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The principle of relativity and accelerated motion

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(Confer Plates XV and XVI, Tables and 2)

Translated by D. H. Delphenich

In the present article, an attempt will be made to adapt the Lorentz transformation from uniform rectilinear motion to certain accelerated motions. That means that an observer that takes part in such accelerated motions is put into a position by means of a suitable comoving reference system to regard himself as being at rest, and phenomena that do not take part in the motion are attributed to apparent accelerations. On epistemological and physical grounds, one then replaces the reference system above with a reference body. However, in order to do that, if the observer is not to perceive his own motion in the changes in the rest form then it is necessary that he should move like a **Born** rigid body. Of the two types of such bodies that **Herglotz** gave, however, when one demands the constancy of the “proper coordinates,” as well as the “non-simultaneous” points of the body, only the latter is to be used, and their worldlines are known to be the paths of a one-parameter group of orthogonal transformations of **Minkowski**’s S_4 . That immediately implies the moving 4-frame of the world-line of the observer as the desired generalization of the comoving system. **Maxwell**’s basic equations remain invariant when one does that, from which one concludes that when once force equilibrium comes about in the reference body, it will continue to exist for the entire motion.

The nature of those accelerated motions is illuminated by the fact that they represent the relativistic generalization of the ones in **Newtonian** mechanics that are performed with constant acceleration, so the *uniform rectilinear motions, free fall, uniform rotation, and combinations of them.* In that way, the aforementioned conservation of equilibrium will become understandable. Now, in order to explain the creation of those distinguished motions for the electron, a model of that will be presented that corresponds to the usual representations of electromagnetic mechanics and therefore, like them (to the extent that **Maxwell**’s equations are applicable to its interior), it suffers from the same defect that has been known since **Abraham** and **Poincaré**, namely, how to explain its cohesion. The hypothesis that is appealed to here asserts the addition of elastic stresses that neutralize the **Maxwell** stresses, such that under force equilibrium, when one also considers the external field, the individual particles of the charge are regarded as indifferent to each other. That will then imply the stated motions as the ones that the electron performs in a spatially and temporally constant external electromagnetic field. In that way, it is assumed that the motion proceeds from minus infinity to plus infinity. If only part of the motion were realized, such as in deflection experiments with cathode rays, then it would be self-evident to assume that deformations of the body occur under the transition into motion, as they do in the motion of the **Born** body, but they are not regarded as a pressure mechanism, but simply as distinguished motions, since, in particular, the ones for the bodies of the second type will be preserved "intrinsically," as will be shown. For the model of the electron above, it also follows that those motions can proceed without radiation which remains a known foundation of the theory of magnetization electrons. However, if the solution of the problem of the natural cohesion of electricity is to be possible without auxiliary elastic stresses then one would expect that it would be only a change in the basic equations.

Upon pursuing the proposed generalization of the Lorentz transformation further, it will be shown in the example of uniform rotations, which are counted among the distinguished motions, that the reference system is distinguished from the comoving system of the older mechanics by only second-order quantities, just as in the usual **Lorentz**ian case, which is why first-order rotational effects cannot at all be consulted as a judgement against the principle of relativity, as has been attempted lately. The fact that such things should occur, and that the observer can perceive his own motion accelerated motion, moreover, is obvious, as was indeed known in the older mechanics.

Finally, we will go into the question of how the moving observer represents the universe. That will show the invariance of the speed of light for his immediate vicinity, but not further beyond it. It will be shown how the speed of light depends upon the force-potential of the apparent acceleration when it is referred to the proper system in the example of **Born**'s hyperbolic falling motion.

In an Appendix, formulas from differential geometry will be given that seem unavoidable in the further construction of **Minkowski**'s kinematics, as well as a discussion of the connection between **Minkowski** world-lines and **Hamilton**'s velocity hodographs ([1]).

Notations.

$x^{(1)}$, $x^{(2)}$, $x^{(3)}$, $x^{(4)}$ are coordinates of S_4 ([2]). Lower indices will never be coordinate indices in what follows. For example, c_1 will mean a vector whose components are $c_1^{(1)}$, $c_1^{(2)}$, $c_1^{(3)}$, $c_1^{(4)}$.

x, y, z, t are coordinates and time in S_3, in which the usual notations of three-dimensional vector analysis will be employed.

The index 4 does not denote a timelike direction in the moving 4-frame; i.e., $c_1^{(1)}$, $c_1^{(2)}$, $c_1^{(3)}$, $c_1^{(4)}$ are the direction cosines of the fourth axis of the 4-frame, but it is, however, spacelike.

([1]) For what follows, cf., in particular, **G. Herglotz**, "Relativitätsprinzip und starrer Körper," Ann. Phys. (Leipzig) (4) **31** (1910), pp. 393. **F. Kottler**, "Über die Raumzeitlinien der Minkowskischen Welt," Wien. Ber. IIa October 1912, §§ 6-8. In the latter paper, one already finds the results in § 5, Point 7 and Appendix 1 to 2 in the present one.

([2]) That notation originates in the absolute differential calculus. **F. Kottler**, *loc. cit.*, §§ 1-3.

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§ 1. – The trajectories of a one-parameter group of orthogonal transformations of Minkowski S_4 .

1. – Any *finite* transformation that depends upon *one* parameter can be constructed from successive applications of its fundamental *infinitesimal* transformation. For a (proper) orthogonal one, the latter has the following form:

$$
(1)\qquad \left\{\begin{aligned}
\frac{dx^{(1)}}{du} &= \varepsilon^{(1)} + \quad * \quad + \varepsilon_2^{(1)} x^{(2)} + \varepsilon_3^{(1)} x^{(3)} + \varepsilon_4^{(1)} x^{(4)}, \\
\frac{dx^{(2)}}{du} &= \varepsilon^{(2)} + \varepsilon_1^{(2)} x^{(1)} + \quad * \quad + \varepsilon_3^{(2)} x^{(3)} + \varepsilon_4^{(2)} x^{(4)}, \\
\frac{dx^{(3)}}{du} &= \varepsilon^{(3)} + \varepsilon_1^{(3)} x^{(1)} + \varepsilon_2^{(3)} x^{(2)} + \quad * \quad + \varepsilon_4^{(3)} x^{(4)}, \\
\frac{dx^{(4)}}{du} &= \varepsilon^{(4)} + \varepsilon_1^{(4)} x^{(1)} + \varepsilon_2^{(4)} x^{(2)} + \varepsilon_3^{(4)} x^{(3)} + \quad *
\end{aligned}\right.
$$

The $\varepsilon^{(h)}$ and $\varepsilon_k^{(h)}$ in this are *constants* that must satisfy only the conditions that:

(a). $\varepsilon_k^{(h)} = -\varepsilon_h^{(k)}$,

(b). $\varepsilon_4^{(1)}, \varepsilon_4^{(2)}, \varepsilon_4^{(3)}$, as well as $\varepsilon^{(4)}$, are pure imaginary, while the others are real,

(c). $\dfrac{\varepsilon^{(4)}}{i} > 0$.

The parameter of the transformation is u .

2. – Equations (1) obviously associate each point with a direction of advance, along which it reaches its consecutive positions along the trajectory, which are associated uniquely by means of the transformation. One then obtains the trajectory by integrating (1) in the form:

$$
(2)\qquad x = x\,(a, b, c, u)\,.
$$

The a, b, c in this are certain integration constants that might perhaps characterize the initial position for $u = 0$.

Those initial positions fill up a triply-extended manifold that successively goes over to ∞^1 other manifolds u = const. under the transformation (1) *that are all congruent to each other.*

3. – Each transformation (1) possesses this distinguished property in general, and only one of them, so (2) just represents an "intrinsic displacement" (*Insichverschiebung*) or "motion" of S_4 . The family of ∞^3 trajectories that belongs to (1) can be displaced into itself. As one would derive directly from the natural equations of differential geometry, each curve of the family will then possess the property:

The three radii of curvature ([1]) *along a curve of the family are constant along a curve of the family* (but not, say, from one curve to another).

That is, the *intrinsic form* of the curve is the same everywhere.

4. – The transformation (1) then produces an entire family of ∞^3 such curves. Conversely, in order for a given family of ∞^3 curves of constant curvature to belong to a transformation (1), it is necessary that certain conditions are fulfilled. If one takes the example of the plane then the transformation that is analogous to (1) will have ∞^1 for its trajectories. The condition for a family of ∞^1 circles to be trajectories of a one-parameter group of orthogonal transformations is known to be that the circles must be concentric. Generally speaking, one has *equidistance* ([2]) as the characteristic property for a family of ∞^3 curves of constant curvature in S_4 to belong to a transformation (1). Namely, if $x_0 = x(a, b, c, u_0)$ and $X_0 = X(A, B, C, U_0)$ are two arbitrary points of two arbitrary curves of the family then if x_0 runs through then curve $x(a, b, c, u)$ and X_0 runs through the curve $X(A, B, C, u)$ then any two associated points will always yield the same separation distance. The fact that two points are associated will then be given by the fact that u_0 and U_0 both experience the same increment Δu. Namely, since we start from the initial positions x_0 (X_0, resp.), which belong to the different parameter values u_0 (U_0, resp.), once we have attached a finite transformation to them, we cannot regard points with the same parameter values as being associated, but points with parameter values u (U, resp.), where:

$$u = u_0 + \Delta u \qquad (U = U_0 + \Delta u, \text{ resp.}),$$

such that the difference:

$$u - u_0 = U - U_0$$

can be treated as constant. If one imagines that the curve $X(A, B, C, u)$ is displaced into itself (whereby, from paragraph **3**, nothing will change in the external appearance of the family of curves) then one can obviously arrive at $U_0 = u_0$, and the statement above will find its justification in that way. The condition of equidistance, and therefore the fact that a transformation (1) can be associated, will then read:

$$\sum_{h=1}^{4}\{X^{(h)}(A,B,C,U_0+\Delta u)-x^{(h)}(a,b,c,u_0+\Delta u)\}^2=\sum_{h=1}^{4}\{X^{(h)}(A,B,C,U_0)-x^{(h)}(a,b,c,u_0)\}^2$$

for the given family of curves of constant curvature, which might also be X_0 and x_0 and Δu.

5. – In regard to the integration of the differential equations (1) and the representation of finite equations for the trajectories, one must obviously appeal to the simplest, but typical, cases. In fact, one can always reduce the matrix $\varepsilon_k^{(h)}$ to certain *canonical forms* by an orthogonal transformation

([1]) Appendix I.

([2]) In the broader sense; hence, it is not merely normal equidistance, as in **Herglotz**, *loc. cit.*

of S_4 ; i.e., one for which as many $\varepsilon_k^{(h)}$ as possible vanish ([1]). In what follows, the integration will only be performed in that canonical form, and the investigation will also be further linked with it. Naturally, one is still free to adapt the results that are obtained in that way to an arbitrary form for $\varepsilon_k^{(h)}$ by means of the inverse orthogonal transformation. One ascertains that canonical form most concisely with the help of the theory of *elementary divisors* ([2]). In that way, one considers the characteristic determinant:

$$D(\rho)=\begin{vmatrix} \rho & \varepsilon_2^{(1)} & \varepsilon_3^{(1)} & \varepsilon_4^{(1)} \\ \varepsilon_1^{(2)} & \rho & \varepsilon_3^{(2)} & \varepsilon_4^{(2)} \\ \varepsilon_2^{(3)} & \varepsilon_2^{(3)} & \rho & \varepsilon_4^{(3)} \\ \varepsilon_2^{(4)} & \varepsilon_2^{(4)} & \varepsilon_3^{(4)} & \rho \end{vmatrix} = \rho^4 + \mathsf{E}\rho^2 + E^2,$$

in which:

$$\mathsf{E} = (\varepsilon_2^{(1)})^2 + (\varepsilon_3^{(1)})^2 + (\varepsilon_3^{(1)})^2 + (\varepsilon_3^{(2)})^2 + (\varepsilon_4^{(2)})^2 + (\varepsilon_4^{(3)})^2$$

and

$$E = \varepsilon_2^{(1)}\varepsilon_4^{(3)} + \varepsilon_3^{(1)}\varepsilon_2^{(4)} + \varepsilon_4^{(1)}\varepsilon_3^{(2)}$$

are the two invariants under orthogonal transformation of the determinant:

$$\left|\varepsilon_k^{(h)}\right|, \quad h, k = 1, 2, 3, 4$$

and determines the roots of:

$$D(\rho) \equiv \rho^4 + \mathsf{E}\,\rho^2 + E^2 = 0 .$$

Let those roots be:

$$-\rho_1 = \rho_2 \qquad (-\rho_3 = \rho_4\text{, resp.}),$$

so their uniqueness or multiplicity, resp. (and in the latter case, their multiplicities as roots of all the subdeterminants of varying degrees) will be a necessity for a classification. When one recalls conditions (a), (b), (c) of paragraph **1** and omits everything that is inessential, one will then find the canonical forms:

(I) All roots are simple. One has nothing but elementary divisors. Notation [1111] :

$$-\rho_1 = +\rho_2 = i\lambda, \qquad -\rho_3 = +\rho_4 = 1,$$

$$dx^{(1)} = -\lambda\, x^{(2)}\, du, \qquad dx^{(2)} = \lambda\, x^{(1)}\, du, \qquad dx^{(3)} = i\lambda\, x^{(4)}\, du, \qquad dx^{(4)} = i\, x^{(3)}\, du .$$

(II) $\rho_3 = \rho_4 = 0$ (double root). All subdeterminants of degree three must then [as a result of condition (a)] vanish at least simply. The double root $\rho = 0$ then possesses two simple elementary divisors: $-\rho_1 = +\rho_2 = i\lambda$, [11 (11)] :

([1]) An example of a reduction to canonical form: the transformation of a second-degree surface to its principal axes.

