"Die Fresnelsche Wellenfläche," Lecture Four of H. Knörrer, C.-G. Schmidt, J. Schwermer, and P. Slodowy, *Mathematische Miniaturen 3: Arithmetik und Geometrie*. *Vier Vorlesungen*, Birkhäuser, Basel, 1986.

The Fresnel wave surface

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Translated by D. H. Delphenich

> *In Iceland, which is an island in the Septrional Sea with a latitude of 66 degrees, one finds a type of crystal, or transparent stone, that is quite remarkable for its shape and other qualities,, but above all, for its strange refractions.*

This statement, which rather reminds one of a fairy tale introduction, is found at the beginning of Chapter 5 in one of the scientific books of Chr. Huyghens, namely, his *Traité de Lumière* [**12**]. The selfsame refraction property of Iceland spar rhombohedra that we will discuss here is the phenomenon of double refraction, which has also been observed in many other crystals: If one lets a light ray fall upon a polished, planar face of such a crystal at an angle then that ray will split into two different light rays that traverse the crystal in different directions (and different velocities of light, in general).

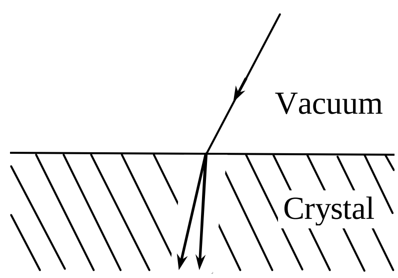

Figure 1.

In the aforementioned *Traité de Lumière*, Huyghens gave a first, more or less satisfactory, explanation for that phenomenon (see [**22**], § 27A). A systematic and quantitative description of the refracting properties of crystals was first arrived at by J. Fresnel between 1820 and 1830. To that end, he introduced a mathematical object – viz., the so-called "Fresnel wave surface" ([1]) – with which, we would mainly like to concern ourselves in this study. In order to describe that surface, we would first like to carry out a small (admittedly, quite unrealistic) Gedanken experiment: Assume that one has a point-like light source that emits monochromatic light. This point-like light source will be switched on for an *infinitely-short* time interval. One will then wait for a certain, finite

([1]) In modern terminology, it is also called a "ray surface." (See [**22**], § 26A).

time interval t in order for the light to spread and then see where the light (i.e., the photons) comes out.

In the event that the experiment is performed in a vacuum, in which no perturbing influences are otherwise present, the result will be the following one: After the time interval t, the light will be found on the outer surface of a ball whose center is found where the light source was switched on, and the radius of the ball is determined from the velocity of light and the time t that one waited after the light source was turned on.

The basis for saying that the surface that light defines after a time t is the outer surface of ball – or, as one says, a *sphere* – lies in the fact that the light propagates *in vacuo* with a constant velocity that is independent of the direction. The velocity of light is independent of direction in many other media (viz., the so-called *optically-isotropic* media), just as it is in the vacuum, and our Gedanken experiment will lead to the same results in those media; namely, the surface that the light defines after a time t will again be a sphere.

However, one can not generally expect that the velocity of light in crystals will be independent of the direction of propagation. For examples, some directions are distinguished by the symmetry of the crystal. If we now perform our experiment inside of a crystal (say, Aragonite) then we will, in fact, get another (and on first glimpse, very surprising) result: The surface that the light defines at time t consists of two shells that both surround the light source, and these two shells come together at some points. This surface is called the *Fresnel wave surface*, and it plays an important role in the investigation of refraction properties in crystals ([2]). In the sequel, we would like to discuss the geometry of that surface somewhat more precisely, explain some phenomena of crystal optics with the help of this geometric investigation, and in conclusion, go into some developments in algebraic geometry that are connected with this and related surfaces.

The basis for the fact that the Fresnel surface consists of two shells that surround the light source is the following: If one gives a general direction to the propagation of light in the crystal then it will follow [perhaps from Maxwell's theory of light ([3])] that only light that is polarized in a certain manner will pass through the crystal in that direction. More precisely, two different polarizations are possible, and these two polarizations correspond to two different velocities of light in that direction (see, perhaps, [**22**], § 25).

In order to describe the Fresnel wave surface quantitatively with more precision, it is necessary to know the possible velocities of light in any given direction. These velocities can be derived from Maxwell's theory of light (see [**22**], § 24-26). We will take the formulas for that theory from the books on theoretical physics with no further discussion and then build our geometric discussion of the Fresnel wave surface upon them.

([2]) The experiment that is described here for the introduction of the Fresnel wave surface is greatly idealized. For example, due to dispersion effects, the incident light in a crystal will first form only after some time has elapsed (cf., the phenomenon of the "precursor." [**22**], § 22). It would then be more realistic to take a light source that is already radiant, which would then emit a stronger impulse after an infinitely-brief time interval.

([3]) Naturally, Fresnel [**5**], who examined these phenomena between 1820 and 1830, did not know of Maxwell's theory of light, which came about in 1860. However, he worked with a theory that was equivalent for that purpose, namely, one in which light was regarded as a wave in a luminous ether (see also [**24**]).

We assume that the point-like wave source of our Gedanken experiment is found at the origin of a Cartesian coordinate system. We describe a direction in space by a vector $x = (x_1, x_2, x_3)$ of length 1, so:

$$x \in S^2 \equiv \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \mid \xi_1^2 + \xi_2^2 + \xi_3^2 = 1\}.$$

The possible velocities of light in the direction x will then be precisely the solutions of the equation:

$$\frac{x_1^2}{(1/a_1^2)-(1/v^2)} + \frac{x_2^2}{(1/a_2^2)-(1/v^2)} + \frac{x_3^2}{(1/a_3^2)-(1/v^2)} = 0, \tag{1}$$

(also $v = \pm a_i$, in the event that $x_i = 0$, resp.)

