the extremals that go through the former surround each part of the arc 01 as a field, while the ones that go through the latter surround each part of the arc 12. From § **41**, that will be possible when H_1 does not vanish along the arcs 01 and 12 (Fig. 21). If one represents both fields by the system of equations:

$$x = \xi_-(t_-, a_-, b_-), \qquad y = \eta_-(t_-, a_-, b_-),$$
$$x = \xi_+(t_+, a_+, b_+), \qquad y = \eta_+(t_+, a_+, b_+)$$

then any point 3 that differs sufficiently little from 1 can be connected with 0 and 2 by extremals of the field. If one regards the point 3 as variable in the vicinity of the position 1 then one will get (§ **35**):

$$\frac{\partial \bar{K}_{03}}{\partial t_-} = G_-\Big|^3,$$

(10)

$$\frac{\partial \bar{K}_{03}}{\partial a_-} = \int_0^3 [\xi_a (G_x - G'_{x'}) + \eta_a (G_y - G'_{y'})]_- \, dt_- + (\xi_a G_{x'} + \eta_a G_{y'})_-\Big|^3,$$

along with an analogous formula in which one replaces *a* with *b*. Therefore, if the transitions from the point 1 to 3 and from the arc 01 to 03 correspond to increments δt_-, δa_-, δb_- then one will have:

$$\bar{K}_{03} - \bar{K}_{01} = \Delta\bar{K}_{01} = G^-_{x'}\,\delta x + G^-_{y'}\,\delta y\,\Big|^1 + A_-\,\delta a_- + B_-\,\delta b_- + [\delta t_-, \delta a_-, \delta b_-]_2 \;,$$

in which one sets:

$$\text{(11)}\qquad \begin{gathered} \delta x = (\xi_a\,\delta a + \xi_b\,\delta b + \xi_t\,\delta t)_-\,, \qquad \delta y = (\eta_a\,\delta a + \eta_b\,\delta b + \eta_t\,\delta t)_-\,, \\ A_- = \int_0^1 [\xi_a (G_x - G'_{x'}) + \eta_a (G_y - G'_{y'})]_-\, dt_- \;, \end{gathered}$$

and B_- arises from A_- when one replaces a with b. Analogously, one has:

$$\Delta\bar{K}_{12} = -G^+_{x'}\,\delta x - G^+_{y'}\,\delta y\,\Big|^1 + A_+\,\delta a_+ + B_+\,\delta b_+ + [\delta t_+, \delta a_+, \delta b_+]_2 \;,$$

in which the quantities δt_+, δa_+, δb_+, A_+, … have analogous meanings for the extremal 12 that δt_+, … do for 01, such that:

$$\text{(12)}\qquad \delta x = (\xi_a\,\delta a + \xi_b\,\delta b + \xi_t\,\delta t)_+\,, \qquad \delta y = (\eta_a\,\delta a + \eta_b\,\delta b + \eta_t\,\delta t)_+\,.$$

Now, should J be an extremum for a given value of K then the broken line that consists of the extremal arcs 03, 32 will be an allowed variation of the broken line 012, as long as one determines the eight quantities δx, δy, $\delta a_\pm$, $\delta b_\pm$, $\delta t_\pm$ in such a way that the equation:

$$\Delta\bar{K}_{01} + \Delta\bar{K}_{12} = 0$$

exists, or:

$$\text{(13)}\qquad \begin{aligned} 0 = \;& (G^-_{x'} - G^+_{x'})\,\delta x + (G^-_{y'} - G^+_{y'})\,\delta y\,\Big|^1 \\ & - A_-\,\delta a_- - B_-\,\delta b_- + A_+\,\delta a_+ + B_+\,\delta b_+ + [\delta a_+, \delta b_+, \delta t_+, \delta a_-, \delta b_-, \delta t_-]_2\,. \end{aligned}$$