([2]) **P. Muth**, *Theorie und Anwendung der Elementarteiler*, Leipzig, Teubner, 1899.

$$dx^{(1)} = -\lambda\, x^{(2)}\, du\,, \qquad dx^{(2)} = \lambda\, x^{(1)}\, du\,, \qquad dx^{(3)} = \beta\, du\,, \quad dx^{(4)} = i\, du\,.$$

(III) $\rho_1 = \rho_2 = 0$, $-\rho_3 = \rho_4 = 1$, as before, it differs only by the various reality properties of $x^{(1)}$, $x^{(2)}$ [$x^{(3)}$, $x^{(4)}$, resp.]. [(11) 11]:

$$dx^{(1)} = \alpha\, du\,, \quad dx^{(2)} = 0\,, \qquad dx^{(3)} = -i\, x^{(4)}\, du\,, \qquad dx^{(4)} = i\, x^{(3)}\, du\,.$$

(IV) $\rho_1 = \rho_2 = \rho_3 = \rho_4 = 0$ (quadruple root). The subdeterminants of degree three must vanish at least simply. The root $\rho = 0$ will then have a triple and a simple elementary divisor [(31)]:

$$dx^{(1)} = (-x^{(3)} - i\, x^{(3)})\, du\,, \qquad dx^{(2)} = 0\,, \qquad dx^{(3)} = x^{(1)}\, du\,, \qquad dx^{(4)} = i\,(x^{(1)} + \alpha)\, du\,.$$

(V) $\rho_1 = \rho_2 = \rho_3 = \rho_4 = 0$ (quadruple root). The subdeterminants of degree three must vanish at least triply, the ones of degree two, at least doubly, and the ones of degree one, simply. One has four elementary divisors [(1111)]:

$$dx^{(1)} = 0, \qquad dx^{(2)} = 0\,, \qquad dx^{(3)} = \beta\, du\,, \quad dx^{(4)} = i\, du\,.$$

The specializations of the "displacement" $\varepsilon^{(1)}$, $\varepsilon^{(2)}$, $\varepsilon^{(3)}$, $\varepsilon^{(4)}$ that appear in these cases were made with hindsight of condition (c) and the essential nature of its preservation.

The integration of (1) will now be easy. It will imply a parametric representation of the type (2) when certain integration constants are chosen in a suitable way. For the results, cf., Tab. 1, col. 2.

§ 2. – Kinematics of worldlines of constant curvatures.

1. – The curves that were found will now yield the desired accelerated motions directly when we interpret them as worldlines in the **Minkowski** sense. For that, it is only necessary that the tangent to such a curve should always have a timelike direction, which is why the integration constants are subject to relevant conditions (Tab. 1, col. 7).

2. – **Minkowski** kinematics is now a pure *differential geometry* of worldlines. In fact, the abandonment of the distinguished role that time plays in ordinary kinematics is necessary if one is to have every right to call kinematics the "geometry of motion." Now, since that is the case for **Minkowski**, one will attempt to call upon the formulas of differential geometry for S_4 for the sake of physical understanding.

3. – **Minkowski**'s equations of motion for a mass-point read:

$$m_0 \frac{d^2 x^{(h)}}{d\tau^2} = K^{(h)}, \quad h = 1, 2, 3, 4.$$

In them, m_0 is the rest mass, $d\tau = dt\sqrt{1-\mathfrak{v}^2/c^2}$ is the element of proper time, and $\mathfrak{v}$ is the ordinary **Newton**ian velocity, i.e.:

$$\mathfrak{v}_x = \frac{dx}{dt}, \quad \mathfrak{v}_y = \frac{dy}{dt}, \quad \mathfrak{v}_z = \frac{dz}{dt}.$$

Finally, K is the **Minkowski** four-force, which is referred to the element of *rest volume*. If $\mathfrak{K}$ is the ordinary **Newton**ian force then, as is known, one has:

$$K^{(1)} = \frac{\mathfrak{K}_x}{\sqrt{1-v^2/c^1}}, \quad \text{etc.,}$$

$$K^{(4)} = \frac{i}{c}\frac{(\mathfrak{K}\,\mathfrak{v})}{\sqrt{1-v^2/c^1}}.$$

Now, it is known that these equations of motion admit the following interpretation:

4. – In S_4 , with the metric:

$$\begin{aligned} d\sigma^2 &= -(dx^{(1)})^2 - (dx^{(2)})^2 - (dx^{(3)})^2 - (dx^{(4)})^2 \\ &= -dx^2 - dy^2 - dz^2 + c^2\,dt^2, \end{aligned}$$

the arc-length along the worldline is:

$$d\sigma = c\,d\tau.$$

The **Minkowski** equations of motion then read:

$$\frac{d^2x}{d\sigma^2} = \frac{K}{m_0\,c^2}.$$

However, the left-hand side of this is known to be (up to sign) nothing but the unit vector of the *principal normal*, divided by the radius of the first curvature ([1]). It then proves to be similar to the three-dimensional centripetal force about the center of curvature of a **Newton**ian trajectory, which makes its magnitude inversely proportional to the radius of curvature.

5. – From § **1**, we know that the first curvature of the worldline that we consider is constant. It then follows from this that:

The magnitude of the Minkowski four-force is constant under a motion of this type.

That magnitude, when divided by m_0 , so:

([1]) Appendix. However, the law of arc-length employs $ds^2 = dx^2 + dy^2 + dz^2 - c^2\,dt^2$, there so $i\,c\,d\tau$, as the arc-length of the worldline.

$$\frac{|K|}{m_0},$$

is called the *proper acceleration* of the mass-point, because when the point is transformed to rest, its acceleration in the new system will represent the magnitude and direction of precisely the **Minkowski** four-force, divided by m_0, whose fourth component will obviously vanish in it.

6. – The interpretation of the second and third curvatures is not as simple as that of the first curvature, which represents the previous $1 / c^2$ times the proper acceleration. Since they vanish for a planar (rectilinear, resp.) motion for the worldlines in S_4 that correspond to the Newtonian S_3, one would suspect that one is dealing with the consecutive position that the accelerations assume. Classical mechanics poses considerations of that sort when it constructs the *Hamiltonian hodograph of the velocity:*

$$\mathfrak{v}_x = \mathfrak{v}_x(t), \qquad \mathfrak{v}_y = \mathfrak{v}_y(t), \qquad \mathfrak{v}_z = \mathfrak{v}_z(t)$$

as a curve. The acceleration then points parallel to the tangent, so the alternating positions of the acceleration will obviously be given by the ordinary three-dimensional curvature (torsion, resp.) of the **Hamilton**ian hodograph.

Now, one easily finds from the formulas of four-dimensional differential geometry ([1]) that:

*The second (third, resp.) curvature at a point of the worldline is equal to the curvature (torsion, resp.) of the associated **Hamilton**ian hodograph times* $1/c$ *of the proper acceleration when that hodograph is considered in a system in which the point is momentarily at rest.*

7. – In order to arrive at an intuitive picture of the nature of the motions considered, we would like to assume, for the moment, that the velocity $\mathfrak{v}$ is small enough that we can set the speed of light $c = \infty$ in comparison to it. **Newton**ian kinematics is obviously true then. The system in which the point is at rest is the ordinary **Newton**ian comoving system where the proper acceleration coincides with the ordinary acceleration, so the hodograph, as viewed from the comoving system also coincides with the ordinary one. One obtains: The motions are performed with constant acceleration, and the **Hamilton**ian hodograph is either a common helix or a circle or a line or a point. That is, however, nothing but *constant acceleration as in **Galilean** free fall or uniform rotation about an axis.* The worldlines of constant curvatures are nothing but the relativistic generalization of *free fall* (possibly a throw) or *uniform rotation* (possibly a screw) or both of them together.

8. – Pursuing that generalization in more detail obviously has great significance for the study of accelerated motion from the standpoint of the principle of relativity. In order to do that, we imagine that the worldlines are given in their simplest form, as in Tab. 1. Let superfluous translations be omitted from the outset by a Lorentz transformation. We shall now interpret them in space:

([1]) Appendix 2, in which the imaginary arc-length is employed, though.

$$x^{(1)} = x, \qquad x^{(2)} = y, \qquad x^{(3)} = z,$$

and let $(1 / i\, c)\, x^{(4)}$ be time t . Result: Tab. 2, col. 1. We now apply the concepts of ordinary **Newton**ian mechanics and determine the **Newton**ian trajectory, the velocity $\mathfrak{v}$, and the acceleration $\dot{\mathfrak{v}}$. Result: Tab. 2, col. 2, 3, 7. We further consider the **Minkowski** equation of motion in that S_3 . We have:

$$\frac{dx^{(1)}}{d\tau} = \frac{\mathfrak{v}_x}{\sqrt{1-\mathfrak{v}^2/c^2}}, \quad \frac{dx^{(2)}}{d\tau} = \frac{\mathfrak{v}_y}{\sqrt{1-\mathfrak{v}^2/c^2}}, \quad \frac{dx^{(3)}}{d\tau} = \frac{\mathfrak{v}_z}{\sqrt{1-\mathfrak{v}^2/c^2}},$$

and obviously:

$$m_0 \frac{d}{dt} \frac{\mathfrak{v}}{\sqrt{1-\mathfrak{v}^2/c^2}} = \mathfrak{K},$$

moreover.

We calculate the differential quotient in the left-hand side:

$$\frac{d}{dt}\frac{\mathfrak{v}}{\sqrt{1-\mathfrak{v}^2/c^2}} = \frac{\dot{\mathfrak{v}}}{\sqrt{1-\mathfrak{v}^2/c^2}} + \frac{\mathfrak{v}\dfrac{(\mathfrak{v}\,\dot{v})}{c^2}}{\left(\sqrt{1-\mathfrak{v}^2/c^2}\right)^3} = \frac{\dot{\mathfrak{v}}(1-\mathfrak{v}^2/c^2) + \mathfrak{v}\,\dfrac{(\mathfrak{v}\,\dot{v})}{c^2}}{\left(\sqrt{1-\mathfrak{v}^2/c^2}\right)^3}.$$

We split $\dot{\mathfrak{v}}$ into two *vectors* $\dot{\mathfrak{v}}_{||}$ ($\dot{\mathfrak{v}}_{\pm}$, resp.) that are parallel (normal, resp.) to $\mathfrak{v}$:

$$\dot{\mathfrak{v}} = \dot{\mathfrak{v}}_{\|} + \dot{\mathfrak{v}}_{\pm}.$$

We obviously find that:

$$\frac{d}{dt}\frac{\mathfrak{v}}{\sqrt{1-\mathfrak{v}^2/c^2}} = \frac{\dot{\mathfrak{v}}_{\pm}}{\sqrt{1-\mathfrak{v}^2/c^2}} + \frac{\dot{\mathfrak{v}}_{\|}(1-\mathfrak{v}^2/c^2) + \dot{\mathfrak{v}}_{\|}(\mathfrak{v}^2/c^2)}{\left(\sqrt{1-\mathfrak{v}^2/c^2}\right)^3}$$

$$= \frac{\dot{\mathfrak{v}}_{\pm}}{\sqrt{1-\mathfrak{v}^2/c^2}} + \frac{\dot{\mathfrak{v}}_{\|}}{\left(\sqrt{1-\mathfrak{v}^2/c^2}\right)^3}.$$

That is nothing but the well-known fact that longitudinal and transverse masses are different. We further split $\mathfrak{K}$ as we did with $\dot{\mathfrak{v}}$:

$$\mathfrak{K} = \mathfrak{K}_{\pm} + \mathfrak{K}_{\|}.$$

It then follows that:

$$\mathfrak{K}_{\perp}\sqrt{1-\mathfrak{v}^2/c^2} = \dot{\mathfrak{v}}_{\perp}\, m_0,$$

$$\mathfrak{K}_{\|}\left(\sqrt{1-\mathfrak{v}^2/c^2}\right)^3 = \dot{\mathfrak{v}}_{\|}\, m_0.$$

If one now forms the resultant $\dot{\mathfrak{v}}$ then one will get:

$$\frac{m_0\,\dot{\mathfrak{v}}}{\sqrt{1-\mathfrak{v}^2/c^2}} = \mathfrak{K} - \frac{(\mathfrak{K}\,\mathfrak{v})\,\mathfrak{v}}{c^2}$$

or

$$m_\perp\,\dot{\mathfrak{v}} = \mathfrak{K} - \frac{(\mathfrak{K}\,\mathfrak{v})\,\mathfrak{v}}{c^2} \qquad \left(\text{where} \quad m_\perp = \frac{m_0}{\sqrt{1-\mathfrak{v}^2/c^2}}\right).$$

That equation shows that when a force $\mathfrak{K}$ acts upon a mass-point m_0 , the acceleration that it produces does not point parallel to the force ([1]). Rather, the mass-point experiences a type of *resistance in the direction of motion* that represents a second-order effect and is proportional to the force component in the direction of motion. It is only in the case where the force acts always parallel or always normal to the motion that the acceleration and force are parallel.

9. – Therefore, assume that we throw a body horizontally with a certain velocity in a gravitational field (Tab. 2, IIIa). Initially, the force is normal to the direction of motion, but that will no longer be the case when the body begins to describe the **Galilei**an parabola, where a sort of frictional resistance will act against it (col. 8). If we decompose it into a vertical (horizontal, resp.) component then the vertical component (whose magnitude naturally increases with $|\mathfrak{v}|$) will reduce the free-fall acceleration, while the horizontal one will *exhaust the velocity that it was thrown with*. If we then let the body fall for an infinitely-long time then it will have attained the free-fall velocity of c (there is certainly no higher one) at the conclusion, while the horizontal velocity would be 0. Thus, the horizontal velocity varies in relativistic mechanics in that way. The path is also *no longer a parabola* accordingly, but a *catenary* that is more strongly curve towards the vertical.