In this, a_1^2, a_2^2, a_3^2 are the so-called *principal dielectric constants* of the crystal, and we have chosen the coordinates and normalization in such a way that the dielectric tensor is in its principal axis form and all natural constants that appear in it are equal to 1. We would like to mainly consider the case in which the principal dielectric constants a_i^2 are all different, say, $0 < a_1 < a_2 < a_3$ ([4]).

Up to a dilatation about the origin (by the factor t), one now obtains the surfaces from the Gedanken experiment above when one transports the possible velocities of light v in each direction x. In other words: Up to a dilatation, the Fresnel wave surface is equal to:

$$F \equiv \{v \cdot x \mid x \in S^2,\ v \text{ and } x \text{ fulfill (1)}\}. \tag{2}$$

One will obtain an explicit equation for the Fresnel wave surface from this by substitution:

$$F = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \mid \tag{3}$$

$$(\xi_1^2 + \xi_2^2 + \xi_3^2)(a_1^2\xi_1^2 + a_2^2\xi_2^2 + a_3^2\xi_3^2) - [a_1^2(a_2^2 + a_3^2)\xi_1^2 + a_2^2(a_3^2 + a_1^2)\xi_2^2 + a_3^2(a_1^2 + a_2^2)\xi_3^2] + a_1^2a_2^2a_3^2$$

$$= 0\}.$$

In order to get some insight into the geometry of F, this fourth-degree equation is generally rather confusing, and we will mostly employ the description that is given by (2), for the most part.

We next consider the set of vectors $x \in S^2$ for which $\pm v$ is the velocity of light in the direction x for various positive values of v. In the event that $v \neq a_1, a_2, a_3$, this will be the intersection of the unit sphere S^2 with the quadratic cone:

$$K_v \equiv \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \mid \frac{\xi_1^2}{(1/a_1^2)-(1/v^2)} + \frac{\xi_2^2}{(1/a_2^2)-(1/v^2)} + \frac{\xi_3^2}{(1/a_3^2)-(1/v^2)} = 0\}.$$

([4]) Two of the principal dielectric constants are equal for calcite; the general case that is considered here might pertain to the crystal Aragonite, for example.

This intersection is empty for $v > a_3$ or $v < a_1$, so K_v would then consist of only the origin [the quadratic form that defines K_v is positive-definite (negative-definite, resp.)]. In the other cases, $K_v \cap S^2$ will be a curve on the unit sphere that consists of two components and will perhaps have the following form:

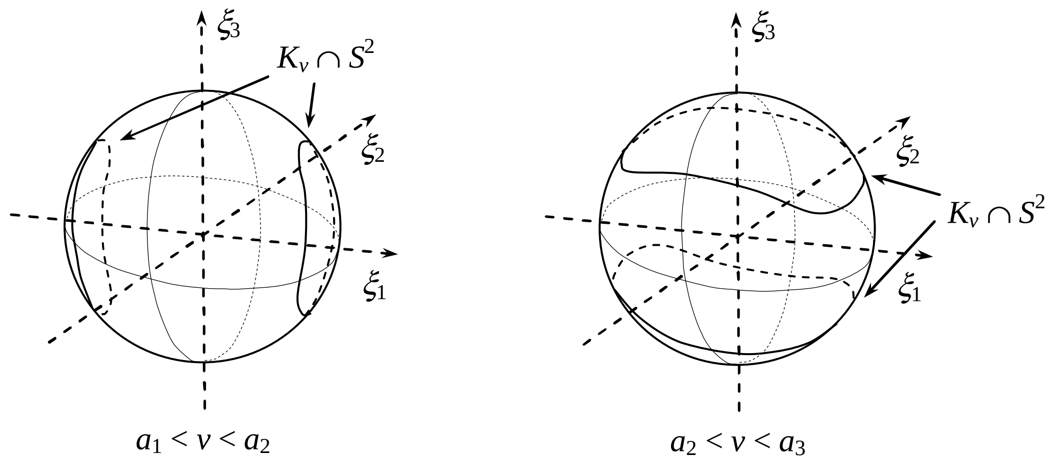

Figure 2.

In case v is equal to one of the values a_i, the set of vectors $x \in S^2$ for which v appears as the velocity of light in the direction x will be equal to the intersection of S^2 with the plane $E_i \equiv \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \mid \xi_i = 0\}$, which we will also correspondingly denote by K_{a_i}.

If one considers all the curves $K_v \cap S^2$ then one will get a covering of the sphere S^2 by a system of curves such that almost every point of S^2 lies on exactly two cures of the system. One also calls such a system of curves a *net* of curves. The following drawing, which was taken from [**23**], shows this net of curves (as seen "from above"):

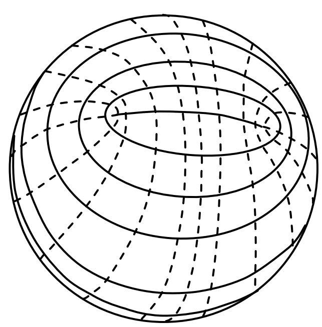

Figure 3.