That equation, in conjunction with the ones that were cited in (11) and (12), can serve to express five of the eight quantities δ in terms of the remaining ones. If one takes the latter to be, e.g., δx, δy, δa_+ then the determinant of the terms that are linear in the quantities δt_-, δa_-, δb_-, δt_+, δb_+ on the right-hand side of the indicated five equations will be:

$$\text{(14)}\qquad \begin{vmatrix} 0 & A_- & B_- & 0 & B_+ \\ \xi^-_t & \xi^-_a & \xi^-_b & 0 & 0 \\ \eta^-_t & \eta^-_a & \eta^-_b & 0 & 0 \\ 0 & 0 & 0 & \xi^+_t & \xi^+_b \\ 0 & 0 & 0 & \eta^+_t & \eta^+_b \end{vmatrix} = \begin{vmatrix} 0 & A_- & B_- \\ \xi^-_t & \xi^-_a & \xi^-_b \\ \eta^-_t & \eta^-_a & \eta^-_b \end{vmatrix} \left[\frac{\partial(\xi, \eta)}{\partial(t, b)}\right]_+ \;.$$

If one adds the second and third row in the first determinant on the right, after multiplying them by $G_{x'}^-$ and $G_{y'}^-$, resp., to the first one and imagines that the derivatives of ξ and η refer to the point 1, as they did in equations (11), then equations (10) will show that the determinant has the value:

$$\frac{\partial(\bar{K}_{01}, \xi_-, \eta_-)}{\partial(t_-, a_-, b_-)},$$

i.e., the quantity Δ_- that refers to the field of the arc 01 will then be non-zero. Moreover, since the arc 12 is also surrounded by a field for which the quantity:

$$\frac{\partial(\xi_+, \eta_+, \bar{K}_{21})}{\partial(t_+, a_+, b_+)}$$

does not vanish, from § **35**, the quantities:

$$\left[\frac{\partial(\xi,\eta)}{\partial(t,a)}\right]_+, \quad \left[\frac{\partial(\xi,\eta)}{\partial(t,b)}\right]_+$$

will not both be equal to zero. Hence, either the second factor on the right-hand side of equation (14) is also non-zero or one can achieve that by switching a_+ and b_+. Thus, one can always express five of the quantities δ in terms of the remaining three (among which δx and δy occur) as power series in those three arguments.

If one now replaces the function symbol G with F in the expressions for A, B then one will get $\mathfrak{A}$, $\mathfrak{B}$, resp. The equation for the extremals then implies that:

$$\lambda_- A_- + \mathfrak{A}_- = \lambda_- A_- + \mathfrak{B}_- = \lambda_+ A_+ + \mathfrak{A}_+ = \lambda_+ A_+ + \mathfrak{B}_+ = 0, \tag{15}$$

and from the previous section, one will have:

$$\lambda_+ = \lambda_- = \lambda$$

for $\mathfrak{C}$ and $\mathfrak{C}_1$. Moreover, one will get an expression that is analogous to the expression $\Delta\bar{K}_{01} + \Delta\bar{K}_{12}$ that was developed in equation (13):

$$\begin{aligned}\Delta\bar{J}_{01} + \Delta\bar{J}_{12} = \quad & (F_{x'}^- - F_{x'}^+)\,\delta x + (F_{y'}^- - F_{y'}^+)\,\delta y\,\Big|^1 \\ & - \mathfrak{A}_-\,\delta a_- - \mathfrak{B}_-\,\delta b_- + \mathfrak{A}_+\,\delta a_+ + \mathfrak{B}_+\,\delta b_+ + [\delta a_\pm, \delta b_\pm, \delta t_\pm]_2 .\end{aligned}$$

If one then multiplies equation (13) by λ and adds it to the last one then, since $H = F + \lambda\, G$, from (15), that will give:

$$\Delta\bar{J}_{01} + \Delta\bar{J}_{12} = (H_{x'}^{-} - H_{x'}^{+})\,\delta x + (H_{y'}^{-} - H_{y'}^{+})\,\delta y\,\Big|^{1} + [\delta a_{\pm}, \delta b_{\pm}, \delta t_{\pm}]_2 \ .$$