10. – If we think of a rotating mass-point that describes a circular path then the centripetal force will continually be perpendicular to the motion, the resistance in the direction of motion will vanish here, the velocity will remain uniform and constant, and the circular motion will therefore be carried over *to relativistic mechanics unchanged.* (Tab. 2, IIa and b)

11. – Now, let a constant gravitational field act suddenly in the direction of the rotational axis at, say, time $t = 0$. The rotating system will begin to fall. In the **Newton**ian sense, it would describe a ballistic parabola on the circular cylinder that is constructed about the rotational axis, as long as one constructs perpendicular to the plane. That is because, in fact, the angular velocity, multiplied by the radius, would represent the horizontal ballistic velocity for the cylinder that was constructed ([2]). From what was said before, we now know that the ballistic velocity will be gradually reduced as a result of the resistance in the direction of motion that now enters in, and a catenary will enter in place of the parabola. If we once more bend the plane into a cylinder then we will next see that

([1]) However, that is the case for the aforementioned case of proper acceleration.

([2]) A representation by winding it up on the plane! Of course, the cylinder is thought of as being composed of infinitely many layers.

in order to preserve the central motion on the previous circles, a continually-decreasing centripetal force is now required, since the rotational velocity will decrease as a result of falling. Furthermore:

If a uniformly-rotating system is suddenly brought into a constant gravitational field then the rotation will slow down while it falls until ultimately it will come to a state of complete rest when the free-fall velocity has attained the speed of light after an infinitely-long time. (Tab. 2, I).

§ 3. – Dynamics of worldlines of constant curvatures.

The foregoing arouses the desire to investigate the creation of those motions for a mass-point and a body, as much as is possible. One will succeed in doing that with the assistance of a hypothesis in the following electromagnetic model of matter:

1. – We imagine an electric charge that is distributed in space with constant volume density ρ, which corresponds to **Thomson**'s picture of the positive atomic nucleus. Let ν be the mass density of the charge, which is likewise assumed to be constant. Let ρ_0, ν_0 be the corresponding rest values. Assume that the matter, or the electric charge that takes its place here, is at rest. If we apply (and we must indeed do this for the time being) **Maxwell**'s equations to the interior of the matter then we will encounter the well-known difficulty that any theory of electromagnetic mechanics must face when one does not postulate, as **Abraham** did, the "rigidity" of electricity from the outset. Namely, one asks the question: "What are the forces that hold our charge together?"

2. – With going into the nature of those forces, we make the assumption about the four-dimensional stress tensor $T^{(hk)}$ of matter that:

$$T^{(hk)} = \nu_0 \frac{dx^{(h)}}{d\tau} \frac{dx^{(k)}}{d\tau} = \nu \frac{dx^{(h)}}{d\tau} \frac{dx^{(k)}}{d\tau} .$$

Nordström ([1]) made that assumption as being the simplest Ansatz for the matter tensor in the case of an incoherent mass-current. The following properties characterize the **Nordström** tensor:

The *relative stresses*:

$$T^{(11)} - T^{(14)} \frac{dx^{(1)}}{dx^{(4)}} = p_{xx} - \mathfrak{g}_x \mathfrak{v}_x ,$$

$$T^{(12)} - T^{(14)} \frac{dx^{(2)}}{dx^{(4)}} = p_{xy} - \mathfrak{g}_x \mathfrak{v}_y ,$$

$$\dots$$

$$T^{(33)} - T^{(34)} \frac{dx^{(3)}}{dx^{(4)}} = p_{zz} - \mathfrak{g}_z \mathfrak{v}_z ,$$

([1]) **G. Nordström**, Phys. Zeit. **11** (1910), pp. 441.

and likewise the *relative energy current:*

$$\frac{c}{i}\left(T^{(41)} - T^{(44)}\frac{dx^{(1)}}{dx^{(4)}}\right) = \mathfrak{S}_s - \mathfrak{v}_z W, \quad \text{etc.}$$

vanish for it; i.e., however, since the relative stresses represent homogeneous linear functions of the elastic rest stresses ([1]): *The rest stresses vanish*.

If one assumes that the ordinary theory of elasticity is valid for the case of rest then that further says that: *The rest deformations vanish.*

The **Nordström** tensor then expresses the idea that nothing but "apparent" deformations of the type of Lorentz contractions and their corresponding stresses will appear in a moving body, which will vanish in a suitable reference system. Clearly, the energy current is therefore *convective*. If the rest stresses were non-zero then generally energy would flow into or out of a closed surface in a medium as a result of the fact that the resultant of the relative surface stresses are not normal to the velocity in general, so the work done by the relative stresses under notion would not vanish.

3. – Now, one can glimpse a formally relativistically-correct generalization of the ***Newton**ian rigid body* in the result that the rest deformations vanish, so the charge of the particle (as seen from a comoving system) is invariably linked with its normal position inside of the body. Just as **Newton**ian mechanics considers a rigid body to be a mass-point, so is the **Nordström** Ansatz an extension of the dynamics of mass-points to the body ([2]).

4. – Of course, what always happens (always under the tentative assumption that **Maxwell**'s equations are valid inside of matter) is that the Maxwell stresses are neutralized, such as by elastic stresses that take over, which is similar to what **Poincaré** did by way of his well-known hydrostatic cohesion pressure for the Lorentz electron with a surface charge and lately also **Nordström** ([3]) by localizing the cohesion pressure on the surface, although what *we* had to worry about ([4]) in order to get complete vanishing of the relative stresses in the interior will not be discussed here since were are merely treating a model for the motions that were considered here ([5]). Later on, we shall see that this model also has the distinguishing property that those motions can proceed *without radiation* (cf., § **5**, paragraph 6).

([1]) **M. Laue**, *Das Relativitätsprinzip*, 2nd ed., pp. 193.

([2]) The **Nordström** tensor is realized in the completely-static system of von **Laue** (*loc. cit.*, pp. 208, *et seq*.) as the mean value over the *total volume*. In our analysis, however, that tensor must obviously be valid for every *volume element*.

([3]) **G. Nordström**, Ann. Phys. (Leipzig) **42** (1913), pp. 540.

([4]) Naturally, that vanishing is true only for the stationary state that is subject to the effect of constant external fields (cf., para. 5). If that field should suddenly change then the stresses (and therefore the rest deformations) would have to appear again.

([5]) The effort expended in exhibiting such elastic stresses (completely ignoring the fact that the **Maxwell** stresses are known to be inconceivable as elastic stresses in an isotropic medium) would hardly be worth it. Obviously, a future alteration of **Maxwell**'s equations for the interior would first depend upon an explanation for the natural cohesion of electric charge.

5. – Assuming that, we would like to examine how the stated model behaves in a *constant external electromagnetic field*. Let $F^{(hk)}$ be the six-vector of the field. With **Lorentz**, we then write the ponderomotive force as:

$$K^{(h)} = \frac{\rho}{c}\sum_{k=1}^{4} F^{(hk)}\frac{dx^{(k)}}{dt} = \frac{\rho_0}{c}\sum_{k=1}^{4} F^{(hk)}\frac{dx^{(k)}}{d\tau}, \quad h = 1, 2, 3, 4,$$

and accordingly, since the divergence of the **Nordström** tensor can be written:

$$\sum_{k=1}^{4}\frac{\partial}{\partial x^{(k)}}\left(\nu\frac{dx^{(h)}}{dt}\frac{dx^{(k)}}{dt}\right) = \nu\frac{d^2x^{(h)}}{dt^2}, \quad h = 1, 2, 3, 4,$$

due to the continuity equation (viz., conservation of mass), it follows that the equations of motion for matter are:

$$\nu\frac{d^2x^{(h)}}{dt^2} = \frac{\rho}{c}\sum_{k=1}^{4} F^{(hk)}\frac{dx^{(k)}}{dt}, \quad h = 1, 2, 3, 4.$$

Integrating this once gives:

$$\nu\frac{dx^{(h)}}{dt} = q^{(h)} + \frac{\rho}{c}\sum_{k=1}^{4} F^{(hk)}x^{(k)}, \quad h = 1, 2, 3, 4,$$

in which the q are integration constants.

However, since $F^{(hk)} = - F^{(kh)}$, that is *precisely the form of the differential equations for an infinitesimal orthogonal transformation, which is what we started from* [§ **1**, equation (1)].

6. – Our model gives us, in fact, the dynamical realization of the motion along worldlines of constant curvatures. The results of integration in § **1** can be adapted immediately when we set:

$$\varepsilon^{(k)}du = \frac{1}{\nu}q^{(h)}, \quad \varepsilon_k^{(h)}du = \frac{\rho}{\nu\, c}F^{(hk)}dt.$$

Correspondingly, a classification by way of the elementary divisors of the simplest type gives the motions as being generated by the *simplest types* of *constant* electromagnetic fields. In particular, the cases in § **1** are:

I.	[1111]	$\lvert\mathfrak{H}\rvert = \mathfrak{H}_z$,	$\lvert\mathfrak{E}\rvert = \mathfrak{E}_z$,	electric and magnetic fields parallel,
II.	[11 (11)]	$\lvert\mathfrak{H}\rvert = \mathfrak{H}_z$,	$\lvert\mathfrak{E}\rvert = 0$,	only a magnetic field,
III.	[(11) 11]	$\lvert\mathfrak{H}\rvert = 0$,	$\lvert\mathfrak{E}\rvert = \mathfrak{E}_z$,	only an electric field,
IV.	[(31)]	$\lvert\mathfrak{H}\rvert = \mathfrak{H}_y$,	$\lvert\mathfrak{E}\rvert = \mathfrak{E}_x$,	electric and magnetic field perpendicular and of equal magnitude,

V. [(1111)] $|\mathfrak{H}| = |\mathfrak{E}| = 0$, no field .

The cases:

$$|\mathfrak{H}| = \mathfrak{H}_y > |\mathfrak{E}| = \mathfrak{E}_x \qquad (|\mathfrak{H}| = \mathfrak{H}_y > |\mathfrak{E}| = \mathfrak{E}_x \text{, resp.})$$

can obviously be obtained from case II (III, resp.) by a Lorentz transformation when one switches the *z*-direction with the *y*-direction (*x*-direction, resp.) in it beforehand. [Invariance of $\mathfrak{H}^2 - \mathfrak{E}^2$ ($\mathfrak{E}\mathfrak{H}$, resp.)]

The results are summarized in Tab. 2, col. 12.

7. – One might perhaps ask whether the results that were found could be applied to the deflection experiments with cathode rays, since the behavior of the external field in them would certainly suggest that. Before making that application, one must be warned that *here* the motion took place in the field for all eternity, while *there*, that is certainly the case only for a small piece of it. Qualitatively, one can generally say that the essence of such an application would be in the fact that the **Galilei** parabola would have to be replaced with the catenary, which, of course, does not even have to be noticeable in practice.

Schott ([1]) considered the motions of the Lorentz electron from another non-relativistic viewpoint as an electric mass-point in a constant electromagnetic field and found the same results.

§ 4. – The family of worldlines of constant curvatures in the representation by the moving 4-frame of a line.

If we take a second look at the foregoing then we will see that in § **1** the worldlines of constant curvatures were based upon the orthogonal transformation of S_4 as the associated family of trajectories, their kinematically-distinguished three and four-dimensional role was discussed in § **2**, and finally in § **3**, they were derived from a hypothetical electromagnetic model.

1. – We shall now move on to examine the physically-distinguished role of these accelerated motions in regard to the generalization of the Lorentz transformation. Since the transition from one point of a curve to the next happens by means of an orthogonal transformation under which the basic equations remain *covariant*, we suspect that there must be a distinguished "comoving" system in which they remain *invariant*. There is, in fact, such a system; it is the *moving 4-frame* of the worldline.

2. – In differential geometry, it is defined as follows:

One axis is the respective *tangent*. Let its direction cosines be:

([1]) **G. A. Schott**, *Electromagnetic radiation*, Cambridge, 1912, pp. 63, *et seq.* and Appendix G, pp. 295, *et seq.*

$$c_1^{(1)}, c_1^{(2)}, c_1^{(3)}, c_1^{(4)}$$

(which is written as the unit vector c_1).

The *principal normal* is perpendicular to the tangent in the *osculating plane*. Let its direction cosines be denoted by:

$$c_2^{(1)}, c_2^{(2)}, c_2^{(3)}, c_2^{(4)} \quad (\text{unit vector}: c_2).$$

The *binormal* is perpendicular to the osculating plane in the *osculating space*. Let its direction cosines be denoted by:

$$c_3^{(1)}, c_3^{(2)}, c_3^{(3)}, c_3^{(4)} \quad (\text{unit vector}: c_3).$$

Finally, the *trinormal* is perpendicular to the *osculating space:*

$$c_4^{(1)}, c_4^{(2)}, c_4^{(3)}, c_4^{(4)} \quad (\text{unit vector}: c_4).$$

The axis frame:

$$[c_1, c_2, c_3, c_4]$$

is called the *moving 4-frame*. Its position varies from place to place along the curve. When one compares the position of a 4-frame at a certain point on the curve with the one at the next point, one will find that the consecutive positions emerge from the foregoing ones by an *infinitesimal orthogonal transformation* (**Frenet** formulas). ([1])

3. – Now, what is the nature of that infinitesimal orthogonal transformation for our curves of *constant curvatures?* Answer: It is nothing but the *infinitesimal transformation itself* that has the curves for its trajectories.

In order to show that, we imagine any radius vector that remains *fixed in the moving 4-frame.* Let x be an arbitrary point of an arbitrary curve of the family, let c_1, c_2, c_3, c_4 be its 4-frame at the position that is characterized by giving the parameter u, so when we define:

(3) $$X^{(h)} = x^{(h)} + \Gamma^{(1)} c_1^{(h)} + \Gamma^{(2)} c_2^{(h)} + \Gamma^{(3)} c_3^{(h)} + \Gamma^{(4)} c_4^{(h)}, \quad h = 1, 2, 3, 4,$$

we will have that:

$$X - x$$

is such a radius vector, as long as the Γ are free of u, so they remain constant for the point X during all of its "intermediate motion." If we now let the point x range along its recorded trajectory then the 4-frame will "move with it" and also the radius vector that points from x to X and is fixed in it. X then describes a curve of constant curvatures ([2]), and it must obviously prove to be identical to a trajectory of the family if both transformations are to be identical. Indeed, there is only one curve of constant curvatures through the initial position of X that continually remains equidistant from

([1]) Appendix 1.