One sees that there are four special points on S^2 that lie along only one curve of the net – namely, $K_{a_i} \cap S^2$ – and we call them the *focal points* of the net. The associated directions will play an important role in what follows. The description of the Fresnel wave surface F that is given by (2) can then be reformulated as follows: One gets F when one determines the two positive values v_1, v_2 for which $x \in K_{v_i} \cap S^2$ (v_1 and v_2 coincide for the focal points) for every point $x \in S^2$, and then carries the lengths v_1, v_2 in the direction x. In particular, for every radius r, one will get the intersection of F with the sphere:

$$S_r \equiv \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \mid \xi_1^2 + \xi_2^2 + \xi_3^2 = r^2\}$$

of radius r around the origin, when one dilates the curve $K_r \cap S^2$ about the origin by the factor r.

The intersections of F with the coordinate planes:

$$E_i \equiv \{(\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3 \mid \xi_i = 0\} \qquad (i = 1, 2, 3)$$

are especially informative.

Consider, say, the intersection of F with the plane E_1. It follows from (1) that for every vector $x \in E_1 \cap S^2$, one of the possible velocities of light in the direction x is equal to a_1 ; $E_1 \cap S^2$ will then contain the circle K of radius a_1 around the origin. From (1), $E_1 \cap F$ will also contains the points $(0, \pm a_3, 0)$ and $(0, 0, \pm a_2)$, neither of which lie along K. The intersection of E_1 with F will then contain at least one further curve K', in addition to the circle K. Since F, and therefore, also $E_1 \cap F$, will be described by a fourth-degree equation and it is well-known that the equation of a circle has degree two, this extra curve K' will likewise be described by an equation of degree ≤ 2. Curves that are described by an equation of degree 1 are lines, and curves that are described by an equation of degree two are conic sections ([5]). Since F, and therefore also $E_1 \cap F$ and K', are compact, it follows that K' is an ellipse. The fact that F is symmetric under reflections in the plane E_i implies that the principal axes of the ellipse K' are precisely the ξ_3-axis and the ξ_2-axis, and the principal axes lengths are a_2 and a_3 . The intersection of E_1 with F is then the union of the circle K and the ellipse K', and since $a_1 < a_2 < a_3$, this circle is contained within the ellipse completely:

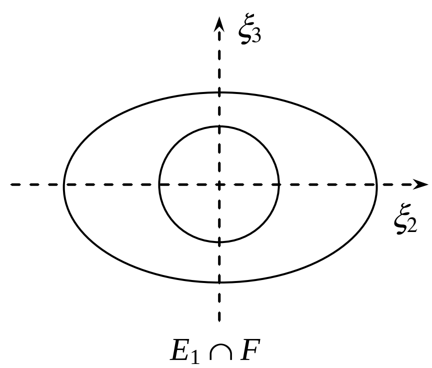

Figure 4.

([5]) Conic sections are ellipses, parabolas, hyperbolas, or line-pairs, and of these, only the ellipses are compact.

Analogously, the intersection of F with E_2 falls upon a circle with radius a_2 and an ellipse with the principal axis lengths a_1 and a_3, and $E_3 \cap F$ is the union of a circle of radius a_3 and an ellipse with principal axis lengths a_1, a_2 . Since $a_1 < a_2 < a_3$, the two aforementioned conic sections will meet only in the case of the plane E_2 .

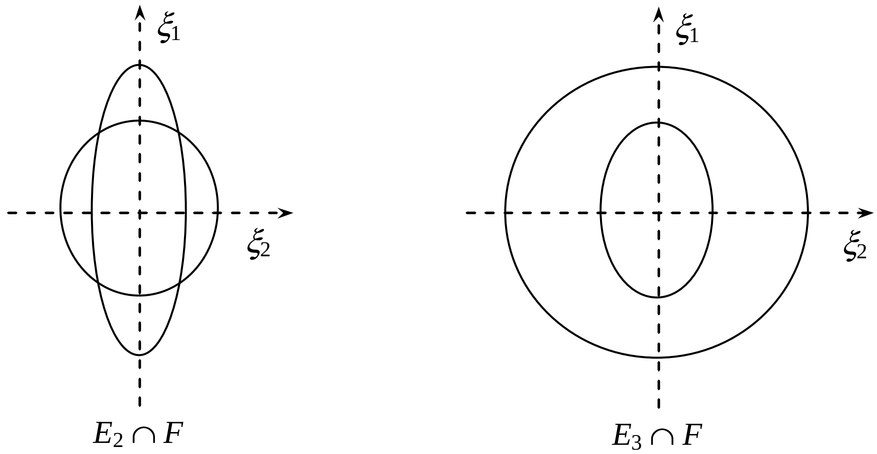

Figure 5.

We have thus gained a rough impression of the geometry of F. Detailed investigation into the Fresnel wave surface were carried out in the last century by, among others, Cauchy, Hamilton, Plücker, Lamé, Cayley, Darboux, Weber ([6]). Here, we would mainly like to refer to a paper of Hamilton ([**9**], nos. 28 and 29) on this topic, since it has especially significant applications to crystal optics.

Hamilton examined the singular points of F, as well as special tangential planes to F. Here, a point of F is called *singular* when F does not have a uniquely-determined tangential plane at that point (more precisely: when the differential of the equation that describes F vanishes at that point).

Theorem *(W. Hamilton 1833).*

(*i*) *F has precisely four singular points, and these singular points all lie in the plane E_2 .*

(*ii*) *There are four planes in space that contact F along a circle. The lines through the origin that are perpendicular to these planes are called the "optical principal axes"* ([7]) *of the crystal; they lie in the plane E_2* ([8]).