From the conditions (11), (12), (13), that quantity must have a fixed sign if the required extremum is to be present. Since only the quantities δx, δy occur in the linear terms, from § **7**, that will give:

$$(H_{x'}^{-} - H_{x'}^{+})\,\delta x + (H_{y'}^{-} - H_{y'}^{+})\,\delta y = 0 \tag{16}$$

for the point 1. According to whether the point 1 moves freely or is constrained to the curve:

$$h\,(x, y) = 0 \ ,$$

one will get the relations:

$$H_{x'}^{-} - H_{x'}^{+} = H_{y'}^{-} - H_{y'}^{+} = 0$$

or

$$\begin{vmatrix} H_{x'}^{-} - H_{x'}^{+} & H_{y'}^{-} - H_{y'}^{+} \\ h_x & h_y \end{vmatrix} = 0 \ .$$

Precisely the same restrictions are then valid for a corner point that were valid in § **43** for the case of an absolute extremum when one replaces F with $F + \lambda\, G$.

Example. Problem IX (§ 34). – One has:

$$H = y\,x' + \lambda\sqrt{x'^2 + y'^2}\ , \qquad H_{x'} = y + \frac{\lambda\, x'}{\sqrt{x'^2 + y'^2}}\ , \qquad H_{y'} = \frac{\lambda\, y'}{\sqrt{x'^2 + y'^2}}\ .$$

Equation (16) then gives:

$$\lambda\left(\frac{x'\,\delta x + y'\,\delta x}{\sqrt{x'^2 + y'^2}}\right)_{-} = \lambda\left(\frac{x'\,\delta x + y'\,\delta x}{\sqrt{x'^2 + y'^2}}\right)_{+} \ . \tag{17}$$

Now, since λ is non-zero, it would then follow for a freely-available corner that:

$$\left(\frac{x'}{\sqrt{x'^2 + y'^2}}\right)_{-} = \left(\frac{x'}{\sqrt{x'^2 + y'^2}}\right)_{+}, \qquad \left(\frac{y'}{\sqrt{x'^2 + y'^2}}\right)_{-} = \left(\frac{y'}{\sqrt{x'^2 + y'^2}}\right)_{+},$$

i.e., the directions of the extremals 01 and 12 would coincide at the point 1, and therefore a discontinuity would not occur. By contrast, if the point 1 were constrained to a curve $\mathfrak{K}$ whose arc-length is Ds, and we were to denote the directions of the extremal 01 from 0 to 1 and the extremal 12 from 1 to 2 at the point 1 by –, +, resp., then equation (17) would imply that:

$$\cos\,(Ds, -) = \cos\,(Ds, +) \ .$$

The arcs 01 and 12 then lie on the same side of the curve $\mathfrak{K}$ and define acute angles with its tangent that open on different sides of it, i.e., they lie like the incident and reflected light rays.

Should, e.g., a closed line of given length be drawn from a given point 5 to an unspecified point 1 on the curve $\mathfrak{K}$ and return to the point 5 in such a way that it encloses the greatest-possible area, then it must consist of two circular arcs of equal radius that define equal angles with the curve $\mathfrak{K}$ in the given sense.

§ 47. – Extending the theorem in § 44 to a relative extremum. Steiner's theorem.