([2]) As one shows with the help of the **Frenet** formulas.

it, which is just the trajectory that belongs to the transformation that goes through that initial position of X. However, if one constructs:

$$(X^{(1)} - x^{(1)})^2 + (X^{(2)} - x^{(2)})^2 + (X^{(3)} - x^{(3)})^2 + (X^{(1)} - x^{(1)})^2 = (\Gamma^{(1)})^2 + (\Gamma^{(2)})^2 + (\Gamma^{(3)})^2 + (\Gamma^{(4)})^2$$

then one will, in fact, find equidistance.

4. – However, we have likewise found a distinguished representation for the total family that belongs to the transformation, as it is represented in the moving 4-frame of *one* curve of the family:

$$X(u) = x(u) + \Gamma^{(1)} c_1(u) + \Gamma^{(2)} c_2(u) + \Gamma^{(3)} c_3(u) + \Gamma^{(4)} c_4(u) .$$

With constant Γ, the total family of trajectories represents an orthogonal transformation that belongs to the curve $x(u)$.

5. – In that way, ∞^3 points of S_4 are linked to each other by the common parameter value u. That association (§ **1**), once it is chosen arbitrarily, will not be perturbed by the transformation. Associated points will each remain characterized by having equal u.

From the discussions in § **1**, however, a different arbitrary association is given. In particular, we emphasize the association:

$$X(u) = x(u) + \Delta^{(1)} c_1(u) + \Delta^{(2)} c_2(u) + \Delta^{(3)} c_3(u) + \Delta^{(4)} c_4(u) ,$$

with Δ that are clearly constant at any rate when all of the points that are associated with x lie in the normal space $[c_2 , c_3 , c_4]$. [That is nothing but **Born**'s rigid body of the second kind (§ **5**).] From the discussions in § **1**, that is *once more entirely the same family of curves as before*, since a worldline of constant curvatures certainly has the same "intrinsic form" everywhere.

6. – Conversely, might ask whether, in fact, all of the families of curves that were exhibited in § **1** admit that representation. As an example, we choose (IIb), Tab. 1:

$$x^{(1)} = a \cos \lambda (u - u_0), \qquad x^{(2)} = a \sin \lambda (u - u_0), \qquad x^{(3)} = x_0^{(3)}, \qquad x^{(4)} = i\,u .$$

One extracts the direction cosines of the 4-frame from cols. 13-16:

$$c_1^{(1)} = -\frac{a\lambda \sin \lambda(u-u_0)}{i\sqrt{1-a^2\lambda^2}}, \quad c_1^{(2)} = \frac{a\lambda \sin \lambda(u-u_0)}{i\sqrt{1-a^2\lambda^2}}, \quad c_1^{(3)} = 0, \quad c_1^{(4)} = \frac{1}{\sqrt{1-a^2\lambda^2}},$$

$$c_2^{(1)} = +\cos \lambda(u-u_0), \qquad c_2^{(2)} = +\sin \lambda(u-u_0), \qquad c_2^{(3)} = 0, \quad c_2^{(4)} = 0,$$

$$c_3^{(1)} = -\frac{\sin\lambda(u-u_0)}{\sqrt{1-a^2\lambda^2}}, \quad c_3^{(2)} = \frac{\cos\lambda(u-u_0)}{\sqrt{1-a^2\lambda^2}}, \qquad c_3^{(3)} = 0, \qquad c_3^{(4)} = \frac{i\,a\lambda}{\sqrt{1-a^2\lambda^2}},$$

$$c_4^{(1)} = 0, \qquad c_4^{(2)} = 0, \qquad c_4^{(3)} = 1, \qquad c_4^{(4)} = 0.$$

If $\Gamma^{(1)}$, $\Gamma^{(2)}$, $\Gamma^{(3)}$, $\Gamma^{(4)}$ mean four completely arbitrary constants ([1]) then one can determine the four constants A, $X_0^{(3)}$, U_0, $U - u$ from:

$$-\frac{a\lambda\,\Gamma^{(1)}}{\sqrt{1-a^2\lambda^2}} - \frac{\Gamma^{(3)}}{\sqrt{1-a^2\lambda^2}} = A\sin\lambda(-U+u+U_0-u_0),$$

$$a + \Gamma^{(2)} = A\cos\lambda(-U+u+U_0-u_0),$$

$$\frac{\Gamma^{(1)}}{\sqrt{1-a^2\lambda^2}} + \frac{i\,\Gamma^{(3)}a\lambda}{\sqrt{1-a^2\lambda^2}} = i\,(U-u),$$

$$x_0^{(3)} + \Gamma^{(4)} = X_0^{(3)}$$

and obtain:

$$X^{(1)} = A\cos\lambda\,(U-U_0), \qquad X^{(2)} = A\sin\lambda\,(U-U_0), \qquad X^{(3)} = X_0^{(3)}, \qquad X^{(4)} = i\,U,$$

which is a new point that obviously describes the trajectory:

$$X\,(A, X_0^{(3)}, U_0, U)$$

of the family (IIb) when u varies. The Γ will then remain constant, since $U - u$ (in the equations above) is certainly treated as a constant (§ **1**).

§ 5. – The moving 4-frame as a "comoving" system in the Lorentzian sense.

We now focus on a certain worldline $x\,(u)$ and let an observer move along it; e.g. (IIb) (Tab. 1): It might rotate with constant angular velocity around the z-axis. We associate it with each moving 4-frame at the instantaneous position $x\,(u)$ as a distinguished reference system.

1. – What does that mean physically? Clearly, the observer must initially be imagined to be at rest as long as it only observes itself. It is then a comoving system in which the proper time τ [or the imaginary arc-length $i\,c\,t$ of the world line $x\,(u)$] is definitive for the determination of time. Now, such an observer cannot sit at a material point, but rather, it must occupy an entire reference body. What body is that? It is a body that participates in the motion of the observer (in a certain sense), namely, a **Born** rigid body of the second kind. In fact, the reference body is given uniquely by an axis-frame that is fixed in it, and by our assumption, the axis-system of the observer was the

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([1]) Naturally, $\Gamma^{(1)}$ must be pure imaginary and $\Gamma^{(2)}$, $\Gamma^{(3)}$, $\Gamma^{(4)}$ pure real.

moving 4-frame, so its three spatial axes are c_2, c_3, c_4 . However, they are fixed directions in the **Born** rigid body of the second kind ([1]), since its worldlines (from **4**) are given by:

$$X = x + \Delta^{(2)} c_2 + \Delta^{(3)} c_3 + \Delta^{(4)} c_4 \qquad \text{(the } \Delta = \text{const.)}, \tag{4}$$

So in particular, $\Delta^{(2)} = 1$, $\Delta^{(3)} = \Delta^{(4)} = 0$ or $\Delta^{(3)} = 1$, $\Delta^{(2)} = \Delta^{(4)} = 0$ or $\Delta^{(4)} = 1$, $\Delta^{(2)} = \Delta^{(3)} = 0$ represent fixed directions in it.

2. – Except for x, the points:

$$X = x + \Delta^{(2)} c_2 + \Delta^{(3)} c_3 + \Delta^{(4)} c_4$$

are obviously "simultaneous" with the observer at x. He measures the spatial "coordinates" $\Delta^{(2)}$, $\Delta^{(3)}$, $\Delta^{(4)}$, where the index 4 does *not* characterize the temporal coordinate this time. Now, if the observer moves then the reference body will move with it, so $\Delta^{(2)}$, $\Delta^{(3)}$, $\Delta^{(4)}$ will remain constant, by definition: *The reference body is always at rest when seen in the proper system of the observer.*

3. – Now, what is going on with the "non-simultaneous" positions of the reference body? Assuming that the observer can perceive them (and he will, in fact, be able to make observations of only non-simultaneous processes with the help of light, strictly speaking, due to the time delay), if he is also to observe a state of rest, he must likewise demand the constancy of the proper coordinates. As the Ansatz (3) shows for non-simultaneous points of the reference body, namely:

$$X = x + \Gamma^{(1)} c_1 + \Gamma^{(2)} c_2 + \Gamma^{(3)} c_3 + \Gamma^{(4)} c_4 ,$$

that is, in fact, fulfilled. Naturally, in so doing, we must endow the observer with a time coordinate, which is obviously:

$$i\, c\, \tau ,$$

such that we have to set:

$$\Gamma^{(1)} = i\, c\, (T' - t) ,$$

where T' is the time of the observed point as evaluated by the proper time of the observer. Obviously one has:

$$T' = \tau$$

for "simultaneous" points, such that the Ansatz (4) is justified.

4. – The reference system that was introduced is then, in fact, a *comoving* one. Now, in what precisely are the distinguished properties of that system based for the worldlines of constant curvatures? Indeed, one can introduce the moving 4-frame as a proper system for an arbitrary motion $x(u)$ and define a reference body of the kind in (4). However, such a representation would

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([1]) **G. Herglotz**, Ann. Phys. (Leipzig) **31** (1910), in particular, pp. 402, *et seq.* **F. Kottler**, *loc. cit.*, § **8**.

not be *reciprocal* initially; i.e., if one were to consider things from X to x and use the moving 4-frame of X as a basis then that would make:

$$x(u) = X(u) + \delta^{(1)} C_1(u) + \delta^{(2)} C_2(u) + \delta^{(3)} C_3(u) + \delta^{(4)} C_4(u),$$

where the δ would no longer be constant, and the x would be distinguished for our reference body. *The δ are found to be constant only for worldlines of constant curvatures.* However, one would naturally have to demand that *equivalence of all points of the reference body* in order to extend the Lorentz transformation, since the observer does not really need to be linked with a definite point of the body ([1]).

5. – Secondly, however – and this is essential – the infinitesimal orthogonal transformation would no longer be constant from one place to another for the observer. As we know, that is only the case for worldlines of constant curvatures.

The physical significance of that situation for our worldlines is illuminated by the following: Assume that electromagnetic forces act between the points of our body. How do they vary during the motion? Answer: *Not at all!* That is because when one refers them to a moving 4-frame in the body, they are represented in terms of the proper coordinates Γ above for the positions of the points of the body. However, they remain constant as long as the points of the body participate in the motion unchanged. Therefore, the electromagnetic forces between the points also remain constant. For that to be true, it is generally necessary that these forces should be in equilibrium; otherwise, the motion would indeed be perturbed, so the Γ could no longer be constant. Once more, in order for that to be true, in general, forces of a different nature than purely electromagnetic ones would be required. However, as a result of the generally-valid picture, they must transform precisely like the electromagnetic one. We express what we have found as follows:

When any sort of force equilibrium prevails in our reference body, it will remain preserved during the entire duration of the motion.

6. – In particular, it follows from that for our hypothetical electromagnetic model (§ **3**) that the motion will proceed without radiation, since the reciprocal actions of the charged particles are zero *once*, and therefore *always*. As is known, an open problem of the theory of magnetization electrons is why no damping due to radiation has been noticed throughout the centuries. However, from the hypothesis above, the uniform rotation that is ascribed to them belongs to the distinguished radiation-free accelerated motions! ([2]).

7. – In conclusion, a thorough proof of the constancy of the electromagnetic field in the moving 4-frame shall still be presented, even though from what was said, it should be superfluous. We shall base it upon the fact that any field can be thought of as being decomposed into the elementary

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([1]) For proper systems in general, cf., Appendix 3.

([2]) On that, cf., the "stationary states" of the electrons of the atomic model of **N. Bohr**, Phil. Mag. **26** (1913), esp., pp. 4.

fields of point-like charges. However, we can calculate the elementary field – say, at X due to x – with the help of the **Schwarzschild** formulas. We then imagine that we are given:

$$X = x + \Gamma^{(1)} c_1 + \Gamma^{(2)} c_2 + \Gamma^{(3)} c_3 + \Gamma^{(4)} c_4 .$$

The points are obviously no longer simultaneous then, since they must act upon each other. Therefore, let:

$$(X^{(1)} - x^{(1)})^2 + (X^{(2)} - x^{(2)})^2 + (X^{(3)} - x^{(3)})^2 + (X^{(4)} - x^{(4)})^2 = (\Gamma^{(1)})^2 + (\Gamma^{(2)})^2 + (\Gamma^{(3)})^2 + (\Gamma^{(4)})^2 = 0 .$$

If one would like to look for the position $\mathfrak{X}$ that X has "simultaneously" with x then one would find that:

$$\mathfrak{X} = x + \Delta^{(2)} c_2 + \Delta^{(3)} c_3 + \Delta^{(4)} c_4 ,$$

in which the $\Delta^{(2)}$, $\Delta^{(3)}$, $\Delta^{(4)}$ are different from the $\Gamma^{(2)}$, $\Gamma^{(3)}$, $\Gamma^{(4)}$. We are not accustomed to such behavior for the usual **Lorentz**ian reference system, since for it the three spatial coordinates of a point at rest will remain the same in each of its positions (the simultaneous ones, as well as the effective ones), so only the time coordinate will vary. However, we cannot just expect a complete analogy with the Lorentz systems. Clearly, that will be hidden to the observer at x, since he certainly can see only the effective ([1]) position $\bar{X}$ in reality, and naturally that position will keep the same spatial coordinates $\bar{\Gamma}^{(2)}$, $\bar{\Gamma}^{(3)}$, $\bar{\Gamma}^{(4)}$ and the same time coordinate difference $\bar{\Gamma}^{(1)}$.

The **Schwarzschild** formulas read ([2]):

$$F^{(hk)} = de \frac{1}{(RV)^2} \left[R \frac{dV}{d\tau} \right]^{(hk)} + de \frac{c^2 - R \frac{dV}{d\tau}}{(RV)^3} [RV]^{(hk)},$$

in which:

de	is the charge at x,
$R = - X + x$	is the radius vector from the reference point to the light point,
$V = \frac{dx}{d\tau}$ and $\frac{dV}{d\tau} = \frac{d^2x}{d\tau^2}$,	
$(RV), \left(R \frac{dV}{d\tau} \right)$	are scalar products,

$$\left[R \frac{dV}{d\tau} \right]^{(hk)} = R^{(h)} \frac{dV^{(k)}}{d\tau} - R^{(k)} \frac{dV^{(h)}}{d\tau} .$$

Now, one has ([3]):

([1]) I.e., the position that transmits the field to him.