The four singular points F are "naturally" the four points of $E_2 \cap F$ at which the circle of radius a_2 and the ellipse with the principal axis lengths a_1, a_3 meet. The

([6]) For a discussion of the further geometric properties of the Fresnel wave surface, as well as references to the classical literature, see, e.g., [**21**], ch. XIX or [**19**], pp. 1740, *et seq.*

([7]) In modern terminology, they are also "optical normal axes."

([8]) Statements (*i*) and (*ii*) in the theorem are closely connected, since – as Hamilton already remarked ([**9**], no. 31) – the dual surface to F is again a Fresnel wave surface (with the constants $1 / a_1$, $1 / a_2$, $1 / a_3$).

associated directions are also the directions that are defined by the focal points of the net $K_v \cap S^2$ of curves on S^2 ([9]).

The second statement of the theorem says the following, in different words: If one brings a plane perpendicular to one of the optical principal axes from the outside to the surface F then that plane will suddenly cut out an entire curve on F, and that curve will be a circle. This is an exceptional phenomenon; in general, a plane contacts a surface in at most finitely many points ([10]).

The following picture shows the intersection of F with the plane E_2, and the two optical principal axes, as well as the intersection of the contact planes of part (*ii*) of the theorem with the plane E_2 :

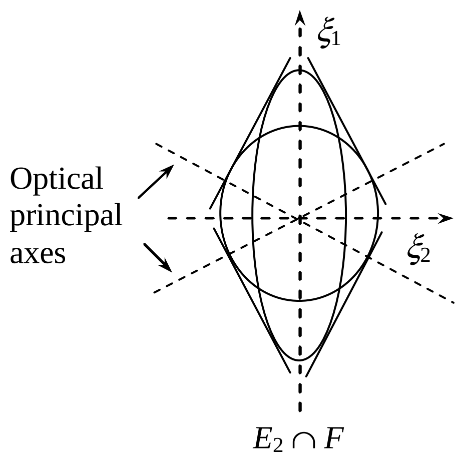

Figure 6.

Some plaster models of the Fresnel wave surface were produced in the last century, in which one can easily recognize some of the facts that were just mentioned. G. Fischer (Düsseldorf) has graciously placed some photographs of these models at my disposal. They are reproduced on the two foregoing pages [†]; we shall give a brief description of these pictures here ([11]):

Photo 1:

In this model, the intermediate space between the outer and inner shell of the Fresnel wave surface F is filled in with plaster. An octant in front and a semi-

([9]) A more precise analysis shows that the singular points of F are so-called *ordinary double points*; i.e., that in a neighborhood U of such a point there are coordinates y_1, y_2, y_3 such that $U \cap F$ will be described by the equation $y_1^2 + y_2^2 - y_3^2 = 0$ in those coordinates.

([10]) F. Zak has recently showed, e.g., that a singularity-free surface in $\mathbb{P}_3(\mathbb{C})$ possesses no tangential planes that are tangential to the surface along an entire curve (see [**6**], § 7).

[†] Translator's note: The photographs are not included here, due to the limited resolution that scanning them would allow, so the reader must refer to the original German book for them.

([11]) The models that are depicted here are found in the model collection of the Mathematical Institute at the University of Göttingen. Many photos of interesting surfaces are presented in the picture volume [**27**]; *inter alia*, photos 1 and 4 of the Fresnel wave surface.

octant in back were removed in order for one to also be able to see the inner shell. The forward intersection surfaces are subsets of the planes E_i – one recognizes pieces of the circles and ellipses in the relevant plane sections of the surface F. One sees singular points of the surface F in the plane E_2, both in front and in back, and in addition, the contact circles in part (*ii*) of Hamilton's theorem are indicated on the surface.

Photo 2:

This picture shows enlargements of the octants that were cut away from the model in (1). One then sees the intersection with the plane E_2 and one of the singular points of F more clearly ([12]).

Photo 3:

One obtains this model when one fills in the space inside of the inner shell of F with plaster. One recognizes (with some difficulty) two vertices on the front side that correspond to two singular points of F.

Photo 4:

Here, a part of model (1) and a part of model (3) are presented next to each other.

We would like to conclude the discussion of the geometry of F with that and go into some phenomena that relate to the refraction of light in crystals. We first consider the following general situation:

Let a light ray be given that comes from the vacuum to a medium; one would like to ascertain the continuation (continuations, resp.) of the light ray (rays, resp.) inside the medium. For the sake of simplicity, we would like to assume that the boundary surface between the vacuum and the medium is a plane and that the velocity of light in the medium possibly depends upon the direction of propagation, but not on position.

There are several equivalent methods for treating the problem of light refraction. To that end, it is most advantageous to apply a variational principle, namely, the so-called *Fermat principle.* That principle yields the following construction prescription:

One thinks of the originally-given ray as having been embedded into a system of parallel rays. One has a system of wave fronts perpendicular to it. One selects one of these wave fronts, which will then be a plane E that is perpendicular to the original ray. In addition, one chooses an auxiliary point p inside of the medium. One next determines the paths of the light that emanates from the wave front E and goes through the point p. One obtains the possible continuation (continuations, resp.) of the originally-given ray by parallel displacement.

([12]) The lines that are indicated on the model are intersection of F with concentric spheres (a family of ellipsoids, resp.) (cf., [**26**], pp. 168).