It still remains for us to examine when a piecewise-linear path 0123 can appear as a solution to the isoperimetric problem when the comparison curves are constrained to a region $\mathfrak{G}$ that is bounded by a barrier and the segment 12 belongs to a wall, but the arcs 01 and 23 do not, such that from § **45**, the latter will be extremals with the same isoperimetric constants. Then let (Fig. 24) the points 0 and 3 be close enough to 1 and 2, resp., that the arcs 01 and 23 can be surrounded by fields whose extremals go through 0 and 3. Place the points 4 and 5 on the barrier in the vicinity of the points 1 and 2, resp. There are then extremals 04 and 53 that belong to the two fields, and the piecewise-linear path 0453 will be an allowed variation of the path 0123 when:

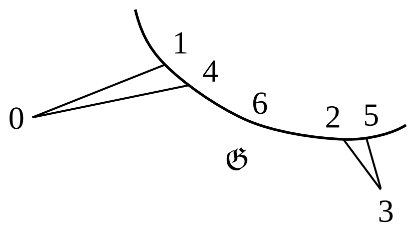

Figure 24.

$$\bar{K}_{01} + K_{12} + \bar{K}_{23} = \bar{K}_{04} + K_{45} + \bar{K}_{53}$$

or

$$K_{12} - K_{45} = \Delta\bar{K}_{01} + \Delta\bar{K}_{23} \,, \tag{18}$$

in which the integrals with no overbar always refer to the barrier and the symbol Δ means the increase during the transition from 01 and 23 to 04 and 53, resp. In order to discuss that requirement, we assign the same meanings to the symbols ξ, η, t_+, $a_\pm$, $b_\pm$, $A_\pm$, $\delta t_\pm$, … for the arcs 01 and 23 that they had for 01 and 12, resp., in the previous section, such that the – sign belongs to the first of the indicated arcs and the + sign belongs to the second one. Accordingly, let:

$$\begin{aligned}
\delta x_- + [\delta t_-, \delta a_-, \delta b_-]_2 &= x_4 - x_1 \,, \\
\delta y_- + [\delta t_-, \delta a_-, \delta b_-]_2 &= y_4 - y_1 \,, \\
\delta x_+ + [\delta t_+, \delta a_+, \delta b_+]_2 &= x_5 - x_6 \,, \\
\delta y_+ + [\delta t_+, \delta a_+, \delta b_+]_2 &= y_5 - y_6 \,,
\end{aligned}$$

and let the expressions δx, δy be linear in δt, δa, δb . One then finds the right-hand side of equation (18) from the formulas for K_{01} that were presented in § **46**. Namely, one has:

(19)
$$\Delta\bar{K}_{01} = \quad G^-_{x'}\,\delta x_- + G^-_{y'}\,\delta y_-\Big|^1 + A_-\,\delta a_- + B_-\,\delta b_- + [\delta t_-, \delta a_-, \delta b_-]_2\,,$$
$$\Delta\bar{K}_{23} = -\,G^+_{x'}\,\delta x_+ - G^+_{y'}\,\delta y_+\Big|^2 + A_+\,\delta a_+ + B_+\,\delta b_+ + [\delta t_+, \delta a_+, \delta b_+]_2\,,$$

and four relations will now exist between the quantities δ:

(20)
$$\delta x_\pm = (\xi_t\,\delta t + \xi_a\,\delta a + \xi_b\,\delta b)_\pm\,,$$
$$\delta y_\pm = (\eta_t\,\delta t + \eta_a\,\delta a + \eta_b\,\delta b)_\pm\,,$$

in which one either takes both sides to have the upper of the ± signs or both of them to have the lower sign. Therefore, if one regards the quantities δt_-, δa_-, δb_-, δb_+, δt_+ here and in equation (18) as being determined by the remaining quantities δ then terms in the five equations that were posed that include those five quantities will have precisely the same determinant as in equations (11), (12), (13) of the previous section. The proof that was given there that this determinant or the one that is obtained from it by exchanging a_+ and b_+ is non-zero will remain valid when one replaces the arc 12 that was considered there with 23. The five equations (18), (20) can then be solved for the quantities $\delta t_\pm$, δa_-, $\delta b_\pm$ as long as a $K_{12} - K_{45}$ is sufficiently small.