([2]) **F. Kottler**, *loc. cit.*, § **4**. However, $V = \frac{dx}{d\tau} \cdot \frac{1}{ic} = \frac{dx}{ds}$ there.

([3]) Appendix 1.

$$\frac{dx}{ds} = \frac{1}{ic}\frac{dx}{d\tau} = c_1, \quad \frac{d^2x}{ds^2} = -\frac{1}{c^2}\frac{d^2x}{d\tau^2} = \frac{c_2}{R_1}$$

for the arc-length $s = i\,c\,t$, where R_1 is the radius of first curvature (referred to the imaginary arc-length s). Therefore, since:

$$R = -\Gamma^{(1)} c_1 - \Gamma^{(2)} c_2 - \Gamma^{(3)} c_3 - \Gamma^{(4)} c_4 ,$$

the field will be:

$$F^{(hk)} = -de\frac{1}{(\Gamma^{(1)})^2}\left\{\frac{\Gamma^{(1)}}{R_1}[c_1\,c_2]^{(hk)} + \frac{\Gamma^{(3)}}{R_1}[c_1\,c_3]^{(hk)} + \frac{\Gamma^{(4)}}{R_1}[c_1\,c_4]^{(hk)}\right\}$$

$$-de\frac{1-\dfrac{\Gamma^{(2)}}{R_1}}{(\Gamma^{(1)})^2}\left\{\Gamma^{(2)}[c_2\,c_1]^{(hk)} + \Gamma^{(3)}[c_3\,c_1]^{(hk)} + \Gamma^{(4)}[c_1\,c_4]^{(hk)}\right\},$$

so:

$$c_1, c_2, c_3, c_4$$

are, in fact, constant in the reference system.

If one calculates the axes of the reference system at X :

$$C_1, C_2, C_3, C_4$$

with the help of the **Frenet** formulas then one will find that, as predicted, they are expressed linearly with constant coefficients in terms of the c_1, c_2, c_3, c_4 . Therefore, the field that originates at x, and likewise from each of the other points, is also constant in the proper system of X, with which, the proof is complete.

§ 6. – Motions relative to a reference body. Generalization of Einstein's law of addition of velocities.

1. – From the foregoing, the system:

$$c_1, c_2, c_3, c_4$$

then represents the generalization of the comoving **Lorentz**ian system to our accelerated motions. If we introduce into:

$$X = x + \Gamma^{(1)} c_1 + \Gamma^{(2)} c_2 + \Gamma^{(3)} c_3 + \Gamma^{(4)} c_4 ,$$

e.g. [1]:

$$\Gamma^{(3)} = X', \qquad \Gamma^{(4)} = Y', \qquad \Gamma^{(2)} = Z', \qquad \Gamma^{(3)} = i\,c\,(T' - \tau) ,$$

then the generalized Lorentz transformation will take the form:

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(1) In that way, either $\Gamma^{(1)} = 0$ ("simultaneous" position) or:

$$\Gamma^{(1)} = -i\sqrt{(\Gamma^{(1)})^2 + (\Gamma^{(2)})^2 + (\Gamma^{(3)})^2} \quad (\text{"effective" position}).$$

$$X - x = i\,c\,(T' - \tau)\,c_1 + Z'c_2 + X'c_3 + Y'c_4 .$$

Obviously, in the "coordinates" X', Y', Z', T', the equilibrium phenomena are represented in the moving reference body by constant quantities. To the comoving observer, as long as his immediate neighborhood participates in the motion, it will seem to be at rest in that coordinate system. Only to the extent that one adopts that viewpoint is there a *relativity* of acceleration. By contrast, for processes that are not "rigidly" coupled to the motion that is mapped out, there is just as little of a relativity of acceleration as there is in classical mechanics, where it is known that in such a case "fictitious" accelerations (**Coriolis**, **Foucault** pendulum, etc.) will appear.

In order to shed more light upon that fact, we would like to treat two of our five types of accelerated motions in more detail, namely, the limiting case of *uniform translation* (V) and the case of *uniform rotation* (II.b). In the former, we will see how the well-known notions are classified in the representation that is given here. In the latter, we will see how the generalized Lorentz transformation that was cited here coincides with closely-related representations and experiments (Sagnac effect).

2. – *Uniform rectilinear translation* (Tab. 1, V). The tangent continually coincides with the straight worldline. Naturally, the three normals are arbitrary, so we choose the "principal normal" (which is naturally an arbitrary terminology here) to be the normal to the plane of the tangent and time axis. (This has the consequence that the z-direction of translation will likewise be represented by the z'-axis in the primed system.) One gets, perhaps:

$$c_3^{(1)} = 1, \qquad c_3^{(2)} = 0, \qquad c_3^{(3)} = 0, \qquad c_3^{(4)} = 0,$$

$$c_4^{(1)} = 0, \qquad c_4^{(2)} = 1, \qquad c_4^{(3)} = 0, \qquad c_4^{(4)} = 0,$$

$$c_2^{(1)} = 0, \qquad c_2^{(2)} = 0, \qquad c_2^{(3)} = \frac{1}{\sqrt{1-\mathfrak{v}^2/c^2}}, \quad c_2^{(4)} = \frac{-\mathfrak{v}_z}{i\,c\sqrt{1-\mathfrak{v}^2/c^2}},$$

$$c_1^{(1)} = 0, \qquad c_1^{(2)} = 0, \qquad c_1^{(3)} = \frac{\mathfrak{v}_z}{i\,c\sqrt{1-\mathfrak{v}^2/c^2}}, \quad c_1^{(4)} = \frac{1}{\sqrt{1-\mathfrak{v}^2/c^2}}.$$

Therefore, when one takes:

$$\Gamma^{(3)} = X', \qquad \Gamma^{(4)} = Y', \qquad \Gamma^{(2)} = Z', \qquad \Gamma^{(1)} = i\,c\,(T' - \tau),$$

one will have:

$$
(5)\qquad\begin{cases}
X-x=\Gamma^{(1)}c_1^{(1)}+\Gamma^{(2)}c_2^{(1)}+\Gamma^{(3)}c_3^{(1)}+\Gamma^{(4)}c_4^{(1)}=\Gamma^{(3)}=X',\\
Y-y=\Gamma^{(1)}c_1^{(2)}+\Gamma^{(2)}c_2^{(2)}+\Gamma^{(3)}c_3^{(2)}+\Gamma^{(4)}c_4^{(2)}=\Gamma^{(4)}=Y',\\
Z-z=\Gamma^{(1)}c_1^{(3)}+\Gamma^{(2)}c_2^{(3)}+\Gamma^{(3)}c_3^{(3)}+\Gamma^{(4)}c_4^{(3)}=\Gamma^{(2)}c_2^{(3)}+\Gamma^{(1)}c_1^{(3)}\\
\qquad =Z'\cdot\dfrac{1}{\sqrt{1-\mathfrak{v}^2/c^2}}+(T'-\tau)\dfrac{\mathfrak{v}_z}{ic\sqrt{1-\mathfrak{v}^2/c^2}},\\
ic(T'-t)=\Gamma^{(1)}c_1^{(4)}+\Gamma^{(2)}c_2^{(4)}+\Gamma^{(3)}c_3^{(4)}+\Gamma^{(4)}c_4^{(4)}=\Gamma^{(2)}c_2^{(4)}+\Gamma^{(1)}c_1^{(4)}\\
\qquad =Z'\cdot\dfrac{-\mathfrak{v}_z}{ic\sqrt{1-\mathfrak{v}^2/c^2}}+ic(T'-\tau)\dfrac{1}{\sqrt{1-\mathfrak{v}^2/c^2}}.
\end{cases}
$$

If one considers (Tab. 2):

$$x=x_0\,,\qquad y=y_0\,,\qquad z=z_0+\mathfrak{v}_z\,t,\qquad t=\frac{\tau}{\sqrt{1-\mathfrak{v}^2/c^2}}$$

then when one sets $x_0 = y_0 = z_0 = 0$, as one is free to do, one will get the usual form of the **Lorentz** formulas:

$$X=X',\qquad Y=Y',\qquad Z=\frac{Z'+\mathfrak{v}_xT'}{\sqrt{1-\mathfrak{v}^2/c^2}},\quad T=\frac{T'+\mathfrak{v}_x/c^2Z'}{\sqrt{1-\mathfrak{v}^2/c^2}}.$$

The system:

$$X',\ Y',\ Z',\ T'$$

is then, in fact, identical to **Lorentz**'s comoving system in this case.

Upon differentiating (5), we will get **Einstein**'s law of the addition of velocities:

$$dX-dx=dX',$$

$$dY-dy=dY',$$

$$dZ-dz=\frac{dZ'+(dT'-d\tau)\mathfrak{v}_x}{\sqrt{1-\mathfrak{v}^2/c^2}},$$

$$dT-dt=\frac{+\mathfrak{v}_x/c^2\,dZ'+(dT'-d\tau)}{\sqrt{1-\mathfrak{v}^2/c^2}}.$$

Obviously, one has:

$$dT=dt$$

(but not also $dT' = d\tau'$, however!), since T and t are universal times. Thus, as before, we further have:

$$dX=dX',\qquad dY=dY',\qquad dZ=\frac{dZ'+\mathfrak{v}_z\,dT'}{\sqrt{1-\mathfrak{v}^2/c^2}},\qquad dt=dT=\frac{+\mathfrak{v}_z/c^2\,dZ'+dT'}{\sqrt{1-\mathfrak{v}^2/c^2}},$$

which implies the known formulas:

$$\frac{dX}{dT} = \frac{\frac{dX'}{dT'}\sqrt{1-\mathfrak{v}^2/c^2}}{1+\mathfrak{v}_z/c^2}, \quad \frac{dY}{dT} = \ldots, \quad \frac{dY}{dT} = \frac{\frac{dZ'}{dT'}+\mathfrak{v}_z}{1+\mathfrak{v}_z/c^2\,\frac{dZ'}{dT'}}.$$

3. – *Uniform rotation* (Tab. 1, II.b). Let the worldline of the observer be:

$$x^{(1)} = a\cos\omega t, \qquad x^{(2)} = a\sin\omega t, \qquad x^{(3)} = x_0^{(3)}, \qquad x^{(3)} = i\,c\,t\,.$$

Tab. 1, cols. 13-16 gives the generalized proper system as:

$$c_1^{(1)} = -\frac{a\,\omega\sin\omega t}{i\,c\sqrt{1-\frac{a^2\omega^2}{c^2}}}, \quad c_1^{(2)} = \frac{a\,\omega\cos\omega t}{i\,c\sqrt{1-\frac{a^2\omega^2}{c^2}}}, \quad c_1^{(3)} = 0, \quad c_1^{(4)} = \frac{1}{\sqrt{1-\frac{a^2\omega^2}{c^2}}},$$

$$c_2^{(1)} = +\cos\omega t, \qquad c_2^{(2)} = +\sin\omega t, \qquad c_2^{(3)} = 0, \qquad c_2^{(3)} = 0,$$

$$c_2^{(1)} = -\frac{\sin\omega t}{\sqrt{1-\frac{a^2\omega^2}{c^2}}}, \quad c_3^{(2)} = \frac{\cos\omega t}{\sqrt{1-\frac{a^2\omega^2}{c^2}}}, \quad c_3^{(3)} = 0, \quad c_4^{(4)} = \frac{i\,a\,\omega}{c\sqrt{1-\frac{a^2\omega^2}{c^2}}},$$

$$c_0^{(1)} = 0\,, \qquad c_4^{(2)} = 0, \qquad c_4^{(3)} = 1, \qquad c_4^{(3)} = 0\,.$$

For the sake of establishing the nature of this reference system ([1]), we introduce the following reference system at the position x (t) along his circular periphery: One axis points from away the center of the circular path (viz., the R-axis), a second one points along the tangent to the path (viz., the Θ-axis), the third is parallel to the axis of rotation (viz., the Z_1-axis), and we once more let the $X^{(4)} = i\,c\,t$-axis function as the time axis. Let the origin be the position of the observer himself (B, Fig. 1).

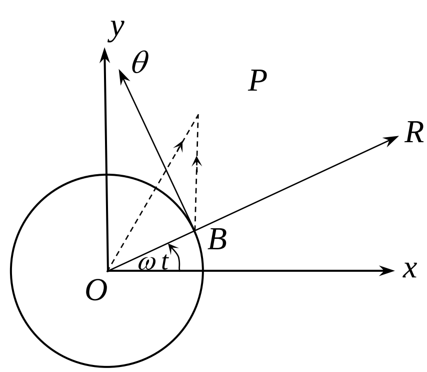

Figure 1.

([1]) That happens more concisely by drawing the 3-frame c_1, c_2, c_3 along the worldline (i.e., helix).