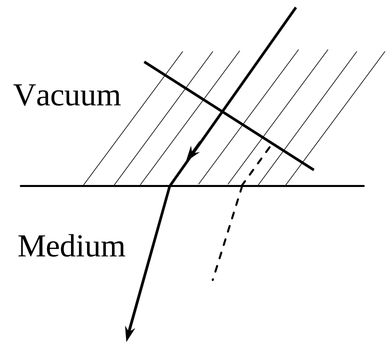

Figure 7.

Now, Fermat's principle (see, perhaps, [**11**], § 2) states that light will choose precisely that path from the wave front E to the point p for which:

(*i*) The path runs along a line segment both inside and outside the medium, and outside of the medium (thus, *in vacuo*) it is parallel to the originally-given light ray.

(*ii*) Among all possible paths of that sort, the paths that are, in fact, chosen by light have the property that the time that light requires between E and p along that path assumes a relative minimum.

In the case for which the medium is optically isotropic – so the velocity of light in the medium is independent of the direction – one sees that the variational problem that was just described has precisely one solution. There will then be precisely one path from E to p that light chooses, and therefore the originally-given ray has a single-valued continuation inside the medium (perhaps as in the drawing above).

Since the velocity of light in a medium is smaller than it is *in vacuo*, the light ray will be bent from the incident direction. A more precise quantitative analysis of this situation with the help of Fermat's principle leads to the known Fresnel formulas for the refraction of light (cf., [**11**], § 3).

In the case of a crystal, the solution of the variational problem above is connected with the geometry of the Fresnel wave surface, since it describes precisely the possible velocities of light inside of the crystal. In general, the variational problem will now have two different solutions ([13]): One relative minimum for the time interval that is required by light from E to p is associated with a direction of propagation and a velocity of light inside the crystal that corresponds to a point on the outer shell of F, and another relative minimum corresponds to a point on the inner shell of F. From p onward, two light rays will emanate from the wave front E. Since each of the light rays that are parallel to the original ray behave the same under refraction by the crystal, one will get two different continuations of the original ray by parallel displacement of the rays through p.

([13]) In the event that two of the principal dielectric constants are different.

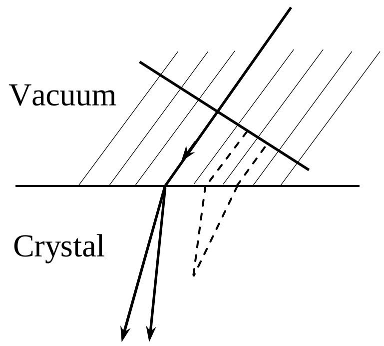

Figure 8.

The incident ray thus splits into two different rays that proceed inside the crystal with differing directions and differing velocities. This is the phenomenon of double refraction that was described in the Introduction.

A special situation arises when the boundary surface between the crystal and the vacuum is perpendicular to one of the optical principal axes. One can choose the boundary surface between the crystal and the vacuum to be a wave front E (= plane perpendicular to the incident system of parallel light rays). We think of the Fresnel wave surface F as being laid through the point p in such a way that E is one of the tangential planes that were mentioned in Hamilton's theorem. One then sees that the time duration for light will be the same for all points of the contact circle $E \cap F$ (points of E, resp.) to p, and that this time duration will be an absolute minimum.

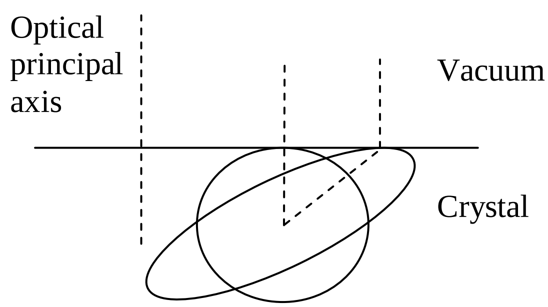

Figure 9. Intersection with the plane E_2.

The variational problem then has an infinitude of solutions in this case and correspondingly the incident light ray splits inside the crystal into an entire cone of light rays.

This phenomenon is called (*internal*) *conical refraction* ([14]) and was predicted by Hamilton in 1832 on the basis of his investigations into the geometry of the wave surface F. The experimental verification of that phenomenon by Hamilton's Dublin colleague, the experimental physicist H. Lloyd, soon proved to be an important confirmation of the wave theory of light, which was in conflict at the time with Newton's propagating particle theory (see, e.g., [**10**]). One also obtains a similar phenomenon when one lets a

([14]) The phenomenon that is actually observed is somewhat more complicated; see, e.g., [**2**], 14.3.4.

light ray that travels inside of a crystal in a direction that is defined by one of the singular points of F leave the crystal into the vacuum (viz., the so-called "external conical refraction").

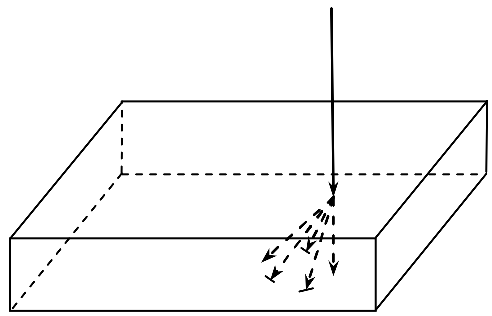

Figure 10.

After this foray into light refraction in crystals, I would now like to go into some more intrinsically mathematical developments that are closely related to the Fresnel wave surface and similar surfaces.