Now let x and y be regular functions of the parameters τ_-, τ_+ along the barrier near the points 1 and 2 that increase in the direction 12 and might take on the increments $D\tau_-$ and $D\tau_+$, resp., when one goes to 4 and 4, resp. One will then have:

$$\delta x_- = \frac{dx}{d\tau_-}\Big|^1 D\tau_- + [D\tau_-]_2\,,$$
$$\delta y_- = \frac{dy}{d\tau_-}\Big|^1 D\tau_- + [D\tau_-]_2\,,$$

and similar equations will exist at the point 2. If one assumes that the integrands F and G are regular in the neighborhood of each element of the path 0123 then one will have:

(21)
$$K_{14} = G\left(x, y, \frac{dx}{d\tau_-}, \frac{dy}{d\tau_-}\right)\Big|^1 D\tau_- + [D\tau_-]_2$$

for positive values of $D\tau_-$ and:

(22)
$$K_{41} = G\left(x, y, \frac{dx}{d\tau_-}, \frac{dy}{d\tau_-}\right)\Big|^1 (-D\tau_-) + [D\tau_-]_2$$

for negatives ones. Now when the point 6 is fixed arbitrarily on the barrier between 1 and 2 and outside of the segments 14, 25, one will have one or the other of the equations:

$$K_{46} - K_{16} = - K_{14} ,$$

according to whether $D\tau_-$ is positive or negative, resp. Hence, from (21), (22), in each case one will have:

$$\Delta K_{16} = K_{46} - K_{16} = - \left. G\left(x, y, \frac{dx}{d\tau_-}, \frac{dy}{d\tau_-} \right) \right|^1 D\tau_- + [D\tau_-]_2 \; . \tag{23}$$

Similarly, that will give:

$$\Delta K_{62} = K_{65} - K_{62} = + \left. G\left(x, y, \frac{dx}{d\tau_+}, \frac{dy}{d\tau_+} \right) \right|^2 D\tau_+ + [D\tau_+]_2 \; . \tag{24}$$

If one substitutes the sum of those expressions:

$$\Delta K_{16} + \Delta K_{62} = \Delta K_{12} = K_{45} - K_{12}$$

in equation (18) then it will follow from (19) that:

$$\begin{aligned} &\left. G\left(x, y, \frac{dx}{d\tau_-}, \frac{dy}{d\tau_-} \right) \right|^1 D\tau_- - \left. G\left(x, y, \frac{dx}{d\tau_+}, \frac{dy}{d\tau_+} \right) \right|^2 D\tau_+ \\ &= \left. G_{x'}^- \frac{dx}{d\tau_-} + G_{y'}^- \frac{dy}{d\tau_-} \right|^1 D\tau_- - \left. G_{x'}^+ \frac{dx}{d\tau_+} - G_{y'}^+ \frac{dy}{d\tau_+} \right|^2 D\tau_+ \\ &+ A_- \,\delta a_- + B_- \,\delta b_- + A_+ \,\delta a_+ + B_+ \,\delta b_+ + [\delta a_\pm , \delta b_\pm , \delta t_\pm , D\tau_\pm]_2 \; . \end{aligned} \tag{25}$$

The right-hand sides of equations (19) go to $\Delta\bar{J}_{01}$ and $\Delta\bar{J}_{23}$ when one replaces G with F and A and B with $\mathfrak{A}$ and $\mathfrak{B}$, resp. If one then multiplies the last equation with the constant λ that belongs to the two curves 01 and 23 and considers equation (15) then that will give:

$$\begin{aligned} &\Delta\bar{J}_{01} + \Delta\bar{J}_{23} + \lambda \left. G\left(x, y, \frac{dx}{d\tau_-}, \frac{dy}{d\tau_-} \right) \right|^1 D\tau_- - \lambda \left. G\left(x, y, \frac{dx}{d\tau_+}, \frac{dy}{d\tau_+} \right) \right|^2 D\tau_+ \\ &= \left. H_{x'}^- \frac{dx}{d\tau_-} + H_{y'}^- \frac{dy}{d\tau_-} \right|^1 D\tau_- - \left. H_{x'}^+ \frac{dx}{d\tau_+} - H_{y'}^+ \frac{dy}{d\tau_+} \right|^2 D\tau_+ + [\delta a_\pm, \delta b_\pm, \delta t_\pm, D\tau_\pm]_2 \; . \end{aligned}$$