One infers from the figure that for an arbitrary point *P* with the coordinates *X*, *Y*, *Z*, *T* that:

$$R = + X \cos \omega t + Y \sin \omega t - a ,$$

$$\Theta = - X \sin \omega t + Y \cos \omega t ,$$

$$Z = z_0 + Z_1 ,$$

$$T = T .$$

The observer *B* has the instantaneous velocity:

$$a\,\omega$$

in the direction of the Θ-axis. That suggests that one can *make it vanish by a Lorentz transformation*, so instead of the system *R*, Θ, *Z*, *T*, one introduces the system *R′*, Θ′, *Z′*, *T′*:

$$(6)\quad \left\{\begin{aligned} R' &= R, \\ \Theta' &= \frac{\Theta - a\,\omega (T - t)}{\sqrt{1 - \frac{a^2\omega^2}{c^2}}}, \\ Z' &= Z_1, \\ T' - \tau &= \frac{-a\,\omega / c^2\,\Theta + (T - t)}{\sqrt{1 - \frac{a^2\omega^2}{c^2}}}. \end{aligned}\right.$$

That is, in fact, the same thing that the moving 4-frame yields, because if one can write $\Gamma^{(1)} = i\,c\,(T' - \tau)$, $\Gamma^{(2)} = R'$, $\Gamma^{(3)} = \Theta'$, $\Gamma^{(4)} = Z'$ then that will give:

$$(6.a)\quad \left\{\begin{aligned} X &= x + \Gamma^{(1)}c_1^{(1)} + \Gamma^{(2)}c_2^{(1)} + \Gamma^{(3)}c_3^{(1)} + \Gamma^{(4)}c_4^{(1)} \\ &= a \cos \omega t - \frac{a\,\omega + (T' - \tau) + \Theta'}{\sqrt{1 - \frac{a^2\omega^2}{c^2}}} \sin \omega t + R' \cos \omega t, \\ Y &= y + \Gamma^{(1)}c_1^{(2)} + \Gamma^{(2)}c_2^{(2)} + \Gamma^{(3)}c_3^{(2)} + \Gamma^{(4)}c_4^{(2)} \\ &= a \sin \omega t + \frac{a\,\omega + (T' - \tau) + \Theta'}{\sqrt{1 - \frac{a^2\omega^2}{c^2}}} \cos \omega t + R' \sin \omega t, \\ Z &= z + \Gamma^{(1)}c_1^{(3)} + \Gamma^{(2)}c_2^{(3)} + \Gamma^{(3)}c_3^{(3)} + \Gamma^{(4)}c_4^{(3)} \\ &= z_0 + Z', \\ i\,c\,T &= i\,c\,t + \Gamma^{(1)}c_1^{(4)} + \Gamma^{(2)}c_2^{(4)} + \Gamma^{(3)}c_3^{(4)} + \Gamma^{(4)}c_4^{(4)} \\ &= i\,c\,t + \frac{(T' - \tau) + a\,\omega / c^2\,\Theta'}{\sqrt{1 - \frac{a^2\omega^2}{c^2}}}. \end{aligned}\right.$$

If one forms R (Θ, resp.) from that as one did above then that will give:

$$R = R', \qquad \Theta = \frac{a\,\omega(T' - \tau) + \Theta'}{\sqrt{1 - \frac{a^2\omega^2}{c^2}}}, \quad Z - z_0 = Z_1 = Z', \qquad T - t = \frac{T' - \tau + a\,\omega / c^2\,\Theta'}{\sqrt{1 - \frac{a^2\omega^2}{c^2}}},$$

and those are, in fact, the inverse formulas for equations (6).

We further derive the addition law for velocities from (6.a):

$$dX = -\omega\,dt \cdot a \sin \omega t + \frac{a\,\omega\,d\tau}{\sqrt{1 - \frac{a^2\omega^2}{c^2}}} \sin \omega t - \frac{a\,\omega(T' - \tau) + \Theta'}{\sqrt{1 - \frac{a^2\omega^2}{c^2}}}\,\omega\,dt \cos \omega t$$

$$- R'\,\omega\,dt \cdot \sin \omega t - \frac{a\,\omega\,dT' + d\Theta'}{\sqrt{1 - \frac{a^2\omega^2}{c^2}}} \sin \omega t - dR' \cos \omega t,$$

$$dY = -\omega\,dt\,a \cos \omega t - \frac{a\,\omega\,d\tau}{\sqrt{1 - \frac{a^2\omega^2}{c^2}}} \cos \omega t - \frac{a\,\omega(T' - \tau) + \Theta'}{\sqrt{1 - \frac{a^2\omega^2}{c^2}}}\,\omega\,dt \cdot \sin \omega t$$

$$- R'\,\omega\,dt \cos \omega t + \frac{a\,\omega\,dT' + d\Theta'}{\sqrt{1 - \frac{a^2\omega^2}{c^2}}} \cos \omega t + dR' \cdot \sin \omega t,$$

$$dZ = dZ',$$

$$dT = dt - \frac{d\tau}{\sqrt{1 - \frac{a^2\omega^2}{c^2}}} + \frac{dT' + a\,\omega / c^2 d\Theta'}{\sqrt{1 - \frac{a^2\omega^2}{c^2}}},$$

from which, when one further considers (6.a) and the fact that:

$$d\tau = dt \sqrt{1 - \frac{a^2\omega^2}{c^2}},$$

one will get:

$$dX + \omega\,dt\,(Y - y) = -\frac{a\,\omega\,dT' + d\Theta'}{\sqrt{1 - \frac{a^2\omega^2}{c^2}}} \sin \omega t + dR' \cos \omega t,$$

$$dY - \omega\, dt\, (X - x) = \frac{a\,\omega\, dT' + d\Theta'}{\sqrt{1-\frac{a^2\omega^2}{c^2}}}\cos\omega t + dR'\sin\omega t,$$

$$dZ = dZ',$$

$$dT = \frac{dT' + a\,\omega / c^2\, d\Theta'}{\sqrt{1-\frac{a^2\omega^2}{c^2}}},$$

so

$$\frac{dX}{dT} + \omega(Y - y) = -\sin\omega t\left(\frac{\frac{d\Theta'}{dT'} + a\,\omega}{1 + \frac{a\,\omega}{c^2}\frac{d\Theta'}{dT'}}\right) + \cos\omega t\left(\frac{\frac{dR'}{dT'}\sqrt{1-\frac{a^2\omega^2}{c^2}}}{1 + \frac{a\,\omega}{c^2}\frac{d\Theta'}{dT'}}\right),$$

$$\frac{dY}{dT} - \omega(X - x) = +\cos\omega t\left(\frac{\frac{d\Theta'}{dT'} + a\,\omega}{1 + \frac{a\,\omega}{c^2}\frac{d\Theta'}{dT'}}\right) + \sin\omega t\left(\frac{\frac{dR'}{dT'}\sqrt{1-\frac{a^2\omega^2}{c^2}}}{1 + \frac{a\,\omega}{c^2}\frac{d\Theta'}{dT'}}\right),$$

$$\frac{dZ}{dT} = \frac{\frac{dZ'}{dT'}\sqrt{1-\frac{a^2\omega^2}{c^2}}}{1 + \frac{a\,\omega}{c^2}\frac{d\Theta'}{dT'}}.$$

If one introduces the vectors in *X*, *Y*, *Z*-space:

$$\mathfrak{R} = (X, Y, Z), \qquad \mathfrak{r} = (x, y, z),$$

$$\mathfrak{u} = (0, 0, \omega), \qquad \mathfrak{t}_1 = (-\sin\omega t, \cos\omega t, 0),$$

$$\mathfrak{n}_1 = (+\cos\omega t, +\sin\omega t, 0), \qquad \mathfrak{z}_1 = (0, 0, 1)$$

then when one considers the fact that $[\mathfrak{u}\,\mathfrak{r}] = a\,\omega\,\mathfrak{t}_1$, one can write, with the usual vector notation:

$$
(7)\quad \left\{
\begin{aligned}
\frac{d\mathfrak{R}}{dT} = [\mathfrak{u}\,\mathfrak{R}] + \mathfrak{t}_1 \cdot \frac{d\Theta'}{dT'} \frac{1 - \dfrac{a^2\omega^2}{c^2}}{}
\\
+ \mathfrak{n}_1 \cdot \frac{dR'}{dT'} \frac{\sqrt{1 - \dfrac{a^2\omega^2}{c^2}}}{1 + \dfrac{a^2\omega^2}{c^2}\dfrac{d\Theta'}{dT'}}
\\
+ \mathfrak{z}_1 \cdot \frac{dZ'}{dT'} \frac{\sqrt{1 - \dfrac{a^2\omega^2}{c^2}}}{1 + \dfrac{a^2\omega^2}{c^2}\dfrac{d\Theta'}{dT'}} .
\end{aligned}
\right.
$$

If one imagines that $c = \infty$, for the moment, then what will result is that:

$$\frac{d\mathfrak{R}}{dT} = [\mathfrak{u}\,\mathfrak{R}] + \frac{d\mathfrak{R}'}{dT'} ,$$

which is the addition law for velocity that is known from classical mechanics, and in which $d'\mathfrak{R}' / dT'$ means the differential quotient that one evaluates in the comoving system, so it is the relative velocity of the point X, Y, Z relative to the rotating system ([1]). (The displacement of the origin of the system from the center to the periphery obviously has no influence on the value of $d\,R' / dT'$.)

If one restricts oneself to first-order terms in a ω / c *then the result will again be the classical law of addition, as one would expect.*

We emphasize the further consequences:

When seen from the rotating system Θ, R', Z', a point at rest in the rest system X, Y, Z will describe a circular path in the opposite sense and with (absolute) angular velocity ω that is equal to that of the observer B, unlike in classical mechanics; rather, that would be the case only when one restricted oneself to first-order quantities,

4. – The invariance of the speed of light can be show for the point B of the rotating system. That no longer needs to be true for the other points of the rotating system when they are considered by B.

In fact, instead of (7), one can write:

([1]) **Abraham-Föppl**, *Theorie der Elektrizität*, Teubner, Leipzig, 1904, § **9**.

$$\frac{d\mathfrak{R}}{dT} - [\mathfrak{u}, \mathfrak{R} - \mathfrak{r}] = \mathfrak{t}_1 \frac{\dfrac{d\Theta'}{dT'} + a\omega}{1 + \dfrac{a\,\omega}{c^2}\dfrac{d\Theta'}{dT'}} + \mathfrak{n}_1 \frac{\dfrac{dR'}{dT'}\sqrt{1 - \dfrac{a\,\omega}{c^2}}}{1 + \dfrac{a\,\omega}{c^2}\dfrac{d\Theta'}{dT'}} + \mathfrak{n}_1 \frac{\dfrac{dZ'}{dT'}\sqrt{1 - \dfrac{a\,\omega}{c^2}}}{1 + \dfrac{a\,\omega}{c^2}\dfrac{d\Theta'}{dT'}}.$$

One recognizes the form of the **Einstein** addition law for a rectilinear translation in the direction $\mathfrak{t}_1$ with a magnitude of $a\ \omega$ on the right-hand side of that. However, the left-hand side is not, as in the **Einstein** law, the absolute velocity as seen from the rest system, but that absolute velocity, minus the velocity:

$$[\mathfrak{u}, \mathfrak{R} - \mathfrak{r}]$$

by which the motion of a point that is fixed in the moving system differs from rectilinear translation. One then distinguishes them by replacing the velocity:

$$\frac{d\mathfrak{R}}{dT}$$

that one sees in the rest system (which is composed of 1. a translation $a\ \omega$ in the direction $\mathfrak{t}_1$, 2. relative motion in the primed system, 3. the difference between the rotation and the translation: $[\mathfrak{u}, \mathfrak{R} - \mathfrak{r}]$) with:

$$\frac{d\mathfrak{R}}{dT} - [\mathfrak{u}, \mathfrak{R} - \mathfrak{r}]$$

then one can obviously adapt the formulas that are true for rectilinear translation to that case, and with that, one will obtain the formula above.

The invariance of the speed of light is true for translation. That is also confirmed in the formulas above (their inverses, resp.) Therefore, that invariance can no longer be true for rotations, except when it does not differ appreciably from translation; that is confirmed by the formulas above when one sets $\mathfrak{R} = \mathfrak{r}$.

The observer will also find the value c for the speed of light in the rotating system as long as he restricts his experiment to the immediate vicinity.

By contrast, for the processes at a greater distance, the speed of light will be influenced by the rotation of the reference body, except when the line of sight of the observer points parallel to the axis of rotation. The path of the light rays will be curved when seen from the rotating system.

The fact that the transformation formulas differ from those of the ordinary relativity principle of classical mechanics only by $a\ \omega / c$ directly implies that the effect that **Sagnac** ([1]) observed,

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([1]) **G. Sagnac**, C. R. Acad. Sci. Paris **27**/X (1913) and **22**/XII (1913); **H. Witte**, Verh. d. Deutsch. Phys. Ges. (1914), no. 3.

namely, that two light rays in a rotating cylinder will interfere at a point *B* when one of them has traversed a polygonal circuit that starts from *B* with the use of several reflections from the walls of the cylinder in one sense, while the other had traversed it in the opposite sense, will have first-order in $a\ \omega / c$, so it cannot be compensated for by the transformations, and it is thus independent of the distinction that exists between the old relativity principle and the new one.

It is obvious, moreover, that the observer on the accelerated reference body cannot conceal his accelerated motion as long as he experiments with processes that are moving relative to the reference body. It is probably pointless to mention that formulas that differ from the ones in the older mechanics by second-order terms cannot vary essentially from the ones that are true there as long as no second-order effects are found.

Appendix

1. – From the differential geometry of worldlines ([1]).

Line through two consecutive points of a curve:	tangent	(unit vector c_1)
Plane through three consecutive curve points:	osculating plane	
Space through four consecutive curve points:	osculating space	
Normal to the tangent in the osculating plane:	principal normal	(c_2)
Normal to the osculating plane in the osculating space:	binormal	(c_3)
Normal to the osculating space:	trinormal	(c_4)
Arc-length of the curve:	distance between two consecutive points	(ds)
Angle between two consecutive tangents:	first contingency angle	($d\omega_1$)
Angle between two consecutive osculating planes:	second contingency angle	($d\omega_2$)
Angle between two consecutive osculating spaces:	third contingency angle	($d\omega_3$)

First curvature:

$$\frac{1}{R_1} = \frac{d\omega_1}{ds}$$

([1]) Cf., **G. Brunel**, Math. Ann. **19** and **G. Landsberg**, Crelle's Journ. **114** and **Kottler**, *loc. cit.*, §§ **7-8**.

Second curvature: $$\frac{1}{R_2} = \frac{d\omega_2}{ds}$$

Third curvature: $$\frac{1}{R_3} = \frac{d\omega_3}{ds}.$$

[All three are continually taken to be positive. In three-dimensional differential geometry, one cares to take the first curvature (which is called *curvature* there, for short) to always be positive, but the second curvature (which is called *torsion* there) is taken to be positive or negative according to whether the screw sense of the curve is negative or positive, resp. With that, the moving 3-frame will always have the same screw sense as the coordinate system then, which is no longer the case for *our* moving 4-frame. Cf., **Landsberg**, *loc. cit.* ([1])]

Moving 4-frame: c_1, c_2, c_3, c_4.