Some time around 1860, the Fresnel wave surface appeared in a completely different context in the theoretical papers of E. Kummer on ray optics. Kummer concerned himself, *inter alia*, with the following problem: Pursue the light rays that emanate from a point-like light source and are then act upon by various optical apparatuses (such as systems of mirrors and lenses). A two-parameter family of light rays (which we will think of as being extended to lines) will come about after all of the reflections and refractions have been performed. Kummer called such a two-parameter family of lines a *ray system*, and he began to systematically investigate all such ray systems [**15**]. In particular, he was interested in the focal surfaces of such ray systems. These are the surfaces B with the property that at least one line of the system will go through each non-singular point of B, and it will be tangential to B at that point. In the optical situation, the focal surfaces will be domains of especially high light intensity.

One can expect few concrete results for general ray systems, and for that reason, Kummer mainly investigated algebraic ray systems (i.e., ray systems that are described by polynomial equations in suitable coordinates) with the property that only a few lines of the ray system will go through a general point of space. One also calls the number of lines of a ray systems that go through a general point in space the *order* of the ray system. (In order to make the definition meaningful, one must also allow complex lines in $\mathbb{C}^3$ – or even better, in projective space $\mathbb{P}_3(\mathbb{C})$ ([15]) – which fulfill the defining equations of the ray system, and thus, consider the associated complex ray system).

We now emphasize that the Fresnel wave surface also appears as the focal surface of a special second-order ray system. Kummer [**16**], [**17**] investigated the focal surfaces of second-order ray systems in general and obtained, *inter alia*, the following result:

([15]) For a discussion of the transition from real, affine space $\mathbb{R}^n$ to complex, projective space $\mathbb{P}_n(\mathbb{C})$, see perhaps [**3**], I.3.

Theorem *(E. Kummer, 1865):*

(*i*) *The focal surface of a second-order ray system in complex projective space* $\mathbb{P}_3(\mathbb{C})$ *is a surface of degree four* ([16]) *with 16 ordinary double points or a degenerate case of such a surface.*

(*ii*) *Any surface of degree four in* $\mathbb{P}_3(\mathbb{C})$ *that has 16 ordinary double points is the focal surface of a second-order ray system.*

Such surfaces are also called *Kummer surfaces* today ([17]); the Fresnel wave surface is then a special case of a Kummer surface ([18]). Twelve of its singular points lie in complex spaces, and indeed four of them lie in the complexifications of the planes E_1, E_3, and the plane at infinity. This result was one of the grounds for the fact that mathematics began to take a more serious interest in the geometry of the complexification of the Fresnel wave surface, and thus in the associated surface in $\mathbb{P}_3(\mathbb{C})$, which we would like to denote by $F_{\mathbb{C}}$ in the sequel.

The examination of the complexification of the Fresnel wave surface also proved to be meaningful on another ground. For some purposes, it is useful for have an explicit parameterization of the surface F at one's disposal. One can show that such a parameterization is not possible with the help of classical functions, such as polynomials and trigonometric or exponential functions; rather, it is necessary to employ the so-called elliptic functions. This leads one into the realm of "function theory," and for that reason it is also obvious that one should consider the complexification of F here. A parameterization of the Fresnel wave surface was given by Weber in 1878 ([**25**], pp. 353); its result can be reformulated into a description of the geometry of $F_{\mathbb{C}}$:

A two-dimensional complex torus is a complex-analytic manifold of the form $A = \mathbb{C}^2 / \Gamma$, where $\Gamma \subset \mathbb{C}^2$ is a subgroup of the form $\Gamma = \mathbb{Z} \cdot \omega_1 + \ldots + \mathbb{Z} \cdot \omega_4$ with vectors $\omega_1, \ldots, \omega_4 \in \mathbb{C}^2$ that are linearly-independent over $\mathbb{R}$. One has the involutory map $i : A \to A$ that is induced by the map $\mathbb{C}^2 \to \mathbb{C}^2$, $x \mapsto -x$ on such a complex torus. One obtains special complex tori when one considers an elliptic curve E (i.e., a complex manifold of the form $E = \mathbb{C} / \mathbb{Z} \cdot \gamma_1 \oplus \mathbb{Z} \cdot \gamma_2$ with vectors $\gamma_1, \gamma_2 \in \mathbb{C}$ that are linearly-independent over $\mathbb{R}$) and defines the product $A \equiv E \times E$.

([16]) The degree of a surface B is the degree of an equation that defines B.

([17]) The result above is also closely connected with the classical investigations into the so-called quadratic line complexes; for a modern presentation, see perhaps [**8**], ch. 6.

([18]) F. Klein, in his book on the history of mathematics in the Nineteenth Century ([**13**], pp. 195) described it as follows: "For the contemporary geometer, the Fresnel surface is no longer an extraordinary construction; it is a special case of the Kummer surface with 16 double points and 16 double planes, which is characterized by its reality behavior and certain symmetries, moreover."

Theorem *(H. Weber):*

There is an elliptic curve such that $F_{\mathbb{C}}$ is isomorphic to $E \times E / i$ as a complex-analytic variety ([19]).

When one makes it explicit, the map $\mathbb{C} \times \mathbb{C} \to A \to A / i \xrightarrow{\infty} F_{\mathbb{C}}$ will yield the parameterization of $F_{\mathbb{C}}$ by elliptic functions. One then gets the parameterization of the real surface F by restricting oneself to the respective "real subspaces."