Now the integral J_{0123} will take on the increase:

$$\Delta\bar{J}_{01} + \Delta\bar{J}_{23} + J_{45} - J_{12} = \Delta\bar{J}_{01} + \Delta\bar{J}_{23} + J_{16} + J_{62}$$

under the variation considered. Furthermore, since one can set:

$$\Delta J_{16} = - \left. F\left(x, y, \frac{dx}{d\tau_-}, \frac{dy}{d\tau_-}\right)\right|^1 D\tau_- + [D\tau_-]_2 ,$$

$$\Delta J_{62} = \left. F\left(x, y, \frac{dx}{d\tau_+}, \frac{dy}{d\tau_+}\right)\right|^2 D\tau_+ + [D\tau_+]_2 ,$$

in analogy with equations (23), (24), one will get the expression:

$$\left. H_{x'}^- \frac{dx}{d\tau_-} + H_{y'}^- \frac{dy}{d\tau_-} - H\left(x, y, \frac{dx}{d\tau_-}, \frac{dy}{d\tau_-}\right)\right|^1 D\tau_- - \left. H_{x'}^+ \frac{dx}{d\tau_+} - H_{y'}^+ \frac{dy}{d\tau_+} + H\left(x, y, \frac{dx}{d\tau_+}, \frac{dy}{d\tau_+}\right)\right|^2 D\tau_+$$

$$+ [\delta t_\pm, \delta a_\pm, \delta b_\pm, D\tau_\pm]_2 ,$$

or more briefly:

$$\Delta J_{0123} = \mathcal{E}^- D\tau_- - \mathcal{E}^+ D\tau_+ + [\delta t_\pm, \delta a_\pm, \delta b_\pm, D\tau_\pm]_2 ,$$

in which the quantity $\mathcal{E}^-$ is defined for the element of the extremal 01 that starts at the point 1 and goes along it and the element of the barrier that corresponds to the increasing values of τ_-, and $\mathcal{E}^+$ is defined for the element of the extremal that points from the point 2 to 3 and in the direction of increasing τ_+.

Should the required extremum be present then ΔJ_{0123} must have a fixed sign for all of the quantities δ, D that are subject to the relations (20), (25). Among them, $D\tau_+$ and $D\tau_-$ remain freely-available. The theorem in § **7** then implies that the linear part of ΔJ_{0123}, which includes none of the variations that are determined by those relations, will vanish for arbitrary values of $D\tau_-$ and $D\tau_+$, i.e.:

$$\mathcal{E}^- = \mathcal{E}^+ = 0 .$$

Therefore, if the extraordinary vanishing of the quantity $\mathcal{E}$ is excluded at the points 1 and 2 then the barrier must contact the extremals that meet those points, such that the result that was proved for absolute extrema will also remain valid here.

Example. Problem IX. – From § **37**, the quantity $\mathcal{E}$ vanishes only in an ordinary way. Therefore, e.g., should a closed line of given length and greatest-possible area be drawn inside of a planar polygon, and if a circular line with the prescribed length has no place inside of the polygon then the desired line can consist of only pieces of the sides of the polygon and the circular arcs of equal radius that contact them.

Problem X. – The quantity $\mathcal{E}$ vanishes, since it differs from the one that appears in Problem IX only by the factor $\sqrt{M}$, in the notation of § **34**, and only in an ordinary way as long as M remains non-zero. If that is true for a region that is bounded by regular curve segments, and should a closed line of given area and greatest-possible length be drawn inside of it, then it can consist of only pieces of the bounding curve pieces and arcs of constant geodetic curvature, and the latter all exhibit the same magnitude of the geodetic curvature and contact the barrier where they meet it. That theorem was presented by **Steiner**.