Consecutive 4-frame: $c_1 + dc_1, c_2 + dc_2, c_3 + dc_3, c_4 + dc_4$.

The transition from the former to the latter is mediated by an *infinitesimal orthogonal transformation* (except for the translation of the origin by ds in the direction of the tangent) that is given by the *Frenet formulas:*

$$\left\{\begin{array}{lcccc} dc_1^{(h)} = & * & c_2^{(h)} d\omega_1 & * & * \\ dc_2^{(h)} = & -c_1^{(h)} d\omega_1 & * & +c_3^{(h)} d\omega_2 & * \\ dc_3^{(h)} = & * & -c_2^{(h)} d\omega_2 & * & +c_4^{(h)} d\omega_3 \\ dc_4^{(h)} = & * & * & -c_3^{(h)} d\omega_3 & * \end{array}\right. \qquad h = 1, 2, 3, 4.$$

In order to bring these into the usual form [(1), § **1**], one merely imagines that a radius vector has been decomposed (by scalar multiplication) into one part along the axes c_1, c_2, c_3, c_4 and one along the axes $c_1 + dc_1, c_2 + dc_2, c_3 + dc_3, c_4 + dc_4$, where both systems of axes are placed at the same origin.

Meaning of the ***Frenet*** *formulas:*

A rotation in the plane $[c_1\, c_2]$ from c_1 to c_2 through an angle of $d\omega_1 = \dfrac{ds}{R_1}$

" " $[c_2\, c_3]$ " c_2 " c_3 " " $d\omega_2 = \dfrac{ds}{R_2}$

([1]) Naturally, the principal normal of a *worldline* points to the *convex* side. (Curvature *hyperbola*, instead of curvature *circle* in the real representation)

″ ″ $[c_3\, c_4]$ ″ c_3 ″ c_4 ″ ″ $d\omega_3 = \dfrac{ds}{R_3}$

Invariants of the transformation {cf., Ε $(du)^2$ [$E\,(du)^2$, resp.] (§ **1**)}:

$$(d\omega_1)^2 + (d\omega_2)^2 + (d\omega_3)^2 = ds^2 \left[\left(\frac{1}{R_1}\right)^2 + \left(\frac{1}{R_2}\right)^2 + \left(\frac{1}{R_3}\right)^2 \right],$$

$$d\omega_1\, d\omega_3 = ds^2 \frac{1}{R_1\, R_3}.$$

Both of them must remain unchanged when one goes to other coordinate systems than c_1, c_2, c_3, c_4.

Worldlines of constant curvatures:

Due to the constancy of R_1, R_2, R_3 along the curve, the infinitesimal orthogonal transformation always the same when referred to the moving 4-frame.

Family of such worldlines that belongs to an orthogonal transformation: These are characterized by the identity of the two invariants:

$$(ds)^2 \left\{ \left(\frac{1}{R_1}\right)^2 + \left(\frac{1}{R_2}\right)^2 + \left(\frac{1}{R_3}\right)^2 \right\} \qquad \left[ds^2 \frac{1}{R_1\, R_3}, \text{resp.} \right]$$

from curve to curve. In regard to that, one notes that the parameter s is not universal, unlike the parameter u that was used in § **1**. However, if $x\,(s)$ [$X\,(S)$, resp.] are two worldlines, referred to their arc-lengths, and any association of s with S is chosen then it is easy to show that:

$$dS = A\, ds,$$

where A is constant along the entire curve $X\,(S)$. If s used as a universal parameter in that sense then it would mean the arc-length of either a single curve of the family or none of them!

Proper time: On formal grounds, the arc-length of the worldline will always be taken to be equal to the proper time τ times $i\,c$ here:

$$s = i\, c\, \tau.$$

(Otherwise, one often chooses $\sigma = c\, t$ to be the real arc-length of the worldline.)

Reality relationships in this representation:

As the cosines of a timelike direction:

$$c_1^{(1)}, c_1^{(2)}, c_1^{(3)} \text{ are pure imaginary}, \quad c_1^{(4)} \text{ is real.}$$

As the cosines of spatial directions:

$$\left.\begin{matrix} c_2^{(1)} & c_2^{(2)} & c_2^{(3)} \\ c_3^{(1)} & c_3^{(2)} & c_3^{(3)} \\ c_4^{(1)} & c_4^{(2)} & c_4^{(3)} \end{matrix}\right\} \text{ are real,} \qquad \left.\begin{matrix} c_2^{(4)} \\ c_3^{(4)} \\ c_4^{(4)} \end{matrix}\right\} \text{ are pure imaginary.}$$

Furthermore:

As a rotation in a plane, the timelike directions include:

$$d\omega_1 \text{ as pure imaginary} \quad \left(\frac{d\omega_1}{ds} = \frac{1}{R_1} \text{ is therefore real}\right)$$

As rotations in the ordinary sense in merely spacelike planes:

$$d\omega_2 \ (d\omega_3, \text{resp.}) \text{ are real} \quad \left(\frac{d\omega_2}{ds} = \frac{1}{R_2}, \frac{d\omega_3}{ds} = \frac{1}{R_3} \text{ is therefore imaginary}\right).$$

Calculating the curvatures and the moving 4-frame.

Let the parameter representation of the worldline be given by:

$$x^{(1)} = x^{(1)}(t), \quad x^{(2)} = x^{(2)}(t), \quad x^{(3)} = x^{(3)}(t), \quad x^{(4)} = x^{(4)}(t) .$$

One forms the first four differential quotients:

$$\frac{dx}{dt}, \quad \frac{d^2x}{dt^2}, \quad \frac{d^3x}{dt^3}, \quad \frac{d^4x}{dt^4},$$

and with them, the matrix:

$$D \equiv \left\| \begin{matrix} \sum \frac{dx}{dt}\frac{dx}{dt} & \sum \frac{dx}{dt}\frac{d^2x}{dt^2} & \sum \frac{dx}{dt}\frac{d^3x}{dt^3} & \sum \frac{dx}{dt}\frac{d^4x}{dt^4} \\ \sum \frac{d^2x}{dt^2}\frac{dx}{dt} & \sum \frac{d^2x}{dt^2}\frac{d^2x}{dt^2} & \sum \frac{d^2x}{dt^2}\frac{d^3x}{dt^3} & \sum \frac{d^2x}{dt^2}\frac{d^4x}{dt^4} \\ \sum \frac{d^3x}{dt^3}\frac{dx}{dt} & \sum \frac{d^3x}{dt^3}\frac{d^2x}{dt^2} & \sum \frac{d^3x}{dt^3}\frac{d^3x}{dt^3} & \sum \frac{d^3x}{dt^3}\frac{d^4x}{dt^4} \\ \sum \frac{d^4x}{dt^4}\frac{dx}{dt} & \sum \frac{d^4x}{dt^4}\frac{d^2x}{dt^2} & \sum \frac{d^4x}{dt^4}\frac{d^3x}{dt^3} & \sum \frac{d^4x}{dt^4}\frac{d^4x}{dt^4} \end{matrix} \right\|$$

Denote its successive principal subdeterminants as follows:

$$D^{\mathrm{I}} = \sum \frac{dx}{dt}\frac{dx}{dt}, \quad D^{\mathrm{II}} = \begin{vmatrix} \sum \frac{dx}{dt}\frac{dx}{dt} & \sum \frac{dx}{dt}\frac{d^2x}{dt^2} \\ \sum \frac{d^2x}{dt^2}\frac{dx}{dt} & \sum \frac{d^2x}{dt^2}\frac{d^2x}{dt^2} \end{vmatrix}, \quad D^{\mathrm{III}} = \ldots, \quad D^{\mathrm{IV}} = \ldots,$$

and finally, denote the subdeterminants of the last rows of each of these principal subdeterminants as follows:

$$D_{11}^{\mathrm{I}} = 1, \quad D_{21}^{\mathrm{II}} = -\sum \frac{dx}{dt}\frac{d^2x}{dt^2}, \quad D_{22}^{\mathrm{II}} = D^{\mathrm{I}} = -\sum \frac{dx}{dt}\frac{dx}{dt}, \quad D^{\mathrm{II}} = \begin{vmatrix} \sum \frac{dx}{dt}\frac{d^2x}{dt^2} & \sum \frac{dx}{dt}\frac{d^3x}{dt^3} \\ \sum \frac{d^2x}{dt^2}\frac{d^2x}{dt^2} & \sum \frac{d^2x}{dt^2}\frac{d^3x}{dt^3} \end{vmatrix},$$

$$D_{32}^{\mathrm{III}} = \ldots, \quad \text{etc.}, \quad D_{41}^{\mathrm{IV}} = \ldots, \quad \text{etc.}$$

One then has:

$$\left(\frac{1}{R_1}\right)^2 = \frac{D^{\mathrm{II}} \cdot D^0}{(D^{\mathrm{I}})^2} \cdot \frac{1}{D^{\mathrm{I}}}, \quad \text{where } D^0 = 1,$$

$$\left(\frac{1}{R_2}\right)^2 = \frac{D^{\mathrm{III}} \cdot D^{\mathrm{I}}}{(D^{\mathrm{II}})^2} \cdot \frac{1}{D^{\mathrm{I}}} = \frac{D^{\mathrm{III}}}{(D^{\mathrm{II}})^2},$$

$$\left(\frac{1}{R_3}\right)^2 = \frac{D^{\mathrm{IV}} \cdot D^{\mathrm{II}}}{(D^{\mathrm{III}})^2} \cdot \frac{1}{D^{\mathrm{I}}},$$

and

$$c_1 = \frac{D_{11}^{\mathrm{I}} \frac{dx}{dt}}{\sqrt{D^{\mathrm{I}} \cdot D^0}},$$

$$c_2 = \frac{D_{21}^{\mathrm{II}} \frac{dx}{dt} + D_{22}^{\mathrm{II}} \frac{d^2x}{dt^2}}{\sqrt{D^{\mathrm{II}} \cdot D^{\mathrm{I}}}},$$

$$c_3 = \frac{D_{31}^{\mathrm{III}}\dfrac{dx}{dt} + D_{32}^{\mathrm{III}}\dfrac{d^2x}{dt^2} + D_{33}^{\mathrm{III}}\dfrac{d^3x}{dt^3}}{\sqrt{D^{\mathrm{III}} \cdot D^{\mathrm{II}}}},$$

$$c_4 = \frac{D_{41}^{\mathrm{IV}}\dfrac{dx}{dt} + D_{42}^{\mathrm{IV}}\dfrac{d^2x}{dt^2} + D_{43}^{\mathrm{IV}}\dfrac{d^3x}{dt^3} + D_{44}^{\mathrm{IV}}\dfrac{d^4x}{dt^4}}{\sqrt{D^{\mathrm{IV}} \cdot D^{\mathrm{III}}}}.$$

2. – The connection between Minkowski's worldlines and Hamilton's velocity hodographs.

Let t be time, so $x(4) = i\,c\,t$, and the matrix will become:

$$D \equiv \left\| \begin{matrix} \mathfrak{v}^2 - c^2 & \mathfrak{v}\dot{\mathfrak{v}} & \mathfrak{v}\ddot{\mathfrak{v}} & \mathfrak{v}\dddot{\mathfrak{v}} \\ \dot{\mathfrak{v}}\mathfrak{v} & \dot{\mathfrak{v}}^2 & \dot{\mathfrak{v}}\ddot{\mathfrak{v}} & \dot{\mathfrak{v}}\dddot{\mathfrak{v}} \\ \ddot{\mathfrak{v}}\mathfrak{v} & \ddot{\mathfrak{v}}\dot{\mathfrak{v}} & \ddot{\mathfrak{v}}^2 & \ddot{\mathfrak{v}}\dddot{\mathfrak{v}} \\ \dddot{\mathfrak{v}}\mathfrak{v} & \dddot{\mathfrak{v}}\dot{\mathfrak{v}} & \dddot{\mathfrak{v}}\ddot{\mathfrak{v}} & \dddot{\mathfrak{v}}^2 \end{matrix} \right\|,$$

which will make:

$$\frac{1}{R_1^2} = \frac{\begin{vmatrix} \mathfrak{v}^2 - c^2 & \mathfrak{v}\dot{\mathfrak{v}} \\ \dot{\mathfrak{v}}\mathfrak{v} & \dot{\mathfrak{v}}^2 \end{vmatrix}}{\left(\mathfrak{v}^2 - c^2\right)^3},$$

$$\frac{1}{R_2^2} = \frac{\begin{vmatrix} \mathfrak{v}^2 - c^2 & \mathfrak{v}\dot{\mathfrak{v}} & \mathfrak{v}\ddot{\mathfrak{v}} \\ \dot{\mathfrak{v}}\mathfrak{v} & \dot{\mathfrak{v}}^2 & \dot{\mathfrak{v}}\ddot{\mathfrak{v}} \\ \ddot{\mathfrak{v}}\mathfrak{v} & \ddot{\mathfrak{v}}\dot{\mathfrak{v}} & \ddot{\mathfrak{v}}^2 \end{vmatrix}}{\left(\begin{vmatrix} \mathfrak{v}^2 - c^2 & \mathfrak{v}\dot{\mathfrak{v}} \\ \dot{\mathfrak{v}}\mathfrak{v} & \dot{\mathfrak{v}}^2 \end{vmatrix}\right)^2},$$

$$\frac{1}{R_3^2} = \frac{\begin{vmatrix} \mathfrak{v}^2 - c^2 & \mathfrak{v}\dot{\mathfrak{v}} & \mathfrak{v}\ddot{\mathfrak{v}} & \mathfrak{v}\dddot{\mathfrak{v}} \\ \dot{\mathfrak{v}}\mathfrak{v} & \dot{\mathfrak{v}}^2 & \dot{\mathfrak{v}}\ddot{\mathfrak{v}} & \dot{\mathfrak{v}}\dddot{\mathfrak{v}} \\ \ddot{\mathfrak{v}}\mathfrak{v} & \ddot{\mathfrak{v}}\dot{\mathfrak{v}} & \ddot{\mathfrak{v}}^2 & \ddot{\mathfrak{v}}\dddot{\mathfrak{v}} \\ \dddot{\mathfrak{v}}\mathfrak{v} & \dddot{\mathfrak{v}}\dot{\mathfrak{v}} & \dddot{\mathfrak{v}}\ddot{\mathfrak{v}} & \dddot{\mathfrak{v}}^2 \end{vmatrix} \begin{vmatrix} \mathfrak{v}^2 - c^2 & \mathfrak{v}\dot{\mathfrak{v}} \\ \dot{\mathfrak{v}}\mathfrak{v} & \dot{\mathfrak{v}}^2 \end{vmatrix}}{\left(\begin{vmatrix} \mathfrak{v}^2 - c^2 & \mathfrak{v}\dot{\mathfrak{v}} & \mathfrak{v}\ddot{\mathfrak{v}} \\ \dot{\mathfrak{v}}\mathfrak{v} & \dot{\mathfrak{v}}^2 & \dot{\mathfrak{v}}\ddot{\mathfrak{v}} \\ \ddot{\mathfrak{v}}\mathfrak{v} & \ddot{\mathfrak{v}}\dot{\mathfrak{v}} & \ddot{\mathfrak{v}}^2 \end{vmatrix}\right)^2 (\mathfrak{v}^2 - c^2)}.$$