More generally, it was shown by Borchardt, Rohn, Klein, and others (see [**14**]) that every Kummer surface is isomorphic to a surface of the form A / i, but in general, A is no longer isomorphic to a product of elliptic curves (see also, [**8**], ch. 6) ([20]). In general, if one has been given a class of complex-analytic (or algebraic) varieties then an obvious question to ask would be: When are two varieties in that class isomorphic? For example, it turns out that two varieties A / i and A' / i with $A = \mathbb{C}^2 / \Gamma$, $A = \mathbb{C}^2 / \Gamma'$ are isomorphic precisely when there is a complex-linear map of $\mathbb{C}^2$ to itself that takes Γ to Γ'. One also calls the problem of gaining some glimpse into the set of all isomorphism classes of all varieties of a certain type the *moduli problem* for varieties of that type. The moduli problem for surfaces of type A / i is then "essentially" reduced to a problem in linear algebra.

The Kummer surfaces (and more generally, the surfaces of type A / i) are a subclass of an even larger class of surface – viz., the so-called $K3$ surfaces ([21]). Examples of such $K3$ surfaces are, perhaps, also non-singular fourth-degree surfaces in $\mathbb{P}_3(\mathbb{C})$. A satisfactory solution of the moduli problem for $K3$ surfaces was first accomplished a few years ago. It can then be shown that, e.g., the set of all algebraic $K3$ surfaces with a certain polarization ([22]) define a 19-dimensional complex-analytic variety M ([23]). This variety M (which one also calls the moduli space of the $K3$ surfaces with this polarization) can be described quite explicitly. One finds a more precise description of this result in, say, the talk by A. Beauville to the Séminaire Bourbaki in1982 [**1**].

([19]) More precisely: E is a double covering of $\mathbb{P}_1(\mathbb{C})$ that is branched over 0, a_1^2, a_2^2, $a_3^2 \in \mathbb{C} \subset \mathbb{P}_1(\mathbb{C})$. Since this result was not formulated explicitly in this way in the classical papers, we shall sketch a geometric proof of it here (see Appendix).

([20]) Conversely, if a two-dimensional complex torus admits a principal polarization ([**8**], ch. 2, 6) then A / i will be isomorphic to a fourth-degree surface in $\mathbb{P}_3(\mathbb{C})$ with 16 ordinary double points, and thus, a Kummer surface.

([21]) More precisely: The minimal de-singularization of a surface of type A / i belongs to the $K3$ surfaces.

([22]) I. e., one singles out a system of algebraic curves in the surface that appear to be precisely hyperplane sections for a suitable embedding of the surface in a projective space.

([23]) I. e., the structure of a complex-analytic variety can be defined on this set "in a natural way." The "naturality" implies, e.g., that the map of the space of coefficients of homogeneous equations of degree four in four variables that define non-singular surfaces in $\mathbb{P}_3(\mathbb{C})$ into the corresponding space M that associates any equation with the isomorphism class of its associated $K3$ surface is holomorphic.

From the standpoint of the moduli problem for *K*3 surfaces, the Fresnel wave surface then corresponds to a point in the 19-dimensional *M* ([24]). This manner of consideration can possibly give the impression that for a modern geometer the Fresnel wave surface is only as interesting or interesting as any other *K*3 surface that corresponds to any other point of the space *M*. Naturally, that is not entirely true: For example, the construction of the moduli space *M* is already based upon the precise knowledge of the geometry of Kummer surfaces. The Kummer surfaces define a "skeleton" for *M*, so-to-speak, from which one can venture into the unknown realms of the general *K*3 surfaces. The precise knowledge of the moduli space *M* also makes it actually possible to carry out a systematic investigation of special *K*3 surfaces (say, ones with higher Picard numbers [**20**] or ones with interesting groups of automorphisms [**4**]) and naturally the Fresnel wave surface belongs to this special *K*3 surface.

This sketch of the developments that were connected with the Fresnel wave surface and Kummer surfaces shows quite clearly that by abstraction, generalization, and defining the links to other intrinsic and extrinsic mathematical issues, the topics and methods that pertain to the investigation of the Fresnel wave surfaces will always lead further away from the original physical problem (viz., the description of the phenomena of crystal optics). On the other hand, many of the modern mathematical methods – such as the ones in moduli theory – have interesting and useful applications to physics ([25]). These statements would prove to go beyond the scope of this lecture; one finds an engaging attempt to do that in the book [**18**] *Mathematics and Physics* of the Soviet mathematician Y. Manin.

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([25]) An example that is close to our circle of topics is the paper of D. Giesecker and E. Trubowitz [**7**] in which the examination of "periodic maps" was employed in the study of Fermi surfaces, which play an important role in the quantum-mechanical description of electrons in metals and crystals.

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Appendix: proof of theorem in footnote 19.

If $f(\xi_1, \xi_2, \xi_3) = 0$ is the equation of F, as in (3), then $F_{\mathbb{C}} \subset \mathbb{P}_3(\mathbb{C})$ will be described by the fourth-degree homogeneous function:

$$\xi_0^4 \cdot f(\xi_1 / \xi_0, \xi_2 / \xi_0, \xi_3 / \xi_0) = 0.$$

One sees that $F_{\mathbb{C}}$ will be taken to itself under the "reflections" $\sigma_i : \mathbb{P}_3(\mathbb{C}) \to \mathbb{P}_3(\mathbb{C})$, $(\xi_1, \dots, \xi_3) \mapsto (\varepsilon_{i0}\, \xi_1, \dots, \varepsilon_{i3}\, \xi_3)$, in which $\varepsilon_{ij} = 1$ when $i \neq j$ and $\varepsilon_{ii} = -1$. $\sigma_0, \dots, \sigma_3$ generate a subgroup G of $PGL(3, \mathbb{C})$ that is isomorphic to $(\mathbb{Z} / 2\,\mathbb{Z})^3$.