If one goes to the limit $c = \infty$ then one will note that the following limiting values exist:

$$\lim_{c=\infty} \frac{c^4}{R_1^2} = \dot{\mathfrak{v}}^2,$$

$$\lim_{c=\infty} \left(-\frac{c^2}{R_2^2}\right) = \frac{\begin{vmatrix} \dot{\mathfrak{v}}^2 & \dot{\mathfrak{v}}\ddot{\mathfrak{v}} \\ \ddot{\mathfrak{v}}\dot{\mathfrak{v}} & \ddot{\mathfrak{v}}^2 \end{vmatrix}}{(\dot{\mathfrak{v}}^2)^2},$$

$$\lim_{c=\infty} \left(-\frac{c^2}{R_3^2}\right) = \frac{\begin{vmatrix} \dot{\mathfrak{v}}^2 & \dot{\mathfrak{v}}\ddot{\mathfrak{v}} & \dot{\mathfrak{v}}\dddot{\mathfrak{v}} \\ \ddot{\mathfrak{v}}\dot{\mathfrak{v}} & \ddot{\mathfrak{v}}^2 & \ddot{\mathfrak{v}}\dddot{\mathfrak{v}} \\ \dddot{\mathfrak{v}}\dot{\mathfrak{v}} & \dddot{\mathfrak{v}}\ddot{\mathfrak{v}} & \dddot{\mathfrak{v}}^2 \end{vmatrix}}{\left(\begin{vmatrix} \dot{\mathfrak{v}}^2 & \dot{\mathfrak{v}}\ddot{\mathfrak{v}} \\ \ddot{\mathfrak{v}}\dot{\mathfrak{v}} & \ddot{\mathfrak{v}}^2 \end{vmatrix}\right)^2} \dot{\mathfrak{v}}^2.$$

However, if one considers the curve in S_3 (**Hamilton**'s velocity hodograph):

$$\mathfrak{v}_x = \mathfrak{v}_x(t), \qquad \mathfrak{v}_y = \mathfrak{v}_y(t), \qquad \mathfrak{v}_z = \mathfrak{v}_z(t),$$

and constructs the matrix $\mathfrak{D}$ from the first three differential quotients:

$$\mathfrak{D} = \begin{Vmatrix} \dot{\mathfrak{v}}^2 & \dot{\mathfrak{v}}\ddot{\mathfrak{v}} & \dot{\mathfrak{v}}\dddot{\mathfrak{v}} \\ \ddot{\mathfrak{v}}\dot{\mathfrak{v}} & \ddot{\mathfrak{v}}^2 & \ddot{\mathfrak{v}}\dddot{\mathfrak{v}} \\ \dddot{\mathfrak{v}}\dot{\mathfrak{v}} & \dddot{\mathfrak{v}}\ddot{\mathfrak{v}} & \dddot{\mathfrak{v}}^2 \end{Vmatrix},$$

which is definitive here for the calculation of the curvatures $1 / \mathfrak{R}_1$ ($1 / \mathfrak{R}_2$, resp) and the moving 3-frame, then one will find that if $\mathfrak{D}'$, $\mathfrak{D}''$, $\mathfrak{D}'''$ are once more the three principal subdeterminants of the matrix $\mathfrak{D}$ then obviously:

$$\lim_{c=\infty} \frac{c^4}{R_1^2} = \mathfrak{D}',$$

$$\lim_{c=\infty} \left(-\frac{c^2}{R_2^2}\right) = \frac{\mathfrak{D}''}{(\mathfrak{D}')^2} = \mathfrak{D}' \cdot \left(\frac{1}{\mathfrak{R}_1}\right)^2,$$

$$\lim_{c=\infty} \left(-\frac{c^2}{R_3^2}\right) = \frac{\mathfrak{D}'''\mathfrak{D}'}{(\mathfrak{D}'')^2} = \mathfrak{D}' \cdot \left(\frac{1}{\mathfrak{R}_2}\right)^2.$$

Naturally, one will find the same connection when one has $\mathfrak{v} = 0$ (so in the proper system; however, one must then substitute the first, second, third *proper* acceleration for $\dot{\mathfrak{v}}$, $\ddot{\mathfrak{v}}$, $\dddot{\mathfrak{v}}$, resp.) or when $\mathfrak{v}$ is small compared to the speed of light, such that $\mathfrak{v} / c$ can be neglected without one needing to appeal to **Newtonian** mechanics (i.e., $c = \infty$) in those two cases.

In particular, if all three of $1 / R_1$, $1 / R_2$, $1 / R_3$ are constant then $\dot{\mathfrak{v}}$, $1 / \mathfrak{R}_1$, $1 / \mathfrak{R}_2$ will be constant. The motion then proceeds with constant acceleration, which will make the **Hamilton**ian hodograph a *common helix* (corresponding to our type I and meaning free fall and uniform rotation around the line of falling) or a circle (II: uniform rotation) or a line (III: free fall) or a point (V: uniform translation). An arbitrary uniform translation can enter into all of those. Since R_1 and R_2 have unequal order in c, IV has no **Newton**ian analogue.

3. Proper systems.

A *proper system* for an arbitrarily-moving material point is obviously one whose time axis coincides with the instantaneous tangent to the worldline of a point, regardless of how the spatial axes are oriented. An observer that moves with the point has every right as a naïve realist to believe that he is at rest, as long as nothing to the contrary take place.

However, such a reference system has no physical reality. Namely, from the standpoint of physical reality, one must replace the system of spatial axes with a reference body with respect to which the first three spatial axes can be defined. Now, if the material point above is to be replaced with a material body then if one is to be able to speak of a "proper system," here as well, in the sense that an observer that is found on the reference body can believe that he is at rest, then it would be necessary for the body to be represented as being "at rest" with respect to that proper system; i.e., that the proper coordinates of each point must be constant, regardless of where the observer stands on the body. However, that is nothing but the **Born** rigid body, because one will have that: Any two of its worldlines can be related to each other in such a way that two corresponding points will remain connected by a constant segment of a line that meets each of the two curves normally (viz., equidistance). If one then introduces any system at one of those points whose time axis coincides with the tangent there then the "simultaneous" position of the second moving point will obviously be given by the aforementioned relationship, and the distance that is thus determined will then remain constant during the entire motion. For any position of the first point, if one then chooses not only the second point, but also yet a third point on a third worldline and a fourth point that lies on a worldline that is "simultaneous with the first one" then, corresponding to the demand above, one will have a proper system with three spatial axes – namely, the three directions from the first point to the other three – that obviously can be made orthogonal by a suitable choice of points and is fixed in the reference body.

That alone is what we call a proper system, insofar as the reference body in whose frame it appears rests in it. **Herglotz** ([1]) has shown that it follows from the definition of equidistance that the worldlines of the **Born** body must be "parallel curves" (**Scheffers**'s terminology) to any of them (first kind) or trajectories of a "motion" of S_4 (second kind).

As far as the **Born** body of the first kind is concerned, its proper system can be easily given ([2]), in which it will be represented by constant proper coordinates. Here, one might only say that

([1]) **G. Herglotz**, *loc. cit.*
([2]) **F. Kottler**, *loc. cit.*, § **7**.

one finds the three spatial axes when one demands that they remain *rigidly linked with the tangent* and also that they are not allowed to rotate around it. If one then refers them to the moving 4-frame – i.e., one sets their direction cosines, which might be denoted by $b_2^{(h)}$, $b_3^{(h)}$, $b_4^{(h)}$, (h = 1, 2, 3, 4), resp., equal to:

$$b_2 = \beta_2^{(2)} c_2 + \beta_2^{(3)} c_3 + \beta_2^{(4)} c_4 \qquad \text{(and analogously for } b_3 \text{ and } b_4\text{)},$$

then naturally the β cannot be constant, since, as we saw, the c_2, c_3, c_4 perform the rotations:

$$d\omega_2 \qquad (d\omega_3\text{, resp.})$$

about the tangent c_1 . Those rotations must then be *compensated* by a suitable change in the β, under which the β must experience the *opposite* orthogonal transformation ([1]) to the axes c_2, c_3, c_4.

However, we still cannot admit the **Born** body of the first kind as a reference body for an actual proper system either when the observer does not notice his own motion, as long as he does not direct his attention to external processes (**Foucault** pendulum, etc.). That is because here as well physical reality does not permit us to focus upon the simultaneous positions of the world, like **Newton** and **Galilei**. Indeed, we see with the help of light, and everything that we see will lag behind our point in time by a latency time when we do not set the speed of light equal to infinity, unlike **Newton** and **Galilei**. However, when an observer on the **Born** body of the first kind considers his non-simultaneous neighborhood, obviously his proper coordinates will be arbitrarily variable, and he must perceive his motion from that.

It is only for the **Born** body of the second kind that the proper coordinates of non-simultaneous positions are also constant, as we indeed infer from the representation in § **4**:

$$X = x + \Gamma^{(1)} c_1 + \Gamma^{(2)} c_2 + \Gamma^{(3)} c_3 + \Gamma^{(4)} c_4 \,.$$

Therefore, we can say exactly that when we also consider the latency time for the immediate neighborhood, there will be only a few accelerated motions for which an observer that does not have access to physical experiments *can* believe that he is *at rest.*

For those accelerated motions, we have further found that once equilibrium comes about during the motion, it must remain preserved for all eternity, which in fact follows from the fact that the proper coordinates of the points of the body are constant.

Of course, such a "universe," as it presents itself to the reference body, exhibits abnormal phenomena for non-equilibrium processes, such as apparent acceleration, changes in the speed of light, etc., when compared to the usual universe.

([1]) Which is naturally the contragredient reciprocal.

4. – In order to show the latter, we would like to present the arc-length of the "universe" as it presents itself to an observer that exhibits a **Born** hyperbolic falling motion. We write ([1]):

$$X = x + i\,c\,(T' - \tau)\,c_1 - Z' c_2 + X' c_3 + Y' c_4$$

and employ the **Frenet** formulas, which read:

$$(s = i\,c\,t) \quad \frac{dc_1}{ds} = \frac{1}{R_1} c_2, \quad \frac{dc_2}{ds} = -\frac{1}{R_1} c_1, \quad \frac{dc_3}{ds} = \frac{dc_4}{ds} = 0$$

here. That gives:

$$\begin{aligned} dX &= dx - ds \cdot c_1 + i\,c\,dT' \cdot c_1 + Z' \cdot \frac{ds}{R_1} c_1 \quad , \\ &\quad + i\,c\,(T' - \tau) \frac{ds}{R_1} c_2 - dZ' \cdot c_2 \quad , \\ &\quad + dX' \cdot c_3 \, , \\ &\quad + dY' \cdot c_4 \, . \end{aligned}$$

Naturally, one has:

$$dx = ds \cdot c_1$$

in this, and we would now like to set:

$$T' = \tau \, ,$$

which clearly includes an association with the aid of which we can dictate the time sequence of events as it presents itself to the observer. That gives:

$$\begin{aligned} dX = \quad & c_1 \cdot \; i\,c\,dT' \left(1 + \frac{Z'}{R_1}\right), \\ - & c_2 \cdot \; dZ', \\ & c_3 \cdot \; dX', \\ & c_4 \cdot \; dY'. \end{aligned}$$

That further gives:

$$dS^2 = (dX')^2 + (dY')^2 + (dZ')^2 - c^2 \left(1 + \frac{Z'}{R_1}\right)^2 (dT')^2 .$$

However, from Tab. 1, col. 8, one has:

$$\frac{1}{R_1} = \frac{1}{b} ,$$

([1]) Here, the assumption that $\Gamma^{(2)} = -Z'$ is made in order for the proper acceleration to point along the *positive* Z'-axis, since the **Minkowski** force $K = -m_0\,(c^2 / R_1)\,c_2$.

in which c^2 / b is the **Minkowski** proper acceleration; one will then have:

$$dS^2 = (dX')^2 + (dY')^2 + (dZ')^2 - \left(c + \frac{Z'c}{b}\right)^2 (dT')^2.$$

Hence, the speed of light that is perceived by the observer will become:

$$c' = c + \frac{Z' c_2}{b} \cdot \frac{1}{c}.$$

However, $+ Z c^2 / b$ is nothing but the "potential" of the **Minkowski** force:

$$K = - m_0 \text{ Grad } \Phi ,$$

when it is considered as an apparent force in the proper system; hence ([1]):

$$c' = c + \Phi / c .$$

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([1]) Cf., **A. Einstein**, Ann. Phys. (Leipzig) **38** (1912), 356-359.