Lemma:

There is a G-invariant isomorphism:

$$\pi : F_{\mathbb{C}} \to \mathbb{P}_1(\mathbb{C}) \times \mathbb{P}_1(\mathbb{C})$$

such that one has:

(*i*) π *induces an isomorphism between* $F_{\mathbb{C}} / G$ *and* $\mathbb{P}_1(\mathbb{C}) \times \mathbb{P}_1(\mathbb{C})$.

(*ii*) π *branches precisely over the divisors:*

$$D_i = \{(v_1, v_2) \in \mathbb{P}_1(\mathbb{C}) \times \mathbb{P}_1(\mathbb{C}) \mid v_1 = (a_i^2, 1) \text{ or } v_2 = (a_i^2, 1)\},$$

$i = 0, \dots, 3$, *in which we set* $a_0 = 0$.

(*iii*) *A path in a small, transversal disc to one of the components of* D_i *with the winding number 1 around that component of* σ_i *will be defined by the homomorphism:*

$$\pi_1(\mathbb{P}_1(\mathbb{C}) \times \mathbb{P}_1(\mathbb{C}) - \bigcup_{i=0}^{3} D_i\,) \to G$$

that belongs to the covering π.

Proof:

The idea for the construction of π comes from the real situation: Every point $\xi \in F$ is of the form $\xi = v \cdot x$ with $x \in S^2$. The point x lies in the quadratic cone K_v and on a further cone $K_{v'}$ with $v' \neq \pm v$, in general (i.e., when x is not a focal point of the net that goes through K_v). The values $|v|$, $|v'|$ are precisely the possible velocities of light in the direction x. If we assign $\xi \in F$ to the pair (v^2, v'^2) then we will get a well-defined map $F \to \mathbb{R} \times \mathbb{R}$. In formulas, this map will be described by:

$$x \mapsto (\xi_1^2 + \xi_2^2 + \xi_3^2, (a_1^2\xi_1^2 + a_2^2\xi_2^2 + a_3^2\xi_3^2)/(a_1^2 a_2^2 a_3^2)) .$$

One easily proves now that this also defines a morphism of $F_{\mathbb{C}}$ to $\mathbb{P}_1(\mathbb{C}) \times \mathbb{P}_1(\mathbb{C})$ that has the desired properties.

Now let $\tau : E \to \mathbb{P}_1(\mathbb{C})$ be the doubly-branched covering of $\mathbb{P}_1(\mathbb{C})$ that is branched over the points $(a_i^2, 1)$, $i = 0, \ldots, 3$, precisely. E is an elliptic curve; we choose the origin of the group structure in E to be the point that lies above $(a_0^2, 1) = (0, 1)$. We let α_i denote the point of E that lies above $(a_i^2, 1)$, so $0 = \alpha_0, \alpha_1, \alpha_2, \alpha_3$ define the group T_2 of doubly-covering points of E. Let $m : E \to E$ be multiplication by 2: $x \mapsto x + x$, and let j $E \to E$ be multiplication by -1. T_2 will then be the kernel of m, and j will be the deck transformation of the covering τ.

$pr \equiv \tau \circ m : E \to \mathbb{P}_1(\mathbb{C})$ is then an eight-fold branched covering of $\mathbb{P}_1(\mathbb{C})$ with $H \equiv T_2 \times \langle j \rangle$ for its deck transformation group (thus, T_2 operates on E by translation). The associated homomorphism $\pi_1(\mathbb{P}_1(\mathbb{C}) - \{a_0^2, \ldots, a_3^2\}) \to H$ defines path on $j \circ \alpha_i$ that encircles that point $(a_i^2, 1)$ once in a small neighborhood of that point. We let p denote the two-fold covering $pr \times pr$: $E \times E \to \mathbb{P}_1(\mathbb{C}) \times \mathbb{P}_1(\mathbb{C})$. It is branched precisely over D_0 … D_3, and its deck transformation group is $H \times H$. If $\tilde{H} \subset H \times H$ is the subgroup of order 8 that is generated by (j, j) and (α_i, α_i) then one will see that there is an isomorphism $H \times H / \tilde{H} \cong G$ such that the diagram:

$$\begin{tikzcd}
& H \times H \arrow[r] & H \times H / \tilde{H} \arrow[dl, "\cong"] \\
\pi_1(\mathbb{P}_1(\mathbb{C}) \times \mathbb{P}_1(\mathbb{C}) - D_0 \cup \ldots \cup D_3) \arrow[r] \arrow[ur] & G &
\end{tikzcd}$$

commutes. As a result, $E \times E / \tilde{H}$ is isomorphic to $F_{\mathbb{C}}$. Furthermore, the subgroup $\tilde{T}_2$ of $H \times H$ that consists of (α_i, α_i), $i = 0, \ldots, 3$ operates on $E \times E$ by translations. Therefore, $A \cong E \times E / \tilde{T}_2$ is a complex torus. Since (j, j) operates on A like multiplication by -1, $F_{\mathbb{C}} = A / i$. It remains to be shown that the torus $A = E \times E / \tilde{T}_2$ is once more isomorphic to $E \times E$. Therefore, let q: $E \times E \to E \times E$ be the map $(x, y) \mapsto (x + y, x - y)$. q is a morphism of complex tori, and one easily shows that $\ker q = \tilde{T}_2$. With that, the assertion is proved